Embedding Problems with Local Conditions

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Section 1. Introduction

This paper proves results on Galois covers of affine varieties in characteristic p, showing that they behave extremely well under embedding problems with p-group kernel. Namely, given such a connected Galois cover $Y \to X$ with Galois group $G = \Gamma/P$, where P is a p-group, is there a connected Galois cover $Z \to X$ with group Γ that dominates $Y \to X$? Moreover, can $Z \to X$ be chosen with prescribed local behavior? For example, if X' is a closed subset of X, and if the restriction $Y' \to X'$ of $Y \to X$ is dominated by a (possibly disconnected) Γ-Galois cover $Z' \to X'$, then can $Z \to X$ above be chosen so as to restrict to Z' over X'? This general type of problem is traditionally called an "embedding problem," since on the function field level a G-Galois extension is being embedded into a Γ-Galois extension. Over an arbitrary field of characteristic $p > 0$ (e.g. finite fields or Laurent series fields), this paper answers several questions of this type in the affirmative.

In particular, if $Y \to X$ is étale, then in the situation above there is such a Z which is étale over Y and extends the given Z' (Theorem 3.11). Moreover this remains true even if $Y \to X$ is ramified, provided that its degree is prime to p (Theorem 4.3). In the case of curves (where X' is a finite set of points), it is true even if $Y \to X$ is merely assumed to be tamely ramified (Theorem 5.14). In this context the local condition corresponds to specifying the residue field extensions over a given finite set of points — and this is a non-trivial condition if the base field k is not algebraically closed. An "adelic" version for curves, in which X' is taken to consist of spectra of local fields rather than points, is also shown (Theorem 5.6).

Problems of this sort have been considered in several papers. The case that X is the spectrum of a global field and X' is adelic was shown in [Ne, Main Theorem]; but there ramification at places outside of X' is permitted and the kernel of $\Gamma \to G$ is allowed to be somewhat more general. (Only the number field case was explicitly treated in [Ne], but the result carries over to the function field case. Cf. also [SW].) In [Ka, Theorem 2.1.5], a related result was shown for projective curves X over separably closed fields, with adelic conditions. (This generalized a result of [Ha1, §2], in the case $G = 1$; cf. [Ka, Theorem 2.1.4].) But there the dominating cover Z need not be connected. On the other hand, in [Se2, Theorem 1] (in connection with the solvable case of the Abhyankar Conjecture), it was shown for $X = \mathbb{A}^1$ over $k = \bar{k}$ that Z can be chosen so as to be connected, if no

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local conditions are imposed. A version for affine curves viewed as rigid analytic spaces (and with X' taken to be an affinoid) appeared in [Ra1, Prop. 4.2.5, Cor. 4.2.6], using the machinery of Runge pairs, as part of the proof of the full Abhyankar Conjecture for $\mathbb{A}^1$. A version motivated by formal schemes appeared in [Ha2, Prop. 4.1], allowing $Y \to X$ to be ramified and of degree prime to p. This result, which relied on the moduli of p-covers in [Ha1], appeared in the context of proving the Abhyankar Conjecture for general affine curves over $k = \bar{k}$ (assuming [Ra1]). Afterwards [Ra2], Raynaud pointed out the relationship with [Ra1, Prop. 4.2.5], and found a rigid analytic version that also allows $Y \to X$ to be ramified and prime to p. This version could then be used instead of [Ha2, Prop. 4.1], to prove the general case of the Abhyankar Conjecture by rigid analytic (rather than formal) methods, assuming the case of $\mathbb{A}^1$. (The opposite is also possible. That is, the rigid approach of [Ra1] to the Abhyankar Conjecture for $\mathbb{A}^1$ could instead be replaced by a formal scheme approach; cf. [HS, Theorem 6].) F. Pop later showed in [Po, Thm. B] how the original version [Ra1, Prop. 4.2.5] could already be used to prove the Abhyankar Conjecture for general curves, and even to prove that embedding problems with quasi-p kernel can be solved properly (i.e. with Z connected) for affine curves over $k = \bar{k}$ (without local conditions).

The results of the current paper can be regarded as generalizations and simplifications of the corresponding results of [Ra1], [Ha2], and [Ra2]. Namely, the base space need not be a curve, and the base field can be arbitrary (of characteristic p) — thus providing arithmetic content. Also the machinery of rigid and formal geometry (including the Runge pairs and the moduli of p-covers) are avoided in the current version, by allowing a more elementary notion of "local condition." Just as the related results in those papers played a role in proving Abhyankar's Conjecture, the results here should be applicable to extensions of that conjecture to more general spaces.

The assertions in this paper are for *affine* varieties, and break down for projective varieties. For example, in Theorem 5.14, the embedding problem is solved by a tamely ramified connected cover of an affine curve; but no assertion is made about the behavior over infinity. In fact, at least one point on the projective completion must be allowed to ramify wildly. This is related to the statement of the "Strong Abhyankar Conjecture" [Ha2, Thm. 6.2], where all but one branch point can be taken to be tame.

The structure of this paper is as follows: Section 2 is purely group theoretic, and provides a cohomological criterion for solving (group-theoretic) embedding problems with p-group kernel and local conditions. The ideas for this section are related to ideas in [Ka], [Se2], and [Ra1]. Sections 3 and 4 then apply this to embedding problems over affine varieties of arbitrary dimension in characteristic p, by using that the appropriate fundamental groups have p-cohomological dimension 1 and infinite p-rank, and by showing an appropriate surjectivity on H^1's. Section 5 turns to results for curves, and shows the adelic result (Theorem 5.6) similarly. In order to show the result in the case that $Y \to X$ is tamely ramified (Theorem 5.14), the strategy is reversed: Theorem 5.6 is combined with group-theoretic results to obtain Theorem 5.14, and from that it follows that the corresponding fundamental group has p-cohomological dimension 1.

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Section 2. Embedding problems with p-kernel.

This section considers embedding problems for profinite groups with p-group kernel. Propositions 2.2 and 2.3 provide cohomological criteria for the existence of solutions satisfying given local conditions. The approaches in [Ka, §2], [Se2, §4] and [Ra1,§4] appear here in a non-geometric setting. Later, in Sections 3-5, the results here will be applied to fundamental groups of affine varieties in characteristic p. We begin with some terminology in this general setting. Here the notion of being "ϕ-solvable" will correspond geometrically to an embedding problem being solvable with given local conditions.

If Π, Γ, G are profinite groups, then an *embedding problem* $\mathcal{E}$ *for* Π consists of a pair of surjective group homomorphisms $(\alpha : \Pi \to G, f : \Gamma \to G)$. A *weak solution* to $\mathcal{E}$ consists of a group homomorphism $\beta : \Pi \to \Gamma$ such that $f\beta = \alpha$. If such a β is surjective, then it is called a *proper solution* to $\mathcal{E}$. We will call $\mathcal{E}$ *weakly* [resp. *properly*] *solvable* if it has a weak [resp. a proper] solution. The *kernel* of $\mathcal{E}$ is defined to be $N := \ker(f)$. We call $\mathcal{E}$ a *finite* embedding problem [resp. a p-embedding problem, an *elementary abelian* p-embedding problem, etc.] if N is a finite group [resp. a p-group, an elementary abelian p-group, etc.]. An elementary abelian p-embedding problem $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ is *irreducible* if the conjugation action of Γ on $P = \ker f$ defines an irreducible representation; or equivalently, if P is a minimal non-trivial normal subgroup of Γ.

Let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be an embedding problem for Π, and let $\phi_1 : \Pi_1 \to \Pi$ be a homomorphism of profinite groups. Write $G_1 = \alpha\phi_1(\Pi_1) \subset G$, $\Gamma_1 = f^{-1}(G_1) \subset \Gamma$, and $f_1 = f|_{\Gamma_1}$. Thus $\phi_1^*(\mathcal{E}) := (\alpha\phi_1 : \Pi_1 \to G_1, f_1 : \Gamma_1 \to G_1)$ is an embedding problem for Π_1, which we call the *pullback* of $\mathcal{E}$ to Π_1. Note that $\mathcal{E}$ and $\phi_1^*(\mathcal{E})$ have the same kernel. If $\beta : \Pi \to \Gamma$ is a weak solution to $\mathcal{E}$, then there is an induced weak solution to $\phi_1^*(\mathcal{E})$, viz. the *pullback* $\phi_1^*(\beta) := \beta\phi_1 : \Pi_1 \to \Gamma_1$. Suppose that $\phi = \{\phi_j\}_{j \in J}$ is a family of homomorphisms $\phi_j : \Pi_j \to \Pi$ of profinite groups. We will say that $\mathcal{E}$ is *weakly* [resp. *properly*] ϕ-*solvable* if for every collection $\{\beta_j\}_{j \in J}$ of weak solutions to the pulled back embedding problems $\phi_j^*(\mathcal{E})$, there is a weak [resp. proper] solution β to $\mathcal{E}$ and elements $n_j \in N = \ker(\mathcal{E})$ such that $\phi_j^*(\beta) = \mathrm{inn}(n_j) \circ \beta_j$ for all $j \in J$. (Here $\mathrm{inn}(n_j) \in \mathrm{Aut}(\Gamma)$ denotes left conjugation by n_j.) Note that weak solutions β_j are considered in this definition, even in the proper case. For a geometric interpretation of these notions, see Proposition 3.1.

The following reduction lemma allows us to restrict attention to the class of finite p-embedding problems that are elementary abelian and irreducible:

Lemma 2.1. *Let $\phi = \{\phi_j\}_{j\in J}$ be a family of homomorphisms $\phi_j : \Pi_j \to \Pi$ of profinite groups. Suppose that every irreducible finite elementary abelian p-embedding problem for Π is weakly [resp. properly] ϕ-solvable. Then so is every finite p-embedding problem for Π.*

Proof. Consider a finite p-embedding problem $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ for Π, together with weak solutions β_j to the pullbacks $\mathcal{E}_j = \phi_j^*(\mathcal{E}) = (\alpha\phi_j : \Pi_j \to G_j, f : \Gamma_j \to G_j)$. We wish to show that $\mathcal{E}$ has a weak [resp. proper] solution β such that $\phi_j^*(\beta)$ agrees with each β_j after conjugation by elements p_j of $P = \ker(\mathcal{E})$. We proceed by induction on the order of P. The desired conclusion is immediate if $P = 1$, and so we assume that P is non-trivial.

Since P is a non-trivial p-group, it has a non-trivial center Z. Since P is normal in Γ, and since Z is characteristic in P, it follows that Z is normal in Γ. Let A be a *minimal* non-trivial normal subgroup of Γ contained in Z. By minimality, A is an elementary abelian p-group (since its subgroup of p-torsion elements is also normal) and Γ acts irreducibly on A via conjugation. Letting $\bar{P} = P/A$ and $\bar{\Gamma} = \Gamma/A$, we obtain exact sequences

$$1 \to \bar{P} \to \bar{\Gamma} \xrightarrow{\bar{f}} G \to 1 \tag{1}$$

and

$$1 \to A \to \Gamma \xrightarrow{g} \bar{\Gamma} \to 1. \tag{2}$$

Consider the embedding problem $\bar{\mathcal{E}} = (\alpha : \Pi \to G, \bar{f} : \bar{\Gamma} \to G)$ for Π, and let $\bar{\mathcal{E}}_j = \phi_j^*(\bar{\mathcal{E}}) = (\alpha\phi_j : \Pi_j \to G_j, \bar{f}_j : \bar{\Gamma}_j \to G_j)$ be the pullback of $\bar{\mathcal{E}}$ from Π to Π_j. Since $\bar{P}$ is strictly smaller than P, it follows by the inductive hypothesis that $\bar{\mathcal{E}}$ is weakly [resp. properly] ϕ-solvable. That is, there is a weak [resp. a proper] solution $\bar{\beta}$ to $\bar{\mathcal{E}}$ that induces the weak solutions $\bar{\beta}_j := g\beta_j$ to $\bar{\mathcal{E}}_j$ up to conjugation by elements $\bar{p}_j \in \bar{P}$ (i.e. $\bar{\beta}_j = \mathrm{inn}(\bar{p}_j)\bar{\beta}\phi_j$). Choose $p_j \in P$ over $\bar{p}_j \in \bar{P}$. Let $\bar{\Gamma}^\circ = \bar{\beta}(\Pi) \subset \bar{\Gamma}$, let $\Gamma^\circ = g^{-1}(\bar{\Gamma}^\circ) \subset \Gamma$, and let $g^\circ = g|_{\Gamma^\circ}$. [Thus $\bar{\Gamma}^\circ = \bar{\Gamma}$, $\Gamma^\circ = \Gamma$, and $g^\circ = g$ in the proper case.] Now $\mathcal{E}^\circ = (\bar{\beta} : \Pi \to \bar{\Gamma}^\circ, g^\circ : \Gamma^\circ \to \bar{\Gamma}^\circ)$ is an irreducible finite elementary abelian p-embedding problem for Π with kernel A, and for each $j \in J$ the map $\mathrm{inn}(p_j^{-1})\beta_j$ is a weak solution to the pullback $\mathcal{E}_j^\circ = \phi_j^*(\mathcal{E}^\circ)$ of $\mathcal{E}^\circ$ from Π to Π_j. So by hypothesis there is a weak solution $\beta : \Pi \to \Gamma^\circ$ [resp. a proper solution $\beta : \Pi \to \Gamma$] to $\mathcal{E}^\circ$ that induces each $\mathrm{inn}(p_j^{-1})\beta_j$ up to A-conjugacy. The map β is then the desired weak [resp. proper] solution to $\mathcal{E}$ inducing the β_j's up to P-conjugacy. $\square$

If $f : \Gamma \to G$ is a (continuous) homomorphism of profinite groups, then there is an induced map $f_* : \mathrm{Hom}(\Pi, \Gamma) \to \mathrm{Hom}(\Pi, G)$, given by $f_*(\gamma) = f \circ \gamma$. Also, if $\phi_1 : \Pi_1 \to \Pi$ is a homomorphism, then there is an induced map $\phi_1^* : \mathrm{Hom}(\Pi, G) \to \mathrm{Hom}(\Pi_1, G)$, given by $\phi_1^*(\gamma) = \gamma \circ \phi_1$ for $\gamma \in \mathrm{Hom}(\Pi, G)$. Consider a family $\phi = \{\phi_j\}_{j \in J}$ of homomorphisms $\phi_j : \Pi_j \to \Pi$. Then there is an induced map $\phi^* : \mathrm{Hom}(\Pi, G) \to \prod_{j \in J} \mathrm{Hom}(\Pi_j, G)$, given by ϕ_j^* on the jth factor. Similarly, if Π acts on an abelian group A, then so does each Π_j (via ϕ_j), and we obtain induced homomorphisms $\phi_j^* : Z^1(\Pi, A) \to Z^1(\Pi_j, A)$ and $\phi^* : Z^1(\Pi, A) \to \prod_{j \in J} Z^1(\Pi_j, A)$ on the corresponding cocycle groups. (Here $\phi_j^*(\gamma) = \gamma \circ \phi_j$ for $\gamma \in Z^1(\Pi, A)$.) These in turn induce homomorphisms $\phi_j^* : H^1(\Pi, A) \to H^1(\Pi_j, A)$ and $\phi^* : H^1(\Pi, A) \to \prod_{j \in J} H^1(\Pi_j, A)$ on cohomology groups. We will say that the family ϕ is *p-dominating* [resp. *strongly p-dominating*] if $\phi^* : H^1(\Pi, P) \to \prod_{j \in J} H^1(\Pi_j, P)$ is surjective [resp. surjective with infinite kernel] for every non-trivial finite elementary abelian p-group P on which Π acts continuously.

As the following two propositions show, (strongly) p-dominating families ϕ are well-behaved, in the sense that every solvable p-embedding problem must also be ϕ-solvable (in the strong or weak sense, respectively). In proving Proposition 2.2, we will need to twist a given weak solution $\gamma \in \mathrm{Hom}(\Pi, \Gamma)$ in order to satisfy the local conditions corresponding to a p-dominating family ϕ. This twisting will be via an appropriate 1-cocycle, and will be defined as follows:

Let A be an abelian normal subgroup of a finite group Γ, let Π be a profinite group, and let $\gamma \in \mathrm{Hom}(\Pi, \Gamma)$. Then Π acts on A via γ and the conjugation action of Γ on A. With respect to this action, we may consider the group $Z^1(\Pi, A)$ of 1-cocycles. If $\alpha \in Z^1(\Pi, A)$ then we may consider the map $\alpha \cdot \gamma : \Pi \to \Gamma$ given by $(\alpha \cdot \gamma)(\pi) = \alpha(\pi)\gamma(\pi)$, for $\pi \in \Pi$. Here $\alpha \cdot \gamma \in \mathrm{Hom}(\Pi, \Gamma)$ because α is a cocycle. If we let $f : \Gamma \to G := \Gamma/A$ be the quotient map, inducing $f_* : \mathrm{Hom}(\Pi, \Gamma) \to \mathrm{Hom}(\Pi, G)$, then $\alpha \mapsto \alpha \cdot \gamma$ is a bijection from $Z^1(\Pi, A)$ to the fibre of f_* containing γ. Under this bijection, if $a \in A = C^0(\Pi, A)$ then $da \in B^1(\Pi, A)$ is sent to $\mathrm{inn}(a^{-1}) \circ \gamma$. Also, the above cocycle group $Z^1(\Pi, A)$ depends only on the fibre of f_* containing γ, and $(\alpha, \gamma') \mapsto \alpha \cdot \gamma'$ defines a "twisting" action of this $Z^1(\Pi, A)$ on that fibre. (In the special case that A is central in Γ, the action of Π on A is trivial. The groups $Z^1(\Pi, A)$, $H^1(\Pi, A)$, $\mathrm{Hom}(\Pi, A)$ then all coincide and are independent of γ, and we obtain a twisting action of this common group on $\mathrm{Hom}(\Pi, \Gamma)$.)

Recall (cf. [Se1, I, 3.4, Proposition 16]) that if p is a prime number and Π is a profinite group, then $\mathrm{cd}_p(\Pi) \le 1$ if and only if every finite p-embedding problem for Π has a weak solution. The proof of the following result is related to ideas in the proofs of [Ka, Theorem 2.1.5] and [Ra1, Prop. 4.2.5] (where they appeared in more geometric contexts).

Proposition 2.2. *Let p be a prime number and let Π be a profinite group. Then the following conditions are equivalent:*

(i) Every finite p-embedding problem for Π is weakly solvable (i.e. $\mathrm{cd}_p(\Pi) \le 1$).

(ii) Every finite p-embedding problem for Π is weakly ϕ-solvable, for every p-dominating family of homomorphisms $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$.

Proof. The implication (ii) $\Rightarrow$ (i) is immediate, by taking the Π_j's to be trivial. So we prove (i) $\Rightarrow$ (ii).

Let $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$ be a p-dominating family of homomorphisms, and let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be a finite p-embedding problem for Π, with kernel P. We wish to show that $\mathcal{E}$ is ϕ-solvable. By Lemma 2.1 it suffices to prove this under the hypothesis that P is an elementary abelian p-subgroup of Γ that properly contains no non-trivial normal subgroups of Γ (corresponding to the representation being irreducible). Let $\gamma : \Pi \to \Gamma$ be a weak solution to $\mathcal{E}$, and assume for each j that $\beta_j : \Pi_j \to \Gamma_j \subset \Gamma$ is a weak solution to the pullback $\mathcal{E}_j = \phi_j^*(\mathcal{E}) = (\alpha\phi_j : \Pi_j \to G_j, f_j : \Gamma_j \to G_j)$. We need to show that $\mathcal{E}$ has a weak solution β that induces each β_j up to P-conjugacy.

As above, we have a natural action of $Z^1(\Pi, P)$ on the fibre of $f_* : \mathrm{Hom}(\Pi, \Gamma) \to \mathrm{Hom}(\Pi, G)$ containing γ. The same assertion holds with Π, Γ replaced by Π_j, Γ_j, and with γ replaced by $\gamma_j = \phi_j^*(\gamma) : \Pi_j \to \Gamma_j$, the pullback of γ from Π to Π_j. Here the action of $Z^1(\Pi_j, P)$ is compatible with that of $Z^1(\Pi, P)$. Since β_j and γ_j are both weak solutions to $\mathcal{E}_j$, they satisfy $f_j\beta_j = \alpha\phi_j = f_j\gamma_j$; and so $\beta_j, \gamma_j \in \mathrm{Hom}(\Pi_j, \Gamma_j)$ lie in the same fibre of $f_{j*} : \mathrm{Hom}(\Pi_j, \Gamma_j) \to \mathrm{Hom}(\Pi_j, G_j)$. Thus there is an element $\rho_j \in Z^1(\Pi_j, P)$ such that $\rho_j \cdot \gamma_j = \beta_j$. Let $\underline{\rho}_j$ be the image of ρ_j in $H^1(\Pi_j, P)$ and let $\underline{\rho}' = \{\underline{\rho}_j\}_j \in \prod_j H^1(\Pi_j, P)$. Now $\phi^* : H^1(\Pi, P) \to \prod_j H^1(\Pi_j, P)$ is surjective (by the p-dominating hypothesis, if $P \neq 1$; and trivially, if $P = 1$). So there is an element $\underline{\rho} \in H^1(\Pi, P)$ such that $\phi^*(\underline{\rho}) = \underline{\rho}'$.

Let $\rho \in Z^1(\Pi, P)$ be a lift of $\underline{\rho}$ (i.e. a representative of the class $\underline{\rho}$, modulo $B^1(\Pi, P)$). Let $\beta = \rho \cdot \gamma \in \mathrm{Hom}(\Pi, \Gamma)$. Then β is a weak solution to $\mathcal{E}$ that induces each β_j up to multiplication by a coboundary $dp_j \in B^1(\Pi_j, P) \subset Z^1(\Pi_j, P)$ (for some $p_j \in C^0(\Pi_j, P) = P$). But multiplication by dp_j is the same as composition by $\mathrm{inn}(p_j^{-1})$, i.e. right conjugation by p_j. So β induces each β_j up to P-conjugacy, as desired. $\square$

Recall that the *p-rank* of a profinite group Π is the dimension of the $\mathbb{F}_p$-vector space $\mathrm{Hom}(\Pi, \mathbb{Z}/p\mathbb{Z})$ of continuous homomorphisms. Equivalently, the p-rank is the rank of the maximum pro-p quotient of Π.

The proof of the following result is related to ideas in [Se2, §4] and in the proof of [Ra1, Prop. 4.2.5].

Theorem 2.3. *Let p be a prime number and let Π be a profinite group of infinite p-rank. Then the following conditions are equivalent:*

(i) *Every finite p-embedding problem for Π is weakly solvable (i.e. $\mathrm{cd}_p(\Pi) \leq 1$).*

(ii) *Every finite p-embedding problem for Π is weakly ϕ-solvable, for every p-dominating family of homomorphisms $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$.*

(iii) *Every finite p-embedding problem for Π is properly solvable.*

(iv) *Every finite p-embedding problem for Π is properly ϕ-solvable, for every strongly p-dominating family of homomorphisms $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$.*

Proof. The equivalence of (i) and (ii) was given in Proposition 2.2, and it is trivial that

(iii) $\Rightarrow$ (i). The implication (iv) $\Rightarrow$ (iii) follows from the assumption that Π has infinite p-rank, by taking the Π_j's to be trivial. So it suffices to show that (ii) $\Rightarrow$ (iv).

So let $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j\in J}$ be a strongly p-dominating family of homomorphisms. We wish to show that if $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ is a p-embedding problem for Π, then $\mathcal{E}$ is properly ϕ-solvable. By Lemma 2.1, it suffices to do this in the case that the kernel of f is an elementary abelian p-group $P = (\mathbb{Z}/p\mathbb{Z})^m$ that properly contains no non-trivial normal subgroups of Γ. That is, we wish to show that for such an $\mathcal{E}$, and for any family of weak solutions β_j to $\mathcal{E}_j = \phi_j^*(\mathcal{E})$, there is a proper solution $\beta : \Pi \to \Gamma$ to $\mathcal{E}$ together with elements $p_j \in P$ such that $\phi_j^*(\beta) = \mathrm{inn}(p_j)\beta_j \in \mathrm{Hom}(\Pi_j, \Gamma)$ for each $j \in J$. This is trivial if $P = 1$; so we may assume $P \neq 1$.

By (ii), there is a weak solution $\beta_0 \in \mathrm{Hom}(\Pi, \Gamma)$ to $\mathcal{E}$ together with elements $p_j \in P$ such that $\phi_j^*(\beta_0) = \mathrm{inn}(p_j)\beta_j$. Since $f\beta_0 = \alpha : \Pi \to G$ is surjective, it follows that Γ is generated by $\beta_0(\Pi)$ and $P = \ker f$. Now $\beta_0(\Pi) \cap P$ is a normal subgroup of $\beta_0(\Pi)$ (since P is normal in Γ) and of P (since P is abelian). Since Γ is generated by $\beta_0(\Pi)$ and P, it follows that $\beta_0(\Pi) \cap P$ is a normal subgroup of Γ.

If $\beta_0(\Pi) \cap P$ is all of P, then the image of β_0 contains P and hence it is all of Γ (again since Γ is generated by $\beta_0(\Pi)$ and P). So in this case β_0 is the desired proper solution and we are done.

Thus we may assume that $\beta_0(\Pi)\cap P$ is strictly contained in P. But by the irreducibility hypothesis on P, it follows that $\beta_0(\Pi)\cap P$ is then trivial. Hence the restriction of $f : \Gamma \to G$ to $\beta_0(\Pi)$ is injective, and thus is an isomorphism onto G (being surjective, since $f\beta_0 = \alpha$). This implies that β_0 factors through G; i.e. β_0 is in the image of $\mathrm{Hom}(G, \Gamma) \to \mathrm{Hom}(\Pi, \Gamma)$.

Since $1 \to P \to \Gamma \to G \to 1$ is a short exact sequence with abelian kernel, there is an induced conjugation action of G on P (by choosing representatives in Γ). This in turn yields actions of Π on P (via $\alpha : \Pi \to G$) and of Π_j on P (via $\alpha \circ \phi_j : \Pi_j \to G$). Let F be the kernel of the induced map $\phi^* : H^1(\Pi, P) \to \prod_j H^1(\Pi_j, P)$, taking cohomology with respect to these actions. Since F is infinite (by the hypothesis on ϕ) while $H^1(G, P)$ is finite, there is an element $\underline{\rho} \in F \subset H^1(\Pi, P)$ that is not in the image of $\alpha^* : H^1(G, P) \to H^1(\Pi, P)$. Let $\rho \in Z^1(\Pi, P)$ be a lift of $\underline{\rho}$. Thus we may consider $\beta := \rho \cdot \beta_0 \in \mathrm{Hom}(\Pi, \Gamma)$. Here β maps to α under $\mathrm{Hom}(\Pi, \Gamma) \to \mathrm{Hom}(\Pi, G)$, because $\beta_0 \mapsto \alpha$ under this map and because $P = \ker(\Gamma \to G)$. That is, β is a weak solution to the embedding problem $\mathcal{E}$. Since $\rho \in F = \ker \phi^* = \bigcap_j \ker \phi_j^*$, we have that $\rho\phi_j = \phi_j^*(\rho) = 1$. So $\phi_j^*(\beta) = \beta\phi_j = \rho\phi_j \cdot \beta_0\phi_j = \beta_0\phi_j = \phi_j^*(\beta_0) = \mathrm{inn}(p_j)\beta_j$, as desired.

It remains to show that $\beta : \Pi \to \Gamma$ is surjective, and thus a proper solution to the embedding problem $\mathcal{E}$. Now β_0 is in the image of $\mathrm{Hom}(G, \Gamma) \to \mathrm{Hom}(\Pi, \Gamma)$, whereas ρ is not in the image of $Z^1(G, P) \to Z^1(\Pi, P)$; so $\beta = \rho \cdot \beta_0$ is not in the image of $\mathrm{Hom}(G, \Gamma) \to \mathrm{Hom}(\Pi, \Gamma)$. Thus the restriction of $f : \Gamma \to G$ to $\beta(\Pi)$ is not injective. That is, $\beta(\Pi) \cap P$ is a non-trivial subgroup of P. But $\beta(\Pi) \cap P$ is normal in Γ. The irreducibility hypothesis thus implies that $\beta(\Pi) \cap P = P$; i.e. $P \subset \beta(\Pi)$. Since Γ is generated by P and $\beta(\Pi)$, it follows that β is surjective. $\square$

Section 3. p-Embedding problems for affine varieties: unramified case.

We now turn to the main theme of this paper, viz. p-embedding problems in a geometric context, for fundamental groups of affine varieties in characteristic p. The group-theoretic results of Section 2 are applied here, and in the following sections, in order to obtain results that assert that covers $Y \to X$ with Galois group $G = \Gamma/P$ (where P is a p-group) can be dominated by Γ-Galois covers $Z \to X$ with $Z \to Y$ étale and with prescribed local behavior. This section considers the case in which $Y \to X$ is étale (Theorem 3.11), while Section 4 allows $Y \to X$ to be ramified (but adds another restriction). Stronger results will be shown in the case of dimension 1, in Section 5.

The link between the group theory of Section 2 and the geometry of these sections is made explicit below in Proposition 3.1, which permits Theorem 2.3 to be applied to fundamental groups to obtain geometric results. To apply Theorem 2.3, it is first observed that the fundamental group of an affine variety in characteristic p has $\mathrm{cd}_p \leq 1$ (Corollary 3.3) and infinite p-rank (Corollary 3.7), and then it is shown that the local conditions are strongly p-dominating in the sense of §2 (Proposition 3.8). As a consequence, if $Y \to X$ is étale, the existence of the desired cover $Z \to X$ is shown (Theorem 3.11), in a result parallel to [Ra1, Corollary 4.2.6] (and indirectly drawing on ideas of [Se2,§4]). In the next section, a variant (Theorem 4.3) is shown in which $Y \to X$ is permitted to be ramified but is required to have degree prime to p.

We begin by recalling some basic terminology. An *étale cover* ("revêtement étale") $f : Y \to X$ is a morphism of schemes that is finite and étale [Gr, I, Def. 4.9]. The *Galois group* $\mathrm{Gal}(Y/X)$ of $Y \to X$ consists of the automorphisms g of Y such that $fg = f$. An étale cover $f : Y \to X$ is *Galois* if X and Y are connected and $\mathrm{Gal}(Y/X)$ acts simply transitively on each generic geometric fibre. If $Y \to X$ is an étale cover (not necessarily connected), if $\iota : G \to \mathrm{Gal}(Y/X)$ is a homomorphism of finite groups, and if G acts simply transitively on each generic geometric fibre (via ι), then we will say that $Y \to X$ and the G-action together constitute a *G-Galois* étale cover.

Let X be a connected locally Noetherian scheme with a geometric base point ξ. A *pointed* étale cover of (X, ξ) consists of an étale cover $f : Y \to X$ and a geometric point $\eta \in Y$ such that $f(\eta) = \xi$. The pointed Galois étale covers of (X, ξ) form an inverse system of pointed schemes, and their Galois groups form an inverse system of groups whose inverse limit is the *algebraic fundamental group* $\Pi = \pi_1(X, \xi)$. (Cf. [Gr, V §7].) If G is a finite group, then there is a bijection between the homomorphisms $\alpha : \Pi \to G$ and the isomorphism classes of pointed G-Galois étale covers $(Y, \eta) \to (X, \xi)$, under which surjective homomorphisms correspond to connected covers. Composing α with conjugation by $g \in G$ has the effect of changing the base point of Y (over ξ) from η to $g(\eta)$, but it does not affect the isomorphism class of the underlying (unpointed) G-Galois cover. Thus isomorphism classes of (unpointed) G-Galois étale covers of X are in bijection with *equivalence* classes of homomorphisms $\alpha : \Pi \to G$, two such homomorphisms being declared equivalent if they differ by an inner automorphism of G.

In the above context, if $f : \Gamma \twoheadrightarrow G$ is a surjection of finite groups and $\alpha : \Pi \to G$ is a surjective homomorphism, then a weak solution $\beta : \Pi \to \Gamma$ to the embedding problem $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ corresponds to a pointed Γ-Galois étale cover $(Z, \zeta) \to (X, \xi)$ that dominates (Y, η). Here Z is connected if and only if β is a proper solution to $\mathcal{E}$. If $\psi : X_1 \to X$ is a morphism of connected schemes, then G-Galois étale covers of X pull back to G-Galois étale covers of X_1 (and such a pullback need not be connected, even if the given cover of X is). On the level of equivalence classes of homomorphisms, pullback may be interpreted as follows:

Let ξ, ξ_1 be geometric base points of X, X_1 respectively. Then there is a natural homomorphism $\psi_* : \pi_1(X_1, \xi_1) \to \pi_1(X, \psi(\xi_1))$. Since X is connected, an isomorphism $\iota_1 : \pi_1(X, \psi(\xi_1)) \xrightarrow{\sim} \pi_1(X, \xi)$ is induced by choosing a geometric point $\tilde{\xi}$ over ξ on the pro-universal cover $(\tilde{X}_{\psi(\xi_1)}, \widetilde{\psi(\xi_1)})$ of $(X, \psi(\xi_1))$. (The choice of $\tilde{\xi}$ corresponds classically to choosing a homotopy class of paths from $\psi(\xi_1)$ to ξ; and varying that choice will vary ι_1 by an inner isomorphism. Cf. [Gr, V.5, V.7].) Composing, we obtain a map $\iota_1 \circ \psi_* : \pi_1(X_1, \xi_1) \to \pi_1(X, \xi)$. The pointed G-Galois étale cover of X corresponding to $\alpha : \pi_1(X, \xi) \to G$ then pulls back to the pointed G-Galois étale cover of X_1 corresponding to $\alpha \circ \iota_1 \circ \psi_* : \pi_1(X_1, \xi_1) \to G$. Forgetting the base points, the unpointed G-Galois étale covers of X pull back to such covers of X_1, as noted above; and this pullback depends only on the cover, i.e. is independent of the choice of ι_1. To the extent that we will focus on unpointed G-Galois covers, we will often suppress the base points and the isomorphism ι_1, and then simplify notation by just writing $\psi_* : \pi_1(X_1) \to \pi_1(X)$ for the map between fundamental groups. Thus the equivalence class of the pullback $\alpha \circ \psi_* : \pi_1(X_1) \to G$ of $\alpha : \pi_1(X) \to G$ will be well defined, corresponding to the pullback of G-Galois covers.

The above remarks, together with the definition of "ϕ-solvable" in Section 2, yield:

Proposition 3.1. *Let $\psi_j : X_j \to X$ (for $j \in J$) be a family of morphisms of connected schemes. Let $\phi_j = \psi_{j*} : \pi_1(X_j) \to \pi_1(X)$ and $\phi = \{\phi_j\}_j$. Let $f : \Gamma \to G$ be a surjective homomorphism of finite groups, let $Y \to X$ be a connected G-Galois étale cover corresponding to a homomorphism $\alpha : \pi_1(X) \to G$, and let $Y_j \to X_j$ be the pullback via ψ_j. Then the following are equivalent:*

(i) *For each choice of Γ-Galois étale covers $Z_j \to X_j$ that dominate $Y_j \to X_j$ (for $j \in J$), there is a [connected] Γ-Galois étale cover $Z \to X$ that dominates $Y \to X$ and pulls back to each $Z_j \to X_j$, up to isomorphism.*

(ii) *The embedding problem $\mathcal{E} = (\alpha : \pi_1(X) \to G, f : \Gamma \to G)$ is weakly [resp. properly] ϕ-solvable.*

Proof. In the statement of the proposition, regard $\pi_1(X)$ and $\pi_1(X_j)$ as the fundamental groups of X, X_j with respect to geometric base points ξ, ξ_j, and choose isomorphisms $\iota_j : \pi_1(X, \psi(\xi_j)) \xrightarrow{\sim} \pi_1(X, \xi)$ as above. Given $Y \to X$, the choice of map $\alpha : \pi_1(X, \xi) \to G$ corresponds to a choice of base point η for Y over ξ; and the composition $\alpha\iota_j\phi_j : \pi_1(X_j, \xi_j) \to G$ corresponds to a choice of base point η_j for Y_j over ξ_j.

Suppose first that condition (i) holds, and let $\beta_j : \pi_1(X_j, \xi_j) \to \Gamma$ be weak solutions of the induced embedding problems $\mathcal{E}_j$ for $\pi_1(X_j, \xi_j)$. Thus $f\beta_j = \alpha\iota_j\phi_j : \pi_1(X_j, \xi_j) \to G$, and β_j corresponds to a pointed Γ-Galois cover $Z_j \to X_j$ that dominates the pointed G-Galois cover $Y_j \to X_j$. By (i), there is a [connected] Γ-Galois étale cover $Z \to X$ that dominates $Y \to X$ and pulls back to each $Z_j \to X_j$, as an unpointed Γ-Galois étale cover of X_j. Choose a base point ζ for Z over η; this corresponds to a weak [resp. proper] solution $\beta : \Pi \to \Gamma$ to the embedding problem $\mathcal{E}$. The composition $\beta\iota_j\phi_j : \pi_1(X_j, \xi_j) \to \Gamma$ is a solution to $\mathcal{E}_j$ corresponding to a pointed Γ-Galois cover of X_j; and by hypothesis, the underlying unpointed Γ-Galois cover agrees with $Z_j \to X_j$ (although the base points might not agree). Thus β_j and $\beta\iota_j\phi_j$ differ by an inner automorphism of Γ — viz. by the element $g_j \in \Gamma$ that takes the base point of one to the base point of the other. But since the reductions to G of these two pointed Γ-Galois covers of X_j are both $(Y_j, \eta_j) \to (X_j, \xi_j)$, it follows that $g_j \in N := \ker(f : \Gamma \to G)$. Thus the weak [resp. proper] solution β to $\mathcal{E}$ induces the given weak solutions β_j to $\mathcal{E}_j$ up to conjugation by elements of N. This shows that (ii) is satisfied.

Conversely, suppose that condition (ii) holds, and let $Z_j \to X_j$ be Γ-Galois étale covers that dominate $Y_j \to X_j$. Choosing a base point ζ_j for Z_j over η_j, we obtain corresponding weak solutions $\beta_j : \pi_1(X_j, \xi_j) \to \Gamma$ to the induced embedding problems $\mathcal{E}_j$. By (ii), there is a weak [resp. proper] solution β to $\mathcal{E}$ that induces each β_j up to conjugacy by N. The map $\beta : \Pi \to \Gamma$ corresponds to a [connected] pointed Γ-Galois cover $Z \to X$ which dominates $Y \to X$ and whose pullback to X_j agrees (as an unpointed Γ-Galois cover) with $Z_j \to X_j$. So (i) is satisfied. □

Via Proposition 3.1, we may obtain results about dominating covers with local conditions by applying Theorem 2.3 to fundamental groups. In order to show that the hypotheses of 2.3 are satisfied, we will first need to verify that certain fundamental groups have p-cohomological dimension ≤ 1. We do so using the following well-known result, which is stated here for the sake of completeness, and whose proof is embedded in those of [Se2, Prop. 1] and [Ka, Lemma 1.4.3]. Cf. also [PS, Thm. 4.13], which provides a detailed proof in the setting of Corollary 3.3(b) below.

Proposition 3.2. *Let X be a connected Noetherian scheme.*

(a) Let $\mathcal{F}$ be the locally constant finite étale sheaf on X associated to a finite $\pi_1(X)$-module F. Then $H^1(\pi_1(X), F) = H^1_{\mathrm{et}}(X, \mathcal{F})$ and $H^2(\pi_1(X), F)$ injects into $H^2_{\mathrm{et}}(X, \mathcal{F})$.

(b) Let ℓ be a prime number. If $\mathrm{cd}_\ell(X) \leq 1$ then $\mathrm{cd}_\ell(\pi_1(X)) \leq 1$.

Proof. (a) Over the pro-universal cover $\tilde{X}$ of X, we have that $\mathcal{F}|_{\tilde{X}}$ is the constant group F. Moreover, for any finite étale $Y \to X$, any $\alpha \in H^1_{\text{et}}(Y, \mathcal{F}|_Y)$ is represented by a finite étale F-Galois cover of Y, and is trivialized over $\tilde{X}$. Thus $H^1_{\text{et}}(\tilde{X}, \mathcal{F}|_{\tilde{X}}) = 0$.

According to the exact sequence of low degree terms [Mi,p. 309] coming from the Hochschild-Serre spectral sequence $H^p(\pi_1(X), H^q_{\text{et}}(\tilde{X}, \mathcal{F})) \Rightarrow H^{p+q}_{\text{et}}(X, \mathcal{F})$ [Mi, I Thm. 2.20, Remark 2.21(b)], we have that

$$
\begin{aligned}
0 \to H^1(\pi_1(X), H^0_{\text{et}}(\tilde{X}, \mathcal{F}|_{\tilde{X}})) \to H^1_{\text{et}}(X, \mathcal{F}) \to H^0(\pi_1(X), H^1_{\text{et}}(\tilde{X}, \mathcal{F}|_{\tilde{X}})) \\
\to H^2(\pi_1(X), H^0_{\text{et}}(\tilde{X}, \mathcal{F}|_{\tilde{X}})) \to H^2_{\text{et}}(X, \mathcal{F})
\end{aligned}
$$

is exact. Since $H^1_{\text{et}}(\tilde{X}, \mathcal{F}|_{\tilde{X}}) = 0$ and $H^0_{\text{et}}(\tilde{X}, \mathcal{F}|_{\tilde{X}}) = F$, we obtain that $H^1(\pi_1(X), F)) \xrightarrow{\sim} H^1_{\text{et}}(X, \mathcal{F})$ and $H^2(\pi_1(X), F) \hookrightarrow H^2_{\text{et}}(X, \mathcal{F})$.

(b) Let F be a finite ℓ-torsion $\pi_1(X)$-module, corresponding to a locally constant ℓ-torsion finite étale sheaf $\mathcal{F}$. Then $H^2(\pi_1(X), F) \hookrightarrow H^2_{\text{et}}(X, \mathcal{F})$ by (a). By hypothesis, $H^2_{\text{et}}(X, \mathcal{F}) = 0$. Hence $H^2(\pi_1(X), F) = 0$. Thus $\text{cd}_\ell(\pi_1(X)) \leq 1$. □

Corollary 3.3. *(a)* (Serre, [Se2, Prop. 1]) *If X is a connected affine curve over a separably closed field k, then* $\text{cd}(\pi_1(X)) \leq 1$.

(b) ([Se3, §2.2], [PS, Thm. 4.13]) *If X is a smooth connected projective curve over a separably closed field k of characteristic $p > 0$, then* $\text{cd}_p(\pi_1(X)) \leq 1$.

(c) If X is a connected Noetherian affine scheme of characteristic $p > 0$ then $\text{cd}_p(\pi_1(X)) \leq 1$.

Proof. (a) Let $p = \text{char}\, k$. For every $\ell \neq p$, we have that $\text{cd}_\ell(X) \leq 1$ by [AGV, IX, Cor. 5.7]. On the other hand if $\ell = p$, then the same conclusion holds by [AGV, X, Thm. 5.1]. So $\text{cd}_\ell(\pi_1(X)) \leq 1$ for all ℓ, by Proposition 3.2(b). That is, $\text{cd}(\pi_1(X)) \leq 1$.

(b) By [AGV, X, Cor. 5.2], $\text{cd}_p(X) \leq \dim X = 1$, so the conclusion follows from the proposition.

(c) By [AGV, X, Thm. 5.1], $\text{cd}_p(X) \leq 1$. So the conclusion again follows. □

In order to verify the hypotheses of Theorem 2.3, we will use Corollary 3.3(c) above and Corollary 3.7 and Proposition 3.8 below. For those results, we need some preparation.

Lemma 3.4. *Let $R \subset S$ be an integral extension of integral domains, and let I be a non-zero ideal of S. Then $I \cap R$ is a non-zero ideal of R.*

Proof. This is a special case of [Bo, V, §2.1, Cor. 1 to Prop. 1], taking $A = R$, $A' = S$, $\mathfrak{a}' = I$, $\mathfrak{p} = (0)$, $\mathfrak{p}' = (0)$. (One can alternatively proceed as in the remark at [La, p.10].) □

Lemma 3.5. *Let X be a Noetherian normal integral scheme and let Σ be the set of points of X of codimension 1. Then $\bigcap_{\xi \in \Sigma} \mathcal{O}_{X,\xi}$ is the ring of global functions on X.*

Proof. If $U = \operatorname{Spec} R$ is any affine open subset of X, then R is a Noetherian integrally closed domain, and hence a Krull domain [Bo, VII, 1.3, Cor. to Lemma 1]. Thus $R = \bigcap_{\xi \in \Sigma_U} \mathcal{O}_{X,\xi}$, where Σ_U is the set of points of U of codimension 1 (corresponding to the height 1 primes of R) [Bo, VII, 1.6, Theorem 4]. Since this is true for each U, the conclusion follows. $\square$

For any ring R of characteristic p, we define the $\mathbb{F}_p$-linear map $\wp : R \to R$ by $\wp(r) = r^p - r$. If R is a domain and $a_1, \ldots, a_m \in R$, then consider the R-algebra S given by adjoining elements $x_1, \ldots, x_m$ subject to $x_i^p - x_i = a_i$ (for $i = 1, \ldots, m$). This extension $R \subset S$ is finite, étale, and P-Galois, where P is an elementary abelian p-group of rank m. Conversely, every P-Galois finite étale extension of R is of this form, by Artin-Schreier theory. Moreover S is a domain if and only if the images of the elements a_i in $R/\wp(R)$ are $\mathbb{F}_p$-linearly independent. Thus the p-rank of $\pi_1(\operatorname{Spec} R)$ is the dimension of $R/\wp(R)$ as an $\mathbb{F}_p$-vector space.

For any ring S and any $n > 0$, the map $\wp : S^n \to S^n$ is defined for the ring S^n, and it is given by $\wp : S \to S$ on each coordinate. If M is a subset of S^n, then we may consider the image of M under $\wp$; this is also a subset of S^n.

Lemma 3.6. *Let k be a field of characteristic p, let R be a finitely generated k-algebra which is an integral domain but not a field. Let S be an integral domain that contains R and is finite as an R-algebra. Let I be a non-zero ideal of R, and let M be a non-zero R-submodule of S^n. Then $IM/(IM \cap \wp(M))$ is an infinite dimensional $\mathbb{F}_p$-vector space.*

Proof. Let $M_1, \ldots, M_n \subset S$ be the images of M under the n projection maps $\pi_j : S^n \to S$. Then each M_j is an R-submodule of S, and some M_j is non-zero. Now $\pi_j(IM \cap \wp(M)) \subset IM_j \cap \wp(M_j)$, and so the map π_j induces a surjective $\mathbb{F}_p$-homomorphism $IM/(IM \cap \wp(M)) \twoheadrightarrow IM_j/(IM_j \cap \wp(M_j))$. Thus $IM/(IM \cap \wp(M))$ is infinite dimensional if $IM_j/(IM_j \cap \wp(M_j))$ is. So replacing M by M_j, it suffices to prove the result under the assumption that M is a non-zero R-submodule of S (i.e. that $n = 1$).

By Noether Normalization [Bo, V, 3.1, Theorem 1], there exist algebraically independent elements $x_1, \ldots, x_d \in R$ such that R is integral over the polynomial ring $T = k[x_1, \ldots, x_d] \subset R$, and such that $J := I \cap T$ is generated by $x_1, \ldots, x_h$ for some $h \geq 0$. Here $d > 0$ since R is not a field. Also, R is finite over T since it is integral over T and is finitely generated as a T-algebra (since it is finitely generated over k). Moreover $J \neq (0)$ by Lemma 3.4, and so $h > 0$; thus $x_1 \in J \subset I$.

Now T is the ring of functions on $\mathbb{A}^d_k \subset \mathbb{P}^d_k$. Let V be the normalization of $\mathbb{P}^d_k$ in the fraction field L of S. Then $\pi : V \to \mathbb{P}^d_k$ is a finite morphism of Noetherian normal integral projective varieties, and $V' := \pi^{-1}(\mathbb{A}^d_k)$ is an affine open subset of V whose ring of functions S' is the integral closure of S. Also, $V - V' = \pi^{-1}(H)$, where $H \subset \mathbb{P}^n_k$ is the hyperplane at infinity; the associated reduced scheme is a union of finitely many divisors D_i on V. Since V is normal, for each i there is a discrete valuation $v_i : L^* \to \mathbb{Z}$ associated to D_i. Note that $v_i(x_1) < 0$, since x_1 has a pole along H.

By Lemma 3.5, if an element $s \in S' = \mathcal{O}(V')$ is regular at the generic point of each D_i, then it is a global function on the projective variety V and hence is constant (i.e. lies in a finite field extension of k). So for each non-constant $s \in S'$, there is an i such that $v_i(s) < 0$. Let a be the smallest (non-negative) integer such that $v_i(m)$ is of the form $-p^a b$ for some i, some non-constant $m \in M \subset S'$, and some positive integer b prime to p. (Here not every element in M is constant, since $M \subset S$ is a non-zero R-module. So the minimum is being taken over a non-empty subset of the non-negative integers, and is thus well defined.) Fix such a choice of i and m corresponding to a. For each positive integer j, consider the element $m_j := x_1^{jp^{a+1}} m \in IM$. Then $v_i(m_j) = jp^{a+1} v_i(x_1) - p^a b$. Thus $p^a \| v_i(m_j)$ (i.e. p^a strictly divides $v_i(m_j)$) and $v_i(m_j) < 0$ for all j, and the integers $v_i(m_j)$ are distinct, since $v_i(x_1) < 0$.

Now consider any non-trivial $\mathbb{Z}/p\mathbb{Z}$-linear combination c of the elements m_j (i.e. any linear combination with at least one non-zero coefficient). Since $p^a \| v_i(m_j) < 0$ and the integers $v_i(m_j)$ are distinct, it follows that c satisfies $p^a \| v_i(c) < 0$. Let $m' \in M$. If m' satisfies $v_i(m') \geq 0$, then $v_i(\wp(m')) \geq 0$ and hence $c \neq \wp(m')$ (since $v_i(c) < 0$). On the other hand, if instead $m' \in M$ satisfies $v_i(m') < 0$, then the minimality of a implies that $p^a | v_i(m') < 0$ and so $p^{a+1} | v_i(\wp(m'))$; so again $c \neq \wp(m')$ (since $p^a \| v_i(c)$). This shows that such a linear combination c does not lie in $\wp(M)$. Thus the elements $m_j \in IM \subset S$ (for $j = 1, 2, \ldots$) are linearly independent modulo $\wp(M)$; and so $IM/(IM \cap \wp(M))$ is an infinite dimensional $\mathbb{F}_p$-vector space. □

Corollary 3.7. *Let k be a field of characteristic p, and let R be a finitely generated k-algebra which is an integral domain but not a field. Then $\pi_1(\operatorname{Spec} R)$ has infinite p-rank.*

Proof. As remarked above, the p-rank of $\pi_1(\operatorname{Spec} R)$ is equal to the dimension of $R/\wp(R)$ as an $\mathbb{F}_p$-vector space. This dimension is infinite by Lemma 3.6, by taking I to be the unit ideal, $M = R = S$, and $n = 1$. □

The following result parallels [Ra1, Prop. 4.2.1], as does its proof (which uses Lemma 3.6). But the proof is able to be much simpler here than for the result in [Ra1], since it deals just with subschemes rather than affinoids. Note that the strategies both here and in [Ra1, §4.2] are inspired by that of [Se2, §4]. (For the definition of a (strongly) p-dominating family $\phi = \{\phi_j\}_{j \in J}$, see Section 2.)

Proposition 3.8. *Let $X = \operatorname{Spec} R$ be an irreducible affine variety of dimension > 0, and of finite type over a field k of characteristic $p > 0$. Let X' be a closed subset, strictly contained in X, and having connected components $X_1, \ldots, X_r$. Let $\phi_j : \pi_1(X_j) \to \pi_1(X)$ be induced by the inclusions $X_j \hookrightarrow X$, and let $\phi = \{\phi_j\}_j$. Then ϕ is strongly p-dominating.*

Proof. Let $\Pi = \pi_1(X)$ and $\Pi_j = \pi_1(X_j)$. Let $P = (\mathbb{Z}/p\mathbb{Z})^n$ be a non-trivial finite elementary abelian p-group together with a continuous action of Π (so $n > 0$). We wish to show that the induced homomorphism $\phi^* : H^1(\Pi, P) \to \prod_j H^1(\Pi_j, P)$ is surjective with infinite kernel.

Under the above action, the finite group P becomes a Π-module, and corresponds to a locally constant finite étale sheaf $\mathcal{F}$ of $\mathbb{Z}/p\mathbb{Z}$-vector spaces over X. (Namely, if U is a connected étale open subset of X, then $\mathcal{F}(U) = P^{\pi_1(U)}$, the subgroup of P fixed by $\pi_1(U)$.) Similarly, for each j the Π_j-module P corresponds to a finite locally constant étale sheaf $\mathcal{F}_j$ of $\mathbb{Z}/p\mathbb{Z}$-vector spaces over X_j. Over X we have the Artin-Schreier exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_{\mathrm{a}} \xrightarrow{\wp} \mathbb{G}_{\mathrm{a}} \to 0$$

of étale sheaves of $\mathbb{Z}/p\mathbb{Z}$-vector spaces [Mi,p.67], where $\wp(a) = a^p - a$. Tensoring over $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{F}$ we obtain an exact sequence

$$0 \to \mathcal{F} \to \mathcal{M} \xrightarrow{\wp} \mathcal{M} \to 0 \tag{1}$$

of étale sheaves on X, where $\mathcal{M}$ is a locally free sheaf of $\mathbb{G}_{\mathrm{a}}$-modules of rank n in the étale topology. Here $\mathcal{M}$ is induced by a locally free sheaf $\mathcal{M}_Z$ of rank n in the Zariski topology [Mi, p.134]; in particular $\mathcal{M}_Z$ is coherent. Since $X = \operatorname{Spec} R$ is affine, and since R is Noetherian (being of finite type over k), there is an equivalence of categories between coherent $\mathcal{O}_X$-modules (in the Zariski sense) and finite R-modules [Ht, II, Prop. 5.4]. Under this equivalence, the locally free Zariski sheaf $\mathcal{M}_Z$ corresponds to a locally free R-module M of rank $n > 0$. Here the map $\wp : \mathcal{M} \to \mathcal{M}$ corresponds to a $\mathbb{Z}/p\mathbb{Z}$-linear map $\wp : M \to M$.

More explicitly, the action of Π on P has a kernel $\Phi \subset \Pi$ which is normal and of finite index, and which corresponds to a Galois finite étale cover $W \to X$ (of Galois group Π/Φ). The action of $\Phi = \pi_1(W)$ on $P = (\mathbb{Z}/p\mathbb{Z})^n$ is trivial, and so $\mathcal{M}|_W$ is free of rank n. Thus $\mathcal{M}(W) = S^n$, where $W = \operatorname{Spec} S$. By the sheaf axiom, M is the equalizer of the two natural maps $\mathcal{M}(W) \to \mathcal{M}(W \times_X W)$; and in particular M is an R-submodule of S^n. Also, the map $\wp : M \to M$ is the restriction of the corresponding map on $\mathcal{M}(W) = S^n$, which is given by the classical map $\wp : S \to S$ on each coordinate.

Since X is affine and since $\mathcal{M}$ is induced by the coherent Zariski sheaf $\mathcal{M}_Z$ corresponding to the R-module M, we have $H^0(X, \mathcal{M}) = M$ and $H^1(X, \mathcal{M}) = 0$ (by comparison of étale and Zariski cohomologies for coherent sheaves [Mi, III 3.8]). So the exact sequence of sheaves (1) gives rise to the exact sequence

$$M \xrightarrow{\wp} M \to H^1(X, \mathcal{F}) \to 0. \tag{2}$$

That is, $H^1(X, \mathcal{F}) = M/\wp(M)$. Similarly, for each j, we have $H^1(X_j, \mathcal{F}_j) = M_j/\wp(M_j)$, where $M_j = M/I_jM$ and I_j is the ideal of X_j. Moreover $H^1(X, \mathcal{F}) = H^1(\Pi, P)$ and $H^1(X_j, \mathcal{F}_j) = H^1(\Pi_j, P)$ by Proposition 3.2(a). So $\prod_{j=1}^r H^1(\Pi_j, P) = \prod_{j=1}^r H^1(X_j, \mathcal{F}_j)$, and ϕ^* may be identified with the map $M/\wp(M) \to \prod_{j=1}^r M_j/\wp(M_j)$. It remains to show that this map is surjective with infinite kernel.

Since $X_1, \ldots, X_r$ are pairwise disjoint closed sets, their ideals $I_1, \ldots, I_r \subset R$ are pairwise relatively prime. So $M \to M' := \prod_{j=1}^r M_j$ is surjective by the Chinese Remainder Theorem. Hence so is the composition $M \to \prod_{j=1}^r M_j \to \prod_{j=1}^r M_j/\wp(M_j)$, which factors through $M/\wp(M) \to \prod_{j=1}^r M_j/\wp(M_j)$. So that latter map is also surjective, as desired.

Finally, we show that the map $M/\wp(M) \to \prod_{j=1}^r M_j/\wp(M_j)$ has infinite kernel. Let $I = \bigcap_{j=1}^r I_j$, so that $M' = M/IM$. Now $I, M \neq 0$; the R-module M is contained in S^n; and S is finite over R. So by Lemma 3.6, $IM/(IM \cap \wp(M))$ is infinite. But $IM/(IM \cap \wp(M))$ is contained in the kernel of $M/\wp(M) \to M'/\wp(M') = \prod_{j=1}^r M_j/\wp(M_j)$. Thus this kernel is indeed infinite. $\square$

Example 3.9. Let $p = 3$, let $P = \mathbb{Z}/3\mathbb{Z}$, let $R = k[x, x^{-1}]$ and let $X = \operatorname{Spec} R$. Also, let $S = k[y, y^{-1}]$ where $y^2 = x$, and let $Y = \operatorname{Spec} S$. Then $C := \operatorname{Gal}(Y/X)$ is cyclic of order 2, and there are two actions of C on P, each inducing an action of $\Pi = \pi_1(X)$ on P and yielding a p-embedding problem $\mathcal{E} = (\alpha : \Pi \to C, f : \Gamma \to C)$.

The first of these two actions is the trivial one, corresponding to the case that Γ is cyclic of order 6. In this case the R-module M in the proof above is R itself, viewed as a submodule of S, and the map $\wp : M \to M$ is just the usual map $\wp$ on R. Here $\ker(\wp) = H^0(\Pi, P) = \mathbb{Z}/3\mathbb{Z}$ and $\operatorname{cok}(\wp) = H^1(\Pi, P) = \operatorname{Hom}(\Pi, P) = R/\wp(R)$, which is isomorphic to $\bigoplus_{(3,n)=1} x^n k$ if k is algebraically closed.

The second of these two actions, in which the generator of C acts on P by $a \mapsto -a$, corresponds to the case that $\Gamma = S_3$. Here M is again a free R-module of rank 1, but the map $\wp : M \to M$ is not the obvious one. This is because M is now the submodule $yR = \bigoplus_{(2,n)=1} y^n k \subset S$. Here $\ker(\wp) = H^0(\Pi, P) = 0$, and $\operatorname{cok}(\wp) = H^1(\Pi, P) = M/\wp(M)$ is isomorphic to $\bigoplus_{(6,n)=1} y^n k$ if k is algebraically closed.

Note that the R-module M is free of rank 1 for each of the two actions above, but the maps $\wp : M \to M$ are different. Thus $\wp$ depends not just on the isomorphism class of M, but also on the embedding of M into S^n (corresponding to the action of Π on A). $\square$

As a result of the above proposition together with Theorem 2.3, we obtain:

Corollary 3.10. *Let X and ϕ be as in Proposition 3.8. Then every finite p-embedding problem for $\pi_1(X)$ is properly ϕ-solvable.*

Proof. The p-rank of $\pi_1(X)$ is infinite, by Corollary 3.7. Also $\operatorname{cd}_p(\pi_1(X)) \le 1$, by Corollary 3.3(c). So condition (i) of Theorem 2.3 holds; hence so does condition (iv) of that result. By Proposition 3.8, ϕ is a strongly p-dominating family. So every finite p-embedding problem for Π is properly ϕ-solvable, by the conclusion of 2.3(iv). $\square$

Reinterpreting the above result in light of Proposition 3.1, we obtain the main result of this section, concerning affine varieties X over an arbitrary field of characteristic $p > 0$:

Theorem 3.11. *Let X be an irreducible affine variety of dimension > 0 and of finite type over a field k of characteristic p. Let P be a p-subgroup of a finite group Γ; let $G = \Gamma/P$;*

and let $Y \to X$ be a connected G-Galois étale cover. Let $X' \subset X$ be a proper closed subset, let $Y' = Y \times_X X'$, and suppose that $Z' \to Y'$ is a P-Galois étale cover such that the composition $Z' \to X'$ is Γ-Galois. Then there is a connected P-Galois étale cover $Z \to Y$ such that the composition $Z \to X$ is Γ-Galois, and such that $Z \times_X X' \approx Z'$ as Γ-Galois covers.

Proof. Let $f : \Gamma \to G$ be the quotient map, and let $\alpha : \pi_1(X) \to G$ be a surjection corresponding to the connected G-Galois étale cover $Y \to X$. Also, let $X_1, \dots, X_r$ be the connected components of X', and let ϕ_j, ϕ be as in the statement of Proposition 3.8. By Corollary 3.10, the finite p-embedding problem $\mathcal{E} = (\alpha : \pi_1(X) \to G, f : \Gamma \to G)$ is properly ϕ-solvable. So the conclusion follows from the implication (ii) $\Rightarrow$ (i) of Proposition 3.1. □

Section 4. p-Embedding problems for affine varieties: ramified case.

This section, like the previous one, considers p-embedding problems with local conditions over affine varieties of characteristic p. That is, we are given a G-Galois cover $Y \to X$ and a group Γ with a normal p-subgroup P such that $\Gamma/P = G$. We then wish to find a Γ-Galois cover $Z \to X$ dominating $Y \to X$, with $Z \to Y$ étale, and with $Z \to X$ having prescribed local behavior. In Section 3, the given cover $Y \to X$ was required to be étale. Here it is permitted to be ramified, but we add the requirement that its degree be prime to p. In a variant of Theorem 3.11, we show here, in Theorem 4.3, that such problems have solutions. While Theorem 3.11 paralleled [Ra1, Cor. 4.2.6], the main result in this section is closer to paralleling the results of [Ha2, §4] and [Ra2].

The strategy here is adapted from that of [Ra2], viz. defining an appropriate fundamental group $\pi_1(Y/X)$. This π_1 is then shown to satisfy the analogs of Cor. 3.3(c), Cor. 3.7 and Prop. 3.8, and hence to satisfy the hypotheses appearing in Theorem 2.3. As a consequence, that group-theoretic result will apply here, and Theorem 4.3 will follow.

We begin by fixing terminology and reviewing concepts concerning covers that are not necessarily étale. Let X be a reduced Noetherian scheme. A morphism $f : Y \to X$ of finite type is *generically étale* if for every irreducible component Y° of Y, the closure of its image $X^\circ := f(Y^\circ)$ is an irreducible component of X, and $Y^\circ \to X^\circ$ is étale at the generic point. A *cover* of X is a finite morphism of schemes $f : Y \to X$ which is generically étale. (If X is irreducible, this is equivalent to the definition in [Ha2, §1]). Thus ramification in codimension ≥ 1 is permitted here. Given a cover $Y \to X$, we define the *Galois group* $\mathrm{Gal}(Y/X)$ exactly as for étale covers (cf. the beginning of Section 3). Similarly, we define the notions of a *Galois* cover and a *G-Galois* cover exactly as in the étale case.

Next, we define the version of π_1 that will be used in proving Theorem 4.3. If X is a connected Noetherian scheme and $Y \to X$ is a Galois cover (not necessarily étale), then we will let $\pi_1(Y/X)$ denote the Galois group of the maximal connected pro-cover of X that is étale over Y. More precisely, if compatible geometric base points are chosen on X and Y, at which $Y \to X$ is étale, then we may consider the inverse system $\{Z_\nu\}_\nu$ of pointed connected étale covers of Y that are Galois over X. The group $\pi_1(Y/X)$ is then defined to be $\varprojlim \mathrm{Gal}(Z_\nu/X)$. Since the pro-universal cover of Y is Galois over X, it follows that there is an exact sequence $1 \to \pi_1(Y) \to \pi_1(Y/X) \to \mathrm{Gal}(Y/X) \to 1$.

Note that above, if X_1 is a connected closed subset of X and Y_1° is a connected component of $Y_1 := Y \times_X X_1$, then there is an induced map $\pi_1(Y_1^\circ/X_1) \to \pi_1(Y/X)$ (determined up to inner automorphism, corresponding to the choice of a base point). This map is compatible with the maps between the respective terms of the above exact sequence and the analogous sequence for $\pi_1(Y_1^\circ/X_1)$.

Proposition 4.1. *Let $X = \operatorname{Spec} R$ be an irreducible affine variety of dimension > 0, and of finite type over a field k of characteristic $p > 0$. Let $\gamma : Y \to X$ be a finite Galois cover of degree prime to p, and let $\tilde{\Pi} = \pi_1(Y/X)$.*

(a) Then $\mathrm{cd}_p(\tilde{\Pi}) \leq 1$.

(b) The group $\tilde{\Pi}$ has infinite p-rank.

(c) Let $X' \subset X$ be a closed subset, strictly contained in X, and having connected components $X_1, \dots, X_r$. For each j, suppose that the pullback $Y_j = Y \times_X X_j \to X$ is generically étale. Let Y_j° be a connected component of Y_j; let $\tilde{\Pi}_j = \pi_1(Y_j^\circ/X_j)$; let $\tilde{\phi}_j : \tilde{\Pi}_j \to \tilde{\Pi}$ be induced by the inclusion $X_j \hookrightarrow X$; and let $\tilde{\phi} = \{\tilde{\phi}_j\}_j$. Then $\tilde{\phi}$ is strongly p-dominating.

Proof. (a) The group $\pi_1(Y)$ is a closed subgroup of $\pi_1(Y/X)$, with quotient group $G := \mathrm{Gal}(Y/X)$. The index $(\pi_1(Y/X) : \pi_1(Y))$ is equal to $|G| = \deg(\gamma)$, which is prime to p. So by [Se1, I.3.3, Prop. 14], these two profinite groups have the same cd_p. The assertion now follows from Corollary 3.3(c).

(c) Consider a continuous action of $\tilde{\Pi}$ on a non-trivial finite elementary abelian p-group P, and the induced action of $\tilde{\Pi}_j$ on P. We wish to show that the induced homomorphism $\tilde{\phi}^* : H^1(\tilde{\Pi}, P) \to \prod_j H^1(\tilde{\Pi}_j, P)$ is surjective with infinite kernel.

Restricting the actions of $\tilde{\Pi}$ and of $\tilde{\Pi}_j$ to the closed subgroups $\pi_1(Y)$ and $\pi_1(Y_j^\circ)$, we may regard P as a module over $\pi_1(Y)$ and over $\pi_1(Y_j^\circ)$. By the Hochschild-Serre spectral sequence $H^p(G, H^q(\pi_1(Y), P)) \Rightarrow H^{p+q}(\tilde{\Pi}, P)$ (cf. [Se1, I 2.6(b)] or [Sh, p.51]), there is an exact sequence

$$H^1(G, P^{\pi_1(Y)}) \to H^1(\tilde{\Pi}, P) \to H^1(\pi_1(Y), P)^G \to H^2(G, P^{\pi_1(Y)}).$$

Here the first and last terms vanish, since G is of order prime to p and since P is a p-group [Se1, I 3.3 Cor. 2]. Thus $H^1(\tilde{\Pi}, P) \to H^1(\pi_1(Y), P)^G$ is an isomorphism. By Proposition 3.2(a) we may identify $H^1(\pi_1(Y), P)$ with $H^1(Y, \mathcal{F}_Y)$, where $\mathcal{F}_Y$ is the locally constant finite p-torsion étale sheaf on Y associated to P (viewed as a $\pi_1(Y)$-module). We thus identify $H^1(\tilde{\Pi}, P)$ with $H^1(Y, \mathcal{F}_Y)^G = H^1(\pi_1(Y), P)^G$.

Similarly, letting $G_j = \mathrm{Gal}(Y_j^\circ/X_j) \subset G$ for each j, the exact sequence

$$1 \to \pi_1(Y_j^\circ) \to \pi_1(Y_j^\circ/X_j) \to G_j \to 1$$

allows us to identify $H^1(\tilde{\Pi}_j, P)$ with $H^1(Y_j^\circ, \mathcal{F}_{Y_j^\circ})^{G_j} = H^1(\pi_1(Y_j^\circ), P)^{G_j}$. For each j, the closed set $Y_j = \gamma^{-1}(X_j)$ is a disjoint union of connected components $Y_{j,1}, \ldots, Y_{j,m_j}$, with $Y_{j,1} = Y_j^\circ$. We may then identify the induced G-module $\mathrm{Ind}_{G_j}^G H^1(\pi_1(Y_j^\circ), P)$ with $\prod_\ell H^1(\pi_1(Y_{j,\ell}), P)$, and thus $H^1(\tilde{\Pi}_j, P) = H^1(\pi_1(Y_j^\circ), P)^{G_j}$ with $\big(\prod_\ell H^1(\pi_1(Y_{j,\ell}), P)\big)^G$.

By Proposition 3.8, the map $\phi_Y^* : H^1(\pi_1(Y), P) \to \prod_{j,\ell} H^1(\pi_1(Y_{j,\ell}), P)$ is surjective. This restricts to a map $H^1(\pi_1(Y), P)^G \to \big(\prod_{j,\ell} H^1(\pi_1(Y_{j,\ell}), P)\big)^G$, which is surjective since the order of G is not divisible by p. (Namely, if $z \in \prod_{j,\ell} H^1(\pi_1(Y_{j,\ell}), P)\big)^G \subset \prod_{j,\ell} H^1(\pi_1(Y_{j,\ell}), P)$ then some $w \in H^1(\pi_1(Y), P)$ maps to z; and then $\dfrac{1}{|G|} \displaystyle\sum_{g \in G} g(w) \in H^1(\pi_1(Y), P)^G$ also maps to z.) This surjectivity and the above identifications

$$H^1(\tilde{\Pi}, P) = H^1(\pi_1(Y), P)^G, \quad H^1(\tilde{\Pi}_j, P) = \big(\prod_\ell H^1(\pi_1(Y_{j,\ell}), P)\big)^G,$$

show that the map $\tilde{\phi}^* : H^1(\tilde{\Pi}, P) \to \prod_j H^1(\tilde{\Pi}_j, P)$ is surjective, as desired. It remains to show that $\tilde{\phi}^*$ has infinite kernel, or equivalently that

$$H^1(Y, \mathcal{F}_Y)^G \to \big(\prod_{j,\ell} H^1(Y_{j,\ell}, \mathcal{F}_{Y_{j,\ell}})\big)^G = H^1(Y', \mathcal{F}_{Y'})^G$$

does, where $Y' = \bigcup Y_{j,\ell} = \gamma^{-1}(X')$.

The exact sequence (2) in the proof of Proposition 3.8, but with Y instead of X, takes the form

$$M_Y \xrightarrow{\wp} M_Y \to H^1(Y, \mathcal{F}_Y) \to 0. \tag{2'}$$

Here M_Y is a finite locally free R_1-module (where $Y = \mathrm{Spec}\, R_1$), and it corresponds to the locally free sheaf $\mathcal{M}_Y = \mathcal{F}_Y \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{G}_{\mathrm{a}}$ on Y. As in the proof of Proposition 3.8, M_Y is a finite R_1-submodule of S^n, for some domain S that is a finite étale extension of R_1, where n is the rank of P. Restricting the sequence (2′) to the G-invariant subspaces yields the sequence

$$M_Y^G \xrightarrow{\wp} M_Y^G \to H^1(Y, \mathcal{F}_Y)^G \to 0 \tag{3}$$

of $\mathbb{Z}/p\mathbb{Z}$-vector spaces. The sequence (3) is again exact because G is of order prime to p (by an averaging argument as above). Here M_Y^G is a finite R-module contained in $M_Y \subset S^n$. Similarly there is an exact sequence

$$M_{Y'}^G \xrightarrow{\wp} M_{Y'}^G \to H^1(Y', \mathcal{F}_{Y'})^G \to 0 \tag{3'}$$

of finite modules over $R' = R/I$, where $I \subset R$ is the ideal of X'; here $M^G_{Y'}$ is a finite R'-submodule of S'^n, where $S' = R' \otimes_R S$.

The sequences (3) and (3$'$) are compatible, and the map $H^1(Y, \mathcal{F}_Y)^G \to H^1(Y', \mathcal{F}_{Y'})^G$ may be identified with the map $M^G_Y/\wp(M^G_Y) \to M^G_{Y'}/\wp(M^G_{Y'})$. The kernel of this map contains $IM^G_Y/(IM^G_Y \cap \wp(M^G_Y))$. So to show that the kernel is infinite, it suffices to show that $IM^G_Y/(IM^G_Y \cap \wp(M^G_Y))$ is. Now $I \neq 0$; M^G_Y is an R-submodule of S^n; and S is finite over R. So by Lemma 3.6, it suffices to show that M^G_Y is non-zero.

Since $\gamma : Y \to X$ is a cover, γ restricts to a finite étale morphism over an affine Zariski open dense subset $U \subset X$. Let $V = \gamma^{-1}(U) \subset Y$. There are induced homomorphisms $\pi_1(U) = \pi_1(V/U) \to \tilde{\Pi}$ and $\pi_1(Y) \to \tilde{\Pi}$, and a commutative diagram

$$\begin{tikzcd} \pi_1(V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ \pi_1(Y) \arrow[r] & \tilde{\Pi} \end{tikzcd}$$

of profinite groups. The action of $\tilde{\Pi}$ on P thus induces actions of $\pi_1(U)$ and $\pi_1(V)$ on P which are compatible with the above actions of $\tilde{\Pi}$ and $\pi_1(Y)$ on P. The corresponding locally constant finite p-torsion étale sheaves $\mathcal{F}_U$ and $\mathcal{F}_V$ on U and V are thus compatible with $\mathcal{F}_Y$; i.e. $\mathcal{F}_Y|_V = \mathcal{F}_V = \gamma^*(\mathcal{F}_U)$. Consider the locally free sheaves $\mathcal{M}_U = \mathcal{F}_U \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{G}_{\mathrm{a}}$ on U and $\mathcal{M}_V = \mathcal{F}_V \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{G}_{\mathrm{a}}$ on V. As in the proof of Proposition 3.8, these two sheaves correspond to finite locally free modules M_U and M_V over the rings of functions of the affine varieties U and V. Since P is non-zero, so are $\mathcal{F}_U$, $\mathcal{M}_U$ and M_U. But M_V and hence M^G_V contains M_U. So M^G_V is non-zero, and thus so is $\mathcal{M}^G_V = \mathcal{M}^G_Y|_V$. So $\mathcal{M}^G_Y$ is non-zero, and hence so is M^G_Y, as desired.

(b) In part (c), take X' to be empty, take $P = \mathbb{Z}/p\mathbb{Z}$, and take the trivial action of $\tilde{\Pi}$ on P. Then part (c) asserts that $H^1(\tilde{\Pi}, P)$ is infinite. But this is just $\mathrm{Hom}(\tilde{\Pi}, \mathbb{Z}/p\mathbb{Z})$. So $\tilde{\Pi}$ has infinite p-rank. $\square$

Corollary 4.2. *Let $\tilde{\Pi}$ and $\tilde{\phi}$ be as in Proposition 4.1. Then every finite p-embedding problem for $\tilde{\Pi}$ is properly $\tilde{\phi}$-solvable.*

Proof. The pro-p-group $\tilde{\Pi}$ has infinite p-rank, by Proposition 4.1(b). Also, $\mathrm{cd}_p(\tilde{\Pi}) \leq 1$, by Proposition 4.1(a). So condition (i) of Theorem 2.3 holds for the group $\tilde{\Pi}$, and hence so does 2.3(iv). By Proposition 4.1(c), $\tilde{\phi}$ is a strongly p-dominating family. So every finite p-embedding problem for $\tilde{\Pi}$ is properly $\tilde{\phi}$-solvable, by the conclusion of 2.3(iv). $\square$

Using this result, we obtain the following analog of Theorem 3.11, in which the G-Galois cover $Y \to X$ is permitted to have ramification, but in which $G = \Gamma/P$ is required to have order prime to p (corresponding to P being a Sylow p-subgroup of Γ):

Theorem 4.3. *Let X be an irreducible affine variety of dimension > 0 and of finite type over a field k of characteristic p. Let P be a p-subgroup of a finite group Γ, and assume that $G = \Gamma/P$ is of order prime to p. Let $Y \to X$ be a connected G-Galois cover, let $X' \subset X$ be a proper closed subset, and assume that $Y' = Y \times_X X'$ is generically étale over X'. Suppose that $Z' \to Y'$ is a P-Galois étale cover such that the composition $Z' \to X'$ is Γ-Galois. Then there is a connected P-Galois étale cover $Z \to Y$ such that the composition $Z \to X$ is Γ-Galois, and such that $Z \times_X X' \approx Z'$ as Γ-Galois covers.*

Proof. We proceed as in the proof of Theorem 3.11. Let $f : \Gamma \to G$ be the quotient map, and let $\tilde{\alpha} : \pi_1(Y/X) \to G = \mathrm{Gal}(Y/X)$ be the canonical map. Also, let $X_1, \dots, X_r$ be the connected components of X' and let $\tilde{\phi}_j, \tilde{\phi}$ be as in the statement of Proposition 4.1. By Corollary 4.2, the finite p-embedding problem $\tilde{\mathcal{E}} = (\tilde{\alpha} : \pi_1(Y/X) \to G, f : \Gamma \to G)$ is properly $\tilde{\phi}$-solvable. Paralleling the implication (ii) $\Rightarrow$ (i) of Proposition 3.1, the Γ-Galois covers $Z_j \to X_j$ correspond to weak solutions β_j to the pullbacks $\tilde{\phi}_j^*(\tilde{\mathcal{E}})$, and the desired cover $Z \to X$ corresponds to the proper solution β to $\tilde{\mathcal{E}}$ that induces the β_j's up to P-conjugacy. □

Remark 4.4. Theorems 3.11 and 4.3 each make an assumption on the given G-Galois cover $Y \to X$: either that it is étale or that it is prime-to-p. If one simply dropped these assumptions (e.g. permitting $Y \to X$ to have wild ramification), then the assertion that the cover $Z \to Y$ can be chosen to be étale would become false. This can be seen, for example, by taking Γ to be cyclic of order p^2; P and $G = \Gamma/P$ to be p-cyclic; $X = \mathbb{A}^1$; and $Y \to X$ a G-Galois cover that is totally ramified over the origin. For then, any Γ-Galois cover $Z \to X$ dominating $Y \to X$ must also be totally ramified over the origin (since its inertia group surjects onto G), and then $Z \to Y$ is not étale. □

While the above remark shows that the separate hypotheses on $Y \to X$ in Theorems 3.11 and 4.3 cannot simply be dropped, in the case that X is a curve there is a natural weaker hypothesis which would lead to a more general assertion containing these two theorems as special cases: That if $Y \to X$ is a connected G-Galois cover of a characteristic p affine variety X having only *tame* ramification, and if $G = \Gamma/P$ for some p-group P, then there is a connected P-Galois étale cover $Z \to Y$ such that $Z \to X$ is Γ-Galois and has given behavior over a given proper closed subset $X' \subset X$. The strategy employed in the proofs of Theorems 3.11 and 4.3 cannot be used directly to prove such an assertion, since one first would need to know that an appropriate version of π_1 has $\mathrm{cd}_p \leq 1$ (but the proof of Proposition 4.1(a) above does not carry over). In the following section, however, we turn this around — proving such an assertion for affine curves X (Theorem 5.14), and then deducing that the corresponding version of π_1 has $\mathrm{cd}_p \leq 1$ (Corollary 5.16). The assertion is shown by first proving an analogous result in an "adelic" situation (Theorem 5.6).

Section 5. p-Embedding problems for affine curves.

The previous two sections showed that p-embedding problems can be solved over characteristic p affine varieties, with prescribed behavior over a given proper closed subset, under appropriate hypotheses. This section will show that in the case of normal affine curves X, even more is true: that such embedding problems can be solved with prescribed behavior over a given finite set of local fields (Theorem 5.6). This gives greater control on the local behavior, and will also lead to a result on tame covers (Theorem 5.14, referred to at the end of the previous section) which subsumes and strengthens the main results of Sections 3 and 4 in the case of curves.

More precisely, suppose that X is a normal curve over an arbitrary field k of characteristic $p > 0$. For each closed point $\xi \in X$, the complete local ring $\hat{\mathcal{O}}_{X,\xi}$ is a complete discrete valuation ring. By the *local field at* ξ we will mean the fraction field $\mathcal{K}_{X,\xi} := \operatorname{frac} \hat{\mathcal{O}}_{X,\xi}$ (or for short, $\mathcal{K}_\xi$). The fundamental group of $\operatorname{Spec}(\mathcal{K}_\xi)$ may be identified with the absolute Galois group $G_{\mathcal{K}_\xi}$ of $\mathcal{K}_\xi$, i.e. the Galois group $\operatorname{Gal}(\mathcal{K}_\xi^{\mathrm{s}}/\mathcal{K}_\xi)$ of the separable closure of $\mathcal{K}_\xi$. If X is connected and $U \subset X$ is a non-empty open subset, the inclusion $\operatorname{Spec}(\mathcal{K}_\xi) \hookrightarrow U$ induces a homomorphism $\phi_\xi : G_{\mathcal{K}_\xi} \to \pi_1(U)$ between the corresponding fundamental groups.

We begin with the following lemmas. Here, as before, for any characteristic p ring A we consider the $\mathbb{F}_p$-linear map $\wp : A \to A$ given by $\wp(a) = a^p - a$.

Lemma 5.1. *Let R be a Noetherian ring of characteristic p that is complete with respect to an ideal I. Let $A \supset R$ be an R-algebra, let M be a finite R-submodule of A, and suppose that $\wp(M) \subset M$. Then $\wp(IM) = IM$ and hence $IM \subset \wp(M)$.*

Proof. Every element of IM is a sum of finitely many elements of the form im (with $i \in I$ and $m \in M$), so we may restrict attention to elements of this form. Also, since $\wp(M) \subset M$, and since M is an R-module (and in particular a $\mathbb{Z}/p\mathbb{Z}$-module), it follows that $F(M) \subset M$, where $F : A \to A$ is the Frobenius map $a \mapsto a^p$.

If $i \in I$ and $m \in M$, then $\wp(im) = i^p m^p - im = i(i^{p-1}m^p - m) \in IM$ since $m^p \in F(M) \subset M$. Thus $\wp(IM) \subset IM$. It remains to show that $IM \subset \wp(IM)$.

Again, say $i \in I$ and $m \in M$. Then $m^{p^j} \in F^j(M) \subset M$ for each non-negative integer j. Since $i \in I$, we then have $i^{p^j-1}m^{p^j} \in I^{p^j-1}M$. Now M is finite over R, and R is I-adically complete; so M is equal to its own I-adic completion [Bo, III, 3.4, Theorem 3(ii)], i.e. M is I-adically complete. So the series $-m - i^{p-1}m^p - i^{p^2-1}m^{p^2} - i^{p^3-1}m^{p^3} - \cdots$ defines a well defined element $m' \in M$. Thus $im' \in IM$. One immediately computes that $\wp(im') = im$. This shows that $IM \subset \wp(IM)$. □

Lemma 5.2. *Let R be a Dedekind domain and let $U = \operatorname{Spec} R_U$ be a dense open subset of $X = \operatorname{Spec} R$. Let A be a normal ring containing R_U, and let M_U be a finite locally free R_U-submodule of A of rank n that is closed under $\wp : A \to A$.*

Then there is a dense open subset $X_0 = \operatorname{Spec} R_0 \subset X$ such that $U \cup X_0 = X$, together with a locally free R-submodule M of M_U having rank n, such that

(i) *The canonical map $M \otimes_R R_U \to M_U$ is an isomorphism;*

(ii) *$M_0 := M \otimes_R R_0 \subset A \otimes_R R_0$ is a free R_0-module of rank n with a basis $B \subset M$; and*

(iii) *M_0 is closed under $\wp$, as is $M_\xi := M \otimes_R \hat{\mathcal{O}}_{X,\xi} \subset A \otimes_R \hat{\mathcal{O}}_{X,\xi}$ for each $\xi \in X - U$.*

Proof. Since M_U is a locally free R_U-module of rank n, there is a dense open subset $U_0 = \operatorname{Spec} R_{U_0}$ of U such that $M_{U_0} := M_U \otimes_{R_U} R_{U_0}$ is free of rank n. Choose a basis $B = \{b_1, \ldots, b_n\}$ for M_{U_0} over R_{U_0}. After multiplying the b_i's by appropriate elements of R, and shrinking U_0 if necessary, we may assume that the basis B consists of elements of $M_U \subset M_{U_0} \subset A$. In particular, $\wp(b_i) \in \wp(M_U) \subset M_U$, and so $b_i^p \in M_U \subset M_{U_0}$ for all i. Write $b_i^p = \sum_j e_{i,j} b_j$ with $e_{i,j} \in R_{U_0}$. After again multiplying the b_i's by appropriate elements of R and shrinking U_0, we may assume that each $e_{i,j} \in R$.

Let $\Sigma = U - U_0$. Then $X_0 := X - \Sigma$ is an affine dense open subset of the affine curve X, say $X_0 = \operatorname{Spec} R_0$. Thus $R_0 \subset R_{U_0}$. Let M_0 be the R_0-submodule of M_{U_0} generated by B. Since the b_i's are R_{U_0}-linearly independent, they are also R_0-linearly independent. So M_0 is a free R_0-module of rank n, and we may identify $M_0 \otimes_{R_0} R_{U_0} = M_{U_0} = M_U \otimes_{R_U} R_{U_0}$. Thus we obtain a locally free coherent sheaf $\mathcal{M}$ on the affine curve X, corresponding (via [Ht, II, Prop. 5.4]) to a locally free R-module M of rank n that induces M_0, M_U, and M_{U_0} (compatibly with the above identifications) over R_0, R_U, and R_{U_0} respectively. Since B generates the R_0-module M_0, it also generates the R_ξ-module $M_\xi = M \otimes_R R_\xi = M_0 \otimes_{R_0} R_\xi$ for each $\xi \in X - U$, where $R_\xi = \hat{\mathcal{O}}_{X,\xi}$. But $b_i^p = \sum_j e_{i,j} b_j \in M_0 \subset M_\xi$ for each $\xi \in X - U$. So M_0 and the M_ξ's are each closed under the Frobenius map F and hence under $\wp$. And since each $b_j \in M_U, M_0$ compatibly, we have $B \subset M$. $\square$

The next result is analogous to Proposition 3.8, but it considers local behavior over local fields (rather than over closed subsets), and it requires X to be a curve. As in Proposition 3.8, the result does not generalize to local schemes like $\operatorname{Spec} k[[t]]$, which are not of finite type over the base field k.

Proposition 5.3. *Let k be a field of characteristic p, and let X be a connected normal affine k-scheme of dimension 1, of finite type over k. Let $U = X - \{\xi_1, \ldots, \xi_r\}$ be a dense open subset of X (where $r \geq 0$). Let $\Pi = \pi_1(U)$, let Π_j be the absolute Galois group of $\mathcal{K}_{\xi_j}$, let $\phi_j : \Pi_j \to \Pi$ be the map induced by $\operatorname{Spec} \mathcal{K}_{\xi_j} \to U$, and let $\phi = \{\phi_j\}_j$. Then ϕ is strongly p-dominating.*

Proof. Let P be a non-trivial finite elementary abelian p-group, say of rank n, together with a continuous action of Π. We wish to show that the induced homomorphism $\phi^* : H^1(\Pi, P) \to \prod_j H^1(\Pi_j, P)$ is surjective with infinite kernel.

Let R be the ring of functions on X, let R_U be the ring of functions on the affine curve U, and let K be the common fraction field of these rings. Let $\bar{X}$ be the set of places of K/k, and identify the closed points of X with the corresponding places. Since X is an affine curve, there is a place $\mathfrak{q} \in \bar{X}$ that is not identified with any point of X. For each place $\mathfrak{p} \in \bar{X}$, let $v_{\mathfrak{p}} : K \to \mathbb{Z}$ be the corresponding discrete valuation. For $\mathfrak{p} = \xi \in X$, v_ξ extends to a discrete valuation $v_\xi : \mathcal{K}_\xi \to \mathbb{Z}$. Let $S = \{\xi_1, \ldots, \xi_r\}$ and $S' = S \cup \{\mathfrak{q}\} \subset \bar{X}$.

As in the proof of 3.8, the action of Π on P induces an exact sequence

$$M_U \xrightarrow{\wp} M_U \to H^1(\Pi, P) \to 0$$

of $\mathbb{Z}/p\mathbb{Z}$-vector spaces, where M_U is a rank n locally free R_U-submodule of S_U^n, and S_U is finite étale over R_U. By the above exact sequence, we may identify $H^1(\Pi, P)$ with $M_U/\wp(M_U)$. Similarly, we may identify $H^1(\Pi_j, P)$ with $M_{U,j}/\wp(M_{U,j})$, where $M_{U,j} = M_U \otimes_{R_U} \mathcal{K}_{\xi_j}$. We may also thus identify $\prod_{j=1}^r H^1(\Pi_j, P)$ with $M'_U/\wp(M'_U)$, where $M'_U = \prod_{j=1}^r M_{U,j}$ (which is a module over $\mathcal{K}' := \prod_j \mathcal{K}_{\xi_j}$ contained in $S_U'^n$, where $S'_U = S_U \otimes_{R_U} \mathcal{K}'$). It thus suffices to show that the natural map $f_U : M_U/\wp(M_U) \to M'_U/\wp(M'_U)$ is surjective with infinite kernel.

Applying Lemma 5.2, we obtain a dense open subset $X_0 = \operatorname{Spec} R_0$ of X such that $U \cup X_0 = X$, and a rank n locally free R-submodule $M \subset M_U$ satisfying (i) - (iii) of that lemma. Thus $R_0 \subset \mathcal{K}', \mathcal{K}_{\xi_j}$, and $B = \{b_1, \ldots, b_n\} \subset M$ is a basis for $M \otimes_R R_0$ and hence for $M'_U = M_U \otimes_{R_U} \mathcal{K}' = M \otimes_R \mathcal{K}'$ and $M_{U,j}$. Let $R_j = \hat{\mathcal{O}}_{X,\xi_j}$ and $M_j = M \otimes_R R_j$. Since M is locally free and hence flat, the inclusions $R \hookrightarrow R_U \hookrightarrow \mathcal{K}'$ and $R \hookrightarrow R' := \prod_j R_j \hookrightarrow \mathcal{K}'$ induce inclusions $M \hookrightarrow M_U \hookrightarrow M'_U$ and $M \hookrightarrow M' := \prod_j M_j = M \otimes_R R' \hookrightarrow M'_U$. Similarly, the exact sequence $0 \to R \xrightarrow{\Delta} R' \times R_U \xrightarrow{\delta} \mathcal{K}'$ induces an exact sequence

$$0 \to M \xrightarrow{\Delta} M' \times M_U \xrightarrow{\delta} M'_U, \tag{1}$$

where Δ is the diagonal map and $\delta(a', a_U) = a' - a_U$.

To show the surjectivity of f_U, let $\bar{x}' \in M'_U/\wp(M'_U)$, and take $x' = (x'_1, \ldots, x'_r) \in M'_U = \prod_{j=1}^r M_{U,j}$ lying over $\bar{x}'$. Thus $x'_j = \sum_i r_{i,j} b_i$, where $r_{i,j} \in \mathcal{K}_{\xi_j}$. For each $i = 1, \ldots, n$, the Strong Approximation Theorem [FJ, Prop. 2.11] implies that there is an element $r_i \in K$ such that $v_{\xi_j}(r_i - r_{i,j}) = 1$ for $1 \le j \le r$ and such that $v_{\mathfrak{p}}(r_i) \ge 0$ for every place $\mathfrak{p} \in \bar{X} - S'$. Thus $r_i \in R_U$ and so the element $x := \sum_i r_i b_i$ lies in M_U. Let I_j be the maximal ideal of R_j. Then $r_i - r_{i,j} \in I_j \subset R_j$ for each i, j, since $v_{\xi_j}(r_i - r_{i,j}) = 1$. So $x - x'_j = \sum_{i=1}^n (r_i - r_{i,j}) b_i \in I_j M_j \subset \wp(M_j) \subset \wp(M_{U,j})$ for each j, by Lemma 5.1 (regarding M_j, M_U as subsets of $M_{U,j}$). Thus $x - x' \in \wp(M'_U)$ (regarding $M_U \subset M'_U$). So $\bar{x}'$ is the image of $\bar{x}$ under $M_U/\wp(M_U) \to M'_U/\wp(M'_U)$, where $\bar{x}$ is the class of x modulo $\wp(M_U)$. This proves the desired surjectivity.

Finally, we show that the kernel of $f_U : M_U/\wp(M_U) \to M'_U/\wp(M'_U)$ is infinite. Let

f, f_U, g, g' be the maps induced by inclusions, in the following commutative diagram:

$$\begin{array}{ccc} M/\wp(M) & \xrightarrow{f} & M'/\wp(M') \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ M_U/\wp(M_U) & \xrightarrow{f_U} & M'_U/\wp(M'_U) \ . \end{array}$$

It suffices to show that $\ker f$ contains an infinite dimensional $\mathbb{Z}/p\mathbb{Z}$-subspace N and that the restriction $g|_N$ has finite kernel — for then, $g(N)$ is an infinite subset of $\ker(f_U)$. Now IM maps into I_jM_j under $M \to M_j$, where $I \subset R$ is the ideal corresponding to the closed subset $X - U$. But $I_jM_j \subset \wp(M_j)$ by Lemma 5.1. This is true for all j, so $N := IM/(IM \cap \wp(M))$ is contained in $\ker f$. This set N is infinite, by Lemma 3.6. It remains to show that $\ker(g|_N)$ is finite.

Viewing $M', M_U \subset M'_U$, and using that $\wp(M') \subset M'$ (by Lemma 5.2(iii)) and $\wp(M_U) \subset M_U$, we have that $\wp(M') \cap \wp(M_U) \subset M' \cap M_U = M$, where the intersection takes place in S'^n_U. By Lemma 5.1, $\ker(g|_N) = (IM \cap \wp(M_U))/(IM \cap \wp(M))$ is contained in the $\mathbb{Z}/p\mathbb{Z}$-vector space $V := (\wp(M') \cap \wp(M_U))/\wp(M) \subset M/\wp(M)$. So it suffices to show that V is finite. Consider the $\mathbb{Z}/p\mathbb{Z}$-vector space

$$\tilde{W} = \{(m', m_U) \in M' \times M_U \,|\, \wp(m') = \wp(m_U) \in S'^n_U\}.$$

There is a surjective $\mathbb{Z}/p\mathbb{Z}$-vector space homomorphism $\tilde{\rho} : \tilde{W} \to V$, given by $\tilde{\rho}(m', m_U) = \wp(m') = \wp(m_U)$ modulo $\wp(M)$. Now if $(m', m_U) \in \tilde{W}$ then $m' \in M' \subset S'^n \subset S'^n_U$ and $m_U \in M^n_U \subset S^n_U \subset S'^n_U$; so $m' - m_U \in Q := \ker(\wp : S'^n_U \to S'^n_U)$. That is, $\delta(W) \subset Q$, where $\delta : M' \times M_U \to M_{U'}$ is as in exact sequence (1) above. Now $S'_U = S_U \otimes_{R_U} \mathcal{K}'$ is finite étale over $\mathcal{K}'$, and thus is a direct sum of finitely many fields of characteristic p. Hence so is S'^n_U. Since $\mathbb{Z}/p\mathbb{Z}$ is the kernel of $\wp$ on any such field, it follows that Q is a finite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space. Moreover by (1), δ induces an injection $\bar{\delta} : (M' \times M_U)/M \to M_{U'}$, where M is included as the diagonal. Thus if we let $W = \tilde{W}/M$ (again, with M as the diagonal), then the restriction of $\bar{\delta}$ to W is an injection $W \to Q$. Thus W is finite. But $\tilde{\rho} : \tilde{W} \to V$ factors through $\tilde{W} \to W$; and the corresponding map $\rho : W \to V$ is surjective since $\tilde{\rho}$ is. It follows that $V = \rho(W)$ is also finite, as desired. □

Remark 5.4. (a) A weak version of the above result holds even if X is projective, provided that the base field k is separably closed. Namely, in this situation, ϕ is p-dominating (rather than strongly p-dominating). Arguing as in [Ka, 2.2.1], this can be shown using the Cartan-Leray spectral sequence $H^p(X, R^q\iota_*\mathcal{F}) \Rightarrow H^{p+q}(U, \mathcal{F})$ [Mi, III Theorem 1.18(a)], where $\iota : U \to X$ is the inclusion and $\mathcal{F}$ is the p-torsion sheaf on U associated to a Π-module P

as above. Namely, identifying $H^1(\mathcal{K}_{\xi_j}, \mathcal{F})$ with $H^1(\mathcal{K}^{\mathrm{h}}_{\xi_j}, \mathcal{F})$ (where $\mathcal{K}^{\mathrm{h}}_{\xi_j}$ is the fraction field of the henselization $\mathcal{O}^{\mathrm{h}}_{\xi_j}$ of $\mathcal{O}_{X,\xi_j}$), the associated exact sequence of low degree terms gives

$$0 \to H^1(X, \iota_*\mathcal{F}) \to H^1(U, \mathcal{F}) \to \prod_j H^1(\mathcal{K}_{\xi_j}, \mathcal{F}) \to H^2(X, \iota_*\mathcal{F})$$

(since $R^1\iota_*\mathcal{F}$ is supported on $X - U$). But $H^2(X, \iota_*\mathcal{F}) = 0$ by Cor. 3.3(b). So the map $H^1(U, \mathcal{F}) \to \prod_j H^1(\mathcal{K}_{\xi_j}, \mathcal{F})$, or equivalently $\phi^* : H^1(\Pi, P) \to \prod_j H^1(\Pi_j, P)$, is surjective.

(b) The argument in (a) above also works in the affine case, using Cor. 3.3(c) instead of Cor. 3.3(b), even over a non-separably closed field, provided that one uses the strict henselization rather than the henselization. This provides a weaker conclusion than 5.3, however. □

Using the above proposition, we obtain the following result, which asserts the existence of proper solutions to p-embedding problems for curves with prescribed behavior over finitely many local fields (rather than over closed subsets, as in Sections 3 and 4).

Corollary 5.5. *Let X, Π, and ϕ be as in Proposition 5.3. Then every finite p-embedding problem for Π is properly ϕ-solvable.*

Proof. The proof is identical to that of Corollary 3.10, except that Proposition 5.3 is cited rather than Proposition 3.8. □

In terms of covers, the above yields:

Theorem 5.6. *Let X be a connected normal affine curve of finite type over a field of characteristic p, let $r \geq 0$, and let $\xi_1, \ldots, \xi_r$ be closed points of X. Let P be a normal p-subgroup of a finite group Γ; let $G = \Gamma/P$; and let $Y \to X$ be a connected normal G-Galois cover which is étale away from $\xi_1, \ldots, \xi_r$. For each j let A_j be a Γ-Galois $\mathcal{K}_{\xi_j}$-algebra, together with an isomorphism $(\operatorname{Spec} A_j)/P \xrightarrow{\sim} Y \times_X \operatorname{Spec} \mathcal{K}_{\xi_j}$ of G-Galois covers. Then there is a connected normal Γ-Galois cover $Z \to X$ which is étale away from $\xi_1, \ldots, \xi_r$, together with compatible isomorphisms $Z/P \xrightarrow{\sim} Y$ as G-Galois covers and $Z \times_X \operatorname{Spec} \mathcal{K}_{\xi_j} \xrightarrow{\sim} \operatorname{Spec} A_j$ as Γ-Galois covers.*

Proof. Let $U = X - \{\xi_1, \ldots, \xi_r\}$ and let $V \to U$ be the restriction of $Y \to X$ over U. Following the proof of Theorem 3.11, let $f : \Gamma \to G$ be the quotient map, let $\alpha : \pi_1(U) \to G$ be a surjection corresponding to the connected G-Galois étale cover $V \to U$, and let ϕ be as in the statement of Proposition 5.3. By Corollary 5.5, the finite p-embedding problem $\mathcal{E} = (\alpha : \pi_1(U) \to G, f : \Gamma \to G)$ is properly ϕ-solvable. So by the implication (ii) $\Rightarrow$ (i) of Proposition 3.1, there is a connected Γ-Galois étale cover $W \to U$ that dominates $V \to U$ and pulls back to each $Z_j \to X_j$ up to isomorphism. Let Z be the normalization of X in W. Then $Z \to X$ is as desired. □

Remark 5.7. (a) In the proof of Corollary 5.5, if one replaces Proposition 5.3 by Remark 5.4(a), then one obtains a proof that if X is a projective curve over a separably closed base field k, then every embedding problem for Π is weakly ϕ-solvable. This in turn implies that a weak form of Theorem 5.6 holds in this case (viz. that the asserted Γ-Galois cover $Z \to X$ exists but need not be connected). This result is essentially [Ka, Theorem 2.1.5]. And in fact, the cover Z cannot in general be chosen to be connected — e.g. if the local extensions A_j are taken to be trivial, and X is the projective line, then Z just consists of disjoint copies of X.

(b) The above remark (a) no longer holds if the base field k is allowed to be arbitrary (rather than separably closed). For example, let $k \subset k'$ be a separable field extension of degree p; let X be the projective k-line; let ξ_1 be the point $x = \infty$ on X and let ξ_2 be the point $x = 0$ on X. Also, let $\Gamma = P$ be cyclic of order p; let G be the trivial group; and let $Y \to X$ be the trivial cover. Let A_1 be the trivial Γ-Galois $\mathcal{K}_{\xi_1}$-algebra $(\mathcal{K}_{\xi_1})^p = \mathrm{Ind}_1^P \mathcal{K}_{\xi_1}$, and let A_2 be the non-trivial Γ-Galois algebra $k'((x))$ over $\mathcal{K}_{\xi_2} = k((x))$. Then in the context of Theorem 5.6, the desired Γ-Galois cover $Z \to X$ does not exist (even if it is permitted to be disconnected), since it would have to be unramified everywhere, hence be purely arithmetic — contradicting the fact that the residue fields k', k over $x = 0, \infty$ would be distinct. $\square$

Using the above theorem, together with the results below, we will obtain (in Theorem 5.14) a strengthening of the results of Sections 3 and 4 in the case that the base space is a curve.

Lemma 5.8. *Let Γ be a finite group with an abelian normal subgroup A, and quotient map $f : \Gamma \twoheadrightarrow G := \Gamma/A$. Suppose that $G = C \rtimes E$, where A and C have relatively prime orders, and suppose also that the exact sequence $1 \to A \to f^{-1}(E) \to E \to 1$ splits. Then the exact sequence $1 \to A \to \Gamma \to G \to 1$ splits.*

Proof. This is equivalent to a theorem of Gaschütz [Hu, I §12, Hauptsatz 17.4(a)]: If A is an abelian normal subgroup of Γ, $A \subset B \subset \Gamma$, and the order of A is relatively prime to the index $(\Gamma : B)$, and if A has a complement in B, then A has a complement in Γ. Namely, in the statement of the lemma, we can take $B = f^{-1}(E)$, whose index in Γ is equal to the order of C. $\square$

If $f : \Gamma \to G$ and $f' : \Gamma' \to G$ are group homomorphisms, then we may form the fibre product of groups, namely $\Gamma \times_G \Gamma' := \{(g, g') \in \Gamma \times \Gamma' \mid f(g) = f'(g')\}$. If f is surjective with kernel N, then the exact sequence $1 \to N \to \Gamma \to G \to 1$ induces an exact sequence $1 \to N \to \Gamma \times_G \Gamma' \to \Gamma' \to 1$. Here the map $\Gamma \times_G \Gamma' \to \Gamma'$ is the second projection, and the map $N \to \Gamma \times_G \Gamma'$ is given by $n \mapsto (n, 1)$.

Lemma 5.9. *Let R be a complete discrete valuation ring of characteristic p, with fraction field K and residue field k. Let G be a finite group, let L be an G-Galois field extension* *of K, and let S be the integral closure of R in L. Suppose that the extension $R \subset S$ is tamely ramified. Let $f : \Gamma \twoheadrightarrow G$ be a surjective homomorphism whose kernel is a finite abelian p-group P.*

Then there is a finite Galois field extension $K \subset K'$ whose corresponding extension of complete d.v.r.'s is étale, such that the induced exact sequence

$$1 \to P \to \Gamma \times_G \mathrm{Gal}(K'L/K) \to \mathrm{Gal}(K'L/K) \to 1 \tag{2}$$

is split (where the compositum $K'L$ is taken in a fixed separable closure of K).

Proof. Let $R \subset R_1$ be the maximal unramified subextension of $R \subset S$, and let K_1 be the fraction field of R_1. Thus the extension $R_1 \subset S$ is totally ramified, and its Galois group C is cyclic of order n prime to p. Let E_1 be the Galois group of R_1 over R. Thus we have an exact sequence

$$1 \to C \to G \to E_1 \to 1.$$

Also, the ring R is isomorphic to $k[[x]]$ by [Ma, Cor. 2 to Theorem 60], because R is a complete regular local ring of dimension 1 containing a field (viz. $\mathbb{F}_p$). Thus R_1 is isomorphic as an R-algebra to $k_1[[x]]$, where k_1 is an E_1-Galois field extension of k.

Let $R_2 = R_1[\zeta_n] = k_1[\zeta_n][[x]]$, where ζ_n is a primitive nth root of unity in the separable closure of K, and where n is the order of C (which is not divisible by p). By Kummer theory, the compositum $R_2S = S[\zeta_n]$ is given as an R_2-algebra by $R_2[z]/(z^n - ux)$, for some unit $u \in R_2$ (where this is the full ring of integers in its fraction field because z is a uniformizer). Let $R_3 = R_2[\sqrt[n]{u}]$. Then $R_3S \approx R_3[y]/(y^n - x)$. Also, since p does not divide n, R_3 is étale over R_1, and hence over R. Moreover R_3S is Galois over S, and the natural surjection $\mathrm{Gal}(R_3S/R) \to \mathrm{Gal}(R_3/R)$ maps $\mathrm{Gal}(R_3S/R[y])$ isomorphically onto $\mathrm{Gal}(R_3/R)$. (Here we regard y as an element of R_3S.) So the short exact sequence

$$1 \to \mathrm{Gal}(R_3S/R_3) \to \mathrm{Gal}(R_3S/R) \to \mathrm{Gal}(R_3/R) \to 1$$

is split. There is a natural isomorphism $\mathrm{Gal}(R_3S/R_3) = \mathrm{Gal}(S/R_1) = C$, and so we may identify $\mathrm{Gal}(R_3S/R)$ with a semidirect product $C \rtimes E_3$, where $E_3 = \mathrm{Gal}(R_3/R)$. The natural surjection $\mathrm{Gal}(R_3S/R) \to \mathrm{Gal}(S/R)$ may thus be identified with a map $C \rtimes E_3 \twoheadrightarrow G$.

The surjections $f : \Gamma \to G$ (with kernel P) and $C \rtimes E_3 \twoheadrightarrow G$ yield an exact sequence

$$1 \to P \to \Gamma \times_G \left(C \rtimes E_3\right) \xrightarrow{g} C \rtimes E_3 \to 1.$$

This restricts to an exact sequence

$$1 \to P \to g^{-1}(E_3) \xrightarrow{g} E_3 \to 1,$$

regarding E_3 as a subgroup of $C \rtimes E_3$. By Corollary 3.3(c), $\mathrm{cd}_p(\pi_1(\mathrm{Spec}\, R)) \le 1$, and so the surjection $\pi_1(\mathrm{Spec}\, R) \twoheadrightarrow E_3$ corresponding to the E_3-Galois étale cover $\mathrm{Spec}\, R_3 \to$

$\operatorname{Spec} R$ must lift to a homomorphism $\pi_1(\operatorname{Spec} R) \to g^{-1}(E_3)$. Since $g^{-1}(E_3)$ is finite, this homomorphism factors through a finite quotient E' of $\pi_1(\operatorname{Spec} R)$. Thus we have a surjection $\alpha : E' \twoheadrightarrow E_3$ which lifts to a map $\beta : E' \to g^{-1}(E_3)$, and the quotient map $\pi_1(\operatorname{Spec} R) \twoheadrightarrow E'$ corresponds to an E'-Galois étale extension R' of R which contains R_3. The surjections $g : g^{-1}(E_3) \twoheadrightarrow E_3$ and $\alpha : E' \twoheadrightarrow E_3$ yield an exact sequence

$$1 \to P \to g^{-1}(E_3) \times_{E_3} E' \to E' \to 1 \tag{3}$$

which has a splitting $(\beta, \mathrm{id}) : E' \to g^{-1}(E_3) \times_{E_3} E'$.

Now R' is étale over R_1 whereas S is totally ramified over R_1. So $R'S$ is a totally ramified C-Galois extension of R'. As in the case of R_3, the Galois group $\operatorname{Gal}(R'S/R)$ may be identified with a semidirect product $C \rtimes E'$, and the natural surjection $\operatorname{Gal}(R'S/R) \to \operatorname{Gal}(S/R)$ may be identified with a map $C \rtimes E' \twoheadrightarrow G$. The surjections $f : \Gamma \to G$ and $C \rtimes E' \twoheadrightarrow G$ then yield an exact sequence

$$1 \to P \to \Gamma \times_G \left(C \rtimes E'\right) \xrightarrow{g'} C \rtimes E' \to 1. \tag{4}$$

Identifying $g'^{-1}(E')$ with $g^{-1}(E_3) \times_{E_3} E'$, the sequence (4) restricts to the split exact sequence (3). So by Lemma 5.8 (with $A = P$), it follows that the sequence (4) splits. Writing $K' = \operatorname{frac}(R')$ and $L = \operatorname{frac}(S)$, we have that $\operatorname{Gal}(K'L/K) = \operatorname{Gal}(R'S/R) = C \rtimes E'$. So the desired conclusion follows. $\square$

Recall (from the beginning of Section 4) that if $Y \to X$ is a Galois cover, then we may consider an associated fundamental group $\pi_1(Y/X)$.

Proposition 5.10. *Let $R \subset S$ be a tamely ramified Galois extension of complete discrete valuation rings of characteristic p. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. Then* $\operatorname{cd}_p(\pi_1(Y/X)) \le 1$.

Proof. Let $\Pi = \pi_1(Y/X)$. The condition $\operatorname{cd}_p(\Pi) \le 1$ is equivalent to the condition that every finite p-embedding problem for $\pi_1(Y/X)$ has a weak solution, by [Se1, I, 3.4, Proposition 16]. By Lemma 2.1, it suffices to restrict attention to finite embedding problems for Π whose kernels are elementary abelian p-groups.

So let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be such an embedding problem, and let $P = \ker f$. Then the surjection α corresponds to a pointed connected G-Galois cover $Z \to X$ which factors as $Z \to Z_0 \to X$, where $Z \to Z_0$ is étale and where $Z_0 \to X$ is a Galois subcover of the tamely ramified cover $Y \to X$. Thus $Z \to X$ is tamely ramified.

Let K, L be the fraction fields of R, S respectively (regarded as subfields of a fixed separable closure of K). By Lemma 5.9 there is a finite Galois field extension $K \subset K'$ such that R' is étale over R, where R' is the integral closure of R in K'; and where the induced exact sequence (2) of 5.9 is split. Let $X' = \operatorname{Spec} R'$. We may give X' the structure of a pointed Galois étale cover of X; then $X' \times_X Z$ is a pointed étale cover of Z. Let Z' be the component of $X' \times_X Z$ containing the base point. Then Z' is étale over Z and Galois over X, with Galois group $G' := \mathrm{Gal}(K'L/K)$. Thus there is a surjection $\alpha' : \Pi \twoheadrightarrow \mathrm{Gal}(K'L/K)$ that induces the given map $\alpha : \Pi \twoheadrightarrow G = \mathrm{Gal}(L/K)$. We then obtain a commutative diagram

$$\begin{tikzcd}
 & & & \Pi \arrow[d, "\alpha'"'] & \\
1 \arrow[r] & P \arrow[r] \arrow[d, "\mathrm{id}"'] & \Gamma \times_G G' \arrow[r, "f'"] \arrow[d, "\lambda'"'] & G' \arrow[r] \arrow[d, "\lambda"'] & 1 \\
1 \arrow[r] & P \arrow[r] & \Gamma \arrow[r, "f"] & G \arrow[r] & 1
\end{tikzcd}$$

with exact rows, where $\lambda : G' \twoheadrightarrow G$ is the natural quotient map; where $\lambda' : \Gamma \times_G G' \twoheadrightarrow \Gamma$ is the first projection and where $\lambda\alpha' = \alpha : \Pi \twoheadrightarrow G$.

Now the upper row is split, so there is a map $\beta' : \Pi \to \Gamma \times_G G'$ such that $f'\beta' = \alpha'$. Thus $\beta := \lambda'\beta' : \Pi \to \Gamma$ satisfies $f\beta = \alpha : \Pi \to G$; i.e. β is a weak solution to the given embedding problem $\mathcal{E}$. □

If X is a regular connected pointed curve, and if $\Sigma \subset X$ is a proper closed subset not containing the base point, then define the *tame fundamental group* $\pi_1^{\mathrm{t}}(X, \Sigma)$ to be the inverse limit of the Galois groups of pointed Galois covers $Y \to X$ with Y regular, tamely ramified over Σ, and étale elsewhere. Thus if X is projective, then this is the same as $\pi_1^{\mathrm{t}}(U)$, in the notation of [Gr], where $U = X - \Sigma$.

Corollary 5.11. *Let R be a complete discrete valuation ring of characteristic p. Let $X = \operatorname{Spec} R$ and let ξ be the closed point of X. Then* $\mathrm{cd}_p\big(\pi_1^{\mathrm{t}}(X, \{\xi\})\big) \le 1$.

Proof. We wish to show that every finite p-embedding problem

$$\mathcal{E}^{\mathrm{t}} = (\alpha^{\mathrm{t}} : \pi_1^{\mathrm{t}}(X, \{\xi\}) \ \to G,\ f : \Gamma \to G)$$

for $\pi_1^{\mathrm{t}}(X, \{\xi\})$ has a weak solution. For such an embedding problem, the surjection α^{t} corresponds to a regular connected G-Galois cover $Y \to X$ that is tamely ramified over ξ. There is a canonical map $\alpha : \pi_1(Y/X) \to G$, corresponding to the cover Y. Since $Y \to X$ is tamely ramified over ξ, the group $\pi_1(Y/X)$ is a quotient of $\pi_1^{\mathrm{t}}(X, \{\xi\})$, say via a map $q : \pi_1^{\mathrm{t}}(X, \{\xi\}) \twoheadrightarrow \pi_1(Y/X)$. Moreover the homomorphism α^{t} factors as $\alpha^{\mathrm{t}} = \alpha q$. By Proposition 5.10, the p-embedding problem $\mathcal{E} := (\alpha : \pi_1(Y/X) \ \to G, f : \Gamma \to G)$ has a weak solution $\beta : \pi_1(Y/X) \to \Gamma$. Thus $\beta q : \pi_1^{\mathrm{t}}(X, \{\xi\}) \to \Gamma$ is a weak solution to $\mathcal{E}^{\mathrm{t}}$. □

As a result, we obtain the following variant of Theorem 5.6:

Proposition 5.12. *Let X be a connected normal affine curve of finite type over a field k of characteristic p, let $r, s \geq 0$, and let $\xi_1, \ldots, \xi_r, \zeta_1, \ldots, \zeta_s$ be distinct closed points of X. Let P be a normal p-subgroup of a finite group Γ; let $G = \Gamma/P$; and let $Y \to X$ be a connected normal G-Galois cover which is tamely ramified over $\zeta_1, \ldots, \zeta_s$ and étale away from $\xi_1, \ldots, \xi_r, \zeta_1, \ldots, \zeta_s$. For each j let A_j be a Γ-Galois $\mathcal{K}_{\xi_j}$-algebra, together with an isomorphism $(\operatorname{Spec} A_j)/P \xrightarrow{\sim} Y \times_X \operatorname{Spec} \mathcal{K}_{\xi_j}$ of G-Galois covers.*

Then there is a connected normal Γ-Galois cover $Z \to X$ which is tamely ramified over $\zeta_1, \ldots, \zeta_s$ and étale away from $\xi_1, \ldots, \xi_r, \zeta_1, \ldots, \zeta_s$, together with compatible isomorphisms $Z/P \xrightarrow{\sim} Y$ as G-Galois covers and $Z \times_X \operatorname{Spec} \mathcal{K}_{\xi_j} \xrightarrow{\sim} \operatorname{Spec} A_j$ as Γ-Galois covers.

Proof. For each $i = 1, \ldots, s$, the pullback $Y_i := Y \times_X \operatorname{Spec} \hat{\mathcal{O}}_{X,\zeta_i} \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\zeta_i}$ is a tamely ramified G-Galois cover of regular curves. By Corollary 5.11, there is a normal tamely ramified Γ-Galois cover $\bar{Z}_i \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\zeta_i}$ that dominates the $G = \Gamma/P$-Galois cover $Y_i \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\zeta_i}$. Let $Z_i = \operatorname{Spec} B_i \to \operatorname{Spec} \mathcal{K}_{\zeta_i}$ be the generic fibre of $\bar{Z}_i \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\zeta_i}$. Thus B_i is a Γ-Galois $\mathcal{K}_{\zeta_i}$-algebra, and $Z_i/P \approx Y \times_X \operatorname{Spec} \mathcal{K}_{\zeta_i}$ as G-Galois algebras. By Theorem 5.6, applied to the set $\Sigma := \{\xi_1, \ldots, \xi_r, \zeta_1, \ldots, \zeta_s\} \subset X$, we obtain a connected normal Γ-Galois cover $Z \to X$ which is étale away from Σ, together with compatible isomorphisms $Z/P \xrightarrow{\sim} Y$ as G-Galois covers, and $Z \times_X \operatorname{Spec} \mathcal{K}_{\xi_j} \xrightarrow{\sim} \operatorname{Spec} A_j$ (for $j = 1, \ldots, r$) and $Z \times_X \operatorname{Spec} \mathcal{K}_{\zeta_i} \xrightarrow{\sim} \operatorname{Spec} B_j$ (for $i = 1, \ldots, s$) as Γ-Galois covers. Since $\bar{Z}_i \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\zeta_i}$ is tamely ramified, $Z \to X$ is as desired. □

Remark 5.13. If k is separably closed, then the assertion of Proposition 5.12 remains true even if X is projective, provided that Z is no longer required to be connected. This follows by replacing Theorem 5.6 by Remark 5.7(a), in the above proof. This variant of 5.12 can be regarded as a generalization of [Ka, Theorem 2.1.6] to the case of more than two branch points (but stated just for one group at a time, rather than for π_1). □

Combining the above proposition with Theorem 3.11, we obtain the following theorem, referred to at the end of Section 4. It implies and subsumes Theorems 3.11 and 4.3 in the case of normal curves X. Namely, those results respectively assume that the given G-Galois cover $Y \to X$ is either étale or is of degree prime-to-p. The result below for curves, though, merely assumes that $Y \to X$ is tamely ramified. Note that the data over the points ξ_j is non-trivial only in the case that the base field k is *not* algebraically closed, which is thus the case of main interest.

Theorem 5.14. *Let X be a connected normal affine curve of finite type over a field k of characteristic p, let $r, s \geq 0$, and let $\xi_1, \ldots, \xi_r, \zeta_1, \ldots, \zeta_s$ be distinct closed points of X. Let P be a p-subgroup of a finite group Γ; let $G = \Gamma/P$; and let $Y \to X$ be a connected normal G-Galois cover which is tamely ramified over $\zeta_1, \ldots, \zeta_s$ and étale elsewhere.*

Let $Z' \to X' := \{\xi_1, \dots, \xi_r\}$ be a Γ-Galois étale cover together with an isomorphism $Z'/P \xrightarrow{\sim} Y \times_X X'$ of G-Galois covers. Then there is a connected normal P-Galois étale cover $Z \to Y$ such that the composition $Z \to X$ is a tamely ramified Γ-Galois cover $Z \to X$ which is étale away from $\zeta_1, \dots, \zeta_s$, and such that $Z \times_X X' \approx Z'$ as Γ-Galois covers.

Proof. Let $U = X - \{\zeta_1, \dots, \zeta_s\}$ and let $V \to U$ be the pullback of $Y \to X$ over U. Applying Theorem 3.11 to $V \to U$ and to the cover $Z' \to X' = \{\xi_1, \dots, \xi_r\}$, we obtain a connected normal Γ-Galois étale cover $W \to U$ that dominates $V \to U$ and whose restriction to X' is $Z' \to X'$. Thus for each j the fibre over $X_j := \{\xi_j\}$ is $Z_j := Z' \times_{X'} X_j \to X_j$. The pullback of $W \to U$ over the complete local ring at ξ_j is a Γ-Galois étale cover of the form $\operatorname{Spec} \bar{A}_j \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$, where $\bar{A}_j$ is a finite product of complete discrete valuation rings. Here the closed fibre of $\operatorname{Spec} \bar{A}_j \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$ is isomorphic to $Z_j \to X_j$ as a Γ-Galois cover, and there is a compatible isomorphism of G-Galois covers of $\operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$ between $(\operatorname{Spec} \bar{A}_j)/P$ and the pullback of Y.

The generic fibre of $\operatorname{Spec} \bar{A}_j \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$ is of the form $\operatorname{Spec} A_j \to \operatorname{Spec} \mathcal{K}_{\xi_j}$, and there is an isomorphism $(\operatorname{Spec} A_j)/P \xrightarrow{\sim} Y \times_X \operatorname{Spec} \mathcal{K}_{\xi_j}$ of G-Galois covers. By Proposition 5.12, we obtain a connected normal Γ-Galois cover $Z \to X$ which is tamely ramified over $\zeta_1, \dots, \zeta_s$ and étale away from $\xi_1, \dots, \xi_r, \zeta_1, \dots, \zeta_s$, and which compatibly induces the G-Galois cover $Y \to X$ modulo P and induces the Γ-Galois covers $\operatorname{Spec} A_j \to \operatorname{Spec} \mathcal{K}_{\xi_j}$ via pullback. The pullback of $Z \to X$ over $\operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$ thus has the same generic fibre $\operatorname{Spec} A_j \to \operatorname{Spec} \mathcal{K}_{\xi_j}$ as the étale cover $\operatorname{Spec} \bar{A}_j \to \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$. Since Z is normal and is finite over X, it follows that these two G-Galois covers of $\operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_j}$ agree. Hence the closed fibre of Z over $X_j = \{\xi_j\}$ agrees with $Z_j \to X_j$ as a Γ-Galois cover (and so $Z \times_X X'$ agrees with Z'), and $Z \to X$ is étale over X'. Thus $Z \to X$ is tamely ramified, and hence so is the intermediate P-Galois cover $Z \to Y$. Hence $Z \to Y$ is étale, since P is a p-group. So Z is as desired. □

Reinterpreting the above results in terms of embedding problems, we obtain:

Corollary 5.15. *Let X be a connected normal affine curve of finite type over a field k of characteristic p, let Σ be proper closed subset of X, let $r \geq 0$, and let $\xi_1, \dots, \xi_r$ be distinct closed points of $X - \Sigma$. Let $G_{k(\xi_j)}$ be the absolute Galois group of the residue field $k(\xi_j)$.*

(a) Let $\phi^{\mathrm{t}} = \{\phi^{\mathrm{t}}_j\}_j$, where $\phi^{\mathrm{t}}_j : G_{k(\xi_j)} \to \pi_1^{\mathrm{t}}(X, \Sigma)$ corresponds to the inclusion $\{\xi_j\} \hookrightarrow X - \Sigma$. Then every finite p-embedding problem for $\pi_1^{\mathrm{t}}(X, \Sigma)$ is properly ϕ^{t}-solvable.

(b) Let $Y \to X$ be a connected normal Galois cover that is tamely ramified over Σ and étale elsewhere. Let $\phi_Y = \{\phi_{Y,j}\}_j$, where $\phi_{Y,j} : G_{k(\xi_j)} \to \pi_1(Y/X)$ corresponds to the inclusion $\{\xi_j\} \hookrightarrow X - \Sigma$. Then every finite p-embedding problem for $\pi_1(Y/X)$ is properly ϕ_Y-solvable.

Proof. (a) Let $\mathcal{E} = (\alpha : \pi_1^{\mathrm{t}}(X, \Sigma) \to G, f : \Gamma \to G)$ be a finite p-embedding problem for $\pi_1^{\mathrm{t}}(X, \Sigma)$. Then the surjection α yields a connected normal G-Galois cover of X which is tamely ramified over Σ and étale elsewhere. A weak solution to the induced embedding problem $\phi^{\mathrm{t}*}(\mathcal{E})$ yields Γ-Galois étale covers $Z_j \to X_j := \{\xi_j\}$ that dominate the pullbacks $Y_j \to X_j$ of $Y \to X$. By Theorem 5.14 there is a connected normal Γ-Galois cover $Z \to X$ that is tamely ramified over Σ and étale elsewhere; that dominates $Y \to X$; and that restricts to each $Z_j \to X_j$. Just as in the remarks prior to Proposition 3.1, such a cover corresponds to a proper solution to $\mathcal{E}$ whose compositions with the component maps $\phi_j^{\mathrm{t}} : G_{k(\xi_j)} \to \pi_1^{\mathrm{t}}(X, \Sigma)$ are conjugate to proper solutions of the pullbacks $\phi_j^{\mathrm{t}*}(\mathcal{E})$. So $\mathcal{E}$ is properly ϕ^{t}-solvable.

(b) Let $\mathcal{E} = (\alpha : \pi_1(Y/X) \to G, f : \Gamma \to G)$ be a finite p-embedding problem for $\pi_1(Y/X)$. Consider a weak solution to $\phi_Y^*(\mathcal{E})$, corresponding to Γ-Galois étale covers $Z_j \to X_j := \{\xi_j\}$ that dominate the pullbacks $Y_j \to X_j$ of $Y \to X$. As in the proof of Proposition 5.10, the surjection α corresponds to a pointed connected normal G-Galois cover $Z \to X$ which factors as $Z \to Z_0 \to X$, where $Z \to Z_0$ is étale and where $Z_0 \to X$ is a Galois subcover of $Y \to X$. Thus $Z \to X$ is tamely ramified over Σ and étale elsewhere; and so it corresponds to a surjection $\alpha^{\mathrm{t}} : \pi_1^{\mathrm{t}}(X, \Sigma) \to G$ (factoring through α). By (a), there is a proper solution to the p-embedding problem $\mathcal{E}^{\mathrm{t}} := (\alpha^{\mathrm{t}} : \pi_1^{\mathrm{t}}(X, \Sigma) \to G, f : \Gamma \to G)$ which up to conjugacy induces the given weak solution to $\phi_Y^*(\mathcal{E})$. Such a solution corresponds to a connected normal Γ-Galois cover $W \to X$ which dominates the G-Galois cover $Z \to X$, such that $W \to Z$ is étale, and which restricts to each $Z_j \to X_j$. Thus the Γ-Galois cover $W \to X$ factors as $W \to Z_0 \to X$, where $W \to Z_0$ is étale (since $W \to Z$ is at most tamely ramified, and is Galois of p-power degree). Hence W corresponds to a proper solution to the given p-embedding problem $\mathcal{E}$, inducing the given weak solution to $\phi_Y^*(\mathcal{E})$ up to conjugacy. So $\mathcal{E}$ is properly ϕ_Y-solvable. □

The following corollary provides a variant of Corollary 3.3(c) in the tame case, and a generalization of Proposition 4.1(a) to the case that the given cover $Y \to X$ need only be tame (rather than prime-to-p). The base space X, however, is assumed here to be a curve.

Corollary 5.16. *Let X be a connected normal affine curve of finite type over a field k of characteristic p, and let Σ be a proper closed subset of X.*

(a) Then $\mathrm{cd}_p(\pi_1^{\mathrm{t}}(X, \Sigma)) \le 1$.

(b) Let $Y \to X$ be a connected normal Galois cover that is tamely ramified over Σ and étale elsewhere. Then $\mathrm{cd}_p(\pi_1(Y/X)) \le 1$.

Proof. Taking $r = 0$ in Corollary 5.15 (so that ϕ^{t} [resp. ϕ_Y] is the empty collection), we obtain that every finite p-embedding problem for $\pi_1^{\mathrm{t}}(X, \Sigma)$ [resp. for $\pi_1(Y/X)$] is properly solvable, and hence weakly solvable. So the assertion that $\mathrm{cd}_p \le 1$ follows from [Se1, I, 3.4, Prop. 16]. □

Remark 5.17. (a) In Sections 3 and 4, it was first proven that $\pi_1(X)$ or $\pi_1(Y/X)$ (in the prime-to-p case) had $\mathrm{cd}_p \le 1$, and then that was used in showing that every finite

p-embedding problem was properly ϕ-solvable (in Theorems 3.11 and 4.3, and Corollaries 3.10 and 4.2). But in the present section, in the case of curves, it was not known *a priori* that the relevant $\mathrm{cd}_p \le 1$. Instead, the prior strategy was reversed above: first proving that every finite p-embedding problem for $\pi_1^{\mathrm{t}}(X)$ and $\pi_1(Y/X)$ (in the tamely ramified case) is properly ϕ-solvable (Theorem 5.14 and Corollary 5.15), and then deducing (in Corollary 5.16) that $\mathrm{cd}_p \le 1$.

(b) It would be interesting to know if Theorem 5.14, and Corollaries 5.15 and 5.16, have higher dimensional analogs (e.g. having hypotheses of tame ramification at points of codimension 1). Such analogs would strengthen the main results of Sections 3 and 4, which assumed either that there was no ramification, or that the given cover $Y \to X$ was of degree prime-to-p. $\square$

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