For $1 \leq k \leq n$, define Boolean Product Polynomial

$$B_{n,k}(X_n) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (x_{i_1} + \cdots + x_{i_k})$$

Example: $B_{4,2}(X_4) = (x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)$

2. $B_{n,1}(X_n) = S_n(X_n)$

Define $B_n(X_n) := \prod_{k=1}^{n} B_{n,k}(X_n)$

$B_{n,k}$ is symmetric by def, so $B_n$ is also symmetric. It's not obvious why it's Schur Pos.

**Explaining some terms:**

Let $X$ be SM: CPX proj. variety, $H^*(X)$ be its cohomology with $\mathbb{Z}$-coeff.

**Cohomology:** Associates $X$ with a graded ring structure, which is a warf invariant.

Example: $H^*(S^n) \cong \mathbb{Z}[x]/x^2$ where $|x| = n$

Explicitly, $H^i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{else} \end{cases}$

$GP^n = \{ \text{lines through origin in } \mathbb{C}^{n+1} \}$

$\cong \mathbb{C}^{n+1}/[z_0, \ldots, z_{n+1}] \cong [x^0, \ldots, x^n]$. 

$H^*(GP^n) \cong \mathbb{Z}[x]/x^{n+1}$ where $|x| = 2$

Explicitly, $H^i(GP^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, \ldots, 2n \\ 0 & \text{else} \end{cases}$
**Vector Bundle**

\[ F \to E \to \mathbb{P} \]

is a fibre bundle if \( \forall \mathbb{G}, \exists \text{ open nbhd } \mathbb{U} \in \mathbb{G} \)

\[ \text{St. } \exists \text{ homeom. } \psi : \pi^{-1}(\mathbb{U}) \to \mathbb{U} \times F. \]

\( E \) is the total space, \( B \) is the base space, and \( F \) is the fibre.

A complex vector bundle is a fibre bundle where \( F \cong \mathbb{C}^k \)

is a k-dim. complex V.S.

\( \mathbb{C}^k \) is a line bundle of \( \mathbb{C} \mathbb{P}^n \), and the fibre of a \( \mathbb{P} \in \mathbb{C} \mathbb{P}^n \) is just \( \{ x \in \mathbb{C}^k \mid x \in F \} \).

---

**Chern Class**

Let \( E \to X \) a complex V.B. of rank \( n \), for \( p \in X \) let \( E_p \cong \mathbb{C}^k \)

be its fibre.

For \( 1 \leq i \leq n \) we have the Chern class \( c_i(\mathcal{E}) \in H^{2i}(X) \).

The sum of the Chern classes in \( H^*(X) \) is the total Chern class

\[ c(\mathcal{E}) := 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots + c_n(\mathcal{E}). \]

If \( E \to X, F \to X \) are two V.B., we can form their direct sum bundle by the rule \( (E \oplus F)_p := E_p \oplus F_p \) at each \( p \in X \). Then the rank would be

\[ \text{rank}(E \oplus F) = \text{rank} E + \text{rank} F. \]
and their total Chern class is
\[ C*(E\oplus F) = C*(E) \cdot C*(F) \]

A line bundle \( E \) is just a V.B. with rank 1. Then it has Chern class \( c_1(E) = x_1 \in H^2(X) \).

Let \( E \to X \) be a rank \( n \) V.B. if \( E = E_1 \oplus \cdots \oplus E_n \), where each \( E_i \) is a line bundle, then it guarantees that total Chern class of \( E \) factors as
\[ C*(E) = (h(x_1)) \cdots (h(x_n)) \]

where \( x_i = c_1(E_i) \in H^2(X) \).

By Splitting Principle, even if we can't write \( E = E_1 \oplus \cdots \oplus E_n \), such factorization still exists.

\( x_1, \ldots, x_n \in H^2(X) \) are the Chern roots of \( E \).

**Chern plethysm**

For \( F \in \text{Sym} \), define \( F(E) \) to be result of

- Plugging in Chern roots \( x_1, \ldots, x_n \) into \( \text{Sym} \) arguments
- Set all other arguments to be 0.

i.e. \( F(E) \) evaluates \( F \) at Chern roots of \( E \).

If \( F \in \Lambda^d \) homog. of degree \( d \), then
\[ F(E) = F(x_1, \ldots, x_n) \in H^{2d}(X) \]
is a poly. of degree $d$ in the $x_i$; the degree $d$ is indep. of rank of $E$.

E.g., $x_1, x_2, x_3$ are Chern roots of $E$. $F = M_{22}$, $d = 4$.

$x_1^3 x_2 + x_1 x_2^3 + x_1^2 x_3^2 \in H^{2\times 4}(X)$.

Rearrange fiberwise operation like tensor or dir. sum induces changes in Chern roots.

Let $E$ has Chern roots $x_1, \ldots, x_n$; $F$ has Chern roots $y_1, \ldots, y_m$.

$\text{By }$, \( F(E \otimes F) = F(x_1, \ldots, x_n, y_1, \ldots, y_m) \).

$E \otimes F$ has Chern roots $x_i y_j$ where $1 \leq i \leq n$, $1 \leq j \leq m$.

So \( F(E \otimes F) = F(\ldots, x_i y_j, \ldots) \), $1 \leq i \leq n$, $1 \leq j \leq m$.

Since $F$ is symm., the ordering doesn't matter.

Let $\Lambda$. Apply the Schur Functor $S^\lambda$ to $E$ to get a new v.b. $S^\lambda(E)$ with fiber $S^\lambda(E)_p = S^\lambda(E_p)$.

The Chern roots of $S^\lambda(E)$ are the multiset sums $\sum_{\alpha \in \lambda} x_{T(\alpha)}$, so

\[ F(S^\lambda(E)) = F(\ldots, \sum_{\alpha \in \lambda} x_{T(\alpha)}, \ldots) \].
EX \( n=(2,1), \ n=3, \) then we have SSYT's

\[
\begin{array}{cccc}
1 & 1 & 2 & 1 \\
2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

Then

\[
F(S'' (E)) = F(2x_1 + x_2, x_1 + 2x_2, 2x_1 + x_3, x_1 + x_2 + x_3,
\]

\[
x_1 + x_2 + x_3, 2x_2 + x_3, x_1 + 2x_3, 2(x_2 + x_3)
\]

For \( F, G \in \Lambda, \) \( \alpha, \beta \in \mathbb{C}, \) we have the laws of poly. evalution:

\[
\begin{cases}
(F \cdot G)(E) = F(E) \cdot G(E) \\
(\alpha F + \beta G)(E) = \alpha F(E) + \beta G(E) \\
\alpha(E) = \alpha
\end{cases}
\]

Comparison with classical Ploteysm:

Let \( F\in \Lambda, \) \( E(t_1, t_2, \ldots) \) any rational function, then

\[
P_r[E] = P_r[E(t_1, t_2, \ldots)] = E(t_1^k, t_2^k, \ldots) \quad k \geq 1.
\]

We also have laws of poly. eval. for the classical Ploteysm.

So \( F[E] \) is def. linear.

For \( \text{v.b.} \ E, \) degree of \( F(E) = \deg(F). \)

For \( E \) a poly. of degree \( e, \) \( \deg(F[E]) = e \cdot \deg(F). \)
\[ F(E) = F[x_1, \ldots, x_n] = F[x_1, \ldots, x_n] \]
\[ F(E + F) = F[x_1, \ldots, x_n, y_1, \ldots, y_m] = F[x_1, \ldots, x_n, y_1, \ldots, y_m] \]

But we don't have a natural interpretation of \( F(E \odot F) \) or \( F(S^\lambda(E)) \) in terms of classical plethysms.

**Schur Positivity of B_{\nu_k}**

Thus let \( E_1, \ldots, E_k \) be V.B., and \( \lambda, \mu^{(1)}, \ldots, \mu^{(k)} \in \text{Par}. \)
\[ \sum C_{\mu^{(1)}, \ldots, \mu^{(k)}}^{\lambda, \nu^{(1)}, \ldots, \nu^{(k)}} \in \mathbb{Z}^+ \text{ (ct.} \]

\[ S \lambda(S^{\mu^{(1)}}(E_1) \otimes \cdots \otimes S^{\mu^{(k)}}(E_k)) = \sum C_{\mu^{(1)}, \ldots, \mu^{(k)}}^{\lambda, \nu^{(1)}, \ldots, \nu^{(k)}} S_{\nu^{(1)}}(E_1) \cdots S_{\nu^{(k)}}(E_k) \]

Thus \( B_{\nu_k}(X^n) \) over \( B_n(X^n) \) are Schur Pos.

**Proof:** let \( E \rightarrow X \) be a rank \( n \) V.B. over \( X \) with chem roots \( x_1, \ldots, x_n \). Then the \( k \)-th exterior power \( \wedge^k E \) has chem roots \( \{ x_{i_1} \wedge \cdots \wedge x_{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n \} \)

By the last Thus, \( S_{\lambda}(\wedge^k E) \) is Schur Pos. for any \( \lambda \in \text{Par} \).

In particular, let \( \lambda = (\lambda_1, \ldots, \lambda_l) \) be a column of length \( \binom{n}{\lambda} \),
\[ S_i(\lambda^k \varepsilon) = M_{\mu}(x_1 + \cdots + x_k, x_1 + \cdots + x_k, \ldots) \]

\[ \prod_{i=1}^{k} \text{Bn}(x_i + \cdots + x_k) = \text{Bn}(x_n) \quad \text{for all } 1 \leq i \leq n \]

Thus Bn is Schur POS, and Bn is too by LR-Rule.

\[ c(\lambda^3) = \prod_{1 \leq i \leq j \leq n} (x_i + x_j) = \sum_{\mu \leq S_n} \mu^{(n)} \cdot S_\mu(\lambda^n) \]

\[ c(\text{Sym}^2 \lambda^3) = \prod_{1 \leq i \leq j \leq n} (x_i + x_j) = \sum_{\mu \leq S_n} \sum_{\nu \leq \mu} 2^{\lambda \cdot \nu} \mu^{(n)} \cdot S_\mu(\lambda^n) \]

\( \mu^{(n)} \) is defined to be the # of reverse flagged fillings of shape \( \mu \).

A filling of shape \( \mu \leq S_n \) is reverse flagged if:

- entries of \( \mu \) either strictly along rows, weakly down columns
- entries in row \( i \) of \( \mu \) lies between 1 and \( n-i \)

\[ \prod_{1 \leq i \leq 3} (x_i + x_j) = S_1(\lambda^3) + S_2(\lambda^3) + 2S_{11}(\lambda^3) + S_{21}(\lambda^3) \]
The network $(H^i, d^i)$ is not exactly $B_{n,2}$ but if we sum up the monomials with degree $(n)$ we get $B_{n,2}$.

Done at the end.

The coefficient of $x^j$ is $\det (l^{n_i-j+i})_{1 \leq i, j \leq n}$.

Goal: Show this determinant = reverse flagged fillings of shape $\mu \leq S_{n-1}$.

By Gessel-Viennot, $\det$ counts # of non-intersecting family of paths from $p_i = (2i-2, n-i)$ to $q_i = (n+i-2, n-i+\mu_i)$

Example: $n=5$, $\mu = (3,1,1,1,0)$

Strictly decreasing row by construction.

Steps between $P_i$ and $Q_i = (n+i-\mu_i - 2) - (2i-2) + (n+i-\mu_i) - (n-i)$

$= n-i$

Biggest can be $n-i$.

Non-intersecting $\iff$ Decreasing on columns.
Thus \( \text{B}_{n-1}(x_n) = \sum_{i=1}^{\infty} \alpha_i x_i \) \( \text{S}_{n} = \sum_{\text{WDA} \in \text{Young}(n)} \text{F}(w) \)

where \( \alpha_i = \# \text{SYT} \) with first ascent being even.

**Proof:** By Torleif. No combinatorial proof yet.

*Analogue of B_{n-1}*

\[
\text{B}_{n-1}(x_i^2) = \prod_{i=1}^{n} (x_1 + \ldots + x_n + 2x_i)
\]

Note \( \text{B}_{n-1}(x_n^{-1}) = \text{B}_{n-1}(x_n) \)

\[
\text{B}_{n-1}(x_n + 1) = \sum_{j=0}^{N} \epsilon_j(x) \cdot h_{(n-1)}(x)
\]

So \( \text{B}_{n-1}(x_n^{-1}) = \text{B}_{n-1}(x_n) = \sum_{j=0}^{N} (-1)^j \epsilon_j(x) \cdot h_{(n-1)}(x) \)

*Here is the "..." Part*
\[ \prod_{i \neq j \in \mathbb{N}} \frac{x_i(Hx_i) - x_j(Hx_j)}{x_i - x_j} \]  

\[ D_n := \prod_{i \neq j \in \mathbb{N}} (x_i - x_j) = \sum_{w \in S_n} \text{sign}(w) (w \cdot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) \]

\[ A_n(f) := \frac{1}{D_n} \sum_{w \in S_n} \text{sign}(w) (w \cdot f) \]

\[ \text{Fact 1: } S_n(X_n) = A_n(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) \]

\[ \Rightarrow (**) = \frac{1}{D_n} \prod_{i \neq j \in \mathbb{N}} \frac{x_i(Hx_i) - x_j(Hx_j)}{x_i - x_j} = \frac{1}{D_n} \prod y_i - y_j \]

by (**) = \[ A_n(\prod_{i \neq j \in \mathbb{N}} x_i^{a_i} (Hx_i)^{b_i}) \]

Claim: \[ A_n(x_1^{a_1} \cdots x_n^{a_n}) = \text{sign}(w) S_n(X_n) \]

Since \( A_n \) counts linearly, \( \prod_{i \neq j \in \mathbb{N}} (Hx_i + x_i) \) is the positive sum of \( a \cdot A_n(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) \).

If \( a_i = a_j \), then \( A_n(x^{a}) = 0 \), so we can assume \( a_i \)'s are distinct. Then \( \exists w \in S_n \) and partition \( M = \{m_1, \ldots, m_n\} \) s.t.

\[ x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = x_1^{M_{m_1} + n - w(1)} \cdots x_n^{M_{m_n} + n - w(n)} \]

Claim: \[ A_n(x_1^{a_1} \cdots x_n^{a_n}) = \text{sign}(w) S_n(X_n) \]
\[ S_n = \frac{1}{\Delta_n} \sum_{\omega \in S_n} \text{sign}(\omega) \sigma(\chi_1^{\mu_1+\nu_1} \chi_2^{\mu_2+\nu_2} \ldots \chi_n^{\mu_n+\nu_n}) \]

\[ A_n(\chi_1^{\mu_1} \ldots \chi_n^{\mu_n}) = \Delta_n \sum_{\omega \in S_n} \text{sign}(\omega) \sigma(\chi_1^{\mu_1(\omega)+\nu_1(\omega)} \chi_2^{\mu_2(\omega)+\nu_2(\omega)} \ldots \chi_n^{\mu_n(\omega)+\nu_n(\omega)}) \]

\[ = \frac{1}{\Delta_n} \sum_{\omega \in S_n} \text{sign}(\omega) \sigma(\chi_1^{\mu_1+\nu_1} \chi_2^{\mu_2+\nu_2} \ldots \chi_n^{\mu_n+\nu_n}) \]

Therefore in (\ref{eqn}), the coeff. of \( \text{sign}(\omega) S_n(\chi_n) \) is

the coeff. of \( A_n(\chi_1^{\mu_1} \chi_2^{\mu_2} \ldots \chi_n^{\mu_n}) \) in \( A_n(\prod_{i=1}^n \chi_i^{\mu_i}(Hx_i)^{\nu_i}) \)

\[ = \sum_{\omega \in S_n} \text{sign}(\omega) \left( \text{coeff. of } x_1^{\mu_1(\omega)+\nu_1(\omega)} \chi_2^{\mu_2(\omega)+\nu_2(\omega)} \ldots \chi_n^{\mu_n(\omega)+\nu_n(\omega)} \text{ in } \prod_{i=1}^n \chi_i^{\mu_i}(Hx_i)^{\nu_i} \right) \]

By Binom. Thm., coeff. of \( x_1^{\mu_1} \chi_2^{\mu_2} \ldots \chi_n^{\mu_n} \) in \( \prod_{i=1}^n (Hx_i)^{\nu_i} \) is

\[ (\nu_1) (\nu_2) \ldots (\nu_n) \]

so coeff. of \( \chi_1^{\mu_1} \ldots \chi_n^{\mu_n} \) in \( \prod_{i=1}^n \chi_i^{\mu_i(\omega)+\nu_i(\omega)}(Hx_i)^{\nu_i} \) is the same as coeff. of \( x_1^{\mu_1(\omega)+\nu_1(\omega)} \chi_2^{\mu_2(\omega)+\nu_2(\omega)} \ldots \chi_n^{\mu_n(\omega)+\nu_n(\omega)} \) in \( \prod_{i=1}^n (Hx_i)^{\nu_i} \)
which is
\[
\binom{n}{k} \left( \frac{1}{a_{n(k) - w_1 + 1}} \right) \left( \frac{1}{a_{n(k) - w_2 + 2}} \right) \cdots \left( \frac{1}{a_{n(k) - w_m + m}} \right) \\
\text{choosing } a_i's \\
\text{choosing } a_i's
\]

so
\[
\binom{n}{k} = \\
\sum_{w \in S_n} \text{sign}(w) \cdot \binom{n-1}{k-1} \left( \frac{1}{a_{n(k) - w_1 + 1}} \right) \left( \frac{1}{a_{n(k) - w_2 + 2}} \right) \cdots \left( \frac{1}{a_{n(k) - w_m + m}} \right)
\]

\[
= \det \begin{pmatrix} a_{i-j+i} \\ \end{pmatrix}_{1 \leq i, j \leq n} \\
(j=1,2,3,...,n)
\]