Advanced Linear Algebra
Math3140/5140, Spring 2023
Assignment 1, due 1/23
(upload on canvas)

1. Which of the following are fields? Justify your answers.

1. The irrational real numbers (with addition and multiplication inherited from the real numbers).
2. The complex numbers of the form $a + bi$ where $a, b \in \mathbb{Q}$ (with addition and multiplication inherited from the complex numbers).
3. The set of real 2×2 matrices of the form
$$\begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}$$
with matrix multiplication and addition.

2. Which of the following statements are true? Justify your answers.

1. $\mathbb{R}$ is a vector space over $\mathbb{Q}$ (with addition and scalar multiplication inherited from $\mathbb{R}$).
2. $\mathbb{Q}$ is a vector space over $\mathbb{R}$ (with addition and scalar multiplication inherited from $\mathbb{R}$).
3. The set of 3×3 matrices with entries in $\mathbb{Q}$ which are magic squares is a vector space over $\mathbb{Q}$. (Here a *magic square* is a matrix such that every row, column, and diagonal sums up to same number.)
4. The set of positive real numbers $\mathbb{R}_{>0}$ is a vector space over $\mathbb{Q}$ with scalar multiplication $a \cdot x := x^a$ and addition $x + y := xy$ for ($a \in \mathbb{Q}$ and $x, y \in \mathbb{R}_{>0}$).

3. Show that for two elements $v, w \in \mathbb{R}^n$, the following are equivalent:

1. $v \neq 0$, and there is no $a \in \mathbb{R}$ so that $w = av$.
2. $w \neq 0$, and there is no $b \in \mathbb{R}$ so that $v = bw$.
3. If $av + bw = 0$ for some $a, b \in \mathbb{R}$, then $a = b = 0$.

4.

1. Find a field with 3 elements.
2. Find a field with 4 elements. (Caution: there are two abelian groups of order 4, only one of them can be endowed with the structure of a field).