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Math 104-231

Homework 8 4/8

$\boxed{\text{Problem 1}}$ Fall 2011 - 14 Which of the following series converge?

(1) $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$ → Integral Test

* the function is:

1) continuous on $[2, \infty)$

2) positive on $[2,\infty)$ [for no input value will you get a negative output]

3) decreasing: $f'(n) = \frac{\frac{n^3}{n} - \ln(n)3n^2}{n^6} = \frac{n^2(1-3\ln(n))}{n^6} = \frac{1-3\ln(n)}{n^4}$

→ $f'(n) < 0$ for all inputs $[2,\infty)$

$$\int_2^{\infty} \frac{\ln x}{x^3}\,dx$$

* integration by parts

$u = \ln x \qquad dv = \frac{1}{x^3}dx$

$du = \frac{1}{x}dx \qquad v = -\frac{1}{2x^2}$

$$\int_2^{\infty} \frac{\ln x}{x^3}dx = \ln(x)\left(-\frac{1}{2x^2}\right) - \int -\frac{1}{2x^2}\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln(x)}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}dx \Big]_2^{\infty}$$

$$= -\frac{\ln(x)}{2x^2} + \frac{1}{2}\left(-\frac{1}{2}x^{-2}\right)\Big]_2^{\infty}$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2}\Big]_2^{\infty}$$

* L'hopitals

$$= \left[\lim_{a\to\infty}\left(-\frac{\ln a}{2a^2} - \frac{1}{4a^2}\right)\right] - \left[\frac{\ln 2}{8} - \frac{1}{16}\right]$$

$$= \left[\lim_{a\to\infty} \frac{\frac{1}{a}}{4a}\right] - \left[\frac{\ln 2}{8} - \frac{1}{16}\right]$$

cont. on BACK

$$\int_2^{\infty}\frac{\ln x}{x^3}dx = \frac{\ln 2}{8} + \frac{1}{16}$$

← Because the integral converges according to ~~integral test~~ $\boxed{\text{(1) converges}}$

Given a series $\sum_{n=1}^{\infty} n$. If ...

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

then ① if $L < 1$, the series converges abs.
② if $L = 1$, the test is inconclusive
③ If $L > 1$, series diverges

(II) $$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}(n!)^2}{(2n)!} \right| = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

* Ratio test

$$a_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!} = \frac{((n+1)n!)^2}{(2n+2)!} = \frac{(n+1)^2(n!)^2}{(2n+2)(2n+1)(2n!)}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2 \cancel{(n!)^2}}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{(n!)^2}} \right|$$

$$= \left| \frac{n^2 + 2n + 1}{4n^2 + 7n + 3} \right|$$

$$\lim_{n\to\infty} \left| \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} \right| = \lim_{n\to\infty} \left| \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} \right| = \frac{1}{4} = L$$

$\boxed{\text{Because } L < 1, \text{ (II) converges absolutely}}$ According to the Ratio test

---

(III) $$\sum_{n=1}^{\infty} \left| \frac{(-1)^n n^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

* Ratio test

$$a_{n+1} = \frac{(n+1)^{(n+1)}}{(n+1)!} = \frac{(n+1)(n+1)^n}{(n+1)n!}, \quad a_n = \frac{n^n}{n!}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^n}{\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n} \right|$$

$$= \left| \frac{(n+1)^n}{n^n} \right|$$

$$= \left| \left(1 + \frac{1}{n}\right)^n \right|$$

$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1$ * series rule: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

see next page

so when $a > 1$, the series diverges ← this means (III) cannot be absolutely convergent

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Problem 1 continued

(II) $\sum_{n=2}^{\infty} \frac{n^3}{\ln(n)} \rightarrow$ * n-th term test

* L'hopital's

$\lim_{n\to\infty} \frac{n^3}{\ln(n)} \rightarrow \frac{\infty}{\infty} = \lim_{n\to\infty} \frac{3n^2}{\frac{1}{n}}$

$= \lim_{n\to\infty} 3n^3 \rightarrow \infty$

Because $\lim_{n\to\infty} \frac{n^3}{\ln(n)}$ goes to infinity $(\neq 0)$, $\boxed{\text{(II) diverges}}$ According to n^{th} term test

(III) $\sum_{n=1}^{\infty} \frac{n}{2^n} \rightarrow$ * Root test

* $\lim_{n\to\infty} n^{1/n} = 1$

$\lim_{n\to\infty} \left|\frac{n}{2^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}$ so $\boxed{\text{(III) converges}}$

According to root test

(IV) $\sum_{n=1}^{\infty} e^{1/n} \rightarrow$ * n^{th} term test

$\lim_{n\to\infty} e^{1/n} = e^{1/\infty} = e^0 = 1$

because $\lim_{n\to\infty} e^{1/n} \neq 0$ $\boxed{\text{(IV) diverges}}$

according to n^{th} term test.

$\boxed{\text{(I) and (III) converge} \quad \boxed{\text{(B)}}}$

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$\boxed{\text{Problem 2}}$ Spring 2010 - 15

Determine if the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)}$ converges or diverges. If it converges, find the sum.

- Using <u>direct comparison test</u> we compare : $\sum_{n=1}^{\infty} \frac{1}{(2n-1)}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$

$$a_i = \frac{1}{2n-1} \quad , \quad b_i = \frac{1}{2n}$$

$$\frac{1}{2n-1} > \frac{1}{2n}$$

- Using p-test we know $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges because $p \le 1$.

- Because $\frac{1}{2n-1} > \frac{1}{2n}$, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ must also <u>diverge</u>

According to the theorem that states :

Let f and g be continuous on $[\overset{1}{a}, \infty)$ with $0 \le \overset{\frac{1}{2x}}{f(x)} \le \overset{\frac{1}{2x-1}}{g(x)}$ for all $x \ge \overset{1}{a}$. Then

① $\int_1^{\infty} \frac{1}{2x-1}\,dx$ diverges if $\underbrace{\int_1^{\infty} \frac{1}{2x}\,dx \text{ diverges}}_{\text{which it does}}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{2n-1} \text{ diverges by the direct comparison test.} \quad \boxed{(f)}}$$

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$\boxed{\text{Problem 3}}$ Spring 2010-16

The series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Ratio Test

Given a series $\sum_{n=1}^{\infty} a_n$, If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then

① If $L<1$, the series converges absolutely
② If $L=1$, the test is inconclusive
③ If $L>1$, the series diverges

$$a_n = \frac{n^n}{n!} \qquad a_{n+1} = \frac{(n+1)^{(n+1)}}{(n+1)!} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}n!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^n}{\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}\right|$$

$$= \left|\frac{(n+1)^n}{n^n}\right|$$

$$= \left|\left(1+\frac{1}{n}\right)^n\right|$$

★ series rule: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1$$

$\boxed{\text{Because } e > L,\ \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges by the ratio test}}$ (b)

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$\boxed{\text{Problem 4}}$ Spring 2010-17

The series $\sum_{n=1}^{\infty} (-1)^n \frac{(n+3)2^{2n}}{3^{n+100}}$

The Ratio Test states:

Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

then 1) if $L < 1$, the series converges absolutely
2) if $L = 1$, the test is inconclusive.
3) if $L > 1$ the series diverges.

$$a_n = \frac{(-1)^n (n+3) 2^{2n}}{3^{n+100}} \qquad a_{n+1} = \frac{(-1)^{(n+1)} (n+4) 2^{2(n+1)}}{3^{n+101}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-1)^{n+1}(n+4)2^{2n+2}}{3^{(n+101)}} \cdot \frac{3^{(n+100)}}{(-1)^n (n+3) 2^{2n}} \right|$$

$$= \left| \frac{(-1)(n+4)2^2}{3^1 (n+3)} \right|$$

$$= \left| \frac{4(n+4)}{3(n+3)} \right|$$

$$= \left| \frac{4n+16}{3n+9} \right|$$

$4/3 > 1$

$$\lim_{n\to\infty} \frac{4n+16}{3n+9} = \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) \cdot \frac{(4n+16)}{(3n+9)} = \lim_{n\to\infty} \frac{4+\frac{16}{n}}{3+\frac{9}{n}} = \frac{4}{3}$$

$\boxed{\text{Because } \frac{4}{3} > L,\ \sum_{n=1}^{\infty} (-1)^n \frac{(n+3)(2^{2n})}{3^{n+100}} \text{ diverges by the ratio test}}$ $\boxed{\text{(C)}}$

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Problem 5 | Fall 2007-13

Determine if the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{7^n}$$

• The series can be separated into two parts as follows:

$$\sum_{n=1}^{\infty} \frac{3^n+4^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$$

→ using the rules of geometric series for $\sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n$;

1^{st} term $= a = \frac{3}{7}$, $r = \frac{3}{7}$

Series sum $= S = \frac{a}{1-r} = \frac{\frac{3}{7}}{1-\frac{3}{7}} = \frac{\frac{3}{7}}{\frac{4}{7}} = \frac{3}{4}$

so $S = \sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n = \frac{3}{4}$

→ using the rules of geometric series for $\sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n$;

1^{st} term $= a = \frac{4}{7}$, $r = \frac{4}{7}$

Series sum $= S = \frac{a}{1-r} = \frac{\frac{4}{7}}{1-\frac{4}{7}} = \frac{\frac{4}{7}}{\frac{3}{7}} = \frac{4}{3}$

so $S = \sum_{n=1}^{\infty} \left(\frac{4}{7}\right)^n = \frac{4}{3}$

thus,

$$\sum_{n=1}^{\infty} \frac{3^n - 4^n}{7^n} = \frac{3}{4} + \frac{4}{3} = \frac{9}{12} + \frac{16}{12} = \frac{25}{12}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{3^n - 4^n}{7^n} = \frac{25}{12} \quad \text{using the geometric series} \quad \boxed{e}}$$

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Problem 6 | Fall 2007 14

How many of the following series converge?

① $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n+2}$ → limit comparison test

• comparison with: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}\left(\frac{\sqrt{n}}{\sqrt{n}}\right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/2}$

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$ diverges by the p-test

• then using limit comp test we find:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{\sqrt{n}+1}{n+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{n+2}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{n+2}{n+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1+\cancel{\frac{2}{n}}^{\,0}}{1+\cancel{\frac{\sqrt{n}}{n}}^{\,0}} \qquad \frac{\sqrt{n}}{n} = \cancel{\frac{1}{\sqrt{n}}}^{\,0}$$

$$= 1$$

because $\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{\sqrt{n}+1}{n+2}} = 1$ and 1 is a finite positive constant

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-test, $\boxed{\text{① diverges}}$ by the limit comparison test

---

② $\sum_{n=1}^{\infty} \frac{2^n+n^2}{3^n+n^3}$ ⟶ we can disregard n^2 and n^3 by intuition as they are insignificant

thus we examine:

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

this series is convergent by geometric series

$\boxed{\text{② is convergent}}$

cont. on back

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Problem 6 (cont.)

③ $\sum_{n=1}^{\infty} \frac{n^2+3}{(n+4)^2} \longrightarrow$

* nth term test

$\lim_{n\to\infty} \frac{n^2+3}{(n+4)^2} \rightarrow \frac{\infty}{\infty}$ * L'hopital's $= \lim_{n\to\infty} \frac{2n}{2(n+4)(1)} \rightarrow \frac{\infty}{\infty}$ * L'hopital's

$= \lim_{n\to\infty} \frac{2}{2} = 1$

Because $\lim_{n\to\infty} \frac{n^2+3}{(n+4)^2} = 1 \; (\neq 0)$, by the nth term test $\boxed{\text{③ diverges}}$

④ $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} \longrightarrow$

* integral test

* the function is

1) continuous on $[2,\infty)$

2) positive on $[2,\infty)$ [for no input value will you get a negative output]

3) decreasing: $f'(n) = \frac{\frac{n^2}{n} - \ln(n)2n}{n^4} = \frac{1-2\ln(n)}{n^3}$

$\rightarrow f'(n) < 0$ for all inputs $[2,\infty)$

$\int_2^{\infty} \frac{\ln x}{x^2}\,dx$

* integration by parts

$u = \ln x \qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$= \ln(x)\left(-\frac{1}{x}\right) - \int_2^{\infty} -\frac{1}{x}\left(\frac{1}{x}\right)dx$

$= -\frac{\ln x}{x} + \int_2^{\infty} \frac{1}{x^2}\,dx$

$= -\frac{\ln x}{x} - \frac{1}{x}\Big]_2^{\infty}$

* L'hopitals = 0

$= \left[\lim_{a\to\infty} \frac{-\ln a}{a} - \frac{1}{a}\right] - \left[-\frac{\ln 2}{2} - \frac{1}{2}\right]$

$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \frac{\ln 2}{2} + \frac{1}{2}$ * according to the integral test $\boxed{\text{④ converges}}$

⑤ $\sum_{n=2}^{\infty} \frac{n^2}{\ln(n)} \longrightarrow$

* nth term test

$\lim_{n\to\infty} \frac{n^2}{\ln(n)} \rightarrow \infty/\infty$ * L'hopitals $= \lim_{n\to\infty} \frac{2n}{\frac{1}{n}} = \lim_{n\to\infty} 2n^2 \rightarrow \infty$

Because $\lim_{n\to\infty} \frac{n^2}{\ln(n)}$ goes to infinity $(\neq 0)$, $\boxed{\text{⑤ diverges}}$ according to the nth term test.

$\boxed{\text{Both ② and ④ converge, so the answer is } \boxed{C}}$

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$\boxed{\text{Problem 7}}$ Fall 2007 - 15

Which statement is true of the following series?

- (I) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots$ converges conditionally
- (II) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!} = \frac{1}{2} - \frac{1}{6} + \frac{1}{20} - \dots$
- (III) $\sum_{n=1}^{\infty} \frac{(-1)^{n} n^{n}}{n!} = 1 - 2 + \frac{9}{2} \dots$

(I) $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges according to the integral and p-tests

$\rightarrow$ this means it cannot be absolutely convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} =$

alternating series test $b_i = \frac{1}{n+1}$

- if the alternating series $\sum_{n=1}^{\infty} a_n$ satisfies:

① $b_{n+1} \le b_n$ for all i:

$$\frac{1}{(n+1)+1} \overset{?}{\le} \frac{1}{n+1} \quad \checkmark \text{ true}$$

② $\lim_{n\to\infty} b_i \overset{?}{=} 0$

$$\lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}} = 0 \quad \checkmark \text{ true}$$

so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ converges $\longrightarrow$ these facts mean that $\boxed{\text{(I) conditionally converges}}$ according to the alternating series test.

see back

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$\boxed{\text{Problem 7 continued}}$ Fall 2007-15

(III) continued

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n!}$$

alternating series test $a_n = (-1)^n b_n$

condition 1 (for convergence)

① $b_{n+1} \leq b_n$ for all i

$$b_n = \frac{n^n}{n!} \qquad b_{n+1} = \frac{(n+1)^{(n+1)}}{(n+1)!}$$

$$b_{n+1} = \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\,n!}$$

So

$b_n < b_{n+1}$ which does NOT follow condition ①

meaning the series does NOT converge by the alternating series test

in addition to previous information found

$\boxed{\text{(III) diverges due to the ratio and alt series tests}}$

(I) conditionally converges
(II) converges absolutely
(III) diverges

$\boxed{a}$

#14. $\sum \frac{\ln n}{n^3} \sim \sum \frac{1}{n^2}$ converge. $p=2$.

$\boxed{B}$

$\sum \frac{n^3}{\ln n}$ diverge. n-th term test

$\sum \frac{n}{2^n}$ ratio test $r = \frac{1}{2} < 1$. converge.

$\sum_1 e^{1/n}$ n-th term test $\to 1$. diverge.

#15. $\sum \frac{1}{2n-1} > \sum \frac{1}{2n} = \frac{1}{2} \cdot (\sum \frac{1}{n})$ diverge. $p=1$

$\boxed{F}$

#16. $\sum \frac{n^n}{n!}$ n-th term test. $\geq \sum 1$. diverge.

$\boxed{B}$

#17. $\sum \frac{(-1)^n (n+3) 2^{2n}}{3^{n+100}} \underset{\text{ratio test.}}{\sim} \left(\frac{4}{3}\right)^n$ $r > 1$. diverge.

$\boxed{C}$

#13. $\sum_{n\geq 1} \frac{3^n + 4^n}{7^n} = \sum_{n\geq 1} \left(\frac{3}{7}\right)^n + \sum \left(\frac{4}{7}\right)^n = \frac{\frac{3}{7}}{1-\frac{3}{7}} + \frac{\frac{4}{7}}{1-\frac{4}{7}}$

$\boxed{E}$

$= \frac{3}{4} + \frac{4}{3} = \frac{25}{12}$

#14. $\sum_n \frac{\sqrt{n+1}}{n+2} \sim \sum \frac{\sqrt{n}}{n} = \sum \frac{1}{n^{\frac{1}{2}}}$ diverge. $p = \frac{1}{2}$.

$\boxed{C}$

$\sum \frac{2^n + n^2}{3^n + n^3} \sim \sum \left(\frac{2}{3}\right)^n$ ratio test. $r = \frac{2}{3} < 1$, converge

#14 - continued:

$$\sum_{n\geq 1} \frac{n^2+3}{(n+4)^2} \sim \sum \frac{n^2+3}{n^2+8n+16} \sim \sum \frac{n^2}{n^2} = \sum 1 \ \underline{\text{diverge.}}$$

n-th term test.

$$\sum \frac{\ln n}{n^2} < \sum \frac{n^{\frac{1}{2}}}{n^2} = \sum \frac{1}{n^{3/2}} \quad \underline{\text{converge.}} \ p=\frac{3}{2}$$

$$\sum \frac{n^2}{\ln n} > \sum n \quad \underline{\text{diverge.}} \ \text{n-th term test.}$$

#15. $\sum_{n\geq 1} (-1)^{n+1} \cdot \frac{1}{n+1}$ <u>converge.</u> by <u>alternating series-test</u>

$\boxed{A}$

$\sum_{n\geq 1} \frac{1}{n+1}$ <u>diverge</u> by p-test $p=1$

so (I) is conditional convergent.

$\sum \frac{(n!)^2}{(2n)!}$ ratio test $r = \frac{1}{4} < 1$. <u>Convergent</u>

so (II) just <u>converge absolutely</u>.

$\sum \frac{(-1)^n n^n}{n!}$ n-th term test. $\frac{n^n}{n!} > 1$.

so (III) <u>diverge</u> anyway

B.	F.	B.	C.	E.	C.	A.
14	15	16	17	13	14	15