Introduction

1. n-space over $\mathbb{R}$

$\mathbb{R}$: real numbers

$\mathbb{R}^n = \{(x_1,\dots,x_n) \mid x_1,\dots,x_n \in \mathbb{R}\}$ set of ordered n-tuples (or vectors) of real numbers

Note:
$(x_1,\dots,x_n) = (y_1,\dots,y_n)$ if and only if $x_i = y_i$ for all i

The x_i are called the <u>components</u> of $x = (x_1,\dots,x_n)$

Small n:

$n=1$ real line $\mathbb{R}^1$

$n=2$ real plane $\mathbb{R}^2$

$n=3$ real space $\mathbb{R}^3$

for small values of n, we have geometric intuition.

But: other concrete interpretations possible, e.g. a bank with n customers, their account balances at a fixed point in time are $x_1,\dots,x_n$.

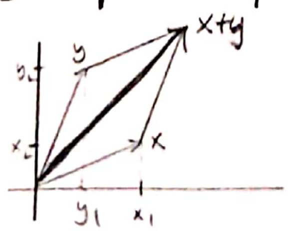
How do we calculate with n-tuples?

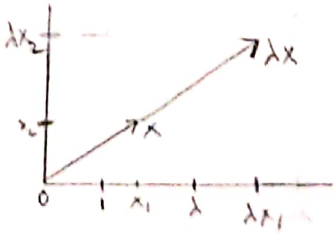
1) For $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we can define $\lambda \cdot x := (\lambda x_1,\dots,\lambda x_n)$

$$\underset{(x_1,\dots,x_n)}{\overset{\parallel}{x}}$$

2) For $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $y = (y_1,\dots,y_n) \in \mathbb{R}^n$ we can define $x + y := (x_1+y_1,\dots,x_n+y_n)$.

Geometric interpretation for $n=2$, view vectors as "arrows" starting at O:





$0 = (0, \ldots, 0)$ is called the <u>origin</u>

<u>Note</u>: $0 + x = x$ for all $x \in \mathbb{R}^n$

We can also set $-x := (-x_1, \ldots, -x_n)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We write $x - y$ for $x + (-y)$.

2. Lines in the plane

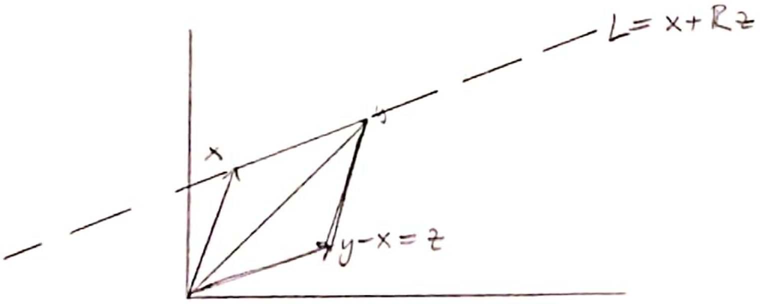
2 distinct points in $\mathbb{R}^2$ determine a <u>line</u>

In vector notation: $x, y \in \mathbb{R}^2$ with $x \neq y$ $\Rightarrow$ $z := y - x$ is not equal to zero.

$$L = \{ v \in \mathbb{R}^2 \mid v = x + \lambda z \text{ for some } \lambda \in \mathbb{R} \}$$

is the set-theoretic description of the line through x and y.

We also write $x + \mathbb{R}z$ for this set.



You may have seen a different description of a line, namely by a <u>linear equation</u> of the form

$$a_1 x_1 + a_2 x_2 = b$$

Here x_1, x_2 are viewed as indeterminates, a_1, a_2 and b are given real numbers

The set of <u>solutions</u> is

$$L = \{ (x_1, x_2) \in \mathbb{R}^2 \mid a_1 x_1 + a_2 x_2 = b \}$$

Note: If $a_1 = a_2 = 0$ then either $L = \emptyset$ (if $b \neq 0$) or $L = \mathbb{R}^2$ (for $b = 0$)

These are "degenerate" cases.

How do we find all elements of L?

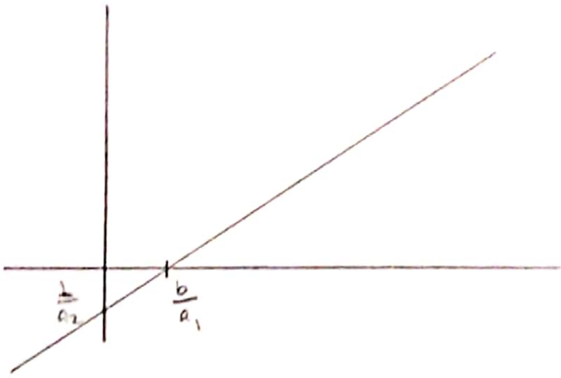
1) If $a_2 = 0$ and $a_1 \neq 0$, rewrite as

$$x_1 = \frac{b}{a_1}$$

to see that L is a line parallel to the x_2-axis

If $a_1 = 0$ and $a_2 \neq 0$, switch roles to get a line parallel to the x_1-axis.

2) If both a_1 and a_2 are nonzero, we can draw L by finding the intersections with the x_1-axis and x_2-axis, respectively:



Two lines in the plane which are different and not parallel intersect in exactly one point. If both lines are given by linear equations, can we determine which case happens?

Ex: 1) $L_1 : \; x_1 - x_2 = 1$

$L_2 : \quad x_2 = 2$

Intersection: insert $x_2 = 2$ in first equation and solve for x_1 to find $x_1 = 1 + 2 = 3$.

So the intersection point is $(3,2)$.

2) $L_1: \quad x_1 - x_2 = 1$

$L_2: \quad x_1 + 3x_2 = 9$

Intersection: subtract first equation from second equation to get $4x_2 = 8$ which gives $x_2 = 2$. Now proceed as in the previous example to find $x_1 = 3$ and the intersection point $(3,2)$.

3) $L_1: \; x_1 - x_2 = 1$

$L_2: \quad 2x_1 - 2x_2 = b$ for some b

Note: If $b=2$, $L_1 = L_2$. If $b \neq 2$, these are parallel, so do not intersect (we see this by drawing a picture)

By calculation, we can also see this:

Subtract 2 times the first equation from the second to get

$$0x_1 - 0x_2 = b-2 \qquad \text{which gives } b=2.$$

Since b was fixed, this is either true in which case we are left with one equation $x_1 - x_2 = 1$ (i.e., $L_1 = L_2$) or it is false and the intersection is empty (geometrically, L_1 and L_2 are parallel).

3. Lines and planes in $\mathbb{R}^3$

Just as in $\mathbb{R}^2$, two distinct points in $\mathbb{R}^3$ determine a line L.

There are two important differences:

1) In general, two lines are skew, that is, without intersection and not parallel

2) A linear equation in $\mathbb{R}^3$ does not determine a line but a plane. To describe a line, we need 2 equations. Geometrically, this means we write the line as the intersection of 2 planes.

Intersection of planes

Ex: $H_1: x_1 + x_2 + x_3 = -6$

$H_2: x_1 + 2x_2 + 3x_3 = -10$

transform this system to

$$x_1 + x_2 + x_3 = -6$$
$$x_2 + 2x_3 = -4$$

by substracting the first equation from the second one

Intersection: line L, points are determined by x_3-coordinate:

$$x_3 = \lambda \Rightarrow x_2 = -2\lambda - 4$$
$$\Rightarrow x_1 = \lambda - 2$$

If we intersect 3 planes, we will usually get a point.

Summary: We saw how to

- calculate with vectors (addition, multiplication by scalar)
- describe subsets of $\mathbb{R}^n$ by equations
- intersect subsets of $\mathbb{R}^n$ using those equations

Goal: Formalize and generalize this.

I. Vector spaces

§1. Basics

1. Fields

Examples of fields $\mathbb{R}, \mathbb{C}, \mathbb{Q}$

Idea: Formalize properties of addition and multiplication so that we can apply them to other objects

Def: A <u>field</u> is a set F with two binary operations

$$+ : F \times F \longrightarrow F \qquad (x,y) \longmapsto x+y \qquad \text{"addition"}$$

$$\cdot : F \times F \longrightarrow F \qquad (x,y) \longmapsto x \cdot y \qquad \text{"multiplication"}$$

that have the following properties:

(A0) There is an element $0 \in F$ for which $0+x = x+0 = x$ for all $x \in F$. (neutral element for addition)

(A1) For all $x, y \in F$, we have $x+y = y+x$ (addition is commutative)

(A2) For all $x, y, z \in F$, we have $(x+y)+z = x+(y+z)$ (addition is associative)

(A3) For every $x \in F$ there is an element $y \in F$ so that $x+y=0$ (inverse element w.r.t. addition)

(M0) There is an element $1 \in F$ for which $1 \cdot x = x \cdot 1 = x$ for all $x \in F$ (neutral element for multiplication)

(M1) For all $x, y \in \mathbb{F}$, we have $x \cdot y = y \cdot x$
(multiplication is commutative)

(M2) For all $x, y, z \in \mathbb{F}$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
(multiplication is associative)

(M3) For every $x \in \mathbb{F} \setminus \{0\}$ there is an element $y \in \mathbb{F}$ so that $x \cdot y = 1$
(inverse w.r.t. multiplication)

(D) For all $x, y, z \in \mathbb{F}$, we have $(x + y) \cdot z = x \cdot z + y \cdot z$
(distributive law)

Remarks:

1) One can show (you should try!) that inverse elements w.r.t. addition and multiplication are unique. This allows us to write

$-x$ for the additive inverse of x and

x^{-1} for the multiplicative inverse of x (when $x \neq 0$)

2) Other rules, e.g. $0 \cdot x = 0$ for all $x \in \mathbb{F}$, are consequences of the rules above.

We also define the notion of a group, which we will need a little later:

Def: A set G with a binary operation $* : G \times G \to G$ is called a group if (A0), (A2) and (A3) are satisfied. If in addition (A1) is also satisfied then $(G, *)$ is an abelian group or commutative group.