

# Representation theory of $S_n$ : a terse summary

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## Introduction and Mashke's theorem

All vector spaces,  $V$ , have field of scalars  $\mathbb{C}$  and are finite dimensional and  $GL(V)$  denotes invertible  $\mathbb{C}$ -linear maps from  $V$  to  $V$ .

A **representation** of  $G$  of degree (or dimension)  $n$  is a homomorphism  $\rho : G \rightarrow GL(V)$ , where  $V$  is an  $n$ -dimensional  $\mathbb{C}$ -space. That is

- $\rho(e) = \text{id}_V$ , identity maps to identity
- $\rho(gh) = \rho(g)\rho(h)$ , group multiplication turns into matrix multiplication

We can think of  $\rho$  as describing a particular way in which  $G$  acts on  $V$  by linear transformations. For this reason, we sometimes write  $G \curvearrowright V$  in place of  $\rho : G \rightarrow GL(V)$  and just say  $g \cdot v$  in place of  $\rho(g)v$ .

A  $G$ -invariant subspace  $W \subseteq V$ , is a vector subspace such that  $\rho(g)w \in W$  for all  $g \in G$  and  $w \in W$ . The **subrepresentation** of  $\rho : G \rightarrow GL(V)$  restricted to  $G$ -invariant  $W$  is the representation  $\rho|_W : G \rightarrow GL(W)$  obtained by restricting  $\rho(g) \in GL(V)$  to only act on  $W$ . An **irreducible representations** (also called **irrep**) is a representation,  $\rho : G \rightarrow GL(V)$ , whose only  $G$ -invariant subspaces are 0 and  $V$ .

A **direct sum** of two representations  $\sigma : G \rightarrow GL(W)$  and  $\pi : G \rightarrow GL(U)$  is defined as  $\sigma \oplus \pi : G \rightarrow GL(W \oplus U)$  obtained in the natural way by taking the direct sum of matrices:

$$(\sigma \oplus \pi)(g) = \begin{bmatrix} \sigma(g) & \\ & \pi(g) \end{bmatrix}.$$

Two representations  $\sigma : G \rightarrow GL(V)$ ,  $\rho : G \rightarrow GL(W)$  are **isomorphic** (denoted  $\rho \cong \sigma$ ) if there is a bijective linear map  $A : V \rightarrow W$ , such that  $A\sigma(g)A^{-1} = \rho(g)$  for all  $g \in G$ . In other words the representations are the same upon a change of basis.

**Proposition 1** (Mashke's theorem). *Take any finite group  $G$  and finite dimensional  $V$ , and any representation  $\rho : G \rightarrow GL(V)$ . Then there are pairwise non-isomorphic irreducible representations,  $\sigma_1, \dots, \sigma_k$  and  $c_1, \dots, c_k \in \mathbb{N}_+$  such that*

$$\rho \cong \sigma_1^{\oplus c_1} \oplus \sigma_2^{\oplus c_2} \oplus \dots \oplus \sigma_k^{\oplus c_k}.$$

*Furthermore, this decomposition into irreps, is unique up to isomorphism and reordering of the  $\sigma_i$ .*

First, we need the following lemma

**Lemma 1.** *If  $W$  is a  $G$ -invariant  $\rho : G \rightarrow GL(V)$ , then there exists a “ $G$ -complement”, a  $G$ -invariant  $U \subseteq V$ , such that  $V = W \oplus U$ . In this case  $\rho \cong \rho|_W \oplus \rho|_U$ .*

*Proof.* Take some inner product on  $V$ ,  $\langle \cdot, \cdot \rangle'$  and replace it with a new one by defining

$$\langle v, w \rangle = \sum_{g \in G} \langle g \cdot v, g \cdot w \rangle'$$

so that

$$\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle,$$

in other words the inner product is  $G$ -invariant. Given  $G$ -invariant subspace  $W \subseteq V$ , one can take its orthogonal complement

$$U = W^\perp = \{v \in V : \langle v, w \rangle = 0, \forall w \in W\},$$

with respect to  $\langle \cdot, \cdot \rangle$ , so that  $V = W \oplus U$ . We claim that  $U$  is also  $G$ -invariant and this holds since if  $u \in U$ , then  $g \cdot u$  is orthogonal to all  $w \in W$  since

$$\langle g \cdot u, w \rangle = \langle g \cdot u, g(g^{-1}w) \rangle = \langle u, g^{-1}w \rangle = 0,$$

since  $g^{-1}w \in W$  by  $G$ -invariance of  $W$ . The isomorphism  $\rho \cong \rho|_W \oplus \rho|_U$  just comes from the (internal) isomorphism of vector spaces  $V = W \oplus U$ .  $\square$

Here are two ways Lemma 1 can fail when  $G$  is not finite, or we are not working over  $\mathbb{C}$ . Let  $G = (\mathbb{R}, +)$ , an infinite group, and consider

$$\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

which is a representation since you can check that  $\rho(a+b) = \rho(a)\rho(b)$  and maps the additive identity to the identity matrix. Now

$$W = \text{Span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a  $G$ -invariant subspace since  $\rho(a) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W$  but has no  $G$ -complement. Similarly, if we are working over a finite field  $\mathbb{F}_p$ , then  $G = (\mathbb{F}_p, +)$  has similar representation

$$\rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

which exploits the necessary fact that  $\rho(p \cdot 1) = \rho(1 + \dots + 1) = \rho(1)^p = \text{id}_2$ . Again the span of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  has no  $G$ -complement.

*(Proof of Maschke's theorem).* The idea is that if  $\rho : G \rightarrow GL(V)$  is irreducible then we are done. Otherwise, it has some  $G$ -invariant subspace  $W \neq 0, V$  and by the lemma  $\rho \cong \sigma \oplus \pi$  for some subrepresentations  $\sigma, \pi$  both of which have dimension less than  $\dim(V)$ . Then by applying the argument to  $\sigma, \pi$  we keep going until  $\rho \cong \sigma_1 \oplus \dots \oplus \sigma_l$ , where all  $\sigma_i$  are irreps (can't be broken further as a direct sum since they have no non-trivial subrepresentations). At this point one can group up the  $\sigma_i$  that are isomorphic. We will see the uniqueness later by using character theory.  $\square$

## Characters and class functions

We saw that all representations break down as a sum of irreducible ones and so it makes sense to understand what the irreps are. They are the fundamental building blocks of representations.

A **class function** is a function  $\alpha : G \rightarrow \mathbb{C}$ , such that agrees on conjugacy classes:  $\alpha(g) = \alpha(hgh^{-1})$  for all  $g, h \in G$ . The set of class functions forms a vector space, which we will call  $\mathcal{C}(G)$ . Let  $c = \dim(\mathcal{C}(G))$  and observe that  $c$  is the number of conjugacy classes of  $G$ .

An important family of class functions are **characters**. The character of representation  $\rho : G \rightarrow GL(V)$  is the class function obtained by taking trace  $\chi_\rho : G \rightarrow GL(V)$ ,  $\chi_\rho(g) := \text{tr}(\rho(g))$ . Notice that if  $\rho \cong \sigma \oplus \pi$ , then  $\chi_\rho(g) = \chi_\sigma(g) + \chi_\pi(g)$ . So if  $\rho \cong \sigma_1^{\oplus c_1} \oplus \sigma_2^{\oplus c_2} \oplus \dots \oplus \sigma_k^{\oplus c_k}$  is a decomposition into irreps, then

$$\chi_\rho = c_1 \chi_{\sigma_1} + \dots + c_k \chi_{\sigma_k}.$$

In other words, all characters are a sum of characters of irreps. Hence, to understand characters it suffices to study characters of irreps. Also notice that *isomorphic representations have the same character*, since  $\text{tr}(A\rho(g)A^{-1}) = \text{tr}(\rho(g))$ .

The columns of the character table have dot product 0 and this follows from orthogonal rows implies orthogonal columns.

The following is one of the many useful values related to a representation you can extract from looking at the character.

**Lemma 2** (Fixed point formula). *Let  $\rho : G \rightarrow GL(V)$  be a representation and let*

$$V^G := \{v \in V : \rho(g)v = v, \forall g \in G\},$$

*the set of vectors fixed by all elements of  $G$ . Then*

$$\dim_{\mathbb{C}}(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) = \frac{1}{|G|} \sum_{g \in G} \text{tr}(\rho(g))$$

*Proof.* Define  $p : V \rightarrow V$ ,  $p = \frac{1}{|G|} \sum_{g \in G} \rho(g)$ . Observe that for  $v \in V^G$ ,  $pv = v$  so it acts as the identity on  $V^G$ . Furthermore, for any  $v \in V$ ,  $pv \in V^G$ , since it is fixed by all  $g \in G$ . Thus,  $p$  is a projection matrix onto  $V^G$ . From linear algebra we have that  $\text{tr}(p) = \dim(\text{im}(p)) = \dim(V^G)$ .  $\square$

**Corollary 1** (Burnside's lemma). *Let  $G$  be a group acting on a set  $X$ . Then*

$$\text{number of orbits} = \frac{1}{|G|} \sum_{g \in G} \text{number of pts fixed by } g$$

*Proof.* Consider the permutation representation and fix an element  $g \in G$ . If  $G$  is a group acting on a set  $X$ , then there is a homomorphism  $\varphi : G \rightarrow \text{Aut}(X)$ . If  $\{e_x\}$  is basis for a vector space  $V$ , then  $G$  acts by  $g \cdot e_x = e_{g \cdot x}$ . Hence the matrix  $M = \rho(g)$  has entries  $[M]_{ii} = 1$  if  $g \cdot x_i = x_i$ . In other words, the trace of  $M = \{x \in X \mid gx = x \forall g \in G\}$  = the number of fixed points. On the other hand, the number of orbits is equal to the dimension of the fixed vectors.  $\square$

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List of some important representation constructions and corresponding characters. Let  $\rho_1 : G \rightarrow GL(V_1)$ ,  $\rho_2 : G \rightarrow GL(V_2)$  be two representations of  $G$ :

- **Contragredient (dual)**: Recall from lin alg, that a vector space  $V$  has a dual  $V^*$ , the set of linear maps from  $l : V \rightarrow \mathbb{C}$ . If  $G$  acts on  $V$  via  $\rho$ , then this induces an action on  $V^*$ , called  $\rho^* : G \rightarrow GL(V^*)$ :

$$(\rho^*(g)l)(h) = l(g^{-1}h)$$

and the associated character is

$$\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$$

- **Direct sums**:  $\chi_{\rho_1 \oplus \rho_2}(g) = \chi_{\rho_1}(g) + \chi_{\rho_2}(g)$
- **(Internal) tensor product**: Let  $\rho_1 \otimes \rho_2 : G \rightarrow GL(V_1 \otimes V_2)$  be the representation defined as  $\rho_1(g) \otimes \rho_2(g)$ , where here  $\otimes$  is the Kronecker product of matrices. Then the trace of this product is  $\text{tr}(\rho_1(g))\text{tr}(\rho_2(g)) = \chi_{\rho_1}(g)\chi_{\rho_2}(g)$

- **Hom:** Consider  $\text{Hom}_{\mathbb{C}}(V_1, V_2)$ , the set of linear maps from  $V_1$  to  $V_2$  and make  $G$  act on it in the following way for  $A : V_1 \rightarrow V_2$

$$g \cdot A := \rho_2(g)A\rho_1(g^{-1}).$$

From linear algebra, we know that  $\text{Hom}_{\mathbb{C}}(V_1, V_2) \cong V_1^* \otimes V_2$ . It turns out that the above is also an isomorphism of  $G$  representations based on how we defined representations  $\otimes$ ,  $*$  and  $\text{Hom}$ . Thus,

$$\chi_{\text{Hom}_{\mathbb{C}}(V_1, V_2)}(g) = \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g)$$

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The vector space  $\mathcal{C}(G)$  admits an inner product known as the **scalar product**, defined as

$$\langle \alpha, \beta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\beta(g^{-1}).$$

Notice  $\langle \alpha, \beta \rangle_G = \langle \beta, \alpha \rangle_G$  It has the following interpretation. For characters of representations, we sometimes abbreviate

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G = [\rho_1, \rho_2]$$

**Lemma 3.** *Let*

$$\text{Hom}_G(V_1, V_2) = \{A \in \text{Hom}_{\mathbb{C}}(V_1, V_2) : A(g \cdot v) = g \cdot (Av)\},$$

*called the intertwiners of representations  $V_1, V_2$ . Then*

$$\dim(\text{Hom}_G(V_1, V_2)) = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G$$

*Proof.* The important realization is that  $\text{Hom}_G(V_1, V_2) = \text{textHom}_{\mathbb{C}}(V_1, V_2)^G$ , the set of linear maps from  $V_1$  to  $V_2$  that are fixed under the  $G$  action as defined in the above list, since

$$A = \rho_2(g)A\rho_1(g^{-1})$$

implies that  $A\rho_1(g) = \rho_2(g)A$  and so  $A(\rho_1(g)v) = (\rho_2(g)A)v$ . Hence by the fixed point formula

$$\dim(\text{Hom}_G(V_1, V_2)) = \frac{1}{|G|} \sum_g \chi_{\text{Hom}_{\mathbb{C}}(V_1, V_2)}(g) = \frac{1}{|G|} \sum_g \chi_{\rho_1}(g^{-1})\chi_{\rho_2}(g) = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle$$

□

**Remark 1.** Every element of a finite group has finite order, i.e. for all  $g \in G$ , there is a  $k > 0$  so that  $g^k = e$ . Hence, for any representation,  $\rho$ ,  $\rho(g)^k = e$ . From linear algebra this means that  $\rho(g)$  is a diagonalizable matrix over  $\mathbb{C}$  and its eigenvalues are roots of unity,  $e^{2\pi it/k}$  for  $t = 0, 1, \dots, k-1$ . Hence  $\chi_{\rho}(g)$  is a sum of roots of unity. In particular, if  $e^{2\pi it/k}$  is an eigenvalue of  $\rho(g)$  then it corresponds to  $e^{-2\pi it/k} = \overline{e^{2\pi it/k}}$  being an eigenvalue of  $\rho(g^{-1})$ . This implies that

$$\chi_{\rho^*}(g) = \chi_{\rho}(g^{-1}) = \overline{\chi_{\rho}(g)}$$

and so

$$\langle \chi_{\rho}, \chi_{\pi} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho}(g)\overline{\chi_{\pi}(g)}.$$

This essentially agrees with the standard inner product over  $\mathbb{C}$  with the exception of the normalizing  $\frac{1}{|G|}$  factor.

**Proposition 2** (Orthonormality of irrep characters). *If  $\rho_1 : G \rightarrow GL(V_1)$  and  $\rho_2 : G \rightarrow GL(V_2)$  are non-isomorphic irreducible representations, then  $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G = 0$ . If they are isomorphic irreps, then  $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle_G = 1$ . In other words, the irrep characters form an **orthonormal set** in  $\mathcal{C}(G)$ .*

*Proof.* In the non-isomorphic case this is the same as showing that  $\text{Hom}_G(V_1, V_2) = \{0\}$ , the set containing only the zero map from  $V_1$  to  $V_2$ . Suppose for contradiction  $f \in \text{Hom}_G(V_1, V_2)$  is non-zero. Since  $f$  is  $G$ -invariant, one has that the  $\ker(f) \subseteq V_1$  is a  $G$ -invariant subspace. Since  $G \curvearrowright V_1$  is irreducible,  $\ker(f) = 0, V_1$  and it cannot be  $V_1$  since we assume  $f \neq 0$ . Hence  $f$  is injective. Similarly  $\text{im}(f) \subseteq V_2$  is a  $G$ -invariant subspace and since  $G \curvearrowright V_2$  is irreducible,  $\text{im}(f) = 0, V_2$  and since  $f \neq 0$ ,  $f$  is surjective. Hence  $f$  is a  $G$ -invariant bijective linear map from  $V_1$  to  $V_2$  making it an isomorphism, contradicting that  $\rho_1, \rho_2$  are non-isomorphic.

For the second part we have to show that  $\text{Hom}_G(V_1, V_2)$  is 1-dimensional if  $V_1$  and  $V_2$  are isomorphic. Since they are isomorphic, this is the same as showing  $\text{Hom}_G(V_1, V_1)$  is 1-dimensional. Let  $f \in \text{Hom}_G(V_1, V_1)$  and let  $\lambda$  be one of its eigenvalues (with corresponding eigenvector  $v$ ). Then  $f - \lambda \cdot I : V_1 \rightarrow V_1$  is also a  $G$ -invariant map. Since  $V_1$  is irreducible, we know that  $\ker(f - \lambda I) = 0, V_1$  and it cannot be  $0$ , since  $v \in \ker(f - \lambda I)$  by definition of eigenvector. Hence,  $f - \lambda I = 0$  and  $f = \lambda I$  and

$$\text{Hom}_G(V_1, V_1) = \text{Span}_{\mathbb{C}}(I),$$

scalar multiples of the identity map (a 1-dimensional space).  $\square$

We can now complete the proof of Mashke's theorem. Let  $\rho$  be a representation of  $G$ , that decomposes into irreps

$$\rho \cong \sigma_1^{\oplus c_1} \oplus \sigma_2^{\oplus c_2} \oplus \dots \oplus \sigma_k^{\oplus c_k},$$

then up to isomorphism and reordering this decomposition is unique, with

$$c_i = [\rho, \sigma_i] = \langle \chi_\rho, \chi_{\sigma_i} \rangle_G = \frac{1}{|G|} \sum_g \chi_\rho(g) \overline{\chi_{\sigma_i}(g)}.$$

Additionally, two representations are isomorphic, if and only if, their characters are the same function.

*Proof.* We have  $\chi_\rho = \sum_i c_i \chi_{\sigma_i}$ . Then

$$\langle \chi_\rho, \chi_{\sigma_j} \rangle = \sum_i c_i \langle \chi_{\sigma_i}, \chi_{\sigma_j} \rangle = c_j,$$

by orthonormality. For the second part, we already saw that isomorphic representations have the same character. Conversely, if two representations,  $\rho_1, \rho_2$ , have the same characters, then for each irreducible  $[\rho_1, \sigma_i] = [\rho_2, \sigma_i]$  and hence we can construct an isomorphism between them by making the map restrict to isomorphisms between their irreducibles.  $\square$

Let  $\Pi(G)$  denote the set of characters of irreps.

**Example: The regular representation.** The regular representation of  $G$ , is defined by considering the vector space with  $G$  as a basis (you can picture  $\{v_g : g \in G\}$  as being the basis vectors), and  $G$  acting on left by multiplication, so  $h \cdot v_g = v_{hg}$ . Let's compute the character of this representation, call it  $\rho$ . Since  $\rho(e)$  is the identity matrix,  $\chi_\rho(e)$  is the dimension of the representation, so  $\chi_\rho(e) = |G|$ . For any  $g \neq e$ , observe that  $gv_h = v_{gh} \neq v_h$  and so  $\text{tr}(\rho(g)) = 0$ . Hence

$$\chi_\rho(g) = \begin{cases} |G|, & g = e \\ 0, & \text{else} \end{cases}.$$

Now for the decomposition into irreducible, if  $\chi_\sigma \in \Pi(G)$  is an irrep character, then

$$[\rho, \sigma] = \frac{1}{|G|} \sum_g \chi_\sigma(g) \overline{\chi_\rho(g)} = \frac{|G|}{|G|} \chi_\sigma(e),$$

which is just the dimension of irrep  $\sigma$ ,  $\dim(\sigma)$ . Hence

$$\rho \cong \bigoplus_{\sigma \in \Pi(G)} \sigma^{\oplus \dim(\sigma)},$$

i.e. each irrep occurs in  $\rho$  with multiplicity being its dimension.

Let  $\Pi(G)$  denote the set of characters of irreps. We saw that it forms an orthonormal set in  $\mathcal{C}(G)$  with respect to  $\langle \cdot, \cdot \rangle_G$ . Now we will show that they in fact form a basis (we just need to show that they span the set of class functions,  $\mathcal{C}(G)$ ).

**Proposition 3.** *The set  $\Pi(G)$  forms a basis for  $\mathcal{C}(G)$ . In other words, the number of irrep characters is precisely  $\dim(\mathcal{C}(G))$  equal to the number of conjugacy classes of the group.*

## 1 Induced representations and restrictions

Suppose  $H$  is a subgroup of  $G$ ,  $H \leq G$ , and  $V$  is a representation of  $G$ , the restriction  $\text{Res}_H^G(V)$  is quite straightforward. It is just  $V$  again now being a representation of  $H$  (we are restricting  $\rho : G \rightarrow GL(V)$  to  $\rho|_H : H \rightarrow GL(V)$ ).

Is there a natural way to go in the other direction? Say you start with a representation of  $H$ ,  $W$  (via  $\rho : H \rightarrow GL(W)$ ), and you want to extend it to  $G$ . You can't undo restriction since information is always lost when restricting but maybe there is a "most natural" way of defining how  $G$  acts on  $W$ . This is called the **induced representation**,  $\text{Ind}_H^G(W)$  or

$$\text{Ind}_H^G(\rho) : G \rightarrow GL(W^{\oplus [G:H]}),$$

which is now a representation of  $[G : H] = |G|/|H|$  copies of  $W$ . We will give an explicit matrix based construction without getting into why this is the most natural one. For each  $g \in G$  we must assign a  $\dim(W) \cdot \frac{|G|}{|H|}$  by  $\dim(W) \cdot \frac{|G|}{|H|}$  matrix. We give this as a block matrix of  $\left(\frac{|G|}{|H|}\right)^2$  blocks (each block being in  $\dim(W)$  by  $\dim(W)$ ):

$$\text{Ind}_H^G(\rho)(g) := [\rho(x^{-1}gy)]_{xH, yH \in G/H},$$

where  $xH, yH$  are cosets (the matrix depends on which coset representatives  $x, y$  you pick but one can change basis so the definition is unique up to representation isomorphism) and we are setting  $\rho(\hat{g}) = 0$  to be the zero block matrix if  $\hat{g} \notin H$  ( $\rho$  only makes sense for elements of  $H$ ). As a consequence the corresponding characters is

$$\chi_{\text{Ind}_H^G(\rho)}(g) = \sum_{xH \in G/H} \chi_\rho(x^{-1}gx),$$

where now the definition really doesn't depend on choice of coset representative and again we are setting  $\chi_\rho(\hat{g}) = 0$  whenever  $\hat{g} \notin H$ . Since this definition is linear in  $\chi_\rho$ , we may extend it to class functions.

**Example:** if  $1_H$  is the 1-dimensional trivial representation of  $H$ , then  $\text{Ind}_H^G(1_H)$  ends up yielding a bunch of permutation matrices. In particular, the permutation matrix  $\text{Ind}_H^G(1_H)(g)$  captures the way  $g$  permutes the cosets  $\{xH \in G/H\}$  (recall these are  $|G/H|$  by  $|G/H|$  matrices). (This example turns out to be very closely related to the  $h_\lambda$  symmetric functions as well as Polya enumeration). In particular, **transitive group actions are precisely induced trivial representations**.

A special case of this is that the regular representation is equal to  $\text{Ind}_e^G(1_e)$ , where  $e$  is the trivial subgroup of  $G$ .

One super important result involving induced representations is **Frobenius reciprocity**.

**Proposition 4. Frobenius reciprocity** *Let  $H \leq G$  and  $\pi : G \rightarrow GL(V)$  and  $\rho : H \rightarrow GL(W)$  be representations. Then the characters satisfy*

$$\langle \text{Ind}_H^G(\chi_\pi), \chi_\rho \rangle_G = \langle \chi_\pi, \text{Res}_H^G(\chi_\rho) \rangle_H$$

*Proof.*

$$\begin{aligned}
\langle \text{Ind}_H^G(\chi_\pi), \chi_\rho \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \text{Ind}_H^G(\chi_\pi)(g) \cdot \overline{\chi_\rho(g)} \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{xH \in G/H} \chi_\pi(x^{-1}gx) \cdot \overline{\chi_\rho(x^{-1}gx)}, \quad \text{since } \chi_\rho \text{ is a } G\text{-character} \\
&= \frac{1}{|G|} \sum_{xH \in G/H} \sum_{\substack{g \in G \\ x^{-1}gx \in H}} \chi_\pi(x^{-1}gx) \cdot \overline{\chi_\rho(x^{-1}gx)} \\
&= \frac{1}{|G|} \sum_{h \in H} \sum_{\substack{g \in G, xH \in G/H \\ x^{-1}gx = h}} \chi_\pi(h) \cdot \overline{\chi_\rho(h)} \\
&= \frac{[G : H]}{|G|} \sum_{h \in H} \chi_\pi(h) \cdot \overline{\chi_\rho(h)} \\
&= \frac{1}{|H|} \sum_{h \in H} \chi_\pi(h) \cdot \overline{\chi_\rho(h)} \\
&= \langle \chi_\pi, \text{Res}_H^G(\chi_\rho) \rangle_H,
\end{aligned}$$

where the number of ways to pick  $xH \in G/H$  and  $g \in G$  to hit given  $h = x^{-1}gx$  is  $[G : H]$  since each distinct  $xH$  gives a different  $g = hx^{-1}$ .  $\square$

**Example:** Consider  $k$ -element subsets of  $[n]$ ,  $\binom{[n]}{k}$  with the natural  $S_n$ -action. Since this is a transitive group action, with stabilizer of  $[k]$  being  $S_{[k]} \times S_{[k+1, n]}$ , we have

$$\binom{[n]}{k} \cong \text{Ind}_{S_{[k]} \times S_{[k+1, n]}}^{S_n}(1).$$

## 2 Symmetric group

We now restrict attention to the symmetric group  $S_n$ . The conjugacy classes are determined by cycle type, a non-increasing sequence  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l)$  with parts  $\lambda_i > 0$  denoting the lengths of the disjoint cycles of an element of the conjugacy class. Hence  $\lambda_1 + \dots + \lambda_l = n$ . We call  $\lambda \vdash n$  a partition of  $n$ . The goal is ultimately to determine the irreps corresponding to these  $\lambda$ .

Let's compute the size of the conjugacy classes. Let  $c_t(\lambda)$  be the number of parts  $\lambda_i$  equal to  $t$ . So for  $\lambda = (3, 3, 2)$ ,  $c_1(\lambda) = 0$ ,  $c_2(\lambda) = 1$  and  $c_3(\lambda) = 2$ . We claim that centralizer of a  $\sigma \in S_n$  with cycle type  $\lambda$  is

$$z_\lambda := \left( \prod_i \lambda_i \right) \left( \prod_t c_t(\lambda)! \right).$$

Then by orbit-centralizer theorem the number of permutations with cycle type  $\lambda$  is  $|S_n|/z_\lambda = n!/z_\lambda$ . Denote this conjugacy class  $C_\lambda$ . We will write  $\lambda = sh(\sigma)$  to say that  $\lambda$  is the shape or cycle type of  $\sigma$

### 2.1 Class functions of symmetric group as a ring

Let  $\mathcal{C}(S_n)$  be the space of class functions  $f : S_n \rightarrow \mathbb{C}$ , which have a basis given by the irreducible representation characters of  $S_n$ ,  $\chi^\lambda : S_n \rightarrow \mathbb{C}$ , where  $\lambda \vdash n$  is a partition (there should be as many characters as conjugacy classes, so there is one irreducible representation character for each partition). If  $f \in \mathcal{C}(G_1)$  and  $g \in \mathcal{C}(G_2)$ , we may define

$$f \boxtimes g \in \mathcal{C}(G_1 \times G_2), \quad f \boxtimes g(x_1, x_2) = f(x_1)g(x_2),$$

a pointwise product. If the two functions are characters  $f = \chi^V$ ,  $g = \chi^W$ , then  $f \otimes g = \chi^{V \otimes W}$  a character of  $G_1 \times G_2$ -representation.

Let  $\mathcal{C}(S) = \bigoplus_n \mathcal{C}(S_n)$ , an infinite dimensional  $\mathbb{C}$ -space. We can give it a ring structure by defining multiplication: for  $f \in \mathcal{C}(S_n)$  and  $g \in \mathcal{C}(S_m)$ , define

$$f * g := \text{Ind}_{S_n \times S_m}^{S_{n+m}} (f \boxtimes g) \in \mathcal{C}(S_{n+m}).$$

Here  $S_n \times S_m$  is viewed as a subgroup of  $S_{n+m}$ , via sending  $(\sigma, \tau)$  to a permutation that makes  $\sigma$  act on  $1, \dots, n$  and  $\tau$  act on  $n+1, \dots, n+m$ . This also turns  $\mathcal{C}(S)$  into a graded ring. This is sometimes called the Grothendieck ring of the tower of symmetric groups,  $(\mathcal{C}(S), *, +)$ . Let's determine what the ring actually is.

**Proposition 5.** *Let  $\mathbb{C}[T_1, T_2, \dots]$  be the ring of polynomials in indeterminants  $T_1, T_2, \dots$ , with  $\deg(T_n) := n$ . Then*

$$\mathcal{C}(S) \cong \mathbb{C}[T_1, T_2, \dots]$$

as graded rings, where the isomorphism, call it  $\Phi$ , maps  $1_{C_\lambda}$  (the indicator of the conjugacy class  $C_\lambda$ ) to  $\frac{1}{z_\lambda} T_{\lambda_1} \cdots T_{\lambda_l}$ .

Let  $T_\lambda := T_{\lambda_1} \cdots T_{\lambda_l}$ . Then we can also write this isomorphism as

$$\Phi(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) T_{sh(\sigma)} = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} f(\sigma) T_\lambda \quad \text{any } \sigma \in C_\lambda.$$

The map  $\Phi$  is a degree preserving isomorphism.

*Proof.* The map is bijective since it maps a basis for  $\mathcal{C}(S_n)$  to a basis for degree  $n$  polynomials in  $T_1, T_2, \dots$ . We see that it is degree preserving since  $\deg(T_\lambda) = \lambda_1 + \lambda_2 + \dots = n$  for  $\lambda \vdash n$ .

Let  $\psi^{(n)} : S_n \rightarrow \mathbb{C}[T_1, T_2, \dots]$  map  $\sigma$  to  $T_{sh(\sigma)}$ . This is a " $\mathbb{C}[T_1, T_2, \dots]$ "-valued class function. If  $\sigma \in S_n$  and  $\tau \in S_m$  and  $\gamma$  is the corresponding permutation in  $S_{n+m}$ , then  $\psi^{(n+m)}(\gamma) = T_{sh(\sigma\tau)} = T_{sh(\sigma)} T_{sh(\tau)} = \psi^{(n)}(\sigma) \psi^{(m)}(\tau)$ . We can interpret this as saying  $\text{Res}_{S_n \times S_m}^{S_{n+m}} (\psi^{(n+m)}) = \psi^{(n)} \boxtimes \psi^{(m)}$ .

Then we have

$$\Phi(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) T_{sh(\sigma)} = \langle f, \psi^{(n)} \rangle_{S_n}.$$

Using Frobenius reciprocity:

$$\begin{aligned} \Phi(f * g) &= \Phi \left( \text{Ind}_{S_n \times S_m}^{S_{n+m}} (f \boxtimes g) \right) = \langle \text{Ind}_{S_n \times S_m}^{S_{n+m}} (f \boxtimes g), \psi^{(n+m)} \rangle_{S_{n+m}} \\ &= \langle f \boxtimes g, \text{Res}_{S_n \times S_m}^{S_{n+m}} (\psi^{(n+m)}) \rangle_{S_n \times S_m} \\ &= \langle f, \psi^{(n)} \rangle_{S_n} \langle g, \psi^{(m)} \rangle_{S_m} \\ &= \Phi(f) \Phi(g) \end{aligned}$$

showing that  $\Phi$  is an isomorphism. □

## 2.2 Symmetric polynomials and Frobenius map

It turns out there is a more combo-friendly way of viewing  $\mathbb{C}[T_1, T_2, \dots]$  that involves using the ring of symmetric functions. **Symmetric polynomials** are polynomials,  $f(x_1, \dots, x_k)$  invariant under re-ordering the variables:  $f(x_1, \dots, x_k) = f(x_{\sigma(1)}, \dots, x_{\sigma(k)})$  for all  $\sigma \in S_k$ . **Symmetric functions** are a generalization of this to an infinite variable set  $f(x_1, x_2, \dots)$ . Let  $\Lambda^{(n)}$  be the set of homogeneous degree  $n$  symmetric functions and  $\Lambda$  the set of all symmetric functions.

One simple basis for  $\Lambda^{(n)}$  is the monomial basis, indexed by partitions  $\lambda \vdash n$ . Essentially, one takes the symmetric function with the least number of terms containing the monomial  $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ :

$$m_\lambda(x_1, x_2, \dots) := \sum_{\substack{i_1 < \dots < i_l \\ \alpha \in \mathcal{R}(\lambda)}} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}$$

where  $\mathcal{R}(\lambda) = \{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(l)}) : \sigma \in S_l\}$  the set of rearrangements of parts in  $\lambda$ .

When expanding a symmetric function in the monomial basis

$$f(x_1, \dots) = \sum_{\lambda \vdash n} a_\lambda m_\lambda(x_1, \dots)$$

we may compute the coefficient  $a_\lambda$ , by simply reading off the coefficient of  $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$ , which is why this *sometimes* the easiest basis to work with.

Another basis is the power sum basis defined as follows. The power sum symmetric functions are

$$p_k(x_1, x_2, \dots) := x_1^k + x_2^k + \dots$$

which are used to define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}.$$

It turns out that  $\{p_\lambda : \lambda \vdash n\}$  form a basis for  $\Lambda^{(n)}$ .

Notice then that

$$\dim(\Lambda^{(n)}) = \#\text{partitions of } n = \#\text{conj classes of } S_n = \#\text{irreps of } S_n$$

It turns out that since the  $p_k$  are algebraically independent,  $\Lambda \cong \mathbb{C}[T_1, T_2, \dots]$ , by sending  $p_k$  to  $T_k$  and notice that  $\deg(p_k) = k = \deg(T_k)$ . This motivates the definition of the **Frobenius map**, which is also degree-preserving ring isomorphism:

$$\mathcal{F} : \mathcal{C}(S) \rightarrow \Lambda$$

that maps  $f \in \mathcal{C}(S_n)$  to

$$\mathcal{F}(f) := \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) \cdot p_{sh(\sigma)}.$$

[Fundamental property of the Frobenius map] Let  $f \in \mathcal{C}(S_n)$  and  $g \in \mathcal{C}(S_m)$ , then

$$\mathcal{F}(f * g) = \mathcal{F}(\text{Ind}_{S_n \times S_m}^{S_{n+m}}(f \boxtimes g)) = \mathcal{F}(f)\mathcal{F}(g)$$

**Examples:** Let's consider the trivial and sign representations. The character of  $\text{triv}_{S_n}$  is 1 on every conjugacy class and so

$$\mathcal{F}(\text{triv}_{S_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} p_{sh(\sigma)} = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$$

Let  $h_n(x_1, x_2, \dots) := \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}$ , which has the following simpler description

$$h_n(x_1, x_2, \dots) = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n},$$

the sum containing every possible degree  $n$  monomial exactly once. This is called the **complete homogeneous symmetric function**. We can define  $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_l}$  and it turns out  $h_\lambda$  is yet another basis for  $\Lambda^{(n)}$  as  $\lambda$  ranges over partitions of  $n$ . As a representation we can think of

$$\mathcal{F}^{-1}(h_\lambda) = \text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}^{S_n}(\text{triv}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}).$$

In the case of the sign representation, notice that if  $\lambda = sh(\sigma)$ , for  $\sigma \in S_n$ , then the sign determined by the number of even length cycles,  $\text{sgn}(\sigma) = (-1)^{n-l(\lambda)}$ , where  $l(\lambda)$  is the number of parts in  $\lambda$ . Then

$$\mathcal{F}(\text{sgn}_{S_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) p_{sh(\sigma)} = \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} \frac{p_\lambda}{z_\lambda}.$$

Let  $e_n(x_1, x_2, \dots) := \sum_{\lambda \vdash n} (-1)^{n-l(\lambda)} \frac{p_\lambda}{z_\lambda}$ , which has the following simpler description

$$e_n(x_1, x_2, \dots) = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} \cdots x_{i_n},$$

and is called the **elementary symmetric function**. Again the  $e_\lambda := e_{\lambda_1} \cdots e_{\lambda_l}$  form a basis for  $\Lambda^{(n)}$  and the corresponding representation is

$$\mathcal{F}^{-1}(e_\lambda) = \text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}^{S_n} (\text{sgn}_{S_{\lambda_1}} \boxtimes \dots \boxtimes \text{sgn}_{S_{\lambda_l}}).$$

Exercise: Show that  $\mathcal{F}(\chi^{\text{reg}}) = h_{(1^n)}$ , the image of the regular representation. Also notice that  $h_{(1^n)} = e_{(1^n)} = p_{(1^n)}$  (you can prove justifying why all three  $\mathcal{F}^{-1}$  of these symmetric functions give the same character,  $\chi^{\text{reg}}$ ).

### 2.3 Transferring the inner product

We have the graded ring isomorphism  $\mathcal{F} : \mathcal{C}(S) \rightarrow \Lambda$  and an inner product on  $\mathcal{C}(S)$ ,  $\langle f, g \rangle_{S_n}$  which can be extended to  $\mathcal{C}(S)$  by making characters from different  $S_n$  orthogonal. But we have no inner product on  $\Lambda$  on symmetric functions. So let's just carry it over. We know that

$$\langle 1_{C_\lambda}, 1_{C_\mu} \rangle_{S_n} = \begin{cases} \frac{|C_\lambda|}{n!} = \frac{1}{z_\lambda}, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}$$

so this induces the following inner product on  $\Lambda$

$$\left\langle \frac{p_\lambda}{z_\lambda}, \frac{p_\mu}{z_\mu} \right\rangle = \langle 1_{C_\lambda}, 1_{C_\mu} \rangle_{S_n},$$

or in other words  $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda, \mu}$ . Since the  $p_\lambda$  form a basis this extends to an inner product on symmetric functions.

Here are two identities involving this new inner product

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda, \mu}$$

and

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu},$$

which is expected since we claimed that  $\mathcal{F}(s_\lambda) = \text{Irr}^\lambda$  and we know that irrep characters are orthonormal. We can go through the proofs of these two identities using the Cauchy kernel method.

---

Here is a clutch example application of this inner product to figuring out the irrep decomposition of  $\mathcal{F}^{-1}(h_\lambda) = H^\lambda = \text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}^{S_n} (\text{triv}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}})$ . Let

$$h_\mu = \sum_{\lambda \vdash n} A_{\lambda, \mu} s_\lambda,$$

for some unknown  $A_{\lambda, \mu}$  (they are unique since the  $s_\lambda$  form a basis). Then take the inner product of both sides with  $s_\nu$  to get

$$\langle h_\mu, s_\nu \rangle = \sum_{\lambda \vdash n} A_{\lambda, \mu} \delta_{\lambda, \nu} = A_{\nu, \mu}.$$

On the other hand,

$$\langle h_\mu, s_\nu \rangle = \langle h_\mu, \sum_{\gamma \vdash n} K_{\nu, \gamma} m_\gamma \rangle = K_{\nu, \mu}.$$

Therefore,

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda, \mu} s_\lambda,$$

and we have gotten the expansion into irreducible:

$$H^\mu \cong \bigoplus_{\lambda \vdash n} (V^\lambda)^{K_{\lambda, \mu}}.$$

*Proof of important lemma*  $\langle h^\lambda, e^{\lambda'} \rangle = 1$ . Assuming we have used the Cauchy kernel method to show  $\langle h_\lambda, m_\mu \rangle$ , we have enough tools to show that  $\langle h_\lambda, e_{\lambda'} \rangle = 1$ , which implies  $[H^\lambda, E^{\lambda'}] = 1$  completing the proofs that the  $\text{Irr}^\lambda$  defined in the previous section are irreducible.

Expanding  $e_\kappa$  in the monomial basis gives

$$e_\kappa = \sum_{\mu \vdash n} B_{\kappa, \mu} m_\mu,$$

where  $B_{\kappa, \mu}$  is the number of matrices containing 0's and 1's with row sums  $\kappa_1, \kappa_2, \dots$  and column sums  $\mu_1, \mu_2, \mu_3 \dots$ . For example if  $\kappa = (3, 2, 1, 1)$  and  $\mu = (4, 1, 1, 1)$ , then

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

are the three matrices contributing to  $B_{\kappa, \mu} = 3$ . These matrices essentially dictate how to build the monomial  $x^{\mu_1} x^{\mu_2} \dots$  out of  $e_{\kappa_1} e_{\kappa_2} \dots$ . So for the first matrix, the first row says: "pick  $x_1^1 x_2^1 x_3^0 x_4^1 = x_1 x_2 x_4$  from  $e_3$ " the second row says "pick  $x_1 x_3$  from  $e_2$ ", and then the last two rows say "pick  $x_1$  from  $e_1$ ". Together the products yields  $x_1^4 x_2 x_3 x_4$  contributing to  $m_\mu$ .

Then

$$\langle h_\lambda, e_{\lambda'} \rangle = \sum_{\mu \vdash n} B_{\lambda', \mu} \delta_{\lambda, \mu} = B_{\lambda', \lambda}.$$

By staring at the boolean matrices for a bit one can see that there is a unique matrix with row sums  $\lambda'$  and column sums  $\lambda$ , with the 1's essentially giving the shape of  $\lambda$ . If for example  $\lambda = (3, 2)$  and  $\lambda' = (2, 2, 1)$ , we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is the only matrix contributing to  $B_{\lambda', \lambda}$ . Hence  $B_{\lambda', \lambda} = 1$ . □

**Proposition 6.** *If  $V$  is an  $S_n$  representation with character  $\chi^V$ , then  $\dim(V)$  is the coefficient of  $x_1 x_2 \dots x_n$  in  $\mathcal{F}(\chi^V)$ .*

*Proof.* Recall that if  $\chi^V$  is the character of a representation then  $\dim(V) = \chi^V(e)$ . In  $\text{Irr}^n$ , the identity has cycle type  $(1^n)$  and since

$$\mathcal{F}(\chi^V) = \sum_{\lambda \vdash n} \text{"}\chi^V(\lambda)\text{"} \frac{p_\lambda}{z_\lambda},$$

where "  $\chi^V(\lambda)$  " means evaluation at any  $\sigma$  of cycle type  $\lambda$ . We have that

$$\dim(V) = \chi^V(e) = \langle \mathcal{F}(\chi^V), p_{(1^n)} \rangle,$$

but since  $p_{(1^n)} = h_{(1^n)}$ , this is the same as  $\langle \mathcal{F}(\chi^V), h_{(1^n)} \rangle$ . Expanding  $\mathcal{F}(\chi^V)$  in the monomial basis, and using the fact that  $\langle h_\mu, m_\lambda \rangle = \delta_{\lambda, \mu}$ . We have that this inner product is the coefficient of  $m_{(1^n)}$  in the monomial expansion. Then this follows by the definition of the monomial basis.  $\square$

Notice that the coefficient of  $x_1 x_2 \cdots x_n$  in  $s_\lambda$  is  $\#\text{SSYT}(\lambda, (1^n)) = \#\text{SYT}(\lambda)$ , which agrees with what we said the dimension of the irrep  $\text{Irr}^\lambda$ .

**Proposition 7.** *The monomial coefficient of  $x^\mu$  in  $\mathcal{F}(V)$  is equal to the  $\dim(V^{S_\mu})$ , the  $S_\mu$ -invariants, where  $S_\mu = S_{\mu_1} \times \cdots \times S_{\mu_l}$ .*

*Proof.*

$$\begin{aligned} \mathcal{F}(V)_{x^\mu} &= [m_\mu] \mathcal{F}(V) \quad \text{expanded in monomial basis} \\ &= \langle \mathcal{F}(V), h_\mu \rangle \\ &= [V, \text{Ind}_{S_\mu}^{S_n}(1)] \\ &= [V|_{S_\mu}, 1_{S_\mu}] \\ &= \dim(V^{S_\mu}) \end{aligned}$$

$\square$

## 2.4 Cauchy Kernel method

We say a pair of basis of symmetric functions,  $\{u_\lambda\}, \{v_\lambda\}$  are dual if  $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda, \mu}$ . Let  $X = x_1, x_2, \dots$  and  $Y = y_1, y_2, \dots$  denote two variable sets.

**Proposition 8** (Cauchy Kernel method). *Consider the dual basis  $\{p_\lambda\}, \{p_\mu/z_\mu\}$ , where indeed  $\langle p_\lambda, p_\mu/z_\mu \rangle = \delta_{\lambda, \mu}$  and define*

$$\Omega_n(X, Y) := \sum_{\lambda \vdash n} p_\lambda(X) \frac{p_\lambda(Y)}{z_\lambda}.$$

*Then any other pair of basis  $\{u_\lambda\}, \{v_\lambda\}$  are dual if and only if*

$$\sum_{\lambda \vdash n} u_\lambda(X) v_\lambda(Y) = \Omega_n(X, Y).$$

*Note that  $\Omega_n(X, Y) = \Omega_n(Y, X)$ .*

*Proof.* We will show this through the following claim: for all  $f \in \Lambda^{(n)}$

$$\langle \Omega_n(X, Y), f(X) \rangle_{\Lambda_X^{(n)}} = f(Y),$$

where we indicate that the inner product is for symmetric functions in variables  $X$  (it sees  $y_i$  as scalars). In other words, taking the inner product with the Cauchy kernel performs a *variable substitution*.

Writing  $f = \sum_{\lambda \vdash n} a_\lambda p_\lambda$ , we have

$$\begin{aligned} \langle \Omega_n(X, Y), f(X) \rangle_{\Lambda_X^{(n)}} &= \sum_{\lambda, \mu \vdash n} p_\lambda(Y) \left\langle \frac{p_\lambda(X)}{z_\lambda}, a_\mu p_\mu(X) \right\rangle \\ &= \sum_{\lambda, \mu \vdash n} p_\lambda(Y) a_\mu \delta_{\lambda, \mu} \\ &= \sum_{\lambda \vdash n} a_\lambda p_\lambda(Y) = f(Y). \end{aligned}$$

Hence if  $\sum_{\lambda \vdash n} u_\lambda(X)v_\lambda(Y) = \Omega_n(X, Y)$ , then

$$\begin{aligned} v_\lambda(Y) &= \langle v_\lambda(X), \Omega_n(X, Y) \rangle \\ &= \sum_{\mu \vdash n} \langle v_\lambda(X), u_\mu(X) \rangle v_\mu(Y) \end{aligned}$$

and since the  $v_\lambda$  forms a basis, comparing coefficients above shows that  $\langle v_\lambda(X), u_\mu(X) \rangle = \delta_{\lambda, \mu}$ .

Conversely,  $\langle v_\lambda(X), u_\mu(X) \rangle = \delta_{\lambda, \mu}$ , then using spanning property of basis we have

$$\Omega_n(X, Y) = \sum_{\lambda} f_\lambda(Y)u_\lambda(X),$$

for some  $f_\lambda(Y) \in \Lambda_Y^{(n)}$ . Then taking the  $\langle \cdot, v_\mu(X) \rangle$  we see that  $f_\lambda(Y) = v_\mu(Y)$  completing the proof.  $\square$

What is the Cauchy kernel? Well we can define the variable set  $XY = \{x_i y_j : i, j = 1, 2, \dots\}$ . Notice that

$$p_k(X)p_k(Y) = \left( \sum_i x_i^k \right) \left( \sum_j y_j^k \right) = \sum_{i,j} (x_i y_j)^k = p_k(XY)$$

and so  $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$  implies that  $p_\lambda(X)p_\lambda(Y) = p_\lambda(XY)$ . Then

$$\Omega_n(X, Y) = \sum_{\lambda \vdash n} p_\lambda(X)p_\lambda(Y)/z_\lambda = \sum_{\lambda \vdash n} p_\lambda(XY)/z_\lambda = h_n(XY).$$

Thus showing that  $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda, \mu}$  and  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}$  is equivalent to showing

$$\sum_{\lambda \vdash n} h_\lambda(X)m_\mu(Y) = h_n(XY) = \sum_{\lambda \vdash n} s_\lambda(X)s_\lambda(Y).$$

## 2.5 The irreps of $S_n$ via representations of polynomials

Consider the polynomial ring  $\mathbb{C}[y_1, \dots, y_n]$ , which forms an infinite dimensional  $S_n$  representation via the action  $\sigma \cdot y_i = y_{\sigma(i)}$ . Recall the Vandermonde determinant

$$\Delta(t_1, \dots, t_k) = \det(t_i^{j-1})_{i,j=1, \dots, k} = \prod_{i < j} (t_i - t_j).$$

Now consider the  $S_n$ -submodule  $\text{Span}_{\mathbb{C}}(\Delta(y_1, \dots, y_n)) \leq \mathbb{C}[y_1, \dots, y_n]$ , which is closed under the  $S_n$  action since

$$\sigma \cdot \Delta(y_1, \dots, y_n) = \text{sgn}(\sigma)\Delta(y_1, \dots, y_n)$$

so this is a 1-dimensional representation and in fact the sign representation of  $S_n$ .

An *injective tableau* of the diagram corresponding to  $\lambda$  is a filling of the cells of the box diagram for  $\lambda$  with the numbers  $1, \dots, n$  all appearing once. For example if  $\lambda = (4, 3, 1)$ , one could take

$$T = \begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 4 & 3 & 1 & \\ \hline 6 & 2 & 7 & 8 \\ \hline \end{array}$$

We will say  $\text{Inj}(\lambda)$  is the set of injective tableau of  $\lambda$ . Let  $\Delta_T$  called **Garnir polynomials** be the product of Vandermonde determinants of variables corresponding to columns of this injective tableau:

$$\begin{aligned} \Delta_T &= \Delta(y_6, y_4, y_5)\Delta(y_2, y_3)\Delta(y_7, y_1)\Delta(y_8) \\ &= (y_6 - y_4)(y_6 - y_5)(y_4 - y_5)(y_2 - y_3)(y_7 - y_1), \end{aligned}$$

where  $\Delta(y_8) = 1$  so columns of height 1 contribute nothing.

Here is the definition for the irreducible representation of  $S_n$  corresponding to partition  $\lambda$ :

$$\text{Irr}^\lambda := \text{Span}_{\mathbb{C}}(\Delta_T(y_1, \dots, y_n) : T \in \text{Inj}(\lambda)).$$

We can discuss on Friday how this is just a different way of picturing the construction in Sagan section 2.3.

So the example from earlier is

$$\text{Irr}^{(1,1,\dots,1)} = \text{Span}_{\mathbb{C}}(\Delta(y_1, \dots, y_n)),$$

since all the  $\Delta_T$  for  $T \in \text{Inj}(\lambda)$  are linearly dependent (they are all  $\pm$  each other).

Note that because  $\text{Irr}^\lambda$  ends up being irreducible you can pick any one  $\Delta_T$  and just take the span of  $\sigma \Delta_T$  ranging over  $\sigma \in S_n$ . You can also define an action of  $S_n$  on  $\text{Inj}(\lambda)$  directly by letting  $\sigma \cdot T$  be the tableau with the cell containing  $i$  being replaced by  $\sigma(i)$  for  $i = 1, \dots, n$  so for example

$$(1\ 3\ 4) \cdot \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline 2 & 5 & 7 & \\ \hline 1 & 4 & 6 & 8 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 2 & 5 & 7 & \\ \hline 3 & 1 & 6 & 8 \\ \hline \end{array}$$

and observe that  $\sigma \cdot \Delta_T = \Delta_{\sigma \cdot T}$ .

As was mentioned for the sign representation, the  $\Delta_T$  are generally not linearly independent. One can rearrange the entries of a column to get a new tableau  $\hat{T}$ , but  $\Delta_{\hat{T}} = \pm \Delta_T$ . One can also swap two columns of equal size to get the same  $\Delta_T$ .

**Proposition 9.** *A basis for  $\text{Irr}^\lambda$  is given by*

$$\{\Delta_T : T \in \text{SYT}(\lambda)\}$$

*Proof.* First, we show that the  $\Delta_T$  span  $\text{Irr}^\lambda$ . Recall that standard Young tableau are tableau with entries increasing down the rows and up columns. Let column strict tableau be tableau that are increasing allong columns and will be denoted by  $\text{CS}(\lambda)$ . By the earlier observation  $\text{Irr}^\lambda$  is spanned by  $\Delta_T$  for  $T \in \text{CS}(\lambda)$ . A crucial step in the proof is establishing the **Garnir relation**.

Suppose that a  $T \in \text{CS}(\lambda)$  is not row strict and so it contains a pair of adjacent values  $a > b$  in some row. Let  $C$  denote the column values in  $a$ 's column and  $C'$ , the column values in  $b$ 's column. Let  $A \subset C$  be the values including and above  $a$  and let  $B \subset C'$  be the value including and below  $b$ . We will abuse notation and let  $\binom{A \cup B}{|A|}$  denote the set of permutation of  $A \cup B$  that leave the values ending up in  $A$  and  $B$  in order (this set does in fact have cardinality  $\binom{|A|+|B|}{|A|}$ ). Then the Garnir relation says that

$$\sum_{\pi \in \binom{A \cup B}{|A|}} \text{sgn}(\pi) \pi \Delta_T = 0.$$

For example if

$$T = \begin{array}{|c|c|c|} \hline 5 & & \\ \hline 4 & 2 & \\ \hline 3 & 1 & 6 \\ \hline \end{array}$$

with  $a = 4$  and  $b = 2$ , then  $A = \{4, 5\}$ ,  $B = \{1, 2\}$  and the possible  $\pi$  consisting of  $\text{id}$ ,  $(42)$ ,  $(124)$ ,  $(542)$ ,  $(5412)$ ,  $(52)(41)$  corresponding to

$$\Delta \begin{array}{|c|c|c|} \hline 5 & & \\ \hline 4 & 2 & \\ \hline 3 & 1 & 6 \\ \hline \end{array} - \Delta \begin{array}{|c|c|c|} \hline 5 & & \\ \hline 2 & 4 & \\ \hline 3 & 1 & 6 \\ \hline \end{array} + \Delta \begin{array}{|c|c|c|} \hline 5 & & \\ \hline 1 & 4 & \\ \hline 3 & 2 & 6 \\ \hline \end{array} + \Delta \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 5 & \\ \hline 3 & 1 & 6 \\ \hline \end{array} - \Delta \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 5 & \\ \hline 3 & 2 & 6 \\ \hline \end{array} + \Delta \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 5 & \\ \hline 3 & 4 & 6 \\ \hline \end{array} = 0$$

The proof of this relation is as follows. Without loss of generality we can forget about columns outside of  $C, C'$  since these can be factored out of the relation and are not affected by any  $\pi$  and so it suffices to show

$$\sum_{\pi \in \binom{A \cup B}{|A|}} \operatorname{sgn}(\pi) \pi \Delta_C \Delta_{C'} = 0.$$

The above sum expands as

$$\sum_{\pi \in \binom{A \cup B}{|A|}} \operatorname{sgn}(\pi) \pi \sum_{\substack{\sigma \in S_C \\ \sigma' \in S_{C'}}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma') x^{\sigma(C)} x^{\sigma'(C')},$$

where if  $C = \{c_1 < c_2 < \dots\}$ , we let  $x^{\sigma(C)} = x_{\sigma(c_1)}^0 x_{\sigma(c_2)}^1 \dots$ . Each  $\sigma \in S_C$  factors uniquely as  $\sigma_A \bar{\sigma}_A$ , where  $\sigma_A \in S_A$  and  $\bar{\sigma}_A \in S_C$  maintaining the relative order of the values in  $A$ . Applying these factorizations we get

$$\sum_{\substack{\pi \in \binom{A \cup B}{|A|} \\ \sigma_A \in S_A \\ \sigma_B \in S_B}} \operatorname{sgn}(\pi \sigma_A \sigma_B) \pi \sigma_A \sigma_B \sum_{\bar{\sigma}_A, \bar{\sigma}_B} \operatorname{sgn}(\bar{\sigma}_A \bar{\sigma}_B) x^{\bar{\sigma}_A C} x^{\bar{\sigma}_B C'},$$

which simplifies as

$$\sum_{\tau \in S_{A \cup B}} \operatorname{sgn}(\tau) \tau \sum_{\bar{\sigma}_A, \bar{\sigma}_B} \operatorname{sgn}(\bar{\sigma}_A \bar{\sigma}_B) x^{\bar{\sigma}_A C} x^{\bar{\sigma}_B C'}$$

and as it turns out  $\sum_{\tau \in S_{A \cup B}} \operatorname{sgn}(\tau) \tau$  kills each monomial  $x^{\bar{\sigma}_A C} x^{\bar{\sigma}_B C'}$ . Indeed by Pigeonhole principal,  $|A \cup B| = |C| + 1$  but there are only  $|C|$  possible exponents for the variables occurring in monomial  $x^{\bar{\sigma}_A C} x^{\bar{\sigma}_B C'}$ . Hence there is a pair  $i, j \in A \cup B$ , with the same degree

$$x^{\bar{\sigma}_A C} x^{\bar{\sigma}_B C'} = \dots x_i^k x_j^k \dots$$

but then for each  $\tau \in S_{A \cup B}$ , we have  $\tau(i, j) \in S_{A \cup B}$  and a sign-reversing involution shows  $\sum_{\tau \in S_{A \cup B}} \operatorname{sgn}(\tau) \tau$  kills the monomial.

The use of the Garnir relation is that it removes the row defect  $a > b$  (all other tableaux in the relation don't have this row strictness failure in this location and none of them pick up new row defects). By repeatedly using this relation we decrease the number of row defects. Thus, any  $\Delta_T$  for  $T \in CS(\lambda)$  is a linear combination of  $\Delta_{T'}$  for  $T' \in SYT(\lambda)$ .

The linear independence of the  $\Delta_T$  for  $T \in SYT(\lambda)$  is fairly straightforward, since  $\Delta_T$  is the only element in the proposed basis that has  $x^T = x^{c_1} \dots x^{c_l}$  as a monomial.  $\square$

As a corollary we immediately get

$$\dim(\operatorname{Irr}^\lambda) = |\operatorname{SYT}(\lambda)|,$$

the number of standard Young tableau.

Exercise: Check that  $\operatorname{Irr}^{(n)}$  is the trivial representation and check that  $\operatorname{Irr}^{(n-1,1)}$  is the standard irreducible representation  $\{a_1 e_1 + \dots + a_n e_n : a_1 + \dots + a_n = 0\}$ .

More generally you can consider the ‘‘hook shape’’:  $\lambda = (n - k, 1^k)$  (here  $1^k$  means 1  $k$ -times:  $(n - k, 1, 1, \dots)$ ). This is a standard way of writing a repeating part of a partition). What are the  $T \in \operatorname{SYT}(\lambda)$  and basis elements  $\Delta_T$  in this case? What is the dimension?

(Tricky): Letting  $W = \langle e_1, \dots, e_n \rangle$  and  $S_n$  acting on the  $e_i$  by permuting, justify why for  $k = 1, \dots, n - 1$

$$\wedge^k W \cong \operatorname{Irr}^{(n-k, 1^k)} \oplus \operatorname{Irr}^{(n-k+1, 1^{k-1})},$$

are isomorphic  $S_n$  representations, where the left hand side is the  $k$ -fold wedge product (spanned by  $e_{i_1} \wedge \dots \wedge e_{i_k}$  with the usual wedge product rules, so like  $(1 \ 2) \cdot e_1 \wedge e_2 = -e_2 \wedge e_1$ ). Notice how we already

saw the special case  $k = 1$ :  $W \cong \text{Irr}^{(n-1,1)} \oplus \text{Irr}^{(n)}$  for the standard representation. By taking dimensions of both sides, the above is the “representation-ification” of the binomial identity:  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ .

One approach that may or may not help is that

$$\wedge^k W \cong \text{Ind}_{S_{n-k} \times S_k}^{S_n} (\text{triv}_{S_{n-k}} \boxtimes \text{sgn}_{S_k}).$$

## 2.6 Proof of irreducibility

Here is the main idea for showing why the  $\text{Irr}^\lambda$  precisely range through the irreducible representations of  $S_n$ .  $\text{Irr}^\lambda$  can be viewed as a submodule of

$$\text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}^{S_n} (\text{triv}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}).$$

Consider the monomials that show up by expanding  $\Delta_T$  for some  $T$ . A monomial that shows up is of the form  $\pm y_1^{a_1} \dots y_n^{a_n}$ , where form some injective tableau  $T$ ,  $i$  appears in row  $a_i + 1$ . Call this monomial  $m(T)$  and let  $H^\lambda$  be the span of such monomials

$$H^\lambda = \text{Span}_{\mathbb{C}}(y_1^{a_1} \dots y_n^{a_n} : T \in \text{Inj}(\lambda), i \text{ in row } a_i + 1)$$

So for

$$T = \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 2 & 5 & 7 \\ \hline 3 & 1 & 6 \\ \hline \end{array}$$

$m(T) = y_2 y_5 y_7 y_4^2$ . Note that  $m(T)$  may arise from  $\Delta_{T'}$  for a different  $T'$  (in particular  $m(T)$  shows up in the expansion of  $\Delta_{T'}$  if and only if there is a permutation only within rows so that  $\sigma T' = T$ ). Also note that you might get the same  $m(T)$  for different  $T$ . In particular  $m(T) = m(T')$  if and only if there is a  $\sigma$  permuting only within rows of  $T$  such that  $\sigma T = T'$ . This essentially shows that

$$H^\lambda \cong \text{Ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}^{S_n} (\text{triv}_{S_{\lambda_1} \times \dots \times S_{\lambda_l}}).$$

and we have  $\text{Irr}^\lambda \subseteq H^\lambda$ .

See page 20 of Sagan for the another way of thinking about the module  $H^\lambda$  via  $\lambda$ -tabloids (page 55-56). Exercise: Show that  $H^{(1^n)}$  is the regular representation of  $S_n$  (example 2.1.7 in Sagan).

It also turns out that  $\text{Irr}^\lambda$  can be viewed as a submodule of

$$E^{\lambda'} = \text{Ind}_{S_{\lambda'_1} \times \dots \times S_{\lambda'_l}}^{S_n} (\text{sgn}_{S_{\lambda'_1}} \boxtimes \dots \boxtimes \text{sgn}_{S_{\lambda'_l}}),$$

where  $\lambda'$  is the conjugate partition of  $\lambda$ . We prove this using character theory starting with the following lemma.

**Lemma 4.** *If  $S_{\{1, \dots, n\}}$  acts on a submodule,  $V$  of  $\mathbb{C}[y_1, \dots, y_n]$  and  $S_{\{n+1, \dots, m\}}$  act on a submodule  $W$  of  $\mathbb{C}[y_{n+1}, \dots, y_m]$  and now let*

$$V.W := \text{Span}_{\mathbb{C}}(\sigma \cdot (fg) : \sigma \in S_{n+m}, f \in V, g \in W),$$

an  $S_{n+m}$  representation. It turns out that

$$[V.W, \text{Ind}_{S_n \times S_m}^{S_{n+m}} (V \otimes W)] \geq [V, V]_{S_n} [W, W]_{S_m}.$$

*Proof.*

$$\begin{aligned} [V.W, \text{Ind}_{S_n \times S_m}^{S_{n+m}} (V \otimes W)]_{S_{n+m}} &= [\text{Res}_{S_n \times S_m}^{S_{n+m}} (V.W), V \otimes W]_{S_n \times S_m} \\ &\geq [\text{Span}(fg : f \in V, g \in W), V \otimes W]_{S_n \times S_m} \\ &= [V \otimes W, V \otimes W]_{S_n \times S_m} \\ &= [V, V]_{S_n} [W, W]_{S_m}. \end{aligned}$$

□

Observe that by definition of  $\Delta_T$ , we have that  $\text{Irr}^\lambda$  may be viewed in light of Lemma 4 as  $\text{Irr}^\lambda = V_1.V_2.V_3 \cdots$ , where  $V_1 = \mathbb{C}\Delta(y_1, \dots, y_{\lambda'_1})$  and so on. Thus, using lemma 4 repeatedly one gets

$$[\text{Irr}^\lambda, E^{\lambda'}] \geq [\text{sgn}, \text{sgn}]_{S_{\lambda'_1}} \cdots = 1.$$

Thus,  $\text{Irr}^\lambda \subseteq H^\lambda$  shares at least one irrep with  $E^{\lambda'}$ . The final step in the proof follows since  $\langle H^\lambda, E^{\lambda'} \rangle = \langle h_\lambda, e_{\lambda'} \rangle = 1$ , i.e. there is a unique irrep occurring both in  $H^\lambda$  and  $E^{\lambda'}$ . But then this irrep has to be  $\text{Irr}^\lambda$  and in particular  $\text{Irr}^\lambda$  has to be an irrep.

## 2.7 Connection to SSYT and Schur functions

**Proposition 10.** *A basis for the  $S_\mu$ -invariants  $(\text{Irr}^\lambda)^{S_\mu}$  is given by*

$$\left\{ \sum_{\sigma \in S_\mu} \sigma T : T \in \text{SYT}(\lambda), \text{Des}(T) \subset \{\mu_1, \mu_1 + \mu_2, \dots\} \right\}.$$

The  $T$  in the above set arise bijection from  $\text{SSYT}(\lambda, \mu)$  by applying standardization. In particular

$$\dim((\text{Irr}^\lambda)^{S_\mu}) = \#\text{SSYT}(\lambda, \mu) = K_{\lambda, \mu}$$

and so

$$\mathcal{F}(\text{Irr}^\lambda) = \sum_{\mu} K_{\lambda, \mu} m_\mu = s_\lambda.$$

*Proof.* In general, the space of  $S_\mu$ -invariants is obtained by applying the averaging operator  $\sum_{\sigma \in S_\mu} \sigma$  to all  $\Delta_T \in \text{Irr}^\lambda$  for  $T \in \text{SYT}(\lambda)$ . The operator will kill  $\Delta_T$  if entries of the same  $\mu$ -block ( $\mu$ -blocks are  $\{1, \dots, \mu_1\}, \{\mu_1 + 1, \dots, \mu_1 + \mu_2\}, \dots$ ) appear in the same column of  $T$ . Since  $T$  is a standard Young tableau this implies that the  $\mu$ -blocks appear as horizontal strips in  $T$  and hence  $T$   $\mu$ -destanderdizes to an element of  $\bar{T} \in \text{SSYT}(\lambda, \mu)$ . Amongst, the various  $T' \in \text{SYT}(\lambda)$  which  $\mu$ -destanderdize to  $\bar{T}$ ,  $T$  being the standerdization of  $T'$  is an  $S_\mu$ -orbit representative.

Hence, this shows that all  $T \in \text{SYT}(\lambda)$ , have  $\Delta_T$  are either killed by the averaging operator or appear in a unique sum in the proposed basis based on the  $\mu$ -destanderdization. By comparing monomials as before, we see linear independence of the proposed basis elements.  $\square$

The deep and surprising fact about Schur functions is that they not only show up naturally in the study of  $S_n$ -representation theory but also as characters in the theory of  $GL_k$ -representation theory (but they will show up in a different way: literally as functions where the  $x_i$  are replaced by eigenvalues of matrices). This connection comes from something called Schur-Weyl duality.

## 3 Principal specialization of $s_\lambda$

Let's show that

$$s_\lambda(1, q, q^2, \dots) = \frac{1}{(1-q) \cdots (1-q^n)} \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)},$$

where  $\text{maj}(T) = \sum_{i \in \text{Des}(T)} i$ . Consider the expression

$$A(\vec{X}, q) = \sum_{\lambda \vdash n} \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} s_\lambda(X) = \sum_{\lambda \vdash n} \sum_{\substack{T \in \text{SYT}(\lambda) \\ S \in \text{SSYT}(\lambda)}} q^{\text{maj}(T)} \cdot x_T.$$

We apply the inverse of RSK to pairs  $(T, S)$ , with  $T \in \text{SYT}(\lambda)$  being treated as an recording tableau and  $S \in \text{SSYT}(\lambda)$  as an insertion tableau, the inverse of this RSK yields pairs

$$\begin{array}{cccc} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{array},$$

where  $w = w_1 w_2 \cdots w_n$  is some word whose content is the same as that of  $S$  (same number of 1's 2's etc.). It is easy to generalize the first part of Theorem 5.18 to running RSK on words (insert the word  $w$  to get  $P(w) = S$  and while recording  $1, 2, \dots, n$  to get  $Q(w) = T$ ) instead of just permutations. The word (not tableau) maj of  $w$  is then  $\text{maj}(w) = \text{maj}(Q(w)) = \text{maj}(T)$ . By ranging over all length  $n$  words we get that RSK has bijectively proved

$$\sum_{\lambda \vdash n} \sum_{\substack{T \in \text{SYT}(\lambda) \\ S \in \text{SSYT}(\lambda)}} q^{\text{maj}(T)} \cdot x_T = \sum_w q^{\text{maj}(w)} \cdot x_w.$$

Now let's group up words by content. Say that  $w$  has  $\alpha_1$  1's,  $\alpha_2$  2's and so on ( $\alpha_i = 0$  is allowed). We may write this as  $w$  being a rearrangement  $w \in \mathcal{R}(1^{\alpha_1}, 2^{\alpha_2}, \dots)$  so

$$\sum_w q^{\text{maj}(w)} \cdot x_w = \sum_{\alpha, \sum_i \alpha_i = n} (x_1^{\alpha_1} x_2^{\alpha_2} \cdots) \sum_{w \in \mathcal{R}(1^{\alpha_1}, 2^{\alpha_2}, \dots)} q^{\text{maj}(w)} = \sum_{\alpha, \sum_i \alpha_i = n} (x_1^{\alpha_1} x_2^{\alpha_2} \cdots) \frac{[n]_q!}{[\alpha_1]_q! [\alpha_2]_q! \cdots},$$

where the last part follows from Chapter 1 and we effectivly drop  $\alpha_i = 0$  since  $[0]_q! = 1$ . Now group further by rearranging each  $\alpha$  to be a partion  $\mu \vdash n$  (so for example  $\alpha = (0, 2, 1, 0, 2, 0, \dots)$  becomes  $(2, 2, 1)$ ). Then the sum becomes

$$\sum_{\mu \vdash n} \frac{[n]_q!}{[\mu_1]_q! [\mu_2]_q! \cdots} \sum_{\alpha \text{ rearrange to } \mu} x_1^{\alpha_1} x_2^{\alpha_2} \cdots = \sum_{\mu \vdash n} \frac{[n]_q!}{[\mu_1]_q! [\mu_2]_q! \cdots} m_\mu(X),$$

where the  $q$ -multinomial coefficient becomes

$$\frac{[n]_q!}{[\mu_1]_q! [\mu_2]_q! \cdots} = \frac{\frac{1-q^n}{1-q} \cdots \frac{1-q}{1-q}}{\frac{1-q^{\mu_1}}{1-q} \cdots \frac{1-q}{1-q}} = \frac{(1-q^n) \cdots (1-q)}{(1-q^{\mu_1}) \cdots (1-q)(1-q^{\mu_2}) \cdots (1-q) \cdots}.$$

which can be simplified as

$$\frac{(q)_n}{(q)_{\mu_1} (q)_{\mu_2} \cdots (q)_{\mu_i}},$$

where  $(q)_n = (1-q^n)(1-q^{n-1}) \cdots (1-q)$ . By the exercise

$$h_m(1, q, q^2, \dots) = \frac{1}{(q)_m}$$

and so the above expression becomes

$$(q)_n h_{\mu_1}(1, q, q^2, \dots) \cdots h_{\mu_i}(1, q, q^2, \dots) = (q)_n h_\mu(1, q, q^2, \dots)$$

and our original expression is thus equal to

$$A(X, q) = (q)_n \sum_{\mu \vdash n} m_\mu(X) h_\mu(1, q, q^2, \dots).$$

But now we can use the Cauchy kernel with  $X = (x_1, x_2, \dots)$  and  $Y = (1, q, q^2, \dots)$  to get

$$(q)_n \sum_{\mu \vdash n} m_\mu(X) h_\mu(1, q, q^2, \dots) = (q)_n \Omega(X, Y) = (q)_n \sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(1, q, q^2, \dots).$$

This equals the original expression

$$\sum_{\lambda \vdash n} \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} s_\lambda(X) = (q)_n \sum_{\lambda \vdash n} s_\lambda(X) s_\lambda(1, q, q^2, \dots)$$

and since  $s_\lambda(X)$  is a basis for symmetric functions (here technically the coefficients are power series in  $q$ ), the coefficients of  $s_\lambda(X)$  above must agree and

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = (q)_n s_\lambda(1, q, q^2, \dots),$$

which is equivalent to Thm 5.19.