Solution to Quiz 4

Recall our problem is to integrate:

$$\int \frac{1}{(1+x^2)^2} dx \tag{1}$$

do the following substitution:

$$x = \tan\theta, dx = \sec^2\theta, \text{note we have}, 1 + x^2 = \sec^2\theta \tag{2}$$

So we may rewrite this integral as:

$$\int \frac{1}{\sec^2\theta} d\theta = \int \cos^2\theta d\theta \tag{3}$$

Here is the issue, you have learnt to use **double angel formula** to solve this question, however if we are dealing with:

$$\int \frac{1}{(1+x^2)^5} dx \tag{4}$$

for example, then after the above substitution, we must be able to integrate:

$$\int \frac{1}{\sec^8\theta} d\theta = \int \cos^8\theta d\theta \tag{5}$$

repeated us of double angel formula is inevitable, though complicated when the degree goes up, this is a possible way to solve the general type problem. Now I would like to present another way to do this, using integration by parts. Set:

$$u = \cos\theta, v = \sin\theta, du = -\sin\theta, dv = \cos\theta \tag{6}$$

$$\int \cos^2\theta d\theta = \int u dv = uv - \int v du = uv + \int sin^2\theta d\theta \tag{7}$$

use the identity $\sin^2\theta + \cos^2\theta = 1$, we may rewrite the last term as $\int 1 - \cos^2\theta d\theta$, this is recursive now, so add on both side of our equation by $\int \cos^2\theta d\theta$, we have:

$$2\int \cos^2\theta d\theta = uv + \int 1 d\theta = \sin\theta\cos\theta + \theta + C \tag{8}$$

finally, you draw the picture and see the following relations hold in your triangle:

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}, \cos\theta = \frac{1}{\sqrt{1+x^2}}, \theta = \tan^{-1}\theta \tag{9}$$

the rest is substitution and simplify the result, we omit that step.

Finally I want to point out several important points that you must know in the future:

1. whenever you do substituion like $x = tan\theta$, you need to change dx into $d\theta$
2. you should know how to use double angle formula
3. you should be able to see integration by parts applied to trigonmetric functions can give you recursive formula
4. remember to change θ back to x in the last step, know how to do this with the auxillery triangle.
5. think about the above problem I sketched, with power 8 for $\cos\theta$, can you see how to really solve it?