

# Weierstrass preparation and algebraic invariants

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## Abstract

We prove a form of the Weierstrass Preparation Theorem for normal algebraic curves over complete discrete valuation rings. While the more traditional algebraic form of Weierstrass Preparation applies just to the projective line over a base, our version allows more general curves. This result is then used to obtain applications concerning the values of  $u$ -invariants, and on the period-index problem for division algebras, over fraction fields of complete two-dimensional rings. Our approach uses patching methods and matrix factorization results that can be viewed as analogs of Cartan's Lemma.

## 1 Introduction

The usual algebraic form of the Weierstrass Preparation Theorem ([Bou72], VII.3.9, Proposition 6) says in particular that if  $T$  is a complete discrete valuation ring then every element of  $T[[x]]$  can be written as a product  $gu$ , where  $g \in T[x]$  and where  $u$  is a unit in  $T[[x]]$ . Thus every divisor on  $\text{Spec}(T[[x]])$  is induced by a divisor on the projective line over  $T$ . There is also an algebraic form of the related Weierstrass Division Theorem.

Just as the original, analytic form of Weierstrass Preparation and its companion division theorem are important tools in the theory of several complex variables, their algebraic forms have been useful in the study of arithmetic curves. In particular, versions of these results have been used in connection with patching and Galois theory; e.g. see [Voe96], Theorem 11.3 and Lemma 11.8.

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Generalized versions of Weierstrass Preparation, which apply to smooth  $T$ -curves that need not be the projective line, were shown in [HH10], Propositions 4.7 and 5.6, in connection with patching; and this was used in [HHK09] to obtain applications to quadratic forms and central simple algebras (see Corollaries 4.17 and 5.10 of that paper). Because of the smoothness restriction, these forms of Weierstrass Preparation could only be applied in the context of one-variable function fields over a complete discretely valued field for which there is a smooth projective model over  $T$ .

The current paper generalizes Weierstrass Preparation further, to  $T$ -curves that need not be smooth. We then apply this to quadratic forms and central simple algebras over related fields, obtaining results about  $u$ -invariants and the period-index problem over fraction fields of complete two-dimensional rings. There has been much interest in these problems; e.g. see the papers [COP02] and [Hu11], which focused on the local case in which the residue field is separably closed or finite. (See also [Hu10].) The method we use here is different, and yields applications with more general residue fields, as well as applications to fraction fields of certain complete rings that are not local.

Weierstrass Preparation can be viewed as a factorization assertion for elements of certain commutative rings. Our approach here involves first proving such an assertion for elements of matrix rings, and then specializing to the  $1 \times 1$  case. The factorization of matrices as products of matrices over smaller base rings plays an important role in the framework of patching problems. Patching permits the construction of algebraic objects over rings or fields of functions, given local data together with agreements on appropriate overfields (e.g. see [Har84] and [HH10]). The matrix factorization result that we prove here is related to [HH10], Theorem 4.6, which was used there to obtain a smooth form of Weierstrass Preparation; and is also related to [HHK09], Theorems 3.4 and 3.6. In obtaining results here for factorization of elements in the context of singular  $T$ -curves, we show that there is an obstruction that does not always vanish, concerning the reduction graph associated to the closed fiber. But by passing to split covers (see [HHK11], Section 5), we are able to prove a Weierstrass-type assertion that suffices for our applications.

The structure of this paper is as follows. In Section 2 we explain how patching relates to factorization, and then prove factorization results for the types of rings and fields that arise in the study of curves over a complete discretely valued field. This is done first for matrices and then for elements. The results of Section 2 are applied in Section 3 to prove our form of Weierstrass Preparation, and then to obtain corollaries that allow one to pass from local to global elements modulo an  $n$ -th power. Applications are given in Section 4, which concern quadratic forms and the  $u$ -invariant (Section 4.1) as well as the period-index problem for division algebras (Section 4.2).

## 2 Patching and factorization

In this paper, we consider curves over complete discretely valued fields, using the setup that was introduced in [HH10]. The approach there used a form of patching to relate structures on a function field  $F$  to structures on certain overfields  $F_\xi$  that arise from geometry, including the realization of the former given the latter.

A key tool in this study involved the factorization of matrices, to pass from locally defined objects to more global ones. This is related to a classical result, Cartan's Lemma. While matrix factorization is simplest over fields, in this paper we will also need to factor matrices over rings, in order to prove our form of Weierstrass Preparation. We begin by recalling some notation and terminology, beginning with the notion of a patching problem.

For any ring  $R$  let  $\mathcal{F}(R)$  denote the category of free  $R$ -modules of finite rank. Given rings  $R \subseteq R_1, R_2 \subseteq R_0$  with  $R = R_1 \cap R_2 \subseteq R_0$ , a (*free module*) *patching problem* for  $(R, R_1, R_2, R_0)$  is an object in the 2-fiber product category  $\mathcal{F}(R_1) \times_{\mathcal{F}(R_0)} \mathcal{F}(R_2)$ . In other words, a patching problem consists of a tuple  $(M_1, M_2, M_0; \nu_1, \nu_2)$  of free  $R_i$ -modules  $M_i$  together with isomorphisms  $\nu_i : M_i \otimes_{R_i} R_0 \rightarrow M_0$  for  $i = 1, 2$ . In the category of patching problems for  $(R, R_1, R_2, R_0)$ , a morphism  $(M_1, M_2, M_0; \nu_1, \nu_2) \rightarrow (M'_1, M'_2, M'_0; \nu'_1, \nu'_2)$  consists of  $R_i$ -module homomorphisms  $f_i : M_i \rightarrow M'_i$  ( $i = 0, 1, 2$ ) such that  $f_0 \circ \nu_i = \nu'_i \circ (f_i \otimes \text{id}) : M_i \otimes_{R_i} R_0 \rightarrow M'_0$  for  $i = 1, 2$ .

In this situation, there is a functor  $\beta : \mathcal{F}(R) \rightarrow \mathcal{F}(R_1) \times_{\mathcal{F}(R_0)} \mathcal{F}(R_2)$  given by base change from  $R$  to  $R_i$ . That is,  $\beta(M) = (M_1, M_2, M_0; \nu_1, \nu_2)$ , where  $M_i = M \otimes_R R_i$  for  $i = 0, 1, 2$ ; and where  $\nu_i$  is the natural map  $(M \otimes_R R_i) \otimes_{R_i} R_0 \rightarrow M \otimes_R R_0$ . A *solution* to a patching problem  $\mathcal{M}$  is a free  $R$ -module  $M$  of finite rank whose image under  $\beta$  is isomorphic to  $\mathcal{M}$ .

The following result appeared in [Har84, Prop. 2.1] and its proof.

**Proposition 2.1.** *In the above situation, the following conditions are equivalent:*

- (i) *The base change functor  $\mathcal{F}(R) \rightarrow \mathcal{F}(R_1) \times_{\mathcal{F}(R_0)} \mathcal{F}(R_2)$  is an equivalence of tensor categories;*
- (ii) *every free module patching problem for  $(R, R_1, R_2, R_0)$  has a solution;*
- (iii) *for every  $n \geq 1$ , every element  $A \in \text{GL}_n(R_0)$  can be written as a product  $A = BC$  with  $B \in \text{GL}_n(R_1)$  and  $C \in \text{GL}_n(R_2)$ .*

*Moreover, under these conditions, the solution to a free module patching problem as above is given by  $M = M_1 \times_{M_0} M_2$ , where the fiber product is taken with respect to the maps  $\nu_1, \nu_2$ .*

*Proof.* For the sake of the reader's convenience, we explain the key part of the proof of the proposition (and of Proposition 2.1 in [Har84]), viz. that (iii) implies (ii). For this, given a free module patching problem defined by modules  $M_i$  and isomorphisms  $\nu_i$ , let  $A \in \text{GL}_n(R_0)$  be the matrix corresponding to the isomorphism  $\nu_2^{-1}\nu_1 : M_1 \otimes_{R_1} R_0 \rightarrow M_2 \otimes_{R_2} R_0$ , with respect to some bases of  $M_1, M_2$  over  $R_1, R_2$ . Let  $B, C$  be as in (iii). Adjusting the chosen

bases by  $B, C$  respectively, the new matrix for  $\nu_2^{-1}\nu_1$  is the identity, and so the new bases have a common image in  $M_0$ . The free  $R$ -module generated by this basis then gives a solution to the patching problem.

The converse implication is obtained by reversing this process. Property (i) clearly implies condition (ii), and it is not hard to show the converse of that implication, using that base change preserves tensor products.  $\square$

The geometric situation that we will consider in this paper is described in the following notation (introduced in [HH10], Section 6; see also [HHK09], Notation 3.3):

**Notation 2.2.** Consider a complete discrete valuation ring  $T$  with uniformizer  $t$ , fraction field  $K$ , and residue field  $k$ ; a one-variable function field  $F$  over  $K$ , with a *normal model*  $\widehat{X}$  of  $F$  over  $T$  (i.e. a projective normal  $T$ -scheme with function field  $F$ ); and a finite non-empty set of closed points  $\mathcal{P}$  of the closed fiber  $X$  of  $\widehat{X}$ , containing all the points where distinct irreducible components of  $X$  meet. We let  $\mathcal{U}$  be the set of connected components of the complement of  $\mathcal{P}$  in  $X$ . For each  $P \in \mathcal{P}$ , we let  $R_P$  be the local ring of  $\widehat{X}$  at  $P$ ; we write  $\widehat{R}_P$  for its completion at its maximal ideal; and write  $F_P$  for the fraction field of  $\widehat{R}_P$ . For each subset  $U$  of  $X$  that is contained in an irreducible component of  $X$  and does not meet other components, let  $R_U$  be the subring of  $F$  consisting of the rational functions on  $\widehat{X}$  that are regular on  $U$ ; write  $\widehat{R}_U$  for its  $t$ -adic completion; and write  $F_U$  for the fraction field of  $\widehat{R}_U$ . Thus  $F$  is a subfield of each  $F_P$  and each  $F_U$ . (In the case  $U = \{P\}$ , note that the field  $F_{\{P\}}$  is strictly contained in  $F_P$ .) A *branch* of the closed fiber  $X$  of  $\widehat{X}$  at a point  $P \in \mathcal{P}$  is a height one prime  $\wp$  of  $\widehat{R}_P$  that contains  $t$ . Let  $\mathcal{B}$  denote the set of all branches at points  $P \in \mathcal{P}$ . The contraction of a branch  $\wp$  to  $R_P$  defines an irreducible component  $X_0$  of  $X$  (which is the closure of a unique  $U \in \mathcal{U}$ ), and we say that  $\wp$  *lies on*  $X_0$ . We let  $R_\wp$  be the local ring of  $\widehat{R}_P$  at  $\wp$ ; we write  $\widehat{R}_\wp$  for its completion; and write  $F_\wp$  for the fraction field of  $\widehat{R}_\wp$ . For a triple  $P, U, \wp$  where  $\wp$  is a branch at a point  $P \in \mathcal{P}$  on the closure of  $U \in \mathcal{U}$ , there are inclusions of  $\widehat{R}_P$  and  $\widehat{R}_U$  into  $\widehat{R}_\wp$ . The induced inclusions of  $F_P$  and  $F_U$  into  $F_\wp$  are compatible with the inclusions  $F \hookrightarrow F_P, F_U$ .

In the situation above, we obtain the following factorization result for matrices. Unlike Proposition 2.1 (which is used in its proof), this result considers a collection of matrices that are to be factored simultaneously. Related results, for fields rather than rings, appeared in [HH10] (Theorems 4.6 and 6.4) and in [HHK09] (Theorems 3.4 and 3.6).

**Proposition 2.3.** *Let  $\widehat{X}$  be a normal connected projective  $T$ -curve, with  $\mathcal{P}, \mathcal{U}, \mathcal{B}$  as in Notation 2.2. Let  $n$  be a positive integer, and suppose that for every branch  $\wp \in \mathcal{B}$  we are given an element  $A_\wp \in \mathrm{GL}_n(\widehat{R}_\wp)$ .*

- (a) *There exist elements  $A_P \in \mathrm{GL}_n(F_P)$  for each  $P \in \mathcal{P}$ , and elements  $A_U \in \mathrm{GL}_n(\widehat{R}_U)$  for each  $U \in \mathcal{U}$ , such that for every branch  $\wp \in \mathcal{B}$  at a point  $P \in \mathcal{P}$  with  $\wp$  lying on the closure of some  $U \in \mathcal{U}$ , we have  $A_\wp = A_P A_U \in \mathrm{GL}_n(F_\wp)$  with respect to the natural inclusions  $\widehat{R}_U, F_P, \widehat{R}_\wp \hookrightarrow F_\wp$ .*

(b) There exist elements  $A'_P \in \mathrm{GL}_n(\widehat{R}_P)$  for each  $P \in \mathcal{P}$ , and elements  $A'_U \in \mathrm{GL}_n(F_U)$  for each  $U \in \mathcal{U}$ , such that for every branch  $\varphi \in \mathcal{B}$  at a point  $P \in \mathcal{P}$  with  $\varphi$  lying on the closure of some  $U \in \mathcal{U}$ , we have  $A_\varphi = A'_P A'_U \in \mathrm{GL}_n(F_\varphi)$  with respect to the natural inclusions  $F_U, \widehat{R}_P, \widehat{R}_\varphi \hookrightarrow F_\varphi$ .

*Proof. Case 1:* We first consider the case when  $\widehat{X} = \mathbb{P}_T^1$  and  $\mathcal{P} = \{\infty\}$  consists of the point  $P$  at infinity on  $\mathbb{P}_k^1$ .

Then  $\mathcal{U} = \{U\}$ , where  $U$  is the affine line over the residue field  $k$  of  $T$ . There is also a single branch  $\varphi$  at  $P$  (on the closure of  $U$ ). As in [HH10], we write  $R_\varnothing$  for the local ring of  $\widehat{X}$  at the generic point of  $X$ , with completion  $\widehat{R}_\varnothing$  and fraction field  $F_\varnothing$ .

For the factorization in (a), by [HH10], Theorem 5.4, there exist  $B_P \in \mathrm{GL}_n(F_P)$  and  $A_\varnothing \in \mathrm{GL}_n(F_\varnothing)$  such that  $A_\varphi = B_P A_\varnothing$ . By [HH10], Theorem 4.6 and the comment just after that, there exist  $C_P \in \mathrm{GL}_n(F_{\{P\}})$  and  $A_U \in \mathrm{GL}_n(\widehat{R}_U)$  such that  $A_\varnothing = C_P A_U$ . Thus  $A_\varphi = A_P A_U$  where  $A_P := B_P C_P \in \mathrm{GL}_n(F_P)$ .

For the factorization in (b), let  $\bar{A}_\varphi$  be the reduction of  $A_\varphi$  modulo  $t$ . Thus  $\bar{A}_\varphi \in \mathrm{GL}_n(k((x^{-1})))$ , where  $k((x^{-1}))$  is the fraction field of the complete local ring  $k[[x^{-1}]]$  at the point at infinity on the projective  $k$ -line. Since every finitely generated projective module over a principle ideal domain is free, it follows from [Har93], Proposition 3 (with  $m = 0$  there) that the quadruple  $(k[x], k[x, x^{-1}], k[[x^{-1}]], k((x^{-1})))$  satisfies condition (ii) of Proposition 2.1, and hence also condition (iii). Thus we may factor  $\bar{A}_\varphi$  as the product of an invertible matrix over  $k[[x^{-1}]]$  and an invertible matrix over  $k[x, x^{-1}]$ . We may lift those matrices to elements of  $\mathrm{GL}_n(\widehat{R}_P)$  and  $\mathrm{GL}_n(T[x, x^{-1}]) \subset \mathrm{GL}_n(F) \subset \mathrm{GL}_n(F_U)$ , respectively. Using these lifts to adjust  $A_\varphi$ , we may assume that  $A_\varphi$  (and hence also its inverse) is congruent to the identity modulo  $t$ . Now by [HH10], Lemma 5.3, the hypotheses of [HH10], Proposition 3.2 are satisfied, with  $\widehat{R}_\varnothing, \widehat{R}_P, \widehat{R}_\varphi$  playing the roles of  $\widehat{R}_1, \widehat{R}_2, \widehat{R}_0$  there and taking  $M_1 = \widehat{R}_1$  there. The conclusion of that proposition then says that  $A_\varphi^{-1} = B_U B_P$  with  $B_U \in \mathrm{GL}_n(F_U)$  and  $B_P \in \mathrm{GL}_n(\widehat{R}_P)$ . Taking  $A'_P = B_P^{-1}$  and  $A'_U = A_U^{-1}$  gives the desired factorization  $A_\varphi = A'_P A'_U$ .

*Case 2: General case.*

Choose a finite  $T$ -morphism  $f : \widehat{X} \rightarrow \mathbb{P}_T^1$  such that  $\mathcal{P} = f^{-1}(\infty)$ , using [HHK11, Proposition 3.3]. Write  $U' = \mathbb{A}_k^1$  and  $P' = \infty \in X' := \mathbb{P}_k^1$ , and let  $\varphi'$  be the branch at infinity on the projective  $k$ -line. Also write  $F'$  for the function field of  $\mathbb{P}_k^1$ . Let  $r = [F : F']$  denote the degree of  $f$ .

For part (a), consider the localization  $\widehat{R}_{P',t}$  of  $\widehat{R}_{P'}$  at the prime ideal  $(t)$ . This is equal to the intersections  $F_{P'} \cap \widehat{R}_{\varphi'} = F_{P'} \cap \widehat{R}_\varnothing$ . Both  $\widehat{R}_{U'}$  and  $\widehat{R}_{P',t}$  are subrings of  $\widehat{R}_{\varphi'}$ , with intersection  $\widehat{R}_{U'} \cap \widehat{R}_{P',t} = \widehat{R}_{U'} \cap F_{U'} \cap F_{P'} \cap \widehat{R}_\varnothing = \widehat{R}_{U'} \cap F' = R_{U'}$ . Also, by Case 1, for any matrix  $A_{\varphi'} \in \mathrm{GL}_n(\widehat{R}_{\varphi'})$  there exist  $A_{P'} \in \mathrm{GL}_n(F_{P'})$  and  $A_{U'} \in \mathrm{GL}_n(\widehat{R}_{U'})$  such that  $A_{\varphi'} = A_{P'} A_{U'}$ . Since  $A_{\varphi'}, A_{U'} \in \mathrm{GL}_n(\widehat{R}_{\varphi'})$ , it follows that the element  $A_{P'}$  lies in  $\mathrm{GL}_n(F_{P'}) \cap \mathrm{GL}_n(\widehat{R}_{\varphi'}) = \mathrm{GL}_n(\widehat{R}_{P',t})$ . So the quadruple of rings  $(R_{U'}, \widehat{R}_{U'}, \widehat{R}_{P',t}, \widehat{R}_{\varphi'})$  satisfies condition (iii) of Proposition 2.1, and hence also condition (ii) on patching problems.

Let  $V \subset X$  be the union of the sets  $U \in \mathcal{U}$ , and let  $R_V$  be the subring of  $F$  consisting of the rational functions on  $\widehat{X}$  that are regular on  $V$ . Thus  $V = f^{-1}(U') = \text{Spec}(R_V/tR_V)$ . For any irreducible polynomial  $g \in k[x]$ , the existence of  $g$ -torsion in  $R_V/tR_V$  is equivalent to the existence of an irreducible component of  $V$  lying in the fiber of  $f$  over the point  $(g)$  on  $U' = \mathbb{A}_k^1$ . But since  $V$  is the union of the irreducible curves  $U \in \mathcal{U}$ , and since the morphism  $f|_V : V \rightarrow U'$  is finite, it follows that no irreducible component of  $V$  lies in a fiber of  $f$ . Thus  $R_V/tR_V$  is a finitely generated torsion-free module over the principal ideal domain  $R_{U'}/tR_{U'} = k[x]$ , and so it is a free module of rank  $r$ . It follows that  $R_V$  is a free module of rank  $r$  over  $R_{U'}$ , by [Bou72], Proposition II.3.2.5. Hence  $\widehat{R}_{U'} \otimes_{R_{U'}} R_V$ ,  $\widehat{R}_{P',t} \otimes_{R_{U'}} R_V$ , and  $\widehat{R}_{\varphi'} \otimes_{R_{U'}} R_V$  are free modules of rank  $r$  over the rings  $\widehat{R}_{U'}$ ,  $\widehat{R}_{P',t}$ ,  $\widehat{R}_{\varphi'}$  respectively, with compatible bases.

There are canonical isomorphisms  $\widehat{R}_{U'} \otimes_{R_{U'}} R_V \xrightarrow{\sim} \prod_{U \in \mathcal{U}} \widehat{R}_U$ ,  $\widehat{R}_{\varphi'} \otimes_{R_{U'}} R_V \xrightarrow{\sim} \prod_{\varphi \in \mathcal{B}} \widehat{R}_\varphi$ ,  $F_{\varphi'} \otimes_{R_{U'}} R_V \xrightarrow{\sim} \prod_{\varphi \in \mathcal{B}} F_\varphi$ , and  $F_{P'} \otimes_{R_{U'}} R_V \xrightarrow{\sim} F_{P'} \otimes_{F'} F \xrightarrow{\sim} \prod_{P \in \mathcal{P}} F_P$ , where the last isomorphism is by [HH10, Lemma 6.2(a)]. Since  $\widehat{R}_{P',t} = F_{P'} \cap \widehat{R}_{\varphi'}$ , we have an exact sequence

$$0 \rightarrow \widehat{R}_{P',t} \rightarrow F_{P'} \times \widehat{R}_{\varphi'} \rightarrow F_{\varphi'}$$

of  $R_{U'}$ -modules, where the first map is the diagonal inclusion and the second map is given by subtraction. Tensoring with the free  $R_{U'}$ -module  $R_V$ , and using the above isomorphisms, we get an exact sequence

$$0 \rightarrow \widehat{R}_{P',t} \otimes_{R_{U'}} R_V \rightarrow \prod_{P \in \mathcal{P}} F_P \times \prod_{\varphi \in \mathcal{B}} \widehat{R}_\varphi \rightarrow \prod_{\varphi \in \mathcal{B}} F_\varphi.$$

Thus we get canonical identifications

$$\widehat{R}_{P',t} \otimes_{R_{U'}} R_V = \prod_{P \in \mathcal{P}} F_P \cap \prod_{\varphi \in \mathcal{B}} \widehat{R}_\varphi = \prod_{P \in \mathcal{P}} (F_P \cap \prod_{\varphi \in \mathcal{B}_P} \widehat{R}_\varphi),$$

where  $\mathcal{B}_P$  is the set of branches at a given  $P \in \mathcal{P}$  (i.e., the set of height one primes in  $\widehat{R}_P$  containing  $t$ ). But for  $P \in \mathcal{P}$ , the intersection  $F_P \cap \prod_{\varphi \in \mathcal{B}_P} \widehat{R}_\varphi$  is equal to the semi-localization  $\widehat{R}_{P,t}$  of  $\widehat{R}_P$  at the set of branches at  $P$ . Thus we also get a canonical identification  $\widehat{R}_{P',t} \otimes_{R_{U'}} R_V \xrightarrow{\sim} \prod_{P \in \mathcal{P}} \widehat{R}_{P,t}$ . Since the intersection of  $\widehat{R}_{U'}$  and  $\widehat{R}_{P',t}$  in  $\widehat{R}_{\varphi'}$  is  $R_{U'}$ , the intersection of  $\prod_{U \in \mathcal{U}} \widehat{R}_U$  and  $\prod_{P \in \mathcal{P}} \widehat{R}_{P,t}$  in  $\prod_{\varphi \in \mathcal{B}} \widehat{R}_\varphi$  equals  $R_V$  by base change.

A collection of elements  $A_\varphi \in \text{GL}_n(\widehat{R}_\varphi)$ , for  $\varphi \in \mathcal{B}$ , defines an element of  $\text{GL}_n(\prod_{\varphi \in \mathcal{B}} \widehat{R}_\varphi)$ . So to prove part (a) it suffices to show that the quadruple of rings

$$(R_V, \prod_{U \in \mathcal{U}} \widehat{R}_U, \prod_{P \in \mathcal{P}} \widehat{R}_{P,t}, \prod_{\varphi \in \mathcal{B}} \widehat{R}_\varphi)$$

satisfies condition (iii) of Proposition 2.1, or equivalently condition (ii). Since these rings are respectively free of rank  $r$  over the rings in the quadruple  $(R_{U'}, \widehat{R}_{U'}, \widehat{R}_{P',t}, \widehat{R}_{\varphi'})$ , a free

module patching problem  $(M_1, M_2, M_0; \nu_1, \nu_2)$  of rank  $n$  with respect to the first quadruple induces a free module patching problem of rank  $nr$  with respect to the second quadruple. As shown above, the latter patching problem has a solution  $M$ . It remains to show that the free  $R_{U'}$ -module  $M$  of rank  $nr$  is also a free  $R_V$ -module of rank  $n$  and that  $M$  induces  $M_1, M_2$  over the rings  $\prod_{U \in \mathcal{U}} \widehat{R}_U$  and  $\prod_{P \in \mathcal{P}} \widehat{R}_{P,t}$ .

First observe that  $M = M_1 \times_{M_0} M_2$ , set-theoretically, by Proposition 2.1 above applied to the quadruple  $(R_{U'}, \widehat{R}_{U'}, \widehat{R}_{P',t}, \widehat{R}_{\varphi'})$ . Since each  $M_i$  is an  $R_V$ -module and since the maps  $\nu_i : M_i \rightarrow M_0$  are  $R_V$ -module homomorphisms,  $M$  is an  $R_V$ -module, compatibly. Since  $M$  is a solution to the patching problem over  $F'$ , we have identifications  $M_1 = \widehat{R}_{U'} \otimes_{R_{U'}} M = \widehat{R}_{U'} \otimes_{R_{U'}} R_V \otimes_{R_V} M = \prod \widehat{R}_U \otimes_{R_V} M$ . That is,  $M$  induces  $M_1$  over  $\prod_{U \in \mathcal{U}} \widehat{R}_U$ . The case of  $M_2$  is similar. Finally, since  $M_1 = \prod \widehat{R}_U \otimes M$  is free of rank  $n$  over  $\prod \widehat{R}_U$ , it follows that  $M_1/tM_1 = M/tM$  is free of rank  $n$  over  $\prod \widehat{R}_U/t\widehat{R}_U = \prod R_U/tR_U = R_V/tR_V$ ; and thus  $M$  is free of rank  $n$  over  $R_V$  by [Bou72], Proposition II.3.2.5. This completes the proof of part (a).

The proof for (b) is similar, but with the roles of  $U, P$  reversed. We replace  $\widehat{R}_{P',t}$  with  $\widehat{R}_{U',t}$ , the localization of  $\widehat{R}_{U'}$  at the prime ideal  $(t)$ . This is equal to each of the intersections  $F_{U'} \cap \widehat{R}_{\varphi'} = F_{U'} \cap \widehat{R}_{\emptyset}$ . For  $U \in \mathcal{U}$  we similarly replace  $\widehat{R}_{P,t}$  with the localization  $\widehat{R}_{U,t}$  of  $\widehat{R}_U$  at its Jacobson radical, which is the unique height one prime containing  $t$ . We obtain canonical isomorphisms as before. The given factorization problem yields a free module patching problem for the quadruple of rings

$$(R_{\mathcal{P}}, \prod_{U \in \mathcal{U}} \widehat{R}_{U,t}, \prod_{P \in \mathcal{P}} \widehat{R}_P, \prod_{\varphi \in \mathcal{B}} \widehat{R}_{\varphi}),$$

where  $R_{\mathcal{P}}$  is the subring of  $F$  consisting of the rational functions on  $\widehat{X}$  that are regular at the points of  $\mathcal{P}$ . These rings are respectively free of rank  $r$  over the rings in the quadruple  $(R_{P'}, \widehat{R}_{U',t}, \widehat{R}_{P'}, \widehat{R}_{\varphi'})$ , and are obtained by tensoring those rings over  $R_{P'}$  with  $R_{\mathcal{P}}$ . The given free module patching problem with respect to the first quadruple induces a free module patching problem with respect to the second quadruple, having a solution  $M$  by Case 1 and Proposition 2.1 as in (a).

Proceeding as in the proof of (a), it remains to show that  $M$  is free of rank  $n$  over  $R_{\mathcal{P}}$ . By hypothesis,  $\prod \widehat{R}_P \otimes M$  is free of rank  $n$  over  $\prod \widehat{R}_P$ , and so is flat over  $\prod \widehat{R}_P$ . Here  $\widehat{R}_P \otimes M$  is the  $\mathfrak{m}_P$ -adic completion of  $M_{\{P\}} := R_{\{P\}} \otimes M$  by [Bou72, Theorem III.3.4.3(ii)]; and thus  $M_{\{P\}}$  is flat over  $R_{\{P\}}$  by [Bou72, Proposition III.5.4.4] (with  $A = B$  in the notation there). Since  $R_{\{P\}}$  is local and  $M$  is finitely generated, it follows that the flat  $R_{\{P\}}$ -module  $M_{\{P\}}$  is free; moreover its rank is  $n$ , since this is the case modulo  $\mathfrak{m}_P$  (because  $\prod \widehat{R}_P \otimes M$  is free of rank  $n$  over  $\prod \widehat{R}_P$ ). Since  $R_{\mathcal{P}}$  is a semi-local ring whose localizations are the rings  $R_{\{P\}}$  for  $P \in \mathcal{P}$ , it follows by [Eis95, Exercise 4.13] (or by [BH93, Lemma 1.4.4]) that  $M$  is free of rank  $n$  over  $R_{\mathcal{P}}$ .  $\square$

**Corollary 2.4.** *Suppose that for every  $U \in \mathcal{U}$  we are given an element  $a_U \in F_U^\times$ . Then there exist  $b \in F^\times$  and elements  $c_U \in \widehat{R}_U^\times$  such that  $a_U = bc_U \in F_U^\times$  for all  $U \in \mathcal{U}$ .*

*Proof.* For each  $U \in \mathcal{U}$ , let  $\eta_U$  be its generic point and let  $t_U \in F$  be a uniformizer at  $\eta_U$ . Choose an affine open subset  $\text{Spec}(R)$  of  $\widehat{X}$  that contains the points  $\eta_U$ . Then  $R$  is Noetherian and integrally closed (since  $\widehat{X}$  is normal), and thus is a Krull domain by [Bou72], Corollary to Lemma 1 in Section VII.1.3. By Theorem 4 of [Bou72], Section VII.1.6, each  $t_U$  defines an essential valuation of  $R$ ; and then by [Bou72], Proposition 9 of Section VII.1.5, there exists  $s \in F^\times$  whose  $t_U$ -adic valuation is the same as that of  $a_U$  for all  $U \in \mathcal{U}$ . Replacing each  $a_U$  by  $s^{-1}a_U$ , we may assume that  $a_U$  is a  $t_U$ -adic unit for every  $U$ . In particular,  $a_U$  is a unit in  $\widehat{R}_\varphi$  for every branch  $\varphi \in \mathcal{B}$  lying on  $U$ .

For each branch  $\varphi \in \mathcal{B}$ , there is a unique  $U \in \mathcal{U}$  such that  $\varphi$  lies on  $U$ ; let  $c_\varphi \in F_\varphi^\times$  be the image of  $a_U$ . Note that  $c_\varphi \in \widehat{R}_\varphi^\times$  because of the above assumption on  $a_U$ . Applying Corollary 2.3(a) to these elements (viewed as  $1 \times 1$  matrices), we obtain elements  $c_P \in F_P^\times$  for all  $P \in \mathcal{P}$ , and elements  $c_U \in \widehat{R}_U^\times$  for all  $U \in \mathcal{U}$ , such that  $c_\varphi = c_P c_U \in F_\varphi^\times$  for each branch  $\varphi$  on  $U$  at  $P$  (with respect to the inclusions of  $F_P$  and  $F_U$  into  $F_\varphi$ ). Set  $b_P = c_P \in F_P^\times$  for every  $P \in \mathcal{P}$ , and set  $b_U = a_U c_U^{-1} \in F_U^\times$  for each  $U \in \mathcal{U}$ . For a branch  $\varphi \in \mathcal{B}$  at  $P \in \mathcal{P}$  lying on  $U \in \mathcal{U}$ , we have  $b_P = c_P = a_U c_U^{-1} = b_U$  in  $\widehat{R}_\varphi^\times$ . Hence the elements  $b_P \in F_P^\times$  (for  $P \in \mathcal{P}$ ),  $b_U \in F_U^\times$  (for  $U \in \mathcal{U}$ ), and the induced elements of  $F_\varphi^\times$  together form an element of the inverse system formed by the groups  $F_U^\times, F_P^\times, F_\varphi^\times$ ; and so define an element  $b$  of the inverse limit, which is  $F^\times$  by [HH10, Proposition 6.3]. Finally,  $a_U = b_U c_U = b c_U$  with respect to the above inclusions.  $\square$

The obvious analog of the above result with the roles of  $\mathcal{U}$  and  $\mathcal{P}$  interchanged would say that if we are given an element  $a_P \in F_P^\times$  for every  $P \in \mathcal{P}$ , then there exist  $b \in F^\times$  and elements  $c_P \in \widehat{R}_P^\times$  such that  $a_P = b c_P \in F_P^\times$  for all  $P \in \mathcal{P}$ . But this assertion is false. The reason is that if  $\varphi, \varphi' \in \mathcal{B}$  are branches lying on a common  $U \in \mathcal{U}$  at points  $P, P' \in \mathcal{P}$ , then the  $\varphi$ -adic valuation  $v_\varphi(a_P)$  of  $a_P = b c_P$  would have to equal the  $\varphi'$ -adic valuation  $v_{\varphi'}(a_{P'})$  of  $a_{P'} = b c_{P'}$ ; viz. both would have to have equal the  $t_U$ -adic valuations of  $b$  (where  $t_U$  is a uniformizer at the generic point  $\eta_U$  of  $U$ , as above). But under this additional hypothesis on the given family  $\{a_P\}$ , the analog holds:

**Corollary 2.5.** *Suppose that for every  $P \in \mathcal{P}$  we are given an element  $a_P \in F_P^\times$ . Suppose also that if  $\varphi, \varphi' \in \mathcal{B}$  are branches at  $P, P' \in \mathcal{P}$  lying on a common  $U \in \mathcal{U}$ , then  $v_\varphi(a_P) = v_{\varphi'}(a_{P'})$ . Then there exist  $b \in F^\times$  and elements  $c_P \in \widehat{R}_P^\times$  such that  $a_P = b c_P \in F_P^\times$  for all  $P \in \mathcal{P}$ .*

*Proof.* For each  $U \in \mathcal{U}$ , let  $n_U$  be the common value of  $v_\varphi(a_P)$  for all  $P \in \mathcal{P}$  lying on the closure of  $U$  and all branches  $\varphi \in \mathcal{B}$  at  $P$  on  $U$ . By [Bou72], Proposition 9 of Section VII.1.5, there exists  $s \in F^\times$  whose  $t_U$ -adic valuation is equal to  $n_U$  for all  $U$ . Thus  $v_\varphi(s) = n_U$  for each branch  $\varphi \in \mathcal{B}$  on  $U$ . Replacing each  $a_P$  by  $s^{-1}a_P$ , we may assume that  $a_P$  is a  $t_U$ -adic unit for every  $P \in \mathcal{P}$  and  $U \in \mathcal{U}$  such that  $P$  lies on the closure of  $U$ . In particular,  $a_P$  is a unit in  $\widehat{R}_\varphi$  for every branch  $\varphi \in \mathcal{B}$  at  $P$ .



Let  $c_\varphi \in \widehat{R}_\varphi^\times$  be the image of  $a_P$ . Applying Corollary 2.3(b), in the  $1 \times 1$  case, to the elements  $c_\varphi^{-1}$ , we obtain elements  $c_P^{-1} \in \widehat{R}_P^\times$  for all  $P \in \mathcal{P}$ , and elements  $c_U^{-1} \in F_U^\times$  for all  $U \in \mathcal{U}$ , such that  $c_\varphi^{-1} = c_P^{-1}c_U^{-1} \in F_\varphi^\times$  (or equivalently,  $c_\varphi = c_Uc_P \in F_\varphi^\times$ ) for each branch  $\varphi$  on  $U$  at  $P$ , with respect to the inclusions of  $F_P$  and  $F_U$  into  $F_\varphi$ . Set  $b_U = c_U \in F_U^\times$  for  $U \in \mathcal{U}$ ; and set  $b_P = a_Pc_P^{-1} \in F_P^\times$  for  $P \in \mathcal{P}$ . Again we have that the elements  $b_P \in F_P^\times$  (for  $P \in \mathcal{P}$ ) and  $b_U \in F_U^\times$  (for  $U \in \mathcal{U}$ ) induce the same elements in  $F_\varphi^\times$ , and so define an element  $b \in F^\times$ . Finally,  $a_P = b_Pc_P = bc_P$  with respect to the above inclusions.  $\square$

**Remark 2.6.** There are variants of the above two results for  $n \times n$  matrices, using that Corollary 2.3 holds for matrices, and not just field elements. It is convenient to state these variants using notations introduced in the proof of Corollary 2.3, viz.  $\widehat{R}_{U,t}$  for the localization of  $\widehat{R}_U$  at its height one prime containing  $t$ , and  $\widehat{R}_{P,t}$  for the semi-localization of  $\widehat{R}_P$  at its branches, i.e., its height one primes containing  $t$ . With this notation, the analog of Corollary 2.4 then asserts that if for each  $U \in \mathcal{U}$  we are given a matrix  $A_U \in \mathrm{GL}_n(\widehat{R}_{U,t})$ , then there exist  $B \in \mathrm{GL}_n(F)$  and  $C_U \in \mathrm{GL}_n(\widehat{R}_U)$  such that  $A_U = BC_U \in \mathrm{GL}_n(\widehat{R}_{U,t})$  for all  $U \in \mathcal{U}$ . Similarly, the analog of Corollary 2.5 says that if for every  $P \in \mathcal{P}$  we are given a matrix  $A_P \in \mathrm{GL}_n(\widehat{R}_{P,t})$ , then there exist  $B \in \mathrm{GL}_n(F)$  and  $C_P \in \mathrm{GL}_n(\widehat{R}_P)$  such that  $A_P = BC_P \in \mathrm{GL}_n(\widehat{R}_{P,t})$  for all  $P \in \mathcal{P}$ .

The reason that we assume here that the given matrices lie in  $\mathrm{GL}_n(\widehat{R}_{U,t})$  (resp. in  $\mathrm{GL}_n(\widehat{R}_{P,t})$ ) rather than simply in  $\mathrm{GL}_n(F_U)$  (resp. in  $\mathrm{GL}_n(F_P)$ ) is that the proofs of the preceding two results involved a reduction step, which relied on the fact that every non-zero element in a discretely valued field is a unit in the valuation ring multiplied by a power of a given uniformizer. The analog of this fact does not hold in the  $n \times n$  case. Assuming that the given matrices are defined over the above smaller rings eliminates the need for that reduction step. Note that it also eliminates the need for the extra hypothesis in Corollary 2.5 when  $n = 1$ .

### 3 Weierstrass preparation

The factorization results of Section 2 apply in particular to the situation in which the elements are given just for some of the  $U \in \mathcal{U}$ , or just for some of the points  $P \in \mathcal{P}$ . Namely, we suitably define the other elements, and then apply the result as stated. In this way, we obtain versions of the Weierstrass Preparation Theorem, extending to the singular case results that had been proven in the smooth case in [HH10] (see Remark 3.2(a) below).

As before, we are in the context of Notation 2.2. We have the following version of the Weierstrass preparation theorem:

**Theorem 3.1.** *Let  $F$  be a one-variable function field over the fraction field of a complete discrete valuation ring  $T$ , and let  $\widehat{X}$  be a normal model for  $F$  over  $T$ . Let  $\mathcal{P}, \mathcal{U}, \mathcal{B}$  be as in Notation 2.2.*

- (a) If  $U \in \mathcal{U}$  and  $a \in F_U$ , then there exist  $b \in F$  and  $c \in \widehat{R}_U^\times$  such that  $a = bc$ .
- (b) If  $\wp \in \mathcal{B}$  and  $a \in F_\wp$ , then there exist  $b \in F$  and  $c \in \widehat{R}_\wp^\times$  such that  $a = bc$ .
- (c) Let  $P \in \mathcal{P}$  and  $a \in F_P$ . Suppose that if  $\wp, \wp' \in \mathcal{B}$  are branches at  $P$  lying on a common  $U \in \mathcal{U}$ , then  $v_\wp(a) = v_{\wp'}(a)$ . Then there exist  $b \in F$  and  $c \in \widehat{R}_P^\times$  such that  $a = bc$ .

*Proof.* We may assume without loss of generality that  $a$  is nonzero in each case; otherwise the assertion is trivially true with  $b = 0$ .

(a) Set  $a_U = a$ , and set  $a_{U'} = 1$  for each  $U' \in \mathcal{U}$  other than  $U$ . The assertion is now immediate from Corollary 2.4.

(b) Consider the irreducible component  $X_0$  of  $X$  on which  $\wp$  lies, and let  $s \in F$  be a uniformizer at the generic point of  $X_0$ . Then  $s$  is also a uniformizer for  $\widehat{R}_\wp$ . Since  $a \neq 0$  we may write  $a = a's^m$  for some  $a' \in \widehat{R}_\wp^\times$  and some  $m \in \mathbb{N}$ . Now let  $b = s^m$  and  $c = a'$ .

(c) Let  $U_1, \dots, U_n \in \mathcal{U}$  be the elements of  $\mathcal{U}$  whose closures contain  $P$ . For  $i = 1, \dots, n$ , let  $I_i \subset R_P$  be the height one prime of  $R_P \subset F$  corresponding to  $U_i$ . By the normality hypothesis, the Noetherian local ring  $R_P$  is a Krull domain. So for each  $i$  there exists  $s_i \in R_P$  whose  $I_i$ -adic valuation is one, and which does not lie in any other  $I_j$  ([Bou72], Proposition 9 of Section VII.1.5). Thus for each branch  $\wp \in \mathcal{B}$  at  $P$  lying on the closure of  $U_i$ ,  $v_\wp(s_i) = 1$  and  $v_\wp(s_j) = 0$  for  $j \neq i$ .

Let  $m_i$  be the  $I_i$ -adic valuation of  $a$ . Thus  $v_\wp(a) = m_i$  for all branches  $\wp \in \mathcal{B}$  at  $P$  lying on the closure of  $U_i$ , and the element  $a_P := a / \prod s_i^{m_i} \in F_P^\times$  has  $\wp$ -adic valuation equal to zero for all  $\wp \in \mathcal{B}$ . Set  $a_{P'} = 1 \in F_{P'}^\times$  for each  $P' \in \mathcal{P}$  other than  $P$ . These elements together satisfy the hypotheses of Corollary 2.5, each having valuation equal to zero at each respective branch. Thus  $a_P = b'c$  for some  $b' \in F^\times$  and  $c \in \widehat{R}_P^\times$ . Setting  $b = b' \prod s_i^{m_i} \in F$  yields the assertion.  $\square$

**Remark 3.2.** (a) In the case that the model  $\widehat{X}$  of  $F$  is smooth over  $T$ , parts (a) and (c) of Theorem 3.1 follow from results in [HH10]. Namely, part (a) is given by Corollary 4.8 of [HH10]. For part (c), by writing  $a \in F_P$  as a ratio of elements in  $\widehat{R}_P$ , we are reduced to the case that  $a$  lies in  $\widehat{R}_P$ . Proposition 5.6 of [HH10] then allows us to write  $a$  as the product of elements in  $\widehat{R}_P^\times$  and  $F_{\{P\}}$ . Applying Corollary 4.8 of [HH10] (with  $U = \{P\}$ ) to the latter element then yields the desired factorization of  $a$ , using that  $\widehat{R}_{\{P\}} \subset \widehat{R}_P$ .

- (b) The extra hypothesis on  $a$  in Theorem 3.1(c) is trivially satisfied if no two branches  $\wp \in \mathcal{B}$  at  $P$  lie on the same  $U \in \mathcal{U}$ . In particular, it always holds in the smooth (unibranched) case considered in the version of Weierstrass Preparation that appeared in [HH10]. It is also satisfied if  $a$  is a unit in  $\widehat{R}_\wp$  for each branch  $\wp \in \mathcal{B}$  at  $P$  (since then the valuations  $v_\wp(a)$  are each equal to zero). This last condition is equivalent to saying that  $a \in \widehat{R}_{P,t}^\times$ , in the notation used in the proof of Case 2 of Proposition 2.3.

- (c) The hypothesis on  $a$  is indeed needed in the general case of Theorem 3.1(c), for a similar reason that a related hypothesis was needed in Corollary 2.5.

Namely, if  $\wp, \wp' \in \mathcal{B}$  are branches at  $P$  lying on the closure of a common  $U \in \mathcal{U}$ , and if there is a global element  $b \in F$  as asserted, then  $\wp$ -adic and  $\wp'$ -adic valuations of  $a$  must each be equal to the  $t_U$ -adic valuation of  $b$ , where  $t_U$  is a uniformizer at the generic point of  $U$ . Hence the existence of an element  $b \in F$  as in the conclusion of the theorem implies that  $v_\wp(a) = v_{\wp'}(a)$ . As an explicit example, we could take  $\widehat{X}$  to be the cover of the projective  $x$ -line over  $T$  given by  $y^2 - x^2(1+x) = t$ , and  $P$  to be the point  $x = y = 0$  on the closed fiber. (Here we assume the residue characteristic is not two.) Let  $z \in \widehat{R}_P$  be a square root of  $1+x$ . Then the two branches  $\wp, \wp'$  at  $P$  respectively correspond to  $y = \pm xz$  along  $t = 0$ . Both branches lie on the unique irreducible component of the closed fiber. Let  $a = y - xz \in F_P$ . Then  $v_\wp(a) = 1$  but  $v_{\wp'}(a) = 0$ . Since these two valuations are unequal, the asserted element  $b \in F$  cannot exist, by the above argument, and the conclusion of Theorem 3.1(c) does not hold for  $a \in F_P$ .

As a consequence, we obtain the following version of Lemma 4.16 of [HHK09] for the case that  $\widehat{X}$  is not necessarily smooth.

**Corollary 3.3.** *With notation as in Theorem 3.1, let  $n > 0$  be a natural number not divisible by the characteristic of the residue field of  $T$ .*

- (a) *For  $U \in \mathcal{U}$  and  $a \in F_U$ , there exist elements  $b \in F$  and  $c \in F_U^\times$  such that  $a = bc^n$ .*
- (b) *For  $\wp \in \mathcal{B}$  and  $a \in F_\wp$ , there exist elements  $b \in F$  and  $c \in F_\wp^\times$  such that  $a = bc^n$ .*
- (c) *Let  $P \in \mathcal{P}$  and  $a \in F_P$ . If  $v_\wp(a) = v_{\wp'}(a)$  for each pair of branches  $\wp, \wp' \in \mathcal{B}$  at  $P$  lying on the closure of a common  $U \in \mathcal{U}$ , then there exist elements  $b \in F$  and  $c \in F_P^\times$  such that  $a = bc^n$ .*

*Proof.* Again, all statements are trivially true if  $a = 0$ , so we assume that  $a \neq 0$  in each case. For part (a) we proceed as in the proof of the analogous case of Lemma 4.16 of [HHK09], but using Theorem 3.1(a) instead of the global Weierstrass Preparation Theorem in the smooth case ([HH10], Proposition 4.7). Namely, write  $a = a_1/a_2$  with  $a_i \in \widehat{R}_U$  and  $a_2 \neq 0$ . By Theorem 3.1(a), there exist  $b_i \in F^\times$  and  $c_i \in \widehat{R}_U^\times$  such that  $a_i = b_i c_i$  for  $i = 1, 2$ . The reduction  $\bar{c}_i \in \widehat{R}_U/t\widehat{R}_U = R_U/tR_U$  of  $c_i$  modulo the uniformizer  $t$  of  $T$  may be lifted to an element  $c'_i \in R_U \subset F$ . Since  $c_i/c'_i \equiv 1$  modulo  $t\widehat{R}_U$ , the element  $c_i/c'_i$  has an  $n$ -th root  $c''_i \in \widehat{R}_U$ , by Hensel's Lemma ([Bou72], Corollary 1 to Theorem III.4.5.2). Here  $c := c''_1/c''_2$  lies in  $F_U^\times$  and  $b := b_1 c'_1/b_2 c'_2$  lies in  $F$ . These elements then satisfy  $a = bc^n$ .

Parts (b) and (c) are proved identically, except that parts (b) and (c) of Theorem 3.1 are used instead of part (a).  $\square$

The extra hypothesis in Theorem 3.1(c) and Corollary 3.3 can be dropped if we allow ourselves to pass to an appropriate finite cover of  $\widehat{X}$ , as we show in Proposition 3.6 and its corollary below.

First, recall from [HHK11, Section 5] that a *split cover* of  $\widehat{X}$  is a finite morphism  $h : \widehat{X}' \rightarrow \widehat{X}$  of normal projective  $T$ -curves such that the fiber over every point  $P$  of  $\widehat{X}$  other than the generic point consists of a disjoint union of copies of  $P$ . Such covers are automatically étale; and for each  $\xi \in \mathcal{P} \cup \mathcal{U}$ , the pullback of  $\widehat{X}'$  to  $\text{Spec}(F_\xi)$  consists of a disjoint union of finitely many copies of  $\text{Spec}(F_\xi)$ . Also recall (from [HHK11, Section 6]) that we may associate to  $\widehat{X}$  a *reduction graph*  $\Gamma(\widehat{X}, \mathcal{P})$  whose vertices are in bijection with  $\mathcal{P} \cup \mathcal{U}$ , and whose edges are in bijection with the set of branches  $\mathcal{B}$ . Namely, the edge associated to  $\wp \in \mathcal{B}$  connects the vertices associated to  $P \in \mathcal{P}$  and  $U \in \mathcal{U}$  when  $\wp$  is a branch at  $P$  lying on the closure of  $U$ . For each split cover  $\pi : \widehat{X}' \rightarrow \widehat{X}$ , taking  $\mathcal{P}' = \pi^{-1}(\mathcal{P})$ , the associated reduction graphs define a covering space  $\Gamma(\widehat{X}', \mathcal{P}') \rightarrow \Gamma(\widehat{X}, \mathcal{P})$ .

If the set  $\mathcal{P}$  contains every closed point of  $X$  at which  $X$  is not unibranched, then this correspondence is a lattice isomorphism between split covers of  $\widehat{X}$  and connected finite covering spaces of  $\Gamma(\widehat{X}, \mathcal{P})$  (Proposition 6.2 of [HHK11]). In particular, there are no non-trivial split covers of  $\widehat{X}$  if the reduction graph is a tree.

**Proposition 3.4.** *Let  $F$  be a one variable function field over the fraction field of a complete discrete valuation ring  $T$ , and let  $\widehat{X}$  be a normal model for  $F$  over  $T$ . Let  $\mathcal{P}$  be a finite non-empty subset of the closed fiber  $X$  that contains all the points where distinct components of  $X$  meet. Then there is a connected split cover  $h : \widehat{X}' \rightarrow \widehat{X}$  such that distinct branches in  $h^{-1}(\mathcal{B})$  at a common point of  $h^{-1}(\mathcal{P})$  lie on the closures of different components in  $h^{-1}(\mathcal{U})$ . Moreover if  $\Gamma(\widehat{X}, \mathcal{P})$  is not a tree, then this cover can be chosen to be abelian of arbitrarily high degree.*

*Proof.* By enlarging the set  $\mathcal{P}$ , we may assume without loss of generality that it contains all the points at which  $X$  is not unibranched (see Hypothesis 5.4 of [HHK11]).

The fundamental group of the graph  $\Gamma = \Gamma(\widehat{X}, \mathcal{P})$  is a free abelian group of rank  $r \geq 0$ , whose abelianization is  $H_1(\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^r$ . Consider the finite set  $\Delta$  of loops in  $\Gamma$  that consist of exactly two vertices (corresponding to some  $P \in \mathcal{P}$  and  $U \in \mathcal{U}$ ) and two edges (corresponding to a choice of two distinct branches at  $P$  on the closure of  $U$ ). The elements of  $\Delta$  induce finitely many non-trivial elements of the abelianization  $\mathbb{Z}^r$ , using the identification  $H_1(\Gamma, \mathbb{Z}) \simeq \mathbb{Z}^r$  and the fact that  $\Gamma$  is a one-dimensional simplicial complex. Choose  $n$  sufficiently large so that none of these elements lies in  $n\mathbb{Z}^r$ , and let  $N$  be the inverse image of  $n\mathbb{Z}^r$  in  $\pi_1(\Gamma)$ . This is a normal subgroup of finite index, corresponding to a finite Galois covering space  $\Gamma'$  of  $\Gamma$ . By construction, none of the elements of  $\Delta$  lift to loops in  $\Gamma'$ . But any loop in  $\Gamma'$  that consists of two vertices and two edges must map to a loop in  $\Gamma$  of the same type, since  $\Gamma' \rightarrow \Gamma$  is a covering space of bipartite graphs. So in fact  $\Gamma'$  contains no loops of this type. Using Proposition 6.2 of [HHK11] (which is possible because of the extra

assumption on  $\mathcal{P}$ ), there is a split cover  $\widehat{X}'$  of  $\widehat{X}$  whose closed fiber  $X'$  gives rise to  $\Gamma'$ . Thus  $\widehat{X}' \rightarrow \widehat{X}$  has the desired property.

For the last assertion, notice that if  $\Gamma$  is not a tree for the original set  $\mathcal{P}$ , then enlarging the set  $\mathcal{P}$  will give a refined graph which is also not a tree. Since the cover  $\widehat{X}'$  constructed above is Galois with group  $\pi_1(\Gamma)/N = \mathbb{Z}^r/n\mathbb{Z}^r$ , it is abelian and its degree  $nr$  can be chosen to be arbitrarily large, by choosing  $n$  large.  $\square$

**Example 3.5.** The simplest non-trivial example of Proposition 3.4 is that of a Tate curve; viz. the generic fiber of  $\widehat{X}$  is an elliptic curve and the special fiber is a rational nodal curve. We let  $\mathcal{P}$  be the set consisting of the nodal point. In this case  $\Gamma(\widehat{X}, \mathcal{P})$  consists of two vertices connected by two edges;  $\widehat{X}$  has non-trivial cyclic split covers of all degrees  $> 1$ ; and distinct branches on any of these covers  $\widehat{X}'$  of  $\widehat{X}$  lie on the closures of distinct components of the closed fiber. Cf. [Sai85], Example 2.7, and [HHK09], Example 4.4. Also compare Remark 3.2(c).

If  $\widehat{X}' \rightarrow \widehat{X}$  is a split cover and  $P'$  is a point on the closed fiber  $X'$  lying over  $P \in X$ , then the associated fields  $F'_{P'}$  and  $F_P$  (see Notation 2.2) are isomorphic under the natural inclusion (see the beginning of Section 5 of [HHK11]). Hence we may identify (elements of) these two fields.

**Proposition 3.6.** *Let  $F$  be a one-variable function field over the fraction field of a complete discrete valuation ring  $T$ , and let  $\widehat{X}$  be a normal model for  $F$  over  $T$ . Then there is a split cover  $\widehat{X}' \rightarrow \widehat{X}$ , with closed fiber  $X'$  and function field  $F'$ , such that the following holds: Given  $P \in \mathcal{P}$  and  $a \in F_P$ , then for each  $P' \in X'$  lying over  $P$ , there exist an element  $b \in F'$  and a unit  $c \in \widehat{R}'_{P'}^\times$  satisfying  $a = bc \in F'_{P'} = F_P$ .*

*Proof.* By Proposition 3.4, there is a split cover  $\widehat{X}' \rightarrow \widehat{X}$  with the property that distinct branches at any point of  $X'$  must lie on distinct components of  $X'$ . Given  $a \in F_P = F'_{P'}$ , the factorization  $a = bc$  now follows from applying Theorem 3.1(c) to  $\widehat{X}'$ , in the special case in which no two branches  $\wp \in \mathcal{B}$  at any point  $P \in \mathcal{P}$  lie on the closure of the same component of the closed fiber (as noted in Remark 3.2(b)).  $\square$

We may now remove the additional assumption in part (c) of Corollary 3.3 above.

**Corollary 3.7.** *With notation as in Proposition 3.6, let  $p \geq 0$  be the characteristic of the residue field of  $T$ . Then there is a split cover  $\widehat{X}' \rightarrow \widehat{X}$ , with closed fiber  $X'$  and function field  $F'$ , such that for every  $n$  not divisible by  $p$ , the following holds: Given  $P \in \mathcal{P}$  and  $a \in F_P$ , then for each  $P' \in X'$  lying over  $P$ , there exist elements  $b \in F'$  and  $c \in F'_{P'}^\times$  such that  $a = bc^n \in F'_{P'} = F_P$ .*

*Proof.* The proof is the same as that of part (c) of Corollary 3.3, using Proposition 3.6 instead of Theorem 3.1.  $\square$

We conclude this section with a result which can be used to show that certain fields are of the form  $F_P$ . This will be useful in the applications in the next section.

**Lemma 3.8.** *Let  $k$  be a field, let  $T = k[[t]]$ , and let  $E$  be a finite separable extension of  $k((x, y))$ , viewed as a  $k$ -algebra. Then there is a connected normal projective  $T$ -curve  $\widehat{X}$  and a closed point  $P$  on  $\widehat{X}$  such that  $E$  is isomorphic to  $F_P$  as a  $k$ -algebra.*

*Proof.* Let  $A$  be the integral closure of  $k[[x, y]]$  in  $E$ . Then  $A$  is unramified over the locus of  $(y + x^n)$  for some positive integer  $n$ . Embedding  $T$  in  $k[[x, y]]$  by sending  $t$  to  $y + x^n$ , we may identify  $k[[x, y]]$  with  $k[[x, t]]$ ; and  $A$  is then unramified over the locus of  $t = 0$ . We may also identify  $k[[x, t]]$  with  $\widehat{R}_{P'}$ , where  $P'$  is the point  $x = t = 0$  on  $\widehat{X}' = \mathbb{P}_T^1$ , the projective  $T$ -line with coordinate  $x$  and function field  $F'$ .

In the case that  $E$  is Galois over  $k((x, y)) = k((x, t))$ , say with group  $G$ , Lemma 5.2 of [HS05] asserts that there is a finite generically separable morphism  $\widehat{X} \rightarrow \widehat{X}'$  of connected normal  $T$ -curves that is  $G$ -Galois as a branched cover (i.e. whose corresponding function field extension is  $G$ -Galois) and whose pullback via  $\text{Spec}(k[[x, t]]) \rightarrow \widehat{X}'$  is isomorphic to  $\text{Spec}(A) \rightarrow \text{Spec}(k[[x, t]])$ . Thus there is a unique closed point  $P$  on  $\widehat{X}$  that lies over the point  $P' \in \widehat{X}'$ , and  $\widehat{R}_P$  is isomorphic to  $A$  as a  $k[[x, t]]$ -algebra with  $G$ -action. Hence  $F_P$  is isomorphic to  $E$  as a  $G$ -Galois field extension of  $k((x, t))$ , and in particular as a  $k$ -algebra.

The general case may now be reduced to the above Galois case. Namely, let  $E^*$  be the Galois closure of the finite separable extension  $E/k((x, t))$ . Let  $G$  be the Galois group of  $E^*/k((x, t))$  and let  $H \subset G$  be the Galois group of  $E^*$  over  $E$ . By the Galois case as above, there is a  $G$ -Galois normal branched cover  $\pi^* : \widehat{X}^* \rightarrow \mathbb{P}_T^1$ , say with function field  $F^*$ , with a unique closed point  $P^*$  on  $\widehat{X}^*$  lying over  $P' \in \widehat{X}'$ , such that  $E^*$  is isomorphic to  $F_{P^*}^*$  over  $F_{P'}^* = k((x, t))$ . Let  $\widehat{X} = \widehat{X}^*/H$ , let  $\pi : \widehat{X} \rightarrow \mathbb{P}_T^1$  be the induced morphism (through which  $\pi^*$  factors), and let  $P \in \widehat{X}$  be the image of  $P^* \in \widehat{X}^*$ . Then  $E$  is isomorphic to  $F_P$  as a  $k((x, t))$ -algebra, and in particular as a  $k$ -algebra.  $\square$

## 4 Applications

Using what was shown in Section 3, we can strengthen results that were obtained in [HHK09] concerning quadratic forms and central simple algebras, by dropping smoothness assumptions.

Our setup is as in Notation 2.2. Given such a discrete valuation ring  $T$  and function field  $F$ , there are many normal models  $\widehat{X}$  of  $F$  over  $T$ , and in particular there are always regular models (see [Abh69], [Lip75]). But in general there need not exist a smooth model of  $F$  over  $T$ . While normality sufficed for some of the results in [HHK09], others required smoothness, because they relied on the form of Weierstrass Preparation that appeared in [HH10], which itself had assumed smoothness. Our more general form of Weierstrass Preparation above (Theorem 3.1) yielded Corollaries 3.3 and 3.7, and these in turn will permit us to generalize results of [HHK09].

## 4.1 Applications to quadratic forms

The classical  $u$ -invariant of a field  $k$ , defined by Kaplansky, is the largest dimension  $u(k)$  of a quadratic form over  $k$  that is anisotropic (i.e. has no non-trivial zeroes). While  $u(k)$  is known for certain fields, in general it is rather mysterious.

In order to obtain results about the value of  $u(k)$  in [HHK09], we introduced a related invariant  $u_s(k)$ . By definition, this is the smallest number  $n$  such that  $u(E) \leq n$  for every finite field extension  $E/k$ , and also such that  $u(E) \leq 2n$  for every finitely generated field extension  $E/k$  of transcendence degree one. (It was pointed out to us by Karim Becher that the value of this invariant remains unchanged if the first of these two conditions is dropped.)

In the geometric situation considered in Notation 2.2, Corollary 4.17 of [HHK09] related the  $u$ -invariant of a field  $F_U$  or  $F_P$  to the values of  $u$  and  $u_s$  on  $k$  and its finite extensions. Using our generalized Weierstrass Preparation Theorem, we can remove the smoothness assumption that was needed in that result in [HHK09]. The proof here parallels the argument in [HHK09], with Lemma 4.16 of [HHK09] replaced by Corollaries 3.3 and 3.7 above.

**Theorem 4.1.** *Let  $T$  be a complete discrete valuation ring with uniformizer  $t$ , whose residue field  $k$  is not of characteristic two. Let  $\widehat{X}$  be a normal projective  $T$ -curve with closed fiber  $X$ , let  $\mathcal{P}$  be a non-empty finite set of closed points of  $\widehat{X}$  that contains every point at which distinct components of  $X$  meet, and let  $\mathcal{U}$  be the set of components of the complement of  $\mathcal{P}$  in  $X$ .*

(a) *If  $\xi \in \mathcal{P} \cup \mathcal{U}$ , then  $u(F_\xi) \leq 4u_s(k)$ .*

(b) *Let  $\tilde{X}$  be the normalization of  $X$ . If  $P \in \mathcal{P}$  is a closed point of  $X$ , then  $u(F_P) \geq 4u(\kappa(\tilde{P}))$  for any  $\tilde{P} \in \tilde{X}$  lying over  $P$ . If  $Q \in \mathcal{U}$  is a closed point of  $X$ , then  $u(F_U) \geq 4u(\kappa(\tilde{Q}))$  for any  $\tilde{Q} \in \tilde{X}$  lying over  $Q$ .*

*Proof.* (a) By the assumption on the characteristic, any quadratic form  $q$  over  $F_\xi$  may be diagonalized. So given a regular quadratic form  $q$  of dimension  $n > 4u_s(k)$  over  $F_\xi$ , we may assume it has the form  $a_1x_1^2 + \cdots + a_nx_n^2$  with  $a_i \in F_\xi^\times$ .

If  $\xi = U \in \mathcal{U}$ , then for each  $i$  we may write  $a_i = a'_i u_i^2$  for some  $a'_i \in F^\times$  and  $u_i \in F_U^\times$ , by Corollary 3.3(a). Hence after adjusting  $x_i$  by a factor of  $u_i$ , we may assume that each  $a_i$  lies in  $F^\times$ . Now  $u_s(K) = 2u_s(k)$  by Theorem 4.10 of [HHK09], where  $K$  is the fraction field of  $T$ ; and so  $n > 2u_s(K)$ . Since  $F$  is a finitely generated field extension of  $K$  of transcendence degree one, it follows from the definition of  $u_s$  that the  $n$ -dimensional form  $q$  is isotropic over  $F$ , and hence over  $F_U$ .

In the case that  $\xi \in \mathcal{P}$ , Corollary 3.7 yields a split cover  $\widehat{X}' \rightarrow \widehat{X}$  with function field  $F'$  such that each  $a_i \in F_P^\times = F_{P'}^\times$  may be written as  $a'_i u_i^2$  for some  $a'_i \in F'^\times$  and  $u_i \in F_{P'}^\times$ , where  $P' \in \widehat{X}'$  lies over  $P$  (where we again identify  $F_P$  with its trivial extension  $F_{P'}$ ). Adjusting  $x_i$  by a factor of  $u_i \in F_{P'}^\times = F_P^\times$ , we may thus assume that each  $a_i$  lies in  $F'^\times$ . Since  $F'$  is a finitely generated field extension of  $K$  of transcendence degree one, and since  $u_s(K) = 2u_s(k)$  as above,  $q$  is isotropic over  $F'$  and hence over  $F_{P'} = F_P$ .

(b) First consider the case that  $\xi = U \in \mathcal{U}$ . The local ring of  $\tilde{X}$  at  $\tilde{Q}$  is a discrete valuation ring with residue field  $\kappa(\tilde{Q})$  and fraction field equal to the function field  $E$  of  $U$ . Also, the localization  $\widehat{R}_{U,t}$  of  $\widehat{R}_U$  at its Jacobson radical is a discrete valuation ring with residue field  $E$  and fraction field  $F_U$ . Lemma 4.9 of [HHK09] then yields the inequalities  $u(F_U) \geq 2u(E) \geq 4u(\kappa(\tilde{Q}))$ , as asserted.

Next, consider the case  $\xi = P \in \mathcal{P}$ . Let  $\tilde{\wp}$  be the unique branch of  $\tilde{X}$  at  $\tilde{P}$ , and let  $\wp$  be the branch of  $X$  at  $P$  that  $\tilde{\wp}$  lies over. Let  $\widehat{R}_{P,\wp}$  be the localization of  $\widehat{R}_P$  at  $\wp$ . This is a discrete valuation ring with residue field  $\kappa(\wp)$  and fraction field  $F_P$ . The complete local ring of  $\tilde{X}$  at  $\tilde{P}$  is a discrete valuation ring with residue field  $\kappa(\tilde{P})$  and fraction field  $\kappa(\tilde{\wp}) \cong \kappa(\wp)$ . Lemma 4.9 of [HHK09] then yields the desired inequalities  $u(F_P) \geq 2u(\kappa(\tilde{\wp})) \geq 4u(\kappa(\tilde{P}))$ .  $\square$

Of course if the closed point  $P$  (resp.  $Q$ ) in Theorem 4.1 (b) is a regular point of the closed fiber  $X$ , then we can simply consider the residue field at the point itself rather than passing to a point on the normalization  $\tilde{X}$ .

As a consequence, we obtain the following (compare Corollary 4.19 of [HHK09], and Question 4.11 of [Hu11]):

**Corollary 4.2.** *Let  $k$  be a field of characteristic unequal to two and let  $E$  be a finite separable extension of  $k((x, t))$ .*

(a) *Then  $u(E) \leq 4u_s(k)$ .*

(b) *If  $u(k) = u_s(k)$  and if every finite extension  $k'$  of  $k$  satisfies  $u(k') = u(k)$ , then  $u(E) = 4u(k)$ .*

*Proof.* Let  $\widehat{X}$  and  $P$  be as in the conclusion of Lemma 3.8 applied to  $E$ , and extend the set  $\{P\}$  to a finite subset  $\mathcal{P} \subset X$  that contains all the points where distinct components of  $X$  meet. Then  $u(E) = u(F_P) \leq 4u_s(k)$  by Theorem 4.1(a), proving part (a). For part (b),  $u(E) \leq 4u_s(k) = 4u(k)$  by (a), while the reverse inequality is given by Theorem 4.1(b).  $\square$

Recall from [Ser73], II.4.5, that for  $d > 0$  a  $C_d$ -field is a field  $k$  such that for every  $m, n$  and every homogeneous polynomial  $f$  over  $k$  of degree  $m$  in  $n$  variables,  $f$  has a non-trivial solution over  $k$  if  $n > m^d$ . As was observed after Corollary 4.12 of [HHK09], if  $k$  is a  $C_d$  field then  $u_s(k) \leq 2^d$ . Hence Corollary 4.2(a) immediately gives:

**Corollary 4.3.** *Let  $k$  be a  $C_d$  field of characteristic unequal to two. Let  $E$  be a finite separable extension of  $k((x, t))$ . Then  $u(E) \leq 2^{d+2}$ .*

More generally, Theorem 4.1(a) implies:

**Corollary 4.4.** *Under the hypotheses of Theorem 4.1, if  $k$  is a  $C_d$  field then  $u(F_\xi) \leq 2^{d+2}$ .*



In the situation of this corollary, observe that if  $u(\kappa(P)) = 2^d$  (resp.  $u(\kappa(Q)) = 2^d$ ), then the opposite inequality also holds, by Theorem 4.1(b). Hence in that case,  $u(F_\xi) = 2^{d+2}$ . In particular, if  $k$  is algebraically closed (resp. finite) and  $\xi \in \mathcal{P} \cup \mathcal{U}$ , then  $u(F_\xi) = 4$  (resp. 8), generalizing [HHK09, Corollary 4.18]. In the situation that  $\xi = P \in \mathcal{P}$ , these special cases were shown in [COP02, Theorem 3.6] and [Hu11, Theorem 1.5], respectively.

We conclude this section with an example.

**Example 4.5.** Recall that  $k$  is an  $m$ -local field with residue field  $k_0$  if there are fields  $k_1, \dots, k_m$  with  $k_m = k$ , such that  $k_i$  is the fraction field of an excellent henselian discrete valuation ring with residue field  $k_{i-1}$  for  $i = 1, \dots, m$ . For example,  $k = k_0((z_1))((z_2)) \cdots ((z_m))$  is such a field.

Now let  $E$  be a finite separable extension of  $k((x, t))$ , where  $k$  is an  $m$ -local field whose residue field  $k_0$  is algebraically closed (resp. finite) of characteristic not 2. Then  $u(E)$  is equal to  $2^{m+2}$  (resp.  $2^{m+3}$ ).

To see this, first note by induction that  $2^m u(k_0) \leq u(k) \leq u_s(k) = 2^m u_s(k_0)$ , using [HHK09], Lemma 4.9 and Corollary 4.12. If  $k_0$  is algebraically closed (resp. finite), then  $u(k_0) = u_s(k_0)$  and so  $u(k) = u_s(k) = 2^m u(k_0)$ , which is equal to  $2^m$  (resp.  $2^{m+1}$ ). Moreover the same holds for any finite extension  $k'$  of  $k$ , since  $k'$  is itself an  $m$ -local field whose residue field is a finite extension of  $k_0$  (by induction). Hence the hypotheses of Corollary 4.2(b) are satisfied, and the conclusion follows.

As an explicit example, if  $E$  is a finite separable extension of  $k_0((u))((x, t))$ , then  $u(E)$  equals 8 (resp. 16).

## 4.2 Applications to central simple algebras

Corollaries 3.3 and 3.7 also have applications to the period-index problem for Brauer groups. Below we use these corollaries to extend results from [HHK09].

Recall (e.g. from [Pie82]) that the Brauer group  $\text{Br}(F)$  of a field  $F$  consists of equivalence classes of central simple  $F$ -algebras. The *period* of a central simple  $F$ -algebra  $A$ , or of its class  $\alpha$ , is the order of the class in the Brauer group. The *index* of  $A$  (or of  $\alpha$ ) is the degree over  $F$  of the division algebra that lies in the class; or equivalently, the minimal degree of a field extension  $L/F$  over which  $A$  splits (i.e. such that  $A \otimes_F L$  is isomorphic to  $\text{Mat}_n(L)$  for some  $n$ ). This is also the greatest common divisor of the degrees of the splitting fields  $L$  of  $A$ . The period always divides the index, and the index always divides some power of the period. The *period-index problem* asks for an exponent  $d$  depending only on  $F$  such that all central simple  $F$ -algebras  $A$  satisfy  $\text{ind}(A) \mid \text{per}(A)^d$ . In asking this question, one often restricts attention to elements whose period is not divisible by a certain prime number.

To help make this question more precise, there is the following terminology ([HHK09], Definition 1.3; see also [Lie09]). The *Brauer dimension* of a field  $k$  (*away from a prime*  $p$ ) is defined to be 0 if  $k$  is separably closed (resp. separably closed away from  $p = \text{char}(k)$ ); i.e. the absolute Galois group of  $k$  is a pro- $p$  group). Otherwise, it is defined to be the smallest

positive integer  $d$  such that  $\text{ind}(A) \mid \text{per}(A)^{d-1}$  for every finite field extension  $E/k$  and every central simple  $E$ -algebra  $A$  (resp. with  $p \nmid \text{per}(A)$ ), and also such that  $\text{ind}(A) \mid \text{per}(A)^d$  for every finitely generated field extension  $E/k$  of transcendence degree one and every central simple  $E$ -algebra  $A$  (resp. with  $p \nmid \text{per}(A)$ ). Note that the latter condition, on transcendence degree one extensions  $E/k$ , is automatically satisfied in the case of fields  $k$  that are separably closed (resp. away from  $p$ ), because in that situation  $\text{Br}(E)$  has no non-trivial prime-to- $p$  torsion (see the paragraph before Proposition 5.2 of [HHK09]).

Corollary 5.10 of [HHK09] related the period-index problem for a field  $k$  to that of  $F_P$  and  $F_U$ , where  $k$  is the residue field of a complete discrete valuation ring  $T$ , and  $F_P$  and  $F_U$  are the fields associated to a point or open subset of the closed fiber of a smooth projective  $T$ -curve. Using Corollaries 3.3 and 3.7 above, that result can now be generalized to the case of normal  $T$ -curves that are not necessarily smooth. Specifically, we have the following result:

**Theorem 4.6.** *Let  $T$  be a complete discrete valuation ring with residue field  $k$  of characteristic  $p \geq 0$ . Let  $\widehat{X}$  be a normal projective  $T$ -curve with closed fiber  $X$ , let  $\mathcal{P}$  be a finite non-empty subset of  $X$  that contains all the points where distinct components of  $X$  meet, and let  $\mathcal{U}$  be the set of components of the complement of  $\mathcal{P}$  in  $X$ . Suppose that  $k$  has Brauer dimension  $d$  away from  $p$ . Then for every  $\xi \in \mathcal{P} \cup \mathcal{U}$  and for all  $\alpha$  in  $\text{Br}(F_\xi)$  with period not divisible by  $p$ , we have  $\text{ind}(\alpha) \mid \text{per}(\alpha)^{d+2}$ . Moreover if  $T$  contains a primitive  $\text{per}(\alpha)$ -th root of unity, then  $\text{ind}(\alpha) \mid \text{per}(\alpha)^{d+1}$ .*

*Proof.* Let  $F$  be the function field of  $\widehat{X}$ , let  $\xi \in \mathcal{P} \cup \mathcal{U}$ , and consider  $\alpha$  in  $\text{Br}(F_\xi)$  as in the assertion. Observe that we may assume that for every  $n > 0$  that is not divisible by  $p$ , and for every  $a \in F_\xi$ , there exist  $a' \in F$  and  $u \in F_\xi^\times$  such that  $a = a'u^n$ . Namely, if  $\xi = U \in \mathcal{U}$  then this property holds automatically by Corollary 3.3(a); and if  $\xi = P \in \mathcal{P}$  then we can achieve this condition by replacing  $\widehat{X}$  by a split cover  $\widehat{X}'$  and replacing  $F$  by the function field  $F'$  of  $\widehat{X}'$ , by Corollary 3.7. (Again,  $F'_{P'} = F_P$  in the notation of Corollary 3.7, because  $\widehat{X}' \rightarrow \widehat{X}$  is a split cover.)

The remainder of the proof is then the same as that of [HHK09], Corollary 5.10, except that the observation in the above paragraph replaces the use of [HHK09], Lemma 4.16. For the convenience of the reader, we sketch the argument.

Writing the Brauer group as a direct product of its primary parts, we may assume that  $\text{per}(\alpha)$  is a prime power  $q^r$ . First consider the case that  $F$  contains a primitive  $q^r$ -th root of unity. Under this hypothesis, it follows from [MS82] that  $\alpha$  can be expressed as a tensor product  $(a_1, b_1)_{q^r} \otimes \cdots \otimes (a_m, b_m)_{q^r}$  of symbol algebras, where each  $a_i, b_i \in F_\xi$ . Applying the observation in the first paragraph of the proof, we may assume that each  $a_i$  and  $b_i$  lie in  $F$ . By [HHK09], Theorem 5.5 (or [Lie09], Theorem 6.3), the fraction field  $K$  of  $T$  has Brauer dimension at most  $d+1$  away from  $p$ ; and so  $\text{ind}(\alpha)$  divides  $\text{per}(\alpha)^{d+1}$ , completing the proof in this case. In the more general case, let  $F' = F[\zeta_{q^r}]$ ; let  $\widehat{X}' = \widehat{X} \times_T T[\zeta_{q^r}]$  (which is étale over  $\widehat{X}$ ); and let  $\alpha'$  be the induced element of  $\text{Br}(F'_{\xi'})$ , for  $\xi'$  on  $\widehat{X}'$  lying over  $\xi$  on  $\widehat{X}$ . Then

$s := [F'_{\xi'} : F_{\xi}(\zeta_q)]$  divides  $q^{r-1}$ ;  $\text{ind}(\alpha)$  divides  $s \text{ind}(\alpha')$ ; and  $\text{per}(\alpha')$  divides  $q^r = \text{per}(\alpha)$ . So applying the previous case to  $\alpha'$ , it follows that  $\text{ind}(\alpha)$  divides  $\text{per}(\alpha)^{d+2}$ .  $\square$

**Corollary 4.7.** *Let  $k$  be a field of characteristic  $p \geq 0$  having Brauer dimension  $d$  away from  $p$ , and let  $E$  be a finite separable extension of  $k((x, t))$ . If the period of  $\alpha \in \text{Br}(E)$  is not divisible by  $p$ , and if  $k$  contains a primitive  $\text{per}(\alpha)$ -th root of unity, then  $\text{ind}(\alpha) \mid \text{per}(\alpha)^{d+1}$ .*

*Proof.* Let  $\widehat{X}$  and  $P$  be as given by Lemma 3.8 applied to  $E$ , and choose a finite subset  $\mathcal{P}$  of the closed fiber  $X$  that contains  $P$  and the points where distinct components of  $X$  meet. Taking  $\xi = P$  in Theorem 4.6 then gives the desired conclusion.  $\square$

**Example 4.8.** Let  $k$  be an  $m$ -local field (see Example 4.5) whose residue field  $k_0$  is separably closed away from  $p$ , where  $p := \text{char}(k_0) = \text{char}(k) \geq 0$ . If  $E$  is a finite separable extension of  $k((x, t))$ , then  $\text{ind}(\alpha)$  divides  $\text{per}(\alpha)^{m+1}$  for elements  $\alpha \in \text{Br}(E)$  for which  $\text{per}(\alpha)$  is not divisible by  $p$ . Namely,  $k_0$  has Brauer dimension 0 away from  $p$ ; and so [HHK09], Corollary 5.7, says that  $k$  has Brauer dimension  $m$  away from  $p$ . Hence the assertion follows from Corollary 4.7.

Similarly, if  $k$  is an  $m$ -local field of characteristic  $p > 0$  whose residue field  $k_0$  is finite, and if  $k_0$  contains a primitive  $\text{per}(\alpha)$ -th root of unity, then  $\text{ind}(\alpha)$  divides  $\text{per}(\alpha)^{m+2}$ .

**Example 4.9.** Let  $k = k_0(u)((z))$ , where  $k_0$  is separably closed. If  $E$  is a finite separable extension of  $k((x, t))$ , then  $\text{ind}(\alpha)$  divides  $\text{per}(\alpha)^3$  in  $\text{Br}(E)$ , provided  $\text{per}(\alpha)$  is not divisible by  $p := \text{char}(k_0) \geq 0$ . To see this, note that any prime-to- $p$  finite extension of  $k_0(u)$  has trivial Brauer group by Tsen's Theorem; and the period equals the index for elements of order not divisible by  $p$  in the Brauer group of any one-variable function field over  $k_0(u)$ , by [deJ04]. So  $k_0(u)$  has Brauer dimension one away from  $p$ . Thus  $k$  has Brauer dimension at most two away from  $p$ , by [HHK09], Corollary 5.6. The assertion now follows from Corollary 4.7.

Similarly, if instead  $k_0$  is a finite field containing a primitive  $\text{per}(\alpha)$ -th root of unity, then  $\text{ind}(\alpha)$  divides  $\text{per}(\alpha)^4$ . This follows by replacing [deJ04] by [Lie11], which allows us to deduce that  $k_0(u)$  has Brauer dimension two away from  $p := \text{char}(k_0)$  and hence that  $k$  has Brauer dimension three away from  $p$ .

Another consequence of Theorem 4.6 is the following generalization of [HHK09, Corollary 5.11], dropping the smoothness assumption on  $\widehat{X}$  that was needed in the earlier result.

**Corollary 4.10.** *Under the hypotheses of Theorem 4.6, if  $k$  is separably closed away from  $p$ , then  $\text{per}(\alpha) = \text{ind}(\alpha)$  for elements in  $\text{Br}(F_{\xi})$  of period not divisible by the characteristic of  $k$ .*

Here, as in Theorem 4.6,  $\xi$  can be in either  $\mathcal{P}$  or  $\mathcal{U}$ . Combining this corollary in the former case with Lemma 3.8 above, we obtain in particular:

**Corollary 4.11.** *Suppose that  $k$  is separably closed away from  $p = \text{char}(k)$ . If  $E$  is a finite separable extension of  $k((x, t))$ , and if the period of  $\alpha \in \text{Br}(E)$  is not divisible by  $p$ , then  $\text{per}(\alpha) = \text{ind}(\alpha)$ .*

This can also be deduced from [COP02], Theorem 2.1, where it was shown that a class in  $\text{Br}(E)$  of period  $n$  not divisible by  $p$  represents a cyclic algebra of index  $n$ , provided that  $k$  is separably closed. This immediately gives the above assertion for such fields  $k$ . More generally, if  $k$  is separably closed away from  $p$ , then the conclusion follows from [COP02], Theorem 2.1, by taking the compositum of  $E$  with the separable closure of  $k$ , and using that the absolute Galois group of  $k$  is a pro- $p$ -group.

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