HW1, Math 3610 Fall 2025

1. a) Show that the series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converges uniformly to a continuous function.

b) Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3} - \int_0^{\pi/2} \sum_{n=1}^{\infty} \frac{\sin((2n+1)x)}{(2n+1)^2}\, dx$$

2 Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \leq |x - y|^2.$$

Prove that $f(x)$ is a constant for all x. [Hint: What is $f'(x)$?]

3 Determine which of the following real series $\sum_{k=1}^{\infty} g_k$ converge, and if so whether the convergence is pointwise or uniformly. Check the continuity of the limit in each case.

a)

$$g_k(x) = \begin{cases} 0, & x \leq k \\ (-1)^k, & x > k \end{cases}.$$

b)

$$g_k(x) = \begin{cases} 1/k^2, & |x| \leq k \\ 1/x^2, & |x| > k \end{cases}.$$

c)

$$g_k(x) = \cos(kx) \frac{(-1)^k}{\sqrt{k}} \text{ on } \mathbb{R}$$

d)

$$g_k(x) = x^k \text{ on } (0,1).$$

4. Let $f_n : [1,2] \to \mathbb{R}$ be defined by $f_n(x) = x/(1+x)^n$.

a) Prove that $\sum_{n=1}^{\infty} f_n(x)$ is convergent for $x \in [1, 2]$.

b) Is it uniformly convergent?

c) Is

$$\int_1^2 \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)\, dx?$$

5. (#7 from p. 317 in the Marsden-Hoffman book). For functions $f : A \subset \mathbb{R}^n \to \mathbb{R}$, form the vector space of bounded, continuous functions $\mathcal{C}_b$ as in section 5.5 in the text. Show that

$$||fg|| \leq ||f||\ ||g||.$$