HW2 - Math 581 - Spring 2024

1. Recall that for a set $T \subseteq \{1, 2, \ldots, n-1\}$, the fundamental quasisymmetric function $F_T(X)$ is defined as

$$F_T(X) = \sum_{\substack{1 \le a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \implies i \notin T}} x_{a_1} \cdots x_{a_n}.$$

Show that for $\alpha \in \mathbb{N}$,

$$F_T(1, q, \ldots q^{\alpha-1}) = q^{\text{pow}} \begin{bmatrix} \alpha + n - 1 - |T| \\ n \end{bmatrix}_q,$$

for a certain nonnegative integer pow.

2. Given a Dyck path γ and the corresponding graph on n vertices G_γ consisting of edges below the path and above the diagonal, we say an *acyclic orientation* of G_γ is a way of assigning a direction to each edge of G_γ so the resulting directed graph has no cycles. Shareshian and Wachs have shown that if $\mathcal{X}_\gamma(X; q) = \sum_\lambda c_\lambda e_\lambda(X)$, where $c_\lambda \in \mathbb{Q}[q]$, then for any $1 \le k \le n$,

$$\sum_{\substack{\lambda \\ \ell(\lambda)=k}} c_\lambda = \sum_{\text{acyclic orientations } \phi \text{ of } G_\gamma \text{ with } k \text{ sinks}} q^{\text{asc}(\phi)},$$

where asc is the number of directed edges (a, b) (from vertex a to vertex b) of ϕ where $a < b$. Show that acyclic orientations of G_γ with k sinks are in bijection with permutations $\sigma = \sigma_1 \cdots \sigma_n$ with no P-descents (i.e, no values of i for which $\sigma_i >_P \sigma_{i+1}$, with P the poset of relations corresponding to squares above the path γ) and k left-to-right P-maxima (i.e, k values of i for which $\sigma_i >_P \sigma_j$ for all $1 \le j < i$). Under your bijection, what does the statistic $\text{asc}(\phi)$ translate to in terms of σ?

3. Show that the family of posets which are $3+1$ and $2+2$ free is the same as the family of posets whose relations are given by the sqaures above a Dyck path γ.

4. As mentioned in class,

$$\omega \mathcal{X}_\gamma(X; q) = \sum_{\sigma \in S_n} q^{\text{inv}_\gamma(\sigma^{-1})} F_{\text{Des}_P(\sigma)}(X),$$

where $\mathrm{Des}_P(\sigma)$ is the set of all i for which $\sigma_i >_P \sigma_{i+1}$. You get a second, closely related F-expansion for $\mathcal{X}_\gamma(X;q)$ by using the theory of P-partitions and looking at acyclic orientations (as explained in class on 3/21 and 4/2). Now given the known expression we proved in class expressing the Schur expansion of $\mathcal{X}_\gamma(X;q)$ in terms of P-tableaux, by replacing each Schur function by its well-known expression as a sum of fundamentals we get a third F-expansion for $\mathcal{X}_\gamma$. Can you find a way of showing any of the first two F-expansions are the same as the third? For example, you might try starting with one of the first two, viewed as a sum over proper colorings of G_γ, and obtain the third by inventing some kind of RSK-like insertion algorithm. I should mention that this idea has been looked at by several researchers in this area and nobody has found such an insertion procedure, so the question is really more of a research problem than a homework problem.

5. Do exercise 6.19, p. 103 of my q,t-Catalan book.