HW 1 - Math 581 - Spring 2024

1. In this problem we work with superwords, which are words in a positive alphabet $\mathcal{A}_+ = \{1, 2, 3, \ldots\}$ and a negative alphabet $\mathcal{A}_- = \{\overline{1}, \overline{2}, \ldots\}$. When standardizing a superword, equal negative letters standardize in the reverse way that equal positive letters do. For example, $\overline{1}\,\overline{1}\,2\,2$ standardizes to 4312 assuming the ordering $1 < 2 < \cdots < n < \overline{n} < \overline{n} - 1 \cdots < \overline{1}$ while it standardizes to 2134 assuming the ordering $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$. For a given composition α of n, let $S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, n\}$. Furthermore, let

$$\tilde{F}_\alpha(X, Y) = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \in \mathcal{A}_- \implies i \in S(\alpha) \\ a_i = a_{i+1} \in \mathcal{A}_+ \implies i \notin S(\alpha)}} \prod_{a_i \in \mathcal{A}_+} x_{a_i} \prod_{a_i \in \mathcal{A}_-} y_{|a_i|}, \tag{1}$$

be the super Gessel fundamental quasisymmetric function, where $|\overline{i}| = i$. Prove that for any ordering of the alphabets and permutation $\beta \in S_n$,

$$\tilde{F}_{\mathrm{runs}(\beta^{-1})}(X, Y) = \sum_{\substack{\text{superwords } \sigma \text{ in } n \text{ letters} \\ \sigma_i \in \mathcal{A}_\pm, \mathrm{stan}(\sigma) = \beta}} \prod_{\sigma_i \in \mathcal{A}_+} x_{\sigma_i} \prod_{\sigma_i \in \mathcal{A}_-} y_{|\sigma_i|}, \tag{2}$$

where $\mathrm{stan}(\sigma)$ is the standardization of σ. Here for any $\tau \in S_n$, $\mathrm{runs}(\tau)$ is the composition obtained by drawing a bar at each descent of τ, and then taking the number of entries between each bar, so for example $\mathrm{runs}(23178465) = (2, 3, 2, 1)$.

2. Show how RSK can be used to prove the "dual Pieri rule", which states that

$$e_k s_\lambda = \sum_{\substack{\beta \\ \beta/\lambda \text{ is a vertical } k\text{-strip}}} s_\beta$$

3. Prove the identity

$$(1 - \sum_{n \ge 1} p_n t^n)^{-1} = \frac{\sum_{n \ge 0} h_n t^n}{1 - \sum_{n \ge 1} (n-1) h_n t^n}$$

(this is Exercise 7.5 in Stanleys "Enumerative Combinatorics" Vol.2).

4. Do exercises 1.18 (p. 12) and 6.12 (p. 100) from my q,t-Catalan book.