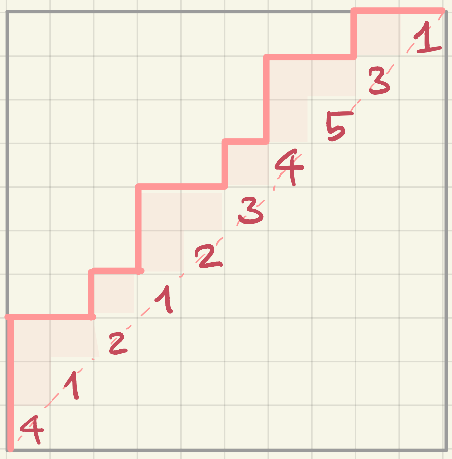
"α-chromatic symmetric functions"

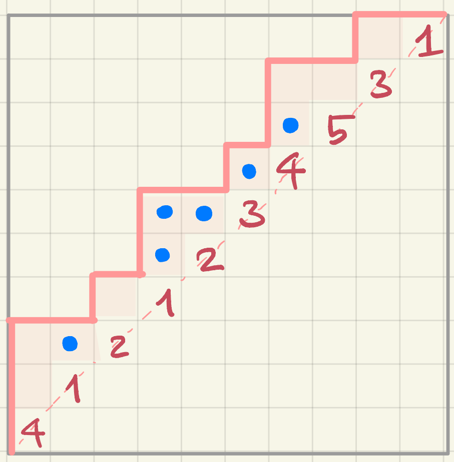
* Overview

There are two families of symmetric polynomials indexed by Dyck paths:



- The chromatic quasisymmetric function

$$X_\pi(x;q) = \sum_{\substack{w \in \mathbb{Z}^n_{>0} \\ \text{proper}}} q^{\mathrm{inv}_\pi(w)} x^w$$



$q^6 x_1^3 x_2^2 x_3^2 x_4^2 x_5$

- The unicellular LLT polynomial

$$LLT_\pi(x;q) = \sum_{w \in \mathbb{Z}^n_{>0}} q^{\mathrm{inv}_\pi(w)} x^w,$$

where

$$\mathrm{inv}_\pi(w) = \left|\{(i,j) : (i,j) \in \mathrm{Area}(\pi),\ w_i < w_j\}\right|.$$

Thm. [Carlsson – Mellit, '2018]

$$X_\pi(x;q) = \frac{LLT_\pi[(q-1)x;q]}{(q-1)^n}$$

← plethystic substitution

Def. (α-chromatic symmetric functions)

$$X^{(\alpha)}_\pi(x;q) = \frac{LLT_\pi[(q^\alpha-1)x;q]}{(q-1)^n}$$

$$= X_\pi\left[\frac{q^\alpha-1}{q-1}x;q\right].$$

For $\alpha \in \mathbb{N}$, $\frac{q^\alpha-1}{q-1} = 1+q+\cdots+q^{\alpha-1}$

Ex.

$$X^{(\alpha)}_{[2,3,3]}(x;1) = (\alpha^3-2\alpha^2+\alpha)m_3 + (3\alpha^3-2\alpha^2)m_{21} + 6\alpha^3 m_{111}$$

Not monomial positive

$$= ([\alpha]\downarrow_3 + [\alpha]\downarrow_2)m_3 + (3[\alpha]\downarrow_3 + 7[\alpha]\downarrow_2 + [\alpha]\downarrow_1)m_{21}$$
$$+ (6[\alpha]\downarrow_3 + 18[\alpha]\downarrow_2 + 6[\alpha]\downarrow_1)m_{111}$$

$$= \left(4\begin{bmatrix}\alpha\\3\end{bmatrix} + 2\begin{bmatrix}\alpha+1\\3\end{bmatrix}\right)m_3 + \left(5\begin{bmatrix}\alpha\\3\end{bmatrix} + 12\begin{bmatrix}\alpha+1\\3\end{bmatrix} + \begin{bmatrix}\alpha+2\\3\end{bmatrix}\right)m_{21}$$
$$+ \left(6\begin{bmatrix}\alpha\\3\end{bmatrix} + 24\begin{bmatrix}\alpha+1\\3\end{bmatrix} + 6\begin{bmatrix}\alpha+2\\3\end{bmatrix}\right)m_{111}$$

Observation.

① Each coeff. is a polynomial in α of degree n whose constant term is zero.

② Coefficients are in

$$\mathrm{Span}\{[\alpha]_q, [\alpha]_q^2, \ldots, [\alpha]_q^n\}$$

$$= \mathrm{Span}\left\{\begin{bmatrix}\alpha\\ n\end{bmatrix}_q, \begin{bmatrix}\alpha+1\\ n\end{bmatrix}_q, \ldots, \begin{bmatrix}\alpha+n-1\\ n\end{bmatrix}_q\right\}$$

$$= \mathrm{Span}\{[\alpha]_{q\downarrow 1}, [\alpha]_{q\downarrow 2}, \ldots, [\alpha]_{q\downarrow n}\},$$

where $[\alpha]_{q\downarrow k} = [\alpha]_q[\alpha-1]_q \cdots [\alpha-k+1]_q$.

* Motivation

- Macdonald polynomials

Thm. [Macdonald, '1988]

Given a partition λ, there exists a unique symmetric polynomial $P_\lambda(x;q,t)$ characterized by

(a) (triangularity)

$$P_\lambda(x;q,t) = m_\lambda(x) + \sum_{\mu<\lambda} \xi_{\lambda\mu}(q,t)\, m_\mu(x)$$

(b) (orthogonality)

$$\langle P_\lambda, P_\mu\rangle_{q,t} = 0 \quad \text{if } \lambda \neq \mu,$$

where $\langle P_\lambda, P_\mu\rangle_{q,t} = \delta_{\lambda\mu}\, z_\lambda \prod_i \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$.

NOTE.

- $q=t$; $P_\lambda(x;t,t) = s_\lambda$
- $t=1$; $P_\lambda(x;q,1) = m_\lambda$
- $q=1$; $P_\lambda(x;1,t) = e_{\lambda'}$
- $\lambda=(1^n)$: $P_{(1^n)}(x;q,t) = e_n(x) = s_{1^n}(x)$
- $q=0$; $P_\lambda(x;0,t) = P_\lambda(x;t)$

; the Hall-Littlewood polynomial

Macdonald also defined the integral form

$$J_\mu(X;q,t) = \prod_{c\in\mu} (1-q^{a(c)} t^{\ell(c)+1}) P_\mu(X;q,t).$$

Macdonald showed that the integral form Macdonald poly. J_μ has the following expansion in terms of $s_\lambda[X(1-t)]$:

$$J_\mu(X;q,t) = \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q,t)\, s_\lambda[X(1-t)].$$

q,t-Kostka poly. $\in \mathbb{N}[q,t]$

The modified Macdonald poly. is defined by

$$H_\mu(X;q,t) = J_\mu\left[\frac{X}{1-t};q,t\right] = \sum_{\lambda\vdash|\mu|} K_{\lambda\mu}(q,t)\, s_\lambda.$$

Macdonald's positivity ~~conjecture~~ thm (Haiman, '02)

$$K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t].$$

* Haglund conjecture

For any $\alpha\in\mathbb{N}$,

$$\left\langle \frac{J_\mu(X;q,q^\alpha)}{(1-q)^n}, s_\lambda(X) \right\rangle \in \mathbb{N}[q].$$

In fact, we conjecture

$$J_\mu(X;q,t) = \sum_{\lambda\vdash n} \left(\sum_{T\in SSYT(\lambda',\mu')} \prod_{c\in\mu} (1-t^{\ell(c)+1} q^{\mathrm{qstat}(c,T)})\, q^{\mathrm{ch}(T)} \right) s_\lambda.$$

$$= \frac{H_\mu[X(1-q^\alpha);q,q^\alpha]}{(1-q)^n}$$

Recall. Carlsson – Mellit relation:

$$\frac{LLT_\gamma[(q-1)X;q]}{(q-1)^n} = X_\gamma(X;q)$$

* Parallel universe?

$$J_\mu\left[\frac{X}{1-t};q,t\right] = H_\mu(X;q,t)$$

$$(1-q)^n X_\gamma\left[\frac{X}{q-1};q\right] = LLT_\gamma(X;q)$$

* Haglund–Wilson Conjecture

For any $\alpha \in \mathbb{N}$,

$$\left\langle \frac{J_\mu(X;q,q^{-\alpha})}{(q-1)^n} ; s_\lambda(X) \right\rangle \in \mathbb{N}[q]$$

after some q^{-*} factor taken care of.

Jaeseong's Conjecture:

$$q^{\alpha \sum_i i\mu_i} \frac{J_\mu(X;q,q^{-\alpha})}{(q-1)^n}$$ is e-positive.

Remark. We have a combinatorial formula for the e-expansion when μ is a hook shape:

Thm. (Haglund–Kim–Oh–Y, '25+)

$n, r \geq 1$, $s \geq 0$, $r+s=n$

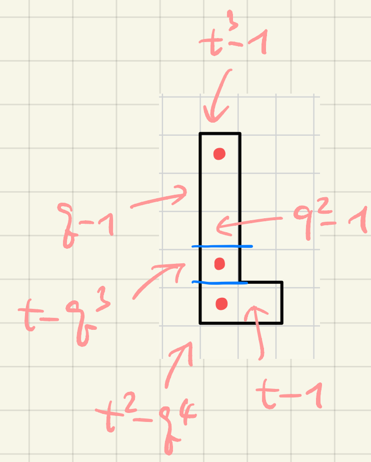
For any $\lambda \vdash n$,

$$[e_\lambda] J_{(r,1^s)}(X;q,t^{-1}) = t^{-r-1-\binom{s+1}{2}} \sum_{\alpha \sim \lambda} wt_{(r,1^s)}(\alpha),$$

where $wt_{(r,1^s)}(\alpha) = \prod_{c \in (r,1^s)'} wt_\alpha(c)$,

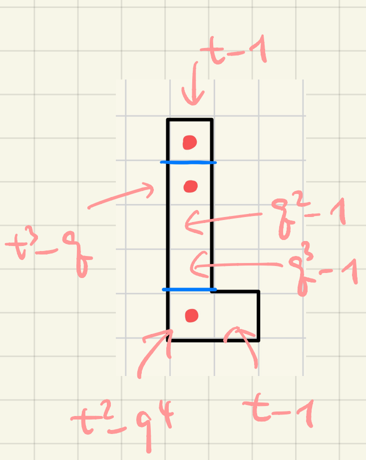
$$wt_\alpha(c) = \begin{cases} t^{\alpha_j} - q^{\ell(c)} & \text{if } c \text{ is the first cell of } \alpha_i \\ t^{a'(c)} q^{\ell(c)} - 1 & \text{otherwise.} \end{cases}$$

Example.



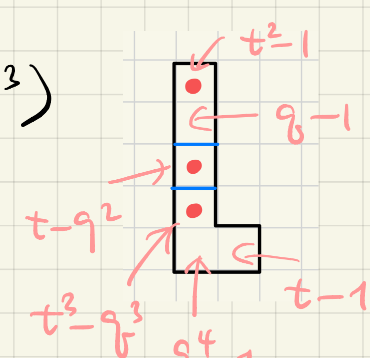
$[e_{321}] J_{51}$

$$= t^{-7}\Big((q-1)(q^2-1)(t-1)(t^3-1)(t-q^3)(t^2-q^4)$$

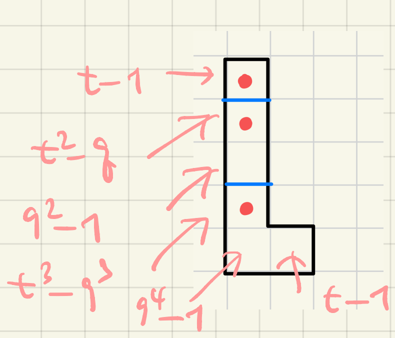


$$+ (q^2-1)(q^3-1)(t-1)^2(t^3-q)(t^2-q^4)$$

$$+ (q-1)(q^4-1)(t-1)(t^2-1)(t-q^2)(t^3-q^3)$$



$$+ (q^2-1)(q^4-1)(t-1)^2(t^2-q)(t^3-q^3) \Big)$$



- (Integral form) Jack polynomials

$$J_\mu^{(\alpha)}(x) := \lim_{t\to 1} \frac{J_\mu(x; t^\alpha, t)}{(1-t)^n}$$

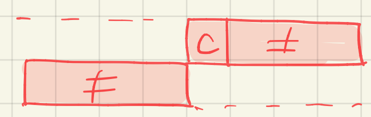
NOTE. If we let

$$\tilde{J}_\mu^{(\alpha)}(x) = \lim_{q\to 1} \frac{J_\mu(x; q, q^\alpha)}{(1-q)^n},$$

then $\tilde{J}_\mu^{(\alpha)} = \alpha^n J_\mu^{(\alpha^{-1})}(x)$.

Thm. (Knop-Sahi, '1997)

$$J_\mu^{(\alpha)}(x) = \sum_{T \in \underbrace{NAF(\mu)}_{\text{nonattacking fillings}}} d_T(\alpha)\, x^T,$$



where

$$d_T(\alpha) = \prod_{\substack{c\in\mu \text{ s.t.} \\ T(c) = T(\mathrm{south}(c))}} (\alpha(a(c)+1) + \ell(c)+1).$$

NOTE. The Schur coeff. are NOT positive.

- Alexandersson-Haglund-Wang Conjecture ('2018)

$$\langle \tilde{J}_\mu^{(\alpha)}(x), s_\lambda \rangle = \sum_{k=0}^{n-1} a_k(\mu,\lambda) \binom{\alpha+k}{n}$$

$$= \sum_{k=1}^{n} b_{n-k}(\mu,\lambda) \underbrace{\binom{\alpha}{k} k!}_{=\alpha\downarrow^k}$$

with $a_k(\mu,\lambda) \in \mathbb{N}$ and $b_{n-k}(\mu,\lambda) \in \mathbb{N}$.

* Coming up next

Recall. $$X_{\pi}^{(\alpha)}(x;q) = \frac{LLT_{\pi}[(q^{\alpha}-1)X;q]}{(q-1)^n}$$

- We apply
  - superization technique
  - XY-technique

  to deal with the plethysm on the right hand side.

- We prove the Schur positivity of $X_{\pi}^{(\alpha)}(x;q)$ in $\left\{\begin{bmatrix}\alpha+k\\n\end{bmatrix}_q\right\}_{0\le k\le n-1}$ and $\left\{\begin{bmatrix}\alpha\\k\end{bmatrix}_q\right\}_{1\le k\le n}$ bases.

  $\Leftarrow$ close relation to the "Rook theory".

- We describe a combinatorial formula for monomial coeff. in $\begin{bmatrix}\alpha+k\\n\end{bmatrix}$-expansion,

  $\alpha\downarrow_k$-expansion when $q=1$.

* Other properties

- p-expansion

* p-expansion

$$\chi_\pi(x;q) = \frac{LLT_\pi[(q-1)x;q]}{(q-1)^n}$$; chromatic quasi-sym.

$$\Rightarrow \chi_\pi^{(\alpha)}(x;q) = \chi_\pi\left[\frac{(q^\alpha-1)x}{q-1};q\right]$$

Say $\chi_\pi(x;q) = \sum_\lambda z_\lambda^{-1} p_\lambda \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_q \, c_\lambda(q)$,

Note that this is already known.

Ref) S-W.

Proposition 7.8. *Let P be a natural unit interval order on $[n]$ with incomparability graph G. Then for all $\mu \in \mathrm{Par}(n)$,*

$$\sum_{\sigma \in \mathcal{N}_{P,\mu}} t^{\mathrm{inv}_G(\sigma)} = \prod_{i=1}^{l(\mu)} [\mu_i]_t \sum_{\sigma \in \widetilde{\mathcal{N}}_{P,\mu}} t^{\mathrm{inv}_G(\sigma)}. \tag{7.13}$$

Then,

$$\chi_\pi^{(\alpha)}(x;q) = \sum_\lambda z_\lambda^{-1} p_\lambda\left[\frac{(q^\alpha-1)x}{q-1}\right] \cdot \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_q \cdot c_\lambda(q)$$

$$= \sum_\lambda z_\lambda^{-1} \cdot \prod_{i=1}^{\ell(\lambda)} \left(\frac{q^{\alpha\cdot\lambda_i}-1}{q^{\lambda_i}-1}\right) \cdot p_\lambda \cdot \prod_{i=1}^{\ell(\lambda)} [\lambda_i]_q \cdot c_\lambda(q)$$

$$= \sum_\lambda z_\lambda^{-1} \cdot \prod_{i=1}^{\ell(\lambda)} \frac{[\alpha\lambda_i]_q}{\cancel{[\lambda_i]_q}} \cdot \prod_{i=1}^{\ell(\lambda)} \cancel{[\lambda_i]_q} \cdot p_\lambda \cdot c_\lambda(q)$$

$$= \boxed{\sum_\lambda z_\lambda^{-1} \prod_{i=1}^{\ell(\lambda)} [\alpha\lambda_i]_q \cdot p_\lambda \cdot c_\lambda(q).}$$

* Symmetry

8

Prop. (HoY, '25+)

For a Dyck path π, if we let

$$X_\pi^{(\alpha)}(x;q) = \sum_{k=0}^{n-1} \sum_{\lambda \vdash n} c_{\pi,\lambda,k}(q) \begin{bmatrix} \alpha+k \\ n \end{bmatrix}_q s_\lambda(x).$$

then

$$c_{\pi,\lambda,k}(q) = q^{\mathrm{area}(\pi)+\binom{n}{2}} c_{\pi,\lambda',n-1-k}(q^{-1}).$$

(pf) Blasiak et al. ('23) proved that

$$\omega(LLT_\pi(x;q)) = q^{\mathrm{area}(\pi)} LLT_\pi(x;q^{-1}).$$

Taking the plethystic substitution of $X \mapsto (q^\alpha-1)X$ on both sides

$$\omega\, LLT_\pi[(q^\alpha-1)X;q] = q^{\mathrm{area}(\pi)} LLT_\pi[(q^\alpha-1)X;q^{-1}]$$
$$= q^{\mathrm{area}(\pi)} LLT_\pi[((q^{-1})^{-\alpha}-1)X;q^{-1}].$$

By the Carlsson-Mellit relation,

$$\omega X_\pi^{(\alpha)}(x;q) = \frac{q^{\mathrm{area}(\pi)} LLT_\pi[((q^{-1})^{-\alpha}-1)X;q^{-1}]}{(-q)^n(q^{-1}-1)^n}$$
$$= (-1)^n q^{\mathrm{area}(\pi)-n} X_\pi^{(-\alpha)}(x;q^{-1}).$$

If we let

$$X_\pi^{(\alpha)}(x;q) = \sum_{k=0}^{n-1} \sum_{\lambda \vdash n} c_{\pi,\lambda,k}(q) \begin{bmatrix} \alpha+k \\ n \end{bmatrix}_q s_\lambda(x),$$

by using the fact that

$$\begin{bmatrix} -\alpha+k \\ n \end{bmatrix}_{q^{-1}} = (-1)^n q^{\binom{n+1}{2}} \begin{bmatrix} \alpha+n-1-k \\ n \end{bmatrix}_q,$$

$$c_{\pi,\lambda,k}(q) = q^{\mathrm{area}(\pi)+\binom{n}{2}} c_{\pi,\lambda',n-1-k}(q^{-1}).$$

$\square$