Representation theory of S_n: a terse summary

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Introduction and Mashke's theorem

All vector spaces, V, have field of scalars $\mathbb{C}$ and are finite dimensional and $GL(V)$ denotes invertible $\mathbb{C}$-linear maps from V to V.

A **representation** of G of degree (or dimension) n is a homomorphism $\rho : G \to GL(V)$, where V is an n-dimensional $\mathbb{C}$-space. That is

- $\rho(e) = \mathrm{id}_V$, identity maps to identity
- $\rho(gh) = \rho(g)\rho(h)$, group multiplication turns into matrix multuplication

We can think of ρ as describing a particular way in which G *acts on* V by linear transformations. For this reason, we sometimes write $G \curvearrowright V$ in place of $\rho : G \to GL(V)$ and just say $g \cdot v$ in place of $\rho(g)v$.

A G-invariant subspace $W \subseteq V$, is a vector subspace such that $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. The **subrepresentation** of $\rho : G \to GL(V)$ restricted to G-invariant W is the representation $\rho|_W : G \to GL(W)$ obtained by restricting $\rho(g) \in GL(V)$ to only act on W. An **irreducible representations** (also called **irrep**) is a representation, $\rho : G \to GL(V)$, whose only G-invariant subspaces are 0 and V.

A **direct sum** of two representations $\sigma : G \to GL(W)$ and $\pi : G \to GL(U)$ is defined as $\sigma \oplus \pi : G \to GL(W \oplus U)$ obtained in the natural way by taking the direct sum of matrices:

$$(\sigma \oplus \pi)(g) = \begin{bmatrix} \sigma(g) & \\ & \pi(g) \end{bmatrix}.$$

Two representations $\sigma : G \to GL(V)$, $\rho : G \to GL(W)$ are **isomorphic** (dentoed $\rho \cong \sigma$) if there is a bijective linear map $A : V \to W$, such that $A\,\sigma(g)\,A^{-1} = \rho(g)$ for all $g \in G$. In other words the representations are the same upon a change of basis.

Proposition 1 (Mashke's theorem)**.** *Take any finite group G and finite dimensional V, and any representation $\rho : G \to GL(V)$. Then there are pairwise non-isomorphic irreducible representations, $\sigma_1, \cdots, \sigma_k$ and $c_1, \cdots, c_k \in \mathbb{N}_+$ such that*

$$\rho \cong \sigma_1^{\oplus c_1} \oplus \sigma_2^{\oplus c_2} \oplus \cdots \oplus \sigma_k^{\oplus c_k}.$$

Furthermore, this ***decomposition into irreps****, is unique up to isomorphism and reordering of the σ_i.*

First, we need the following lemma

Lemma 1. *If W is a G-invariant $\rho : G \to GL(V)$, then there exists a "G-compliment", a G-invariant $U \subseteq V$, such that $V = W \oplus U$. In this case $\rho \cong \rho|_W \oplus \rho|_U$.*

Proof. Take some inner product on V, $\langle \cdot, \cdot \rangle'$ and replace it with a new one by defining

$$\langle v, w \rangle = \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle'$$

so that

$$\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle,$$

in other words the inner product is G-invariant. Given G-invariant subspace $W \subseteq V$, one can take its orthogonal complement

$$U = W^{\perp} = \{v \in V : \langle v, w \rangle = 0, \ \forall w \in W\},$$

with respect to $\langle , \rangle$, so that $V = W \oplus U$. We claim that U is also G-invariant and this holds since if $u \in U$, then $g \cdot u$ is orthogonal to all $w \in W$ since

$$\langle g \cdot u, w \rangle = \langle g \cdot u, g(g^{-1}w) \rangle = \langle u, g^{-1}w \rangle = 0,$$

since $g^{-1}w \in W$ by G-invariance of W. The isomorphism $\rho \cong \rho|_W \oplus \rho|_U$ just comes from the (internal) isormorphism of vector spaces $V = W \oplus U$. $\square$

Here are two ways Lemma 1 can fail when G is not finite, or we are not working over $\mathbb{C}$. Let $G = (\mathbb{R}, +)$, an infinite group, and consider

$$\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

which is a representation since you can check that $\rho(a + b) = \rho(a)\rho(b)$ and maps the additive identity to the identity matrix. Now

$$W = \text{Span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a G-invariant subspace since $\rho(a)\binom{1}{0} = \binom{1}{0} \in W$ but has no G-compliment. Similarly, if we are working over a finite field $\mathbb{F}_p$, then $G = (\mathbb{F}_p, +)$ has similar representation

$$\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

which exploits the necessary fact that $\rho(p \cdot 1) = \rho(1 + \cdots + 1) = \rho(1)^p = \text{id}_2$. Again the span of $\binom{1}{0}$ has no G-compliment.

(Proof of Mashke's theorem). The idea is that if $\rho : G \to GL(V)$ is irreducible than we are done. Otherwise, it has some G-invariant subspace $W \neq 0, V$ and by the lemma $\rho \cong \sigma \oplus \pi$ for some subrepresentations σ, π both of which have dimension less than $\dim(V)$. Then by applying the argument to σ, π we keep going until $\rho \cong \sigma_1 \oplus \cdots \oplus \sigma_l$, where all σ_i are irreps (can't be broken further as a direct sums since they have no non-trivial subrepresentations). At this point once can group up the σ_i that are isomorphic. We will see the uniqueness later by using character theory. $\square$

Characters and class functions

We saw that all representations break down as a sum of irreducible ones and so it makes sense to understand what the irreps are. They are the fundamental building blocks of representations.

A **class function** is a function $\alpha : G \to \mathbb{C}$, such that agrees on conjugacy classes: $\alpha(g) = \alpha(hgh^{-1})$ for all $g, h \in G$. The set of class functions forms a vector space, which we will call $\mathcal{C}(G)$. Let $c = \dim(\mathcal{C}(G))$ and observe that c is the number of conjugacy classes of G.

An important family of class functions are **characters**. The character of representation $\rho : G \to GL(V)$ is the class function obtained by taking trace $\chi_\rho : G \to GL(V)$, $\chi_\rho(g) := \text{tr}(\rho(g))$. Notice that if $\rho \cong \sigma \oplus \pi$, then $\chi_\rho(g) = \chi_\sigma(g) + \chi_\pi(g)$. So if $\rho \cong \sigma_1^{\oplus c_1} \oplus \sigma_2^{\oplus c_2} \oplus \cdots \oplus \sigma_k^{\oplus c_k}$ is a decomposition into irreps, then

$$\chi_\rho = c_1 \, \chi_{\sigma_1} + \cdots + c_k \, \chi_{\sigma_k}.$$

In other words, all characters are a sum of characters of irreps. Hence, to understand characters it suffices to study characters of irreps. Also notice that *isomorphic representations have the same character*, since $\text{tr}(A\rho(g)A^{-1}) = \text{tr}(\rho(g))$.

The columns of the character table have dot product 0 and this follows from orthogonal rows implies orthogonal columns.

The following is one of the many useful values related to a representation you can extract from looking at the character.

Lemma 2 (Fixed point formula). *Let $\rho : G \to GL(V)$ be a representation and let*

$$V^G := \{v \in V : \rho(g)v = v, \forall g \in G\},$$

the set of vectors fixed by all elements of G. Then

$$\dim_{\mathbb{C}}(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \frac{1}{|G|} \sum_{g \in G} tr(\rho(g))$$

Proof. Define $p : V \to V$, $p = \frac{1}{|G|} \sum_{g \in G} \rho(g)$. Observe that for $v \in V^G$, $pv = v$ so it acts as the identity on V^G. Furthermore, for any $v \in V$, $pv \in V^G$, since it is fixed by all $g \in G$. Thus, p is a projection matrix onto V^G. From linear algebra we have that $\text{tr}(p) = \dim(\text{im}\,(p)) = \dim(V^G)$. ☐

Corollary 1 (Burnside's lemma). *Let G be a group acting on a set X. Then*

$$\textit{number of orbits} = \frac{1}{|G|} \sum_{g \in G} \textit{number of pts fixed by } g$$

.

Proof. Consider the permutation representation and fix an element $g \in G$. If G is a group acting on a set X, then there is a homomorphism $\varphi : G \to Aut(X)$. If $\{e_x\}$ is basis for a vector space V, then G acts by $g.e_x = e_{g.x}$ Hence the matrix $M = \rho(g)$ has entries $[M]_{ii} = 1$ if $g.x_i = x_i$. In other words, the trace of $M = \{x \in X \mid gx = x \forall g \in G\}$ = the number of fixed points. On ther other hand, the number of orbits is equal to the dimension of the fixed vectors. ☐

List of some important representation constructions and corresponding characters. Let $\rho_1 : G \to V_1$,$\rho_2 : G \to GL(V_2)$ be two representations of G:

- **Contragradient (dual)**: Recall from lin alg, that a vector space V has a dual V^*, the set of linear maps from $l : V \to \mathbb{C}$. If G acts on V via ρ, then this induces an action on V^*, called $\rho^* : G \to GL(V^*)$:

$$(\rho^*(g)l)(h) = l(g^{-1}h)$$

  and the associated character is

$$\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$$

- **Direct sums**: $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$
- **(Internal) tensor product**: Let $\rho_1 \otimes \rho_2 : G \to GL(V_1 \otimes V_2)$ be the representation defined as $\rho_1(g) \otimes \rho_2(g)$, where here $\otimes$ is the Kronecker product of matrices. Then the trace of the this product is $\text{tr}(\rho_1(g))\text{tr}(\rho_2(g)) = \chi_{\rho_1}(g)\chi_{\rho_2}(g)$

- **Hom**: Consider $\text{Hom}_{\mathbb{C}}(V_1, V_2)$, the set of linear maps from V_1 to V_2 and make G act on it in the following way for $A : V_1 \to V_2$
$$g \cdot A := \rho_2(g) A \rho_1(g^{-1}).$$
From linear algebra, we know that $\text{Hom}_{\mathbb{C}}(V_1, V_2) \cong V_1^* \otimes V_2$. It turns out that the above is also an isomorphism of G representations based on how we defined representations $\otimes$, * and Hom. Thus,
$$\chi_{\text{Hom}_{\mathbb{C}}(V_1,V_2)}(g) = \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g)$$

The vector space $\mathcal{C}(G)$ admits an inner product known as the **scalar product**, defined as

$$\langle \alpha, \beta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\beta(g^{-1}).$$

Notice $\langle \alpha, \beta \rangle_G = \langle \beta, \alpha \rangle_G$ It has the following interpretation. For characters of representations, we sometimes abbreviate

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G = [\rho_1, \rho_2]$$

Lemma 3. *Let*

$$Hom_G(V_1, V_2) = \{A \in Hom_{\mathbb{C}}(V_1, V_2) : A(g \cdot v) = g \cdot (Av)\},$$

called the intertwiners of representations V_1, V_2. Then

$$\dim(Hom_G(V_1, V_2)) = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G$$

Proof. The important realization is that $\text{Hom}_G(V_1, V_2) =$
$textHom_{\mathbb{C}}(V_1, V_2)^G$, the set of linear maps from V_1 to V_2 that are fixed under the G action as defined in the above list, since

$$A = \rho_2(g) A \rho_1(g^{-1})$$

implies that $A\rho_1(g) = \rho_2(g)A$ and so $A(\rho_1(g)v) = (\rho_2(g)A)v$. Hence by the fixed point formula

$$\dim(\text{Hom}_G(V_1, V_2)) = \frac{1}{|G|} \sum_g \chi_{\text{Hom}_{\mathbb{C}}(V_1,V_2)}(g) = \frac{1}{|G|} \sum_g \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g) = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$$

□

Remark 1. Every element of a finite group has finite order, i.e. for all $g \in G$, there is a $k > 0$ so that $g^k = e$. Hence, for any representation, ρ, $\rho(g)^k = e$. From linear algebra this means that $\rho(g)$ is a diagonalizable matrix over $\mathbb{C}$ and its eigenvalues are roots of unity, $e^{2\pi i t/k}$ for $t = 0, 1, \ldots, k-1$. Hence $\chi_\rho(g)$ is a sum of roots of unity. In particular, if $e^{2\pi i t/k}$ is an eigenvalue of $\rho(g)$ then it corresponds to $e^{-2\pi i t/k} = \overline{e^{2\pi i t/k}}$ being an eigenvalue of $\rho(g^{-1})$. This implies that

$$\chi_{\rho^*}(g) = \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$$

and so

$$\langle \chi_\rho, \chi_\pi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g)\overline{\chi_\pi(g)}.$$

This essentially agrees with the standard inner product over $\mathbb{C}$ with the exception of the normalizing $\frac{1}{|G|}$ factor.

Proposition 2 (Orthonormality of irrep characters)**.** *If $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are non-isomorphic irreducible representations, then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G = 0$. If they are isomorphic irreps, then $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G = 1$. In other words, the irrep characters form an **orthonormal set** in $\mathcal{C}(G)$.*

Proof. In the non-isomorphic case this is the same as showing that $\mathrm{Hom}_G(V_1, V_2) = \{0\}$, the set containing only the zero map from V_1 to V_2. Suppose for contradiction $f \in \mathrm{Hom}_G(V_1, V_2)$ is non-zero. Since f is G-invariant, one has that the $\ker(f) \subseteq V_1$ is a G-invariant subspace. Since $G \curvearrowright V_1$ is irreducible, $\ker(f) = 0, V_1$ and it cannot be V_1 since we assume $f \neq 0$. Hence f is injective. Similarly $\mathrm{im}\,(f)V_2$ is a G-invariant subspace and since $G \curvearrowright V_2$ is irreducible, $\mathrm{im}\,(f) = 0, V_2$ and since $f \neq 0$, f is surjective. Hence f is a G-invariant bijective linear map from V_1 to V_2 making it an isomorphism, contradicting that ρ_1, ρ_2 are non-isomorphic.

For the second part we have to show that $\mathrm{Hom}_G(V_1, V_2)$ is 1-dimensional if V_1 and V_2 are isomorphic. Since they are isomorphic, this is the same as showing $\mathrm{Hom}_G(V_1, V_1)$ is 1-dimensional. Let $f \in \mathrm{Hom}_G(V_1, V_1)$ and let λ be one of its eigenvalues (with corresponding eigenvector v). Then $f - \lambda \cdot I : V_1 \to V_1$ is also a G-invariant map. Since V_1 is irreducible, we know that $\ker(f - \lambda I) = 0, V_1$ and it cannot 0, since $v \in \ker(f - \lambda I)$ by definition of eigenvector. Hence, $f - \lambda I = 0$ and $f = \lambda I$ and

$$\mathrm{Hom}_G(V_1, V_1) = \mathrm{Span}_{\mathbb{C}}(I),$$

scalar multiples of the identity map (a 1-dimensional space). ☐

We can now complete the proof of Mashke's theorem Let ρ be a representation of G, that decomposes into irreps

$$\rho \cong \sigma_1^{\oplus c_1} \oplus \sigma_2^{\oplus c_2} \oplus \cdots \oplus \sigma_k^{\oplus c_k},$$

then up to isomorphism and reoordering this decomposition is unique, with

$$c_i = [\rho, \sigma_i] = \langle \chi_\rho, \chi_{\sigma i} \rangle_G = \frac{1}{|G|} \sum_g \chi_\rho(g) \overline{\chi_{\sigma_i}(g)}.$$

Additionally, two representations are isomorphic, if and only if, their characters are the same function.

Proof. We have $\chi_\rho = \sum_i c_i \chi_{\sigma_i}$. Then

$$\langle \chi_\rho, \chi_{\sigma_j} \rangle = \sum_i c_i \langle \chi_{\sigma_i}, \chi_{\sigma_j} \rangle = c_j,$$

by orthonormality. For the second part, we already saw that isomorphic representations have the same character. Conversely, if two representations, ρ_1, ρ_2, have the same characters, then for each irreducible $[\rho_1, \sigma_i] = [\rho_2, \sigma_i]$ and hence we can construct an isomorphism between them by making the map restrict to isomorphisms between their irreducibles. ☐

Let $\Pi(G)$ denote the set of characters of irreps.

Example: The regular representation. The regular representation of G, is defined by considering the vector space with G as a basis (you can picture $\{v_g : g \in G\}$ as being the basis vectors), and G acting on left by multiplication, so $h \cdot v_g = v_{hg}$. Let's compute the character of this representation, call it ρ. Since $\rho(e)$ is the identity matrix, $\chi_\rho(e)$ is the dimension of the representation, so $\chi_\rho(e) = |G|$. For any $g \neq e$, observe that $gv_h = v_{gh} \neq v_h$ and so $\mathrm{tr}(\rho(g)) = 0$. Hence

$$\chi_\rho(g) = \begin{cases} |G|, & g = e \\ 0, & \text{else} \end{cases}.$$

Now for the decomposition into irreducible, if $\chi_\sigma \in \Pi(G)$ is an irrep character, then

$$[\rho, \sigma] = \frac{1}{|G|} \sum_g \chi_\sigma(g) \overline{\chi_\rho(g)} = \frac{|G|}{|G|} \chi_\sigma(e),$$

which is just the dimension of irrep σ, $\dim(\sigma)$. Hence

$$\rho \cong \bigoplus_{\sigma \in \Pi(G)} \sigma^{\oplus \dim(\sigma)},$$

i.e. each irrep occurs in ρ with multiplicity being its dimension.

Let $\Pi(G)$ denote the set of characters of irreps. We saw that it forms an orthonormal set in $\mathcal{C}(G)$ with resepect to $\langle,\rangle_G$. Now we will show that they in fact form a basis (we just need to show that they span the set of class functions, $\mathcal{C}(G)$).

Proposition 3. *The set $\Pi(G)$ forms a basis for $\mathcal{C}(G)$. In other words, the number of irrep characters is precisely $\dim(\mathcal{C}(G))$ equal to the number of conjugacy classes of the group.*

1 Induced representations and restrictions

Suppose H is a subgroup of G, $H \leq G$, and V is a representation of G, the restriction $\mathrm{Res}_H^G(V)$ is quite straightforward. It is just V again now being a representation of H (we are restricting $\rho : G \to GL(V)$ to $\rho|_H : H \to GL(V)$).

Is there a natural way to go in the other direction? Say you start with a representation of H, W (via $\rho : H \to GL(W)$), and you want to extend it to G. You can't undo restriction since information is always lost when restricting but maybe there is a "most natural" way of defining how G acts on W. This is called the **induced representation**, $\mathrm{Ind}_H^G(W)$ or

$$\mathrm{Ind}_H^G(\rho) : G \to GL(W^{\oplus[G:H]}),$$

which is now a representation of $[G : H] = |G|/|H|$ copies of W. We will give an explicit matrix based construction without getting into why this is the most natural one. For each $g \in G$ we must assign a $\dim(W) \cdot \frac{|G|}{|H|}$ by $\dim(W) \cdot \frac{|G|}{|H|}$ matrix. We give this as a block matrix of $\left(\frac{|G|}{|H|}\right)^2$ blocks (each block being in $\dim(W)$ by $\dim(W)$):

$$\mathrm{Ind}_H^G(\rho)(g) := [\rho(x^{-1}\, g\, y)]_{xH, yH \in G/H},$$

where xH, yH are cosets (the matrix depends on which coset representatives x, y you pick but one can change basis so the definition is unique up to representation isomorphism) and we are setting $\rho(\hat{g}) = 0$ to be the zero block matrix if $\hat{g} \notin H$ (ρ only makes sense for elements of H). As a consequence the corresponding characters is

$$\chi_{\mathrm{Ind}_H^G(\rho)}(g) = \sum_{xH \in G/H} \chi_\rho(x^{-1} g x),$$

where now the definition really doesn't depend on choice of coset representative and again we are setting $\chi_\rho(\hat{g}) = 0$ whenever $\hat{g} \notin H$. Since this definition is linear in χ_ρ, we may extend it to class functions.

Example: if 1_H is the 1-dimensional trivial representation of H, then $\mathrm{Ind}_H^G(1_H)$ ends up yielding a bunch of permutation matrices. In particular, the permutation matrix $\mathrm{Ind}_H^G(1_H)(g)$ captures the way g permutes the cosets $\{xH \in G/H\}$ (recall these are $|G/H|$ by $|G/H|$ matrices). (This example turns out be very closely related to the h_λ symmetric functions as well as Polya enumeration). In particular, **transitive group actions are precisely induced trivial representations**.

A special case of this is that the regular representation is equal to $\mathrm{Ind}_e^G(1_e)$, where e is the trivial subgroup of G.

One super important result involving induced representations is **Frobenius reciprocity**.

Proposition 4. *Frobenius reciprocity Let $H \leq G$ and $\rho : G \to GL(V)$ and $\pi : H \to GL(W)$ be representations. Then the characters satisfy*

$$\langle Ind_H^G(\chi_\pi), \chi_\rho \rangle_G = \langle \chi_\pi, Res_H^G(\chi_\rho) \rangle_H$$

Proof.

$$\begin{aligned}
\langle \mathrm{Ind}_H^G(\chi_\pi), \chi_\rho \rangle_G &= \frac{1}{|G|} \sum_{g\in G} \mathrm{Ind}_H^G(\chi_\pi)(g) \cdot \overline{\chi_\rho(g)} \\
&= \frac{1}{|G|} \sum_{g\in G} \sum_{xH\in G/H} \chi_\pi(x^{-1}gx) \cdot \overline{\chi_\rho(x^{-1}gx)}, \qquad \text{since } \chi_\rho \text{ is a } G\text{-character} \\
&= \frac{1}{|G|} \sum_{xH\in G/H} \sum_{\substack{g\in G \\ x^{-1}gx\in H}} \chi_\pi(x^{-1}gx) \cdot \overline{\chi_\rho(x^{-1}gx)} \\
&= \frac{1}{|G|} \sum_{h\in H} \sum_{\substack{g\in G, xH\in G/H \\ x^{-1}gx=h}} \chi_\pi(h) \cdot \overline{\chi_\rho(h)} \\
&= \frac{[G:H]}{|G|} \sum_{h\in H} \chi_\pi(h) \cdot \overline{\chi_\rho(h)} \\
&= \frac{1}{|H|} \sum_{h\in H} \chi_\pi(h) \cdot \overline{\chi_\rho(h)} \\
&= \langle \chi_\pi, \mathrm{Res}_H^G(\chi_\rho) \rangle_H,
\end{aligned}$$

where the number of ways to pick $xH \in G/H$ and $g \in G$ to hit given $h = x^{-1}gx$ is $[G : H]$ since each distinct xH gives a different $g = xhx^{-1}$. ☐

Example: Consider k-element subsets of $[n]$, $\binom{[n]}{k}$ with the natural S_n-action. Since this is a transitive group action, with stabilizer of $[k]$ being $S_{[k]} \times S_{[k+1,n]}$, we have

$$\binom{[n]}{k} \cong \mathrm{Ind}_{S_{[k]}\times S_{[k+1,n]}}^{S_n}(1).$$

2 Symmetric group

We now restrict attention to the symmetric group S_n. The conjugacy classes are determined by cycle type, a non-increasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ with parts $\lambda_i > 0$ denoting the lengths of the disjoint cycles of an element of the conjugacy class. Hence $\lambda_1 + \cdots + \lambda_l = n$. We call $\lambda \vdash n$ a partition of n. The goal is ultimately to determine the irreps corresponding to these λ.

Let's compute the size of the conjugacy classes. Let $c_t(\lambda)$ be the number of parts λ_i equal to t. So for $\lambda = (3, 3, 2)$, $c_1(\lambda) = 0$, $c_2(\lambda) = 1$ and $c_3(\lambda) = 2$. We claim that centralizer of a $\sigma \in S_n$ with cycle type λ is

$$z_\lambda := \left(\prod_i \lambda_i\right)\left(\prod_t c_t(\lambda)!\right).$$

Then by orbit-centralizer theorem the number of permutations with cycle type λ is $|S_n|/z_\lambda = n!/z_\lambda$. Denote this conjugacy class C_λ. We will write $\lambda = sh(\sigma)$ to say that λ is the shape or cycle type of σ

2.1 Class functions of symmetric group as a ring

Let $\mathcal{C}(S_n)$ be the space of class functions $f : S_n \to \mathbb{C}$, which have a basis given by the irreducible representation characters of S_n, $\chi^\lambda : S_n \to \mathbb{C}$, where $\lambda \vdash n$ is a partition (there should be as many characters as conjugacy classes, so there is one irreducible representation character for each partition). If $f \in \mathcal{C}(G_1)$ and $g \in \mathcal{C}(G_2)$, we may define

$$f \boxtimes g \in \mathcal{C}(G_1 \times G_2),\ f \boxtimes g(x_1, x_2) = f(x_1)g(x_2),$$

a pointwise product. If the two functions are characters $f = \chi^V$, $g = \chi^W$, then $f \otimes g = \chi^{V\otimes W}$ a character of $G_1 \times G_2$-representation.

Let $\mathcal{C}(S) = \oplus_n \mathcal{C}(S_n)$, an infinite dimensional $\mathbb{C}$-space. We can give it a ring structure by defining multiplication: for $f \in \mathcal{C}(S_n)$ and $g \in \mathcal{C}(S_m)$, define

$$f * g := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(f \boxtimes g)) \in \mathcal{C}(S_{n+m}).$$

Here $S_n \times S_m$ is viewed as a subgroup of S_{n+m}, via sending (σ, τ) to a permutation that makes σ act on $1, \cdots, n$ and τ act on $n+1, \cdots, n+m$. This also turns $\mathcal{C}(S)$ into a graded ring. This is sometimes called the Grothendieck ring of the tower of symmetric groups, $(\mathcal{C}(S), *, +)$. Let's determine what the ring actually is.

Proposition 5. *Let $\mathbb{C}[T_1, T_2, \cdots]$ be the ring of polynomials in indeterminants $T_1, T_2, \cdots$, with $\deg(T_n) := n$. Then*

$$\mathcal{C}(S) \cong \mathbb{C}[T_1, T_2, \cdots]$$

as graded rings, where the isomorphism, call it Φ, maps 1_{C_λ} (the indicator of the conjugacy class C_λ) to $\frac{1}{z_\lambda} T_{\lambda_1} \cdots T_{\lambda_l}$.

Let $T_\lambda := T_{\lambda_1} \cdots T_{\lambda_l}$. Then we can also write this isomorphism as

$$\Phi(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) T_{sh(\sigma)} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} f(\sigma) T_\lambda \quad \textit{any } \sigma \in C_\lambda.$$

The map Φ is a degree preserving isomorphism.

Proof. The map is bijective since it maps a basis for $\mathcal{C}(S_n)$ to a basis for degree n polynomials in $T_1, T_2, \ldots$ We see that it is degree preserving since $\deg(T_\lambda) = \lambda_1 + \lambda_2 + \cdots = n$ for $\lambda \vdash n$.

Let $\psi^{(n)} : S_n \to \mathbb{C}[T_1, T_2, \cdots]$ map σ to $T_{sh(\sigma)}$. This is a "$\mathbb{C}[T_1, T_2, \cdots]$"-valued class function. If $\sigma \in S_n$ and $\tau \in S_m$ and γ is the corresponding permutation in S_{n+m}, then $\psi^{(n+m)}(\gamma) = T_{sh(\sigma\tau)} = T_{sh(\sigma)} T_{sh(\tau)} = \psi^{(n)}(\sigma)\psi^{(m)}(\tau)$. We can interpret this as saying $\mathrm{Res}_{S_n \times S_m}^{S_{n+m}}(\psi^{(n+m)}) = \psi^{(n)} \boxtimes \psi^{(m)}$.

Then we have

$$\Phi(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) T_{sh(\sigma)} = \langle f, \psi^{(n)} \rangle_{S_n}.$$

Using Frobenius reciprocity:

$$\begin{aligned}
\Phi(f * g) = \Phi\left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(f \boxtimes g)\right) &= \langle \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(f \boxtimes g), \psi^{(n+m)} \rangle_{S_{n+m}} \\
&= \langle f \boxtimes g, \mathrm{Res}_{S_n \times S_m}^{S_{n+m}}(\psi^{(n+m)}) \rangle_{S_n \times S_m} \\
&= \langle f, \psi^{(n)} \rangle_{S_n} \langle g, \psi^{(m)} \rangle_{S_m} \\
&= \Phi(f)\Phi(g)
\end{aligned}$$

showing that Φ is an isomorphism. ☐

2.2 Symmetric polynomials and Frobenius map

It turns out there is a more combo-friendly way of viewing $\mathbb{C}[T_1, T_2, \cdots]$ that involves using the ring of symmetric functions. **Symmetric polynomials** are polynomials, $f(x_1, \cdots, x_k)$ invariant under re-ordering the variables: $f(x_1, \cdots, x_k) = f(x_{\sigma(1)}, \cdots x_{\sigma(k)})$ for all $\sigma \in S_k$. **Symmetric functions** are a generalization of this to an infinite variable set $f(x_1, x_2, \cdots)$. Let $\Lambda^{(n)}$ be the set of homogeneous degree n symmetric functions and Λ the set of all symmetric functions.

One simple basis for $\Lambda^{(n)}$ is the monomial basis, indexed by partitions $\lambda \vdash n$. Essentially, one takes the symmetric function with the least number of terms containing the monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$:

$$m_\lambda(x_1, x_2, \cdots) := \sum_{\substack{i_1 < \cdots < i_l \\ \alpha \in \mathcal{R}(\lambda)}} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

where $\mathcal{R}(\lambda) = \{(\lambda_{\sigma(1)}, \cdots \lambda_{\sigma(l)}) : \sigma \in S_l\}$ the set of rearrangements of parts in λ.

When expanding a symmetric function in the monomial basis

$$f(x_1, \ldots) = \sum_{\lambda \vdash n} a_\lambda m_\lambda(x_1, \ldots)$$

we may compute the coefficient a_λ, by simply reading off the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$, which is why this *sometimes* the easiest basis to work with.

Another basis is the power sum basis defined as follows. The power sum symmetric functions are

$$p_k(x_1, x_2, \cdots) := x_1^k + x_2^k + \cdots$$

which are used to define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}.$$

It turns out that $\{p_\lambda : \lambda \vdash n\}$ form a basis for $\Lambda^{(n)}$.

Notice then that

$$\dim(\Lambda^{(n)}) = \#\text{partitions of } n = \#\text{conj classes of } S_n = \#\text{irreps of } S_n$$

It turns out that since the p_k are algebraically independent, $\Lambda \cong \mathbb{C}[T_1, T_2 \cdots]$, by sending p_k to T_k and notice that $\deg(p_k) = k = \deg(T_k)$. This motivates the definition of the **Frobenius map**, which is also degree-preserving ring isomorphism:

$$\mathcal{F} : \mathcal{C}(S) \to \Lambda$$

that maps $f \in \mathcal{C}(S_n)$ to

$$\mathcal{F}(f) := \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \cdot p_{sh(\sigma)}.$$

[Fundamental property of the Frobenius map] Let $f \in \mathcal{C}(S_n)$ and $g \in \mathcal{C}(S_m)$, then

$$\mathcal{F}(f * g) = \mathcal{F}(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (f \boxtimes g)) = \mathcal{F}(f)\mathcal{F}(g)$$

Examples: Let's consider the trivial and sign representations. The character of triv_{S_n} is 1 on every conjugacy class and so

$$\mathcal{F}(\mathrm{triv}_{S_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} p_{sh(\sigma)} = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$$

Let $h_n(x_1, x_2, \cdots) := \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$, which has the following simpler description

$$h_n(x_1, x_2, \cdots) = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n},$$

the sum containing every possible degree n monomial exactly once. This is called the **complete homogeneous symmetric function**. We can define $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_l}$ and it turns out h_λ is yet another basis for $\Lambda^{(n)}$ as λ ranges over partitions of n. As a representation we can think of

$$\mathcal{F}^{-1}(h_\lambda) = \mathrm{Ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}}^{S_n} (\mathrm{triv}_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}}).$$

In the case of the sign representation, notice that if $\lambda = sh(\sigma)$, for $\sigma \in S_n$, then the sign determined by the number of even length cycles, $\text{sgn}(\sigma) = (-1)^{n-l(\lambda)}$, where $l(\lambda)$ is the number of parts in λ. Then

$$\mathcal{F}(\text{sgn}_{S_n}) = \frac{1}{n!}\sum_{\sigma \in S_n} \text{sgn}(\sigma) p_{sh(\sigma)} = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} \frac{p_\lambda}{z_\lambda}.$$

Let $e_n(x_1, x_2, \cdots) := \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} \frac{p_\lambda}{z_\lambda}$, which has the following simpler description

$$e_n(x_1, x_2, \cdots) = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} \cdots x_{i_n},$$

and is called the **elementary symmetric function**. Again the $e_\lambda := e_{\lambda_1} \cdots e_{\lambda_l}$ form a basis for $\Lambda^{(n)}$ and the corresponding representation is

$$\mathcal{F}^{-1}(e_\lambda) = \text{Ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}}^{S_n} (\text{sgn}_{S_{\lambda_1}} \boxtimes \cdots \boxtimes \text{sgn}_{S_{\lambda_l}}).$$

Exercise: Show that $\mathcal{F}(\chi^{\text{reg}}) = h_{(1^n)}$, the image of the regular representation. Also notice that $h_{(1^n)} = e_{(1^n)} = p_{(1^n)}$ (you can prove justifying why all three $\mathcal{F}^{-1}$ of these symmetric functions give the same character, χ^{reg}).

2.3 Transferring the inner product

We have the graded ring isomorphism $\mathcal{F} : \mathcal{C}(S) \to \Lambda$ and an inner product on $\mathcal{C}(S)$, $\langle f, g \rangle_{S_n}$ which can be extended to $\mathcal{C}(S)$ by making characters from different S_n orthogonal. But we have no inner product on Λ on symmetric functions. So let's just carry it over. We know that

$$\langle 1_{C_\lambda}, 1_{C_\mu} \rangle_{S_n} = \begin{cases} \frac{|C_\lambda|}{n!} = \frac{1}{z_\lambda}, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}$$

so this induces the following inner product on Λ

$$\langle \frac{p_\lambda}{z_\lambda}, \frac{p_\mu}{z_\mu} \rangle = \langle 1_{C_\lambda}, 1_{C_\mu} \rangle_{S_n},$$

or in other words $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}$. Since the p_λ form a basis this extends to an inner product on symmetric functions.

Here are two identities involving this new inner product

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$$

and

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu},$$

which is expected since we claimed that $\mathcal{F}(s_\lambda) = \text{Irr}^\lambda$ and we know that irrep characters are orthonormal. We can go through the proofs of these two identities using the Cauchy kernel method.

Here is a clutch example application of this inner product to figuring out the irrep decomposition of $\mathcal{F}^{-1}(h_\lambda) = H^\lambda = \text{Ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}}^{S_n} (\text{triv}_{S_{\lambda_1} \times \cdots \times S_{\lambda_l}})$. Let

$$h_\mu = \sum_{\lambda \vdash n} A_{\lambda,\mu} s_\lambda,$$

for some unknown $A_{\lambda,\mu}$ (they are unique since the s_λ form a basis). Then take the inner product of both sides with s_ν to get

$$\langle h_\mu, s_\nu \rangle = \sum_{\lambda \vdash n} A_{\lambda,\mu} \delta_{\lambda,\nu} = A_{\nu,\mu}.$$

On the other hand,

$$\langle h_\mu, s_\nu\rangle = \langle h_\mu, \sum_{\gamma\vdash n} K_{\nu,\gamma} m_\gamma\rangle = K_{\nu,\mu}.$$

Therefore,

$$h_\mu = \sum_{\lambda\vdash n} K_{\lambda,\mu} s_\lambda,$$

and we have gotten the expansion into irreducible:

$$H^\mu \cong \bigoplus_{\lambda\vdash n} (V^\lambda)^{K_{\lambda,\mu}}.$$

Proof of important lemma $\langle h^\lambda, e^{\lambda'}\rangle = 1$. Assuming we have used the Cauchy kernel method to show $\langle h_\lambda, m_\mu\rangle$, we have enough tools to show that $\langle h_\lambda, e_{\lambda'}\rangle = 1$, which implies $[H^\lambda, E^{\lambda'}] = 1$ completing the proofs that the Irr^λ defined in the previous section are irreducible.

Expanding e_κ in the monomial basis gives

$$e_\kappa = \sum_{\mu\vdash n} B_{\kappa,\mu} m_\mu,$$

where $B_{\kappa,\mu}$ is the number of matrices containing 0's and 1's with row sums $\kappa_1, \kappa_2, \ldots$ and column sums $\mu_1, \mu_2, \mu_3 \ldots$. For example if $\kappa = (3,2,1,1)$ and $\mu = (4,1,1,1)$, then

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

are the three matrices contributing to $B_{\kappa,\mu} = 3$. These matrices essentially dictate how to build the monomial $x^{\mu_1} x^{\mu_2} \cdots$ out of $e_{\kappa_1} e_{\kappa_2} \cdots$. So for the first matrix, the first row says: "pick $x_1^1 x_2^1 x_3^0 x_4^1 = x_1 x_2 x_4$ from e_3" the second row says "pick $x_1 x_3$ from e_2", and then the last two rows say "pick x_1 from e_1". Together the products yields $x_1^4 x_2 x_3 x_4$ contributing to m_μ.

Then

$$\langle h_\lambda, e_{\lambda'}\rangle = \sum_{\mu\vdash n} B_{\lambda',\mu} \delta_{\lambda,\mu} = B_{\lambda',\lambda}.$$

By staring at the boolean matrices for a bit one can see that there is a unique matrix with row sums λ' and column sums λ, with the 1's essentially giving the shape of λ. If for example $\lambda = (3,2)$ and $\lambda' = (2,2,1)$, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is the only matrix contributing to $B_{\lambda',\lambda}$. Hence $B_{\lambda',\lambda} = 1$. □

Proposition 6. *If V is an S_n representation with character χ^V, then $\dim(V)$ is the coefficient of $x_1 x_2 \cdots x_n$ in $\mathcal{F}(\chi^V)$.*

Proof. Recall that if χ^V is the character of a representation then $\dim(V) = \chi^V(e)$. In Irr^n, the identity has cycle type (1^n) and since

$$\mathcal{F}(\chi^V) = \sum_{\lambda\vdash n} \text{“}\chi^V(\lambda)\text{”} \frac{p_\lambda}{z_\lambda},$$

where "$\chi^V(\lambda)$" means evaluation at any σ of cycle type λ. We have that

$$\dim(V) = \chi^V(e) = \langle \mathcal{F}(\chi^V), p_{(1^n)}\rangle,$$

but since $p_{(1^n)} = h_{(1^n)}$, this is the same as $\langle \mathcal{F}(\chi^V), h_{(1^n)} \rangle$. Expanding $\mathcal{F}(\chi^V)$ in the monomial basis, and using the fact that $\langle h_\mu, m_\lambda \rangle = \delta_{\lambda,\mu}$. We have that this inner product is the coefficient of $m_{(1^n)}$ in the monomial expansion. Then this follows by the definition of the monomial basis. □

Notice that the coefficient of $x_1 x_2 \cdots x_n$ in s_λ is $\#\mathrm{SSYT}(\lambda, (1^n)) = \#\mathrm{SYT}(\lambda)$, which agrees with what we said the dimension of the irrep Irr^λ.

Proposition 7. *The monomial coefficient of x^μ in $\mathcal{F}(V)$ is equal to the* $\dim(V^{S_\mu})$, *the S_μ-invariants, where $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_l}$.*

Proof.

$$\begin{aligned} \mathcal{F}(V)_{x^\mu} &= [m_\mu]\mathcal{F}(V) \quad \text{expanded in monomial basis} \\ &= \langle \mathcal{F}(V), h_\mu \rangle \\ &= [V, \mathrm{Ind}_{S_\mu}^{S_n}(1)] \\ &= [V|_{S_\mu}, 1_{S_\mu}] \\ &= \dim(V^{S^\mu}) \end{aligned}$$

□

2.4 Cauchy Kernel method

We say a pair of basis of symmetric functions, $\{u_\lambda\}, \{v_\lambda\}$ are dual if $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$. Let $X = x_1, x_2, \ldots$ and $Y = y_1, y_2, \ldots$ denote two variable sets.

Proposition 8 (Cauchy Kernel method)**.** *Consider the dual basis $\{p_\lambda\}$, $\{p_\mu / z_\mu\}$, where indeed $\langle p_\lambda, p_\mu / z_\mu \rangle = \delta_{\lambda,\mu}$ and define*

$$\Omega_n(X, Y) := \sum_{\lambda \vdash n} p_\lambda(X) \frac{p_\lambda(Y)}{z_\lambda}.$$

Then any other pair of basis $\{u_\lambda\}$, $\{v_\lambda\}$ are dual if and only if

$$\sum_{\lambda \vdash n} u_\lambda(X) v_\lambda(Y) = \Omega_n(X, Y).$$

Note that $\Omega_n(X, Y) = \Omega_n(Y, X)$.

Proof. We will show this through the following claim: for all $f \in \Lambda^{(n)}$

$$\langle \Omega_n(X, Y), f(X) \rangle_{\Lambda_X^{(n)}} = f(Y),$$

where we indicate that the inner product is for symmetric functions in variables X (it sees y_i as scalars). In other words, taking the inner product with the Cauchy kernel performs a *variable substitution.*

Writing $f = \sum_{\lambda \vdash n} a_\lambda p_\lambda$, we have

$$\begin{aligned} \langle \Omega_n(X, Y), f(X) \rangle_{\Lambda_X^{(n)}} &= \sum_{\lambda, \mu \vdash n} p_\lambda(Y) \left\langle \frac{p_\lambda(X)}{z_\lambda}, a_\mu p_\mu(X) \right\rangle \\ &= \sum_{\lambda, \mu \vdash n} p_\lambda(Y) a_\mu \delta_{\lambda,\mu} \\ &= \sum_{\lambda \vdash n} a_\lambda p_\lambda(Y) = f(Y). \end{aligned}$$

Hence if $\sum_{\lambda \vdash n} u_\lambda(X)v_\lambda(Y) = \Omega_n(X,Y)$, then

$$\begin{aligned} v_\lambda(Y) &= \langle v_\lambda(X), \Omega_n(X,Y)\rangle \\ &= \sum_{\mu \vdash n} \langle v_\lambda(X), u_\mu(X)\rangle v_\mu(Y) \end{aligned}$$

and since the v_λ forms a basis, comparing coefficients above shows that $\langle v_\lambda(X), u_\mu(X)\rangle = \delta_{\lambda,\mu}$.

Conversely, $\langle v_\lambda(X), u_\mu(X)\rangle = \delta_{\lambda,\mu}$, then using spanning property of basis we have

$$\Omega_n(X,Y) = \sum_\lambda f_\lambda(Y)u_\lambda(X),$$

for some $f_\lambda(Y) \in \Lambda_Y^{(n)}$. Then taking the $\langle \cdot, v_\mu(X)\rangle$ we see that $f_\lambda(Y) = v_\mu(Y)$ completing the proof. $\square$

What is the Cauchy kernel? Well we can define the variable set $XY = \{x_iy_j : \ i,j = 1,2,\ldots\}$. Notice that

$$p_k(X)p_k(Y) = \left(\sum_i x_i^k\right)\left(\sum_j y_j^k\right) = \sum_{i,j}(x_iy_j)^k = p_k(XY)$$

and so $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$ implies that $p_\lambda(X)p_\lambda(Y) = p_\lambda(XY)$. Then

$$\Omega_n(X,Y) = \sum_{\lambda \vdash n} p_\lambda(X)p_\lambda(Y)/z_\lambda = \sum_{\lambda \vdash n} p_\lambda(XY)/z_\lambda = h_n(XY).$$

Thus showing that $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda,\mu}$ and $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}$ is equivalent to showing

$$\sum_{\lambda \vdash n} h_\lambda(X)m_\mu(Y) = h_n(XY) = \sum_{\lambda \vdash n} s_\lambda(X)s_\lambda(Y).$$

2.5 The irreps of S_n via representations of polynomials

Consider the polynomial ring $\mathbb{C}[y_1, \cdots, y_n]$, which forms an infinite dimensional S_n representation via the action $\sigma \cdot y_i = y_{\sigma(i)}$. Recall the Vandermonde determinant

$$\Delta(t_1, \cdots, t_k) = \det(t_i^{j-1})_{i,j=1,\cdots,k} = \prod_{i<j}(t_i - t_j).$$

Now consider the S_n-submodule $\mathrm{Span}_{\mathbb{C}}(\Delta(y_1, \cdots, y_n)) \leq \mathbb{C}[y_1, \cdots, y_n]$, which is closed under the S_n action since

$$\sigma \cdot \Delta(y_1, \cdots, y_n) = \mathrm{sgn}(\sigma)\Delta(y_1, \cdots, y_n)$$

so this is a 1-dimensional representation and in fact the sign representation of S_n.

An *injective tableau* of the diagram corresponding to λ is a filling of the cells of the box diagram for λ with the numbers $1, \ldots, n$ all appearing once. For example if $\lambda = (4,3,1)$, one could take

$T =$

<table>
<tr><td>5</td><td></td><td></td><td></td></tr>
<tr><td>4</td><td>3</td><td>1</td><td></td></tr>
<tr><td>6</td><td>2</td><td>7</td><td>8</td></tr>
</table>

We will say $\mathrm{Inj}(\lambda)$ is the set of injective tableau of λ. Let Δ_T called **Garnir polynomials** be the product of Vandermonde determinants of variables corresponding to columns of this injective tableau:

$$\begin{aligned} \Delta_T &= \Delta(y_6, y_4, y_5)\Delta(y_2, y_3)\Delta(y_7, y_1)\Delta(y_8) \\ &= (y_6 - y_4)(y_6 - y_5)(y_4 - y_5)(y_2 - y_3)(y_7 - y_1), \end{aligned}$$

where $\Delta(y_8) = 1$ so columns of height 1 contribute nothing.

Here is the definition for the irreducible representation of S_n corresponding to partition λ:

$$\mathrm{Irr}^\lambda := \mathrm{Span}_{\mathbb{C}}(\Delta_T(y_1, \dots, y_n) : T \in \mathrm{Inj}(\lambda)).$$

We can discuss on Friday how this is just a different way of picturing the construction in Sagan section 2.3.

So the example from earlier is

$$\mathrm{Irr}^{(1,1,\cdots,1)} = \mathrm{Span}_{\mathbb{C}}(\Delta(y_1, \dots, y_n)),$$

since all the Δ_T for $T \in \mathrm{Inj}(\lambda)$ are linearly dependent (they are all $\pm$ each other).

Note that because Irr^λ ends up being irreducible you can pick any one Δ_T and just take the span of $\sigma\Delta_T$ ranging over $\sigma \in S_n$. You can also define an action of S_n on $\mathrm{Inj}(\lambda)$ directly by letting $\sigma \cdot T$ be the tableau with the cell containing i being replaced by $\sigma(i)$ for $i = 1, \dots, n$ so for example

$$(1\ 3\ 4) \cdot \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-3} 2 & 5 & 7 \\ \hline 1 & 4 & 6 & 8 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 4 \\ \cline{1-3} 2 & 5 & 7 \\ \hline 3 & 1 & 6 & 8 \\ \hline \end{array}$$

and observe that $\sigma \cdot \Delta_T = \Delta_{\sigma \cdot T}$.

As was mentioned for the sign representation, the Δ_T are generally not linearly independent. One can rearrange the entries of a column to get a new tableau $\hat{T}$, but $\Delta_{\hat{T}} = \pm\Delta_T$. One can also swap two columns of equal size to get the same Δ_T.

Proposition 9. *A basis for* Irr^λ *is given by*

$$\{\Delta_T : T \in SYT(\lambda)\}$$

Proof. First, we show that the Δ_T span Irr^λ. Recall that standard Young tableau are tableau with entries increasing down the rows and up columns. Let column strict tableau be tableau that are increasing allong columns and will be denoted by $\mathrm{CS}(\lambda)$. By the earlier observation Irr^λ is spanned by Δ_T for $T \in CS(\lambda)$. A crucial step in the proof is establishing the **Garnir relation**.

Suppose that a $T \in \mathrm{CS}(\lambda)$ is not row strict and so it contains a pair of adjacent values $a > b$ in some row. Let C denote the column values in a's column and C', the column values in b's column. Let $A \subset C$ be the values including and above A and let $B \subset C'$ be the value including and below b. We will abuse notation and let $\binom{A \cup B}{|A|}$ denote the set of permutation of $A \cup B$ that leave the values ending up in A and B in order (this set does in fact have cardinality $\binom{|A|+|B|}{|A|}$)). Then the Garnir relation says that

$$\sum_{\pi \in \binom{A \cup B}{|A|}} \mathrm{sgn}(\pi)\pi\Delta_T = 0.$$

For example if

$$T = \begin{array}{|c|c|c|} \hline 5 \\ \cline{1-2} 4 & 2 \\ \hline 3 & 1 & 6 \\ \hline \end{array}$$

with $a = 4$ and $b = 2$, then $A = \{4, 5\}$, $B = \{1, 2\}$ and the possible π consiting of id, (42), (124), (542), (5412), $(52)(41)$ corresponding to

$$\Delta \begin{array}{|c|c|c|} \hline 5 \\ \cline{1-2} 4 & 2 \\ \hline 3 & 1 & 6 \\ \hline \end{array} - \Delta \begin{array}{|c|c|c|} \hline 5 \\ \cline{1-2} 2 & 4 \\ \hline 3 & 1 & 6 \\ \hline \end{array} + \Delta \begin{array}{|c|c|c|} \hline 5 \\ \cline{1-2} 1 & 4 \\ \hline 3 & 2 & 6 \\ \hline \end{array} + \Delta \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-2} 2 & 5 \\ \hline 3 & 1 & 6 \\ \hline \end{array} - \Delta \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-2} 1 & 5 \\ \hline 3 & 2 & 6 \\ \hline \end{array} + \Delta \begin{array}{|c|c|c|} \hline 2 \\ \cline{1-2} 1 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} = 0$$

The proof of this relation is as follows. Without loss of generality we can forget about columns outside of C, C' since these can be factored out of the relation and are not affected by any π and so it suffices to show

$$\sum_{\pi \in \binom{A\cup B}{|A|}} \operatorname{sgn}(\pi)\pi\Delta_C\Delta_{C'} = 0.$$

The above sum expands as

$$\sum_{\pi \in \binom{A\cup B}{|A|}} \operatorname{sgn}(\pi)\pi \sum_{\substack{\sigma\in S_C\\ \sigma'\in S_{C'}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma')x^{\sigma(C)}x^{\sigma'(C')},$$

where if $C = \{c_1 < c_2 < \cdots\}$, we let $x^{\sigma(C)} = x^0_{\sigma(c_1)}x^1_{\sigma(c_2)}\cdots$. Each $\sigma \in S_c$ factors uniquely as $\sigma_A\bar{\sigma}_A$, where $\sigma_A \in S_A$ and $\bar{\sigma}_A \in S_C$ maintaining the relative order of the values in A. Applying these factorizations we get

$$\sum_{\substack{\pi \in \binom{A\cup B}{|A|}\\ \sigma_A\in S_A\\ \sigma_B\in S_B}} \operatorname{sgn}(\pi\sigma_A\sigma_B)\pi\sigma_A\sigma_B \sum_{\bar{\sigma}_A,\bar{\sigma}_B} \operatorname{sgn}(\bar{\sigma}_A\bar{\sigma}_B)x^{\bar{\sigma}_A C}x^{\bar{\sigma}_B C'},$$

which simplifies as

$$\sum_{\tau\in S_{A\cup B}} \operatorname{sgn}(\tau)\tau \sum_{\bar{\sigma}_A,\bar{\sigma}_B} \operatorname{sgn}(\bar{\sigma}_A\bar{\sigma}_B)x^{\bar{\sigma}_A C}x^{\bar{\sigma}_B C'}$$

and as it turns out $\sum_{\tau\in S_{A\cup B}} \operatorname{sgn}(\tau)\tau$ kills each monomial $x^{\bar{\sigma}_A C}x^{\bar{\sigma}_B C'}$. Indeed by Pigeonhole principal, $|A\cup B| = |C|+1$ but there are only $|C|$ possible exponents for the variables occuring in monomial $x^{\bar{\sigma}_A C}x^{\bar{\sigma}_B C'}$. Hence there is a pair $i, j \in A \cup B$, with the same degree

$$x^{\bar{\sigma}_A C}x^{\bar{\sigma}_B C'} = \cdots x_i^k x_j^k \cdots$$

but then the for each $\tau \in S_{A\cup B}$, we have $\tau(i\ j) \in S_{A\cup B}$ and a sign-reversing involution shows $\sum_{\tau\in S_{A\cup B}} \operatorname{sgn}(\tau)\tau$ kills the monomial.

The use of the Garnir relation is that it removes the row defect $a > b$ (all other tableaux in the relation don't have this row strictness failure in this location and none of them pick up new row defects). By repeatedly using this relation we decrease the number of row defects. Thus, any Δ_T for $T \in CS(\lambda)$ is a linear combination of $\Delta_{T'}$ for $T' \in SYT(\lambda)$.

The linear independence of the Δ_T for $T \in \mathrm{SYT}(\lambda)$ is fairly straightforward, since Δ_T is the only element in the proposed basis that has $x^T = x^{C_1}\cdots x^{C_l}$ as a monomial. ☐

As a corollary we immediately get

$$\dim(\mathrm{Irr}^\lambda) = |\mathrm{SYT}(\lambda)|,$$

the number of standard Young tableau.

Exercise: Check that $\mathrm{Irr}^{(n)}$ is the trivial representation and check that $\mathrm{Irr}^{(n-1,1)}$ is the standard irreducible representation $\{a_1e_1 + \cdots + a_ne_n : a_1 + \cdots + a_n = 0\}$.

More generally you can consider the "hook shape": $\lambda = (n-k, 1^k)$ (here 1^k means 1 k-times: $(n-k, 1, 1, \ldots)$. This is a standard way of writing a repeating part of a partition). What are the $T \in \mathrm{SYT}(\lambda)$ and basis elements Δ_T in this case? What is the dimension?

(Tricky): Letting $W = \langle e_1, \ldots, e_n\rangle$ and S_n acting on the e_i by permuting, justify why for $k = 1, \ldots, n-1$

$$\wedge^k W \cong \mathrm{Irr}^{(n-k,1^k)} \oplus \mathrm{Irr}^{(n-k+1,1^{k-1})},$$

are isomorphic S_n representations, where the left hand side is the k-fold wedge product (spanned by $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with the usual wedge product rules, so like $(1\ 2)\cdot e_1 \wedge e_2 = -e_2 \wedge e_1$). Notice how we already saw the special case $k=1$: $W \cong \mathrm{Irr}^{(n-1,1)} \oplus \mathrm{Irr}^{(n)}$ for the standard representation. By taking dimensions of both sides, the above is the "representation-ification" of the binomial identity: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

One approach that may or may not help is that

$$\wedge^k W \cong \mathrm{Ind}_{S_{n-k}\times S_k}^{S_n}(\mathrm{triv}_{S_{n-k}} \boxtimes \mathrm{sgn}_{S_k}).$$

2.6 Proof of irreducibility

Here is the main idea for showing why the Irr^λ precisely range through the irreducible representations of S_n. Irr^λ can be viewed as a submodule of

$$\mathrm{Ind}_{S_{\lambda_1}\times\cdots\times S_{\lambda_l}}^{S_n}(\mathrm{triv}_{S_{\lambda_1}\times\cdots\times S_{\lambda_l}}).$$

Consider the monomials that show up by expanding Δ_T for some T. A monomial that shows up is of the form $\pm y_1^{a_1}\cdots y_n^{a_n}$, where form some injective tableau T, i appears in row a_i+1. Call this monomial $m(T)$ and let H^λ be the span of such monomials

$$H^\lambda = \mathrm{Span}_{\mathbb{C}}(y_1^{a_1}\cdots y_n^{a_n} : \; T \in \mathrm{Inj}(\lambda), \; i \text{ in row } a_i+1)$$

So for

$$T = \begin{array}{|c|ccc}\hline 4 & \multicolumn{1}{|c}{} & & \\ \hline 2 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{7} \\ \hline 3 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

$m(T) = y_2y_5y_7y_4^2$. Note that $m(T)$ may arise from $\Delta_{T'}$ for a different T' (in particular $m(T)$ shows up in the expansion of $\Delta_{T'}$ if and only if there is a permutation only within rows so that $\sigma T' = T$). Also note that you might get the same $m(T)$ for different T. In particular $m(T) = m(T')$ if and only if there is a σ permuating only within rows of T such that $\sigma T = T'$. This essentially shows that

$$H^\lambda \cong \mathrm{Ind}_{S_{\lambda_1}\times\cdots\times S_{\lambda_l}}^{S_n}(\mathrm{triv}_{S_{\lambda_1}\times\cdots\times S_{\lambda_l}}).$$

and we have $\mathrm{Irr}^\lambda \subseteq H^\lambda$.

See page 20 of Sagan for the another way of thinking about the module H^λ via λ-tabloids (page 55-56). Exercise: Show that $H^{(1^n)}$ is the regular representation of S_n (example 2.1.7 in Sagan).

It also turns out that Irr^λ can be viewed as a submodule of

$$E^{\lambda'} = \mathrm{Ind}_{S_{\lambda'_1}\times\cdots\times S_{\lambda'_l}}^{S_n}(\mathrm{sgn}_{S_{\lambda'_1}} \boxtimes \cdots \boxtimes \mathrm{sgn}_{S_{\lambda'_l}}),$$

where λ' is the conjugate partition of λ. We prove this using character theory starting with the following lemma.

Lemma 4. *If $S_{\{1,\dots,n\}}$ acts on a submodule, V of $\mathbb{C}[y_1,\dots,y_n]$ and $S_{\{n+1,\dots,m\}}$ act on a submodule W of $\mathbb{C}[y_{n+1},\dots,y_m]$ and now let*

$$V.W := \mathit{Span}_{\mathbb{C}}(\sigma\cdot(fg) : \; \sigma \in S_{n+m}, \; f \in V, \; g \in W),$$

an S_{n+m} representation. It turns out that

$$[V.W, \mathit{Ind}_{S_n\times S_m}^{S_{n+m}}(V\otimes W)] \geq [V,V\rangle_{S_n}[W,W\rangle_{S_m}.$$

Proof.

$$\begin{aligned}
[V.W, \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(V\otimes W)]_{S_{n+m}} &= [\mathrm{Res}_{S_n\times S_m}^{S_{n+m}}(V.W), V\otimes W]_{S_n\times S_m} \\
&\geq [\mathrm{Span}(fg : \; f\in V, \; g \in W), V\otimes W]_{S_n\times S_m} \\
&= [V\otimes W, V\otimes W]_{S_n\times S_m} \\
&= [V,V]_{S_n}[W,W]_{S_m}.
\end{aligned}$$

□

Observe that by definition of Δ_T, we have that Irr^λ may be viewed in light of Lemma 4 as $\mathrm{Irr}^\lambda = V_1.V_2.V_3\cdots$, where $V_1 = \mathbb{C}\Delta(y_1,\cdots,y_{\lambda'_1})$ and so on. Thus, using lemma 4 repeatedly one gets

$$[\mathrm{Irr}^\lambda, E^{\lambda'}] \geq [\mathrm{sgn},\mathrm{sgn}]_{S_{\lambda'_1}}\cdots = 1.$$

Thus, $\mathrm{Irr}^\lambda \subseteq H^\lambda$ shares at least one irrep with $E^{\lambda'}$. The final step in the proof follows since $\langle H^\lambda, E^{\lambda'}\rangle = \langle h_\lambda, e_{\lambda'}\rangle = 1$, i.e. there is a unique irrep occurring both in H^λ and $E^{\lambda'}$. But then this irrep has to be Irr^λ and in particular Irr^λ has to be an irrep.

2.7 Connection to SSYT and Schur functions

Proposition 10. *A basis for the S_μ-invariants $(Irr^\lambda)^{S_\mu}$ is given by*

$$\{\sum_{\sigma\in S_\mu}\sigma T : T\in SYT(\lambda),\quad Des(T)\subset\{\mu_1,\mu_1+\mu_2,\dots\}\}.$$

The T in the above set arise bijection from $SSYT(\lambda,\mu)$ by applying standardization. In particular

$$\dim((Irr^\lambda)^{S_\mu}) = \#SSYT(\lambda,\mu) = K_{\lambda,\mu}$$

and so

$$\mathcal{F}(Irr^\lambda) = \sum_\mu K_{\lambda,\mu}m_\mu = s_\lambda.$$

Proof. In general, the space of S_μ-invariants is obtained by applying the averaging operator $\sum_{\sigma\in S_\mu}\sigma$ to all $\Delta_T \in \mathrm{Irr}^\lambda$ for $T\in \mathrm{SYT}(\lambda)$. The operator will kill Δ_T if entries of the same μ-block (μ-blocks are $\{1,\dots\mu_1\},\{\mu_1+1,\dots,\mu_1+\mu_2\},\dots$) appear in the same column of T. Since T is a standard Young tableau this implies that the μ-blocks appear as horizontal strips in T and hence T μ-destanderdizes to an element of $\bar{T}\in SSYT(\lambda,\mu)$. Amongst, the various $T'\in SYT(\lambda)$ which μ-destanderdize to $\bar{T}$, T being the standerdization of T is an S_μ-orbit represenative.

Hence, this shows that all $T\in\mathrm{SYT}(\lambda)$, have Δ_T are either killed by the averaging operator or appear in a unique sum in the proposed basis based on the μ-destanderdization. By comparing monomials as before, we see linear independence of the propsed basis elements. □

The deep and surprising fact about Schur functions is that they not only show up naturally in the study of S_n-representation theory but also as characters in the theory of GL_k-representation theory (but they will show up in a different way: literally as functions where the x_i are replaced by eigenvalues of matrices). This connection comes from something called Schur-Weyl duality.

3 FI-modules

The category **FI** denotes the category with finite sets as objects and injective maps as morphisms. A skeleton for this category consists of the finite sets $[n]$ for $n\in\mathbb{N}$. Let $\iota_{n,m}(i) = i$ be the standard injection from $[n]$ to $[m]$ and $\iota_n = \iota_{n,n+1}$. An **FI**-module is a functor from **FI** to the category of $\mathbb{C}$-modules. A really simple example would be the functor F, mapping sets $F(S)$ to free $\mathbb{C}$-modules on S and injective maps $\iota: S\to T$ get mapped to the linear maps defined by where they take the basis elements.

In reality, we only really care about what the functor does a skeleton. Let $V_n = F([n])$ and let $\varphi_n = F(\iota_n)$ so that $\varphi_{n,m} := F(\iota_{n,m}) = \varphi_{m-1}\varphi_{m-2}\cdots\varphi_n$. We can write this down as

$$V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2}\cdots \tag{1}$$

We also have to consider the injective maps from $[n]$ back to itself, which are bijection $\sigma\in S_n$. Each of these induces a linear map $F(\sigma)$ on V_n and we see that and since $F(\mathrm{id}_n) = \mathrm{id}_{V_n}$ and the maps compose correctly, we have a representation of S_n on V_n. Since all injective maps from $[n]$ to $[m]$, can be thought of as composing $\sigma \in S_m$ with $\iota_{n,m}$, we have captured all the morphisms of this skeleton and so we me structurally view **FI**-modules as a sequence of S_n-modules V_n, with linear maps $\varphi_n : V_n \to V_{n+1}$.

What are the necessary and sufficient condition for such a sequence $(V_\bullet, \varphi_\bullet)$ to arise from some **FI**-module?

Proposition 11. *The sequence $(V_\bullet, \varphi_\bullet)$ with V_n an S_n-module arises from some **FI**-module if and only if*

- $\varphi_n : V_n \to V_{n+1}$ *is S_n-equivariant upon naturally viewing $S_n \subset S_{n+1}$ (i.e. $\varphi_n(\sigma \cdot v) = \sigma \cdot \varphi_n(v)$ for all $\sigma \in S_n$ and $v \in V_n$)*
- $s_{n+1} \cdot \varphi_{n+1}\varphi_n = \varphi_{n+1}\varphi_n$*, where s_{n+1} is the elementary transposition interchanging $n+1$ and $n+2$*

Proof. The necessity of the first and second condition follow from functoriality. Observe that in **FI**, we have $\iota_n\sigma = \hat{\sigma}\iota_n$ for $\sigma \in S_n$, where $\hat{\sigma} \in S_{n+1}$ agrees with σ on $[n]$ and fixes $n+1$. For the second condition observe that $(n+1\ n+2) \circ \iota_{n,n+2} = \iota_{n,n+2}$ in **FI**.

We show that these conditions are sufficient. Any injective map $[n] \to [m]$ may be uniquely written as $\sigma \circ \iota_{n,m}$ for $\sigma \in S_m$ such that $\sigma(n+1) < \cdots < \sigma(m)$. We define a functor that sends these maps to $\sigma \cdot \varphi_{m-1} \cdots \varphi_n : V_n \to V_m$. It is relatively straightforward to verify that composition is treated correctly by the functor. Consider composing two such maps $\sigma\iota_{n,m}$ and $\tau\iota_{m,l}$ for $\sigma \in S_m$ with $\sigma(n+1) < \cdots < \sigma(m)$ and $\tau \in S_l$ with $\tau(m+1) < \cdots < \tau(l)$. This yields $\tau\sigma\iota_{n,l}$, which may not have the property that $\tau\sigma(n+1) < \cdots < \tau\sigma(l)$. But we may factor it as $\tau\sigma = \gamma\pi$, where $\gamma(n+1) < \cdots < \gamma(l)$ and π is a permutation fixing $1, \cdots, n$ (the permutation π untwists all the indices $n+1, \cdots, l$ to make γ ascending on these indices). Then

$$\tau\sigma\iota_{n,m} = \gamma\pi\iota_{n,m} = \gamma\iota_{n,m}.$$

Similarly,

$$\tau \cdot \varphi_{m,l} \circ \sigma \cdot \varphi_{n,m} = \tau\sigma\varphi_{n,l}$$

follows by the first condition (we keep commuting σ towards the left and

$$\tau\sigma \cdot \varphi_{n,m} = \gamma\pi \cdot \varphi_{n,m} = \gamma \cdot \varphi_{n,m},$$

follows since we may write π as a product of elementary transpositions s_i for $i = n+1, \ldots, m-1$. Using the first condition, these s_i may be successively commuted to the right past the φ_j until the encounter φ_i (which they may not be commuted past since $s_i \notin S_i$). At this point, we may use the fact that $i \geq n+1$, to ensure that there is a φ_{i-1} immediately right of φ_i and we are looking at $s_i\varphi_i\varphi_{i-1} = \varphi_i\varphi_{i-1}$, by the second condition. Thus, we eliminate all the s_i making up π, as claimed.

This establishes a valid functor on the skeleton of **FI**, which may easily extend to all of **FI** non-canonically by choosing an arbitrary bijection, f_S, from $[n]$ to every finite n-element set, S. Then declare that the functor maps that set to V_S with dimension n and pick an arbitrary linear bijection from V_n to V_S, g_S. For any injection $\iota : S \to T$, we may write ι uniquely as $f_T\iota' f_S^{-1}$, where $\iota' : [n] \to [m]$ is an injection and we may declare that the functor sends ι to $g_T F(\iota') g_S^{-1}$. □

3.1 Examples

Let $\mathbb{C}[X_n] := \mathbb{C}[x_1, \cdots, x_n]$. Under the standard S_n-actions on polynomial and standard injective maps $\varphi_n : \mathbb{C}[x_1, \cdots, x_n] \to \mathbb{C}[x_1, \cdots, x_{n+1}]$, the polynomial algebra is an **FI**-module (the second condition holds since swapping x_{n+1} and x_{n+2} has no impact on a polynomial in $x_1, \ldots, x_n$). For future reference let us compute the graded Frobenius image (with q tracking the degree) of $\mathbb{C}[X_n]$. From a character perspective, we may observe what S_n does to the monomial basis. Since it acts a permutation representation the character is just the number of fixed points and we see that

$$\mathcal{F}(\mathbb{C}[X_n]; q) = \sum_{k \geq 0} q^k \sum_{\mu \vdash n} \frac{p_\mu[X]}{z_\mu} \cdot \#(\text{degree k monomials fixed by a type } \mu\, \sigma) = \sum_{\mu \vdash n} \frac{p_\mu[X]}{z_\mu} p_\mu[\frac{1}{1-q}] = h_n \left[\frac{X}{1-q}\right].$$

Similarly, if we had more variable sets like $\mathbb{C}[X_n, Y_n]$ and tracked both x and y degree via q and t, we would get

$$\mathcal{F}(\mathbb{C}[X_n, Y_n]; q, t) = h_n \left[\frac{X}{(1-q)(1-t)}\right].$$

Note that this yields the Schur decompositions of $\mathbb{C}[X_n]$ since

$$h_n \left[\frac{X}{1-q}\right] = \sum_{\lambda \vdash n} s_\lambda[X] s_\lambda[1/(1-q)]$$

with

$$s_\lambda[1/(1-q)] = \frac{1}{(1-q^n)\cdots(1-q)} \sum_{T \in SYT(\lambda)} q^{\mathrm{maj}(T)} = q^{n(\lambda)} \prod_{c \in \lambda} \frac{1}{1-q^{h(c)}}.$$

We can see directly that each graded component of $(\mathbb{C}[X_n], \varphi_n)$ is URMS, since $n(\lambda[m]) = n(\lambda) + |\lambda|$ is independent of m and for any fixed q-degree i all hook lengths of cells in the bottom row will eventually be greater than i, so that for $n \geq |\lambda| + \lambda_1 + i$

$$[q^i] s_{\lambda[n]}[1/(1-q)] = [q^{i-n(\lambda)-|\lambda|}] \prod_{c \in \lambda} \frac{1}{1-q^{h(c)}}$$

is independent of n and the multiplicity of $\mathrm{Irr}^{\lambda[n]}$ in the i-th graded component of $\mathbb{C}[X_n]$ has stabilized. Note that the dimension of the i-th graded component is $\binom{i+n-1}{i}$ which is a degree i polynomial in n.

For the next example consider Specht **FI**-module, $(\mathrm{Irr}^{\lambda[\bullet]}, \varphi_\bullet)$, where φ_n maps the a Garnir polynomial $\varphi_n(\Delta_T) = \Delta_{T^+} = \Delta_T$, where T^+ is T with cell labelled $n+1$ added to the end of the bottom row (which doesn't actually change the polynomial). This is a sub-**FI**-submodule of $\mathbb{C}[X_\bullet]$. Call the dimension of the $\mathrm{Irr}^{\lambda[n]}$, $f^{\lambda[n]}$ and notice that it is also a polynomial in n. One way to see this is through the hook length formula. Alternatively, we can use finite differences and see that

$$\begin{aligned} f^{\lambda[n+1]} - f^{\lambda[n]} &= \#\{T \in SYT(\lambda[n+1]) : \text{ bottom row doesn't end in } n+1\} \\ &= \sum_{\substack{\nu \vdash |\lambda|-1 \\ \nu \subset \lambda}} f^{\nu[n]} \end{aligned}$$

so we can see by induction that $f^{\lambda[n]}$ is a polynomial in n for $n \geq |\lambda| + \lambda_1$ of degree $|\lambda|$.

The next example we will consider is called a free **FI**-module and consists of $\mathrm{Ind}_{S_{|\lambda|} \times S_{n-|\lambda|}}^{S_n}(\mathrm{Irr}^\lambda \otimes \mathrm{Irr}^{(n-|\lambda|)}) = \mathbb{C}[S_n] \otimes_{\mathbb{C}[S_{|\lambda|} \times S_{n-|\lambda|}]} \mathrm{Irr}^\lambda \otimes \mathrm{Irr}^{(n-|\lambda|)}$, with the natural injective map between successive extensions of scalars. These have Frobenius image

$$s_\lambda h_{n-|\lambda|} = \sum_{\lambda/\nu \text{ horz strip}} s_{\nu[n]}.$$

3.2 The main results

We say that an **FI**-module is finitely generated if there is a finite set of vectors such that all $v \in V_n$ may be obtained via linear combinations and the use of φ_k and $\sigma \in S_n$. Equivalently, there is some n, such that all $m \geq n$ have $V_m = S_m \cdot \varphi_{n,m}(V_n)$ (in which case we say the **FI**-module is generated in degree n).

Proposition 12 (Chruch, Ellenburg, Farb)**.** *The category of **FI**-modules is noetherian, that is, any sub-**FI**-module of a finitely generated **FI**-module is finitely generated.*

Proposition 13 (Church, Ellenburg, Farb)**.** *A **FI**-module is finitely generated if and only if the maps φ_n are eventually injective and the modules V_n are URMS (uniformally representation multiplicity stable, meaning there is an N, such that for $n \geq N$ the multiplicity of Irr$^{\lambda[n]}$ in V_n is independent of n).*

Proof. Suppose that the **FI**-module is generated in degree N. To see that the connective maps are eventually injective, consider the kernels of these maps $\ker(\varphi_m)$ which form a sub-**FI**-module. Using the noethrian property, these are finitely generated, so there is some M, such that for all $m \geq M$, $\ker(\varphi_m) = S_m \cdot \varphi_{M,m}(\ker(\varphi_M)) = 0$. Thus, the maps are injective for $m \geq M$.

Now replace N by $\max(N, M)$, so that for $n \geq N$, φ_n is injective and $V_{n+1} = S_{n+1} \cdot \varphi_n(V_n)$. Without loss of generality, we may identify V_n with $\varphi_n(V_n) \subset V_{n+1}$. One can use the universal property of induced modules to see that for $n \geq N$

$$V_n = S_n \cdot \varphi_{N,n}(V_N)$$

is some quotient of

$$\mathrm{Ind}_{S_N \times S_{n-N}}^{S_n}(V_N \otimes \mathrm{Irr}^{(n-N)}).$$

But decomposing V_N into Specht modules and using the Pieri rule we see that these modules are URMS and by taking quotients we have established an upper bound the multiplicities of each $\mathrm{Irr}^{\lambda[n]}$ with a finite set of supporting λ. Using the monotonicity lemma in my paper, we get that multiplicities of the V_n must themselves eventually stabilize.

To see the converse, if one has that the maps are eventually injective and URMS say by degree n, then by Corollary 2.1.1 of my paper the **FI**-module is generated in degree n. ☐