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Permanence criteria for semi-free profinite groups

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Dedicated to Moshe Jarden on the occasion of his 65th birthday

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Abstract We introduce the condition of a profinite group being semi-free, which is more general than being free and more restrictive than being quasi-free. In particular, every projective semi-free profinite group is free. We prove that the usual permanence properties of free groups carry over to semi-free groups. Using this, we conclude that if k is a separably closed field, then many field extensions of $k((x, y))$ have free absolute Galois groups.

Keywords Free profinite group · semi-free profinite group · absolute Galois groups

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1 Introduction and results

A central problem is Galois theory is to understand the absolute Galois groups of fields, and a key aspect is to find fields with free absolute Galois groups. For example, if C is an algebraically closed field, then $K = C(x)$ is such a field. This was proved for $C = \mathbb{C}$ by Douady; and in the general case by Pop [19] and the third author [9], with another proof later by Jarden and the second author [8]. The major conjecture in

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this context, Shafarevich's conjecture, asserts that the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of the rational numbers $\mathbb{Q}$ has a free absolute Galois group.

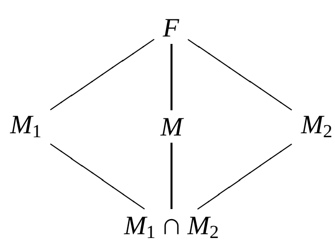
In [11], the third author and K. Stevenson suggest a strategy for proving the freeness of a profinite group: breaking the argument into two simpler pieces, viz. quasi-freeness and projectivity. This strategy was carried out in [10] in the context of a two-variable Laurent series field $K = k((x,y))$. For any base field k, the absolute Galois group $\mathrm{Gal}(K)$ is quasi-free [11], though it is not free since it is not projective. In [10] the third author proves that the commutator subgroup of a quasi-free group is quasi-free, and hence $\mathrm{Gal}(K^{\mathrm{ab}})$ is quasi-free. Now, if in addition k is separably closed, then $\mathrm{Gal}(K^{\mathrm{ab}})$ is also projective. Therefore $\mathrm{Gal}(K^{\mathrm{ab}})$ is free, for such k. This can be viewed as an analog of Shafarevich's conjecture.

In the above situation, it is key that the commutator subgroup of a quasi-free group is quasi-free. This leads to the question of when a closed subgroup of a quasi-free group is quasi-free, particularly in the case of projective subgroups. Since closed subgroups inherit projectivity, this question generalizes the corresponding classical question about free subgroups of a free profinite group. A partial answer is given in [23], where Ribes, Stevenson, and Zalesskii prove that an open subgroup of a quasi-free group is quasi-free.

The classical question — when is a closed subgroup of a free group itself free — has been dealt with in numerous papers, e.g. [5,13,15,16,18]. The second author has used twisted wreath products in [5] to attack this question. Not only does this approach reprove many of the previously known results, but it also proves the so-called 'Diamond Theorem' (see [4, Theorem 25.4.3]):

Theorem *Let F be a free profinite group of infinite rank m. Let M_1, M_2 be normal subgroups of F and let M be a subgroup of F such that $M_1 \cap M_2 \le M$ but $M_1 \not\le M$ and $M_2 \not\le M$. Then M is free of rank m.*

(The diagram



suggests the name Diamond Theorem.) Recently the first author proved this theorem for finite $m \ge 2$ [2].

It would thus be desirable to carry over this and other permanence properties of free profinite groups to the class of quasi-free profinite groups. However, our methods seem to work well only after a slight modification of the notion: We say that a profinite group of infinite rank m is *semi-free* if every nontrivial finite split embedding problem for it has *m independent* proper solutions. (See Section 2 below.)

The modified notion is in some ways more natural. First we have

a. free groups are semi-free (Theorem 3.6),

b. semi-free groups are quasi-free, but not vice-versa (Proposition 6.1), and
c. the absolute Galois group of $k((x,y))$ is semi-free (Theorem 7.1).

Moreover, we are able to prove the following theorem (where case VI corresponds to the Diamond Theorem above). Also, as Example 6.5 below shows, not all of these properties hold for the class of quasi-free groups.

Main Theorem *Let F be a semi-free profinite group of infinite rank m and let M be a closed subgroup of F. Then, in each of the following cases the group M is semi-free of rank m.*

I. $(F : M) < \infty$.
II. *$F/\hat{M}$ is finitely generated, where $\hat{M} = \bigcap_{\sigma \in F} M^\sigma$ is the normal core of M.*
III. $\mathrm{weight}(F/M) < m$ *(the definition of weight is recalled at Section 5.1.5).*
IV. *M is a proper subgroup of finite index of a closed normal subgroup of F.*
V. *M is normal in F, and F/M is abelian.*
VI. *There exist closed normal subgroups M_1, M_2 of F such that $M_1 \cap M_2 \leq M$ but $M_1 \nleq M$ and $M_2 \nleq M$.*
VII. *M contains a closed normal subgroup N of F such that F/N is pronilpotent and $(F : M)$ is divisible by at least two primes.*
VIII. *M is sparse in F (see Definition 5.1).*
IX. $(F : M) = \prod p^{\alpha(p)}$*, where $\alpha(p) < \infty$ for all p.*

The proof of Main Theorem is in Section 5.

This theorem gives rise to new constructions of fields having free absolute Galois groups; see Section 8. One of them generalizes the construction of fields with free absolute Galois groups discussed above in the second paragraph of the introduction. Another was provided by Jarden, using ideas of Pop.

We conclude the introduction with some ideas of the proof. The goal is to prove that M is semi-free, i.e. that an arbitrary finite split embedding problem $\mathcal{E}_1$ for M has many independent proper solutions. We know that M is a subgroup of a semi-free group F, so we wish to transfer the solvability problem to F. The first thing we do is to induce a split embedding problem $\mathcal{E}$ for F with the property that a weak solution of $\mathcal{E}$ induces a weak solution to $\mathcal{E}_1$ (see Proposition 4.6 for the exact definition of $\mathcal{E}$). The embedding problem $\mathcal{E}$ is constructed using a *twisted wreath product* (see Definition 4.1).

Now $\mathcal{E}$ has many independent proper solutions because F is semi-free. Each one of these proper solutions, say ψ, induces a solution ν of $\mathcal{E}_1$. (Here $\nu = \pi \circ \psi|_M$, where π is the Shapiro map; see Definition 3.2.) We encounter two difficulties: (1) ν is not necessarily a *proper* solution; (2) for two distinct proper solutions $\psi_1 \neq \psi_2$ of $\mathcal{E}$ we may get that $\nu_1 = \nu_2$.

We extract from [5] a condition under which ν remains a proper solution. This settles the first difficulty. To treat (2), we use that fact that in our situation, ψ_1, ψ_2 are not only distinct, but also independent. Hence the image of $\psi_1 \times \psi_2$ is also a wreath product (Lemma 4.4). This fact leads us to generalize the work in [5], and find a necessary conditions for any two independent proper solutions ψ_1, ψ_2 to induce independent proper solutions ν_1, ν_2, as needed for M to be semi-free. See Proposition 4.6 b.

Note that this strategy does not apply to the corresponding problem for quasi-free groups, where the distinct proper solutions for a split embedding problem need not be independent, and since the image of $\psi_1 \times \psi_2$ for distinct solutions ψ_1, ψ_2 of $\mathcal{E}$ need not be a twisted wreath product in the absence of independence. By avoiding this difficulty, our focus on semi-free groups permits us to show that many subgroups of semi-free groups are semi-free (and in particular quasi-free); and that if such a subgroup is also projective then it is free (see Theorem 3.6).

2 Independent subgroups and solutions of embedding problems

Definition 2.1 Let F be a profinite group.

a. Open subgroups $M_1, \dots, M_n$ of F are F**-independent** if

$$(F : \bigcap_{i=1}^{n} M_i) = \prod_{i=1}^{n} (F : M_i).$$

If $M_1, \dots, M_n$ are normal in F, this is equivalent to

$$F / \bigcap_{i=1}^{n} M_i \cong \prod_{i=1}^{n} F/M_i$$

b. A family $\mathcal{M}$ of open subgroups of F is F**-independent** if every finite subset of $\mathcal{M}$ is F-independent.

The notion of F-independence coincides with independence with respect to the Haar probability measure on F [4, Section 18.3]. There is also the following equivalent characterization of independence: Open subgroups $M_1, \dots, M_n$ are F-independent if and only if F acts transitively on $\prod_{i=1}^{n} F/M_i$. This criterion can be used to obtain alternative short proofs of parts c and d in Proposition 2.2 below.

A key example of independence occurs in the case of a Galois field extension L/K. If $F = \mathrm{Gal}(L/K)$ and $L_1, \dots, L_n$ are the fixed fields of $M_1, \dots, M_n$ in L, then by the Galois correspondence, $M_1, \dots, M_n$ are F-independent if and only if $L_1, \dots, L_n$ are linearly disjoint over K.

The following properties can be either proven directly or deduced from the corresponding properties of linear disjointness of fields:

Proposition 2.2 *Let $M_1, \dots, M_n$ be open subgroups of a profinite group F.*

a. $(F : \bigcap_{i=1}^{n} M_i) \le \prod_{i=1}^{n} (F : M_i)$.

b. Let $M_1 \le N_1 \le F$. Then M_1, M_2 are F-independent if and only if N_1, M_2 are F-independent and $M_1, N_1 \cap M_2$ are N_1-independent.

c. The subgroups $M_1, \dots, M_n$ are F-independent if and only if $M_1, \dots, M_{n-1}$ are F-independent and $\bigcap_{i=1}^{n-1} M_i$, M_n are F-independent.

d. Let $M_i \le N_i \le F$ for each $1 \le i \le n$. If $M_1, \dots, M_n$ are F-independent, then so are $N_1, \dots, N_n$.

e. Suppose $M_1 \triangleleft F$. Then M_1, M_2 are F-independent if and only if $F = M_1 M_2$.

Proof (a) This follows by induction from the case $n = 2$, which is standard.

(b) First assume M_1, M_2 are F-independent. Then, since $(N_1 \cap M_2 : M_1 \cap M_2) \le (N_1 : M_1)$ we have

$$
\begin{aligned}
(F : N_1 \cap M_2) &= \frac{(F : M_1 \cap M_2)}{(N_1 \cap M_2 : M_1 \cap M_2)} = \frac{(F : M_1)(F : M_2)}{(N_1 \cap M_2 : M_1 \cap M_2)} \\
&= \frac{(F : N_1)(N_1 : M_1)(F : M_2)}{(N_1 \cap M_2 : M_1 \cap M_2)} \ge (F : N_1)(F : M_2).
\end{aligned}
$$

Therefore equality holds by (a), and N_1, M_2 are F-independent. Similarly, since $(N_1 : N_1 \cap M_2) \le (F : M_2)$ we have

$$
\begin{aligned}
(N_1 : M_1 \cap (N_1 \cap M_2)) &= \frac{(F : M_1 \cap M_2)}{(F : N_1)} = \frac{(F : M_1)(F : M_2)}{(F : N_1)} \\
&\ge (N_1 : M_1)(N_1 : N_1 \cap M_2),
\end{aligned}
$$

so $M_1, N_1 \cap M_2$ are N_1-independent by (a). Conversely,

$$
\begin{aligned}
(F : M_1 \cap M_2) &= (F : N_1)(N_1 : M_1 \cap (N_1 \cap M_2)) = (F : M_1)(N_1 : N_1 \cap M_2) \\
&= (F : M_1)\frac{(F : N_1 \cap M_2)}{(F : N_1)} = (F : M_1)(F : M_2).
\end{aligned}
$$

(c) By part (a),

$$
(F : \bigcap_{i=1}^{n} M_i) \le (F : \bigcap_{i=1}^{n-1} M_i)(F : M_n) \le \prod_{i=1}^{n} (F : M_i).
$$

So $(F : \bigcap_{i=1}^n M_i) = \prod_{i=1}^n (F : M_i)$ if and only if the above two inequalities are equalities, and the assertion follows.

(d) Since $(\bigcap_i M_i : \bigcap_i N_i) \le \prod_i (M_i : N_i)$ we have

$$
(F : \bigcap_i N_i) = \frac{(F : \bigcap_i M_i)}{(\bigcap_i M_i : \bigcap_i N_i)} \ge \frac{\prod_i (F : M_i)}{\prod_i (M_i : N_i)} = \prod_i (F : N_i),
$$

so equality holds by (a).

(e) We have $(M_1 M_2 : M_1) = (M_2 : M_1 \cap M_2)$. Thus

$$
\begin{aligned}
(F : M_1)(F : M_2) &= (F : M_1 M_2)(M_2 : M_1 \cap M_2)(F : M_2) \\
&= (F : M_1 M_2)(F : M_1 \cap M_2).
\end{aligned}
$$

□

Recall that an **embedding problem** for a profinite group F is a pair of epimorphisms of profinite groups

$$
(\varphi\colon F \to G, \alpha\colon H \to G). \tag{1}
$$

The embedding problem is called **finite** if H and G are finite. It is called **split** (respectively **nontrivial**) if α splits (respectively is not an isomorphism). We abbreviate 'finite split embedding problem' and write 'FSEP'. A **(weak) solution** for an embedding problem is a homomorphism $\psi\colon F \to H$ with $\alpha \circ \psi = \varphi$. A solution is said to be **proper** if it is surjective.

Definition 2.3 We call solutions of a finite embedding problem (1) **independent** if their kernels are Kerφ-independent.

We now introduce a criterion for the independence of proper solutions of finite embedding problems in terms of fiber products of groups.

Let $\{\alpha_i : H_i \to G \mid i \in I\}$ be a family of epimorphisms of profinite groups. Their **fiber product** with respect to the α_i's is defined by

$$\bigtimes_G H_i = \{h \in \prod H_i \mid \alpha_i(h_i) = \alpha_j(h_j)\ \forall i, j \in I\}.$$

(Here $h_i = h(i)$ is the value of h at i.) This is a closed subgroup of $\prod H_i$, hence a profinite group. The projection on the i-th coordinate, $\mathrm{pr}_i : \bigtimes_G H_i \to H_i$, is surjective. The fiber product is equipped with a canonical epimorphism $\alpha^I = \alpha_i \circ \mathrm{pr}_i : \bigtimes_G H_i \to G$, which is independent of $i \in I$.

In particular, if I is a finite set, say $I = \{1, \dots, n\}$, then

$$\bigtimes_G H_i = H_1 \times_G \cdots \times_G H_n = \{(h_1, \cdots, h_n) \in \prod H_i \mid \alpha_1(h_1) = \cdots = \alpha_n(h_n)\}.$$

Fiber products are associative:

Lemma 2.4 *Let $\alpha_i : H_i \to G_0$, $i = 1, \dots, n$, and $\beta : G \to G_0$ be epimorphisms of finite groups. Then the natural map $(\bigtimes_{G_0} H_i) \times_{G_0} G \to \bigtimes_G (H_i \times_{G_0} G)$ is an isomorphism.*

Proof An element in $(\bigtimes_{G_0} H_i) \times_{G_0} G$ is of the form $((h_1, \dots, h_n), g)$, where the elements $h_i \in H_i$ and $g \in G$ all have the same image in G_0. An element in $\bigtimes_G (H_i \times_{G_0} G)$ is of the form $((h_1, g) \dots, (h_n, g))$, for such elements $h_i \in H_i$ and $g \in G$, because the fiber product is taken over G. The map that takes $((h_1, \dots, h_n), g)$ to $((h_1, g) \dots, (h_n, g))$ is clearly an isomorphism. □

A key property, in our setting, of fiber products is that solutions ψ_i of embedding problems $(\varphi : F \to G, \alpha_i : H_i \to G)$, $i \in I$, induce a canonical solution, $\psi^I = \prod \psi_i$, of the embedding problem $(\varphi : F \to G, \alpha^I : \bigtimes_G H_i \to G)$. More precisely, $(\psi^I(x))_i = \psi_i(x)$ for each $x \in F$; e.g., if $I = \{1, \dots, n\}$, then $\psi^I(x) = (\psi_1(x), \cdots, \psi_n(x))$. We obtain the original solutions via the projection on the coordinates, i.e. $\psi_i = \mathrm{pr}_i \circ \psi^I$ for each $i \in I$. In particular, taking $F = G$ and $\varphi = \mathrm{id}$, we see that if all the α_i's split, so does α^I.

Given a single epimorphism $\alpha : H \to G$ and a set I, we write H_G^I for the fiber product $\bigtimes_G H_i$, where $H_i = H$ and $\alpha_i = \alpha$ for each $i \in I$.

Lemma 2.5 *Let I be a set and let $\mathcal{E} = (\varphi : F \to G, \alpha : H \to G)$ be a finite embedding problem for a profinite group F. Put $\mathcal{E}^I = (\varphi : F \to G, \alpha^I : H_G^I \to G)$. Then solutions $\{\psi_i\}_{i \in I}$ of $\mathcal{E}$ are independent and proper if and only if the solution $\psi^I = \prod \psi_i$ of $\mathcal{E}^I$ is proper.*

Proof We first assume that I is finite, $I = \{1, \dots, n\}$. If one of the ψ_i's is not surjective, then ψ^I is not surjective. Hence, we may assume that $\psi_1, \dots, \psi_n$ are surjective. Let $K = \mathrm{Ker}\varphi$ and $M_i = \mathrm{Ker}\psi_i$, $i = 1, \dots, n$. By the definition of ψ^I we have $\mathrm{Ker}\psi^I = \bigcap_{i=1}^n M_i$. Since $|H_G^I| = |H|^n / |G|^{n-1}$, we get that ψ^I is surjective if and only if $(F :$

$\bigcap_{i=1}^n M_i) = |H|^n/|G|^{n-1}$. But $(F : \bigcap_{i=1}^n M_i) = (F : K)(K : \bigcap_{i=1}^n M_i) = |G|(K : \bigcap_{i=1}^n M_i)$; hence ψ^I is surjective if and only if $(K : \bigcap_{i=1}^n M_i) = |H|^n/|G|^n = \prod_{i=1}^n (K : M_i)$, as desired.

In the general case H_G^I is the inverse limit of H_G^J, where J runs through the finite subsets of I and the epimorphisms $\pi^J\colon H_G^I \to H_G^J$ are given by the restriction of coordinates from I to J. Obviously, $\psi^J = \pi^J \circ \psi^I$, for each J. Hence ψ^I is proper if and only if all ψ^J's are proper. By the first paragraph of this proof this happens if and only if the ψ_i's are independent and proper. $\square$

3 Semi-free profinite groups

Definition 3.1 A profinite group F is **quasi-free** if there exists an infinite cardinal m such that every nontrivial FSEP for F has exactly m distinct proper solutions (see [10, 11, 23]). By [23, Lemma 1.2] such a group is of rank m.

In the following definition we give a stronger variant of quasi-freeness.

Definition 3.2 A profinite group F is **semi-free**[1] if it is a profinite group of infinite rank m and every nontrivial FSEP for F has m independent proper solutions.

Remark 3.3 Definition of semi-free finitely generated profinite group: One might consider saying that a group F of finite rank m is semi-free if every FSEP for F has a proper solution. But this condition will *never* be satisfied, since F cannot surject onto finite groups that have rank greater than m (i.e. that cannot be generated by m or fewer elements). As an alternative, one might say that a group F of rank m is semi-free if every FSEP $(\varphi\colon F \to G, \alpha\colon H \to G)$ is properly solvable *provided* that H has rank at most m (this condition on H being automatic if m is infinite). For m finite, this condition would imply that any finite group H generated by m elements is a quotient of F (taking $G = 1$). But then F is free [4, Lemma 17.7.1]. Thus a finitely generated profinite group is semi-free (in this sense) if and only if it is free. For this reason, we restrict our attention to groups of infinite rank.

Remark 3.4 In Definition 3.2, it would suffice to assume just that rank F is at most m. More precisely, let F be a profinite group and let m be an infinite cardinal. Assume that rank $F \le m$ and every nontrivial FSEP for F has m independent proper solutions. Then rank $F = m$, and thus F is semi-free.

Indeed, consider any nontrivial FSEP and let $\{\psi_i \mid i < m\}$ a set of independent proper solutions. Then $\mathrm{Ker}\psi_i \ne \mathrm{Ker}\psi_j$ for all $i \ne j$. This implies that F has at least m open subgroups, the set $\{\mathrm{Ker}\psi_i \mid i < m\}$, and hence rank $F \ge m$ (see [4, Proposition 17.1.2]). Therefore rank $F = m$, as needed.

Clearly, every semi-free group is quasi-free. One might suspect that the opposite is also true. If $m = \aleph_0$, then for both notions it suffices to have one proper solution of any nontrivial FSEP (see the lemma below), and hence they are equivalent. If $m > \aleph_0$, then there are quasi-free groups that are not semi-free. We postpone the discussion of this to Section 6.

[1] a term coined by Moshe Jarden as an alternative to "strongly quasi-free", which we initially used.

Lemma 3.5 *Let F be a countably generated profinite group. Then F is semi-free of rank $\aleph_0$ if and only if every FSEP for F is properly solvable.*

Proof Let $\mathcal{E} = (\varphi_0\colon F \to G, \alpha_0\colon H \to G)$ be a nontrivial FSEP. For each integer $n > 0$, let $\alpha_{n-1}\colon H_G^n \to H_G^{n-1}$ be the projection map. Inductively, we can find solutions $\varphi_n\colon F \to H_G^n$ of the FSEP

$$\mathcal{E}_n = (\varphi_{n-1}\colon G \to H_G^{n-1}, \alpha_{n-1}\colon H_G^n \to H_G^{n-1}).$$

Then $\varphi := \varprojlim \varphi_n\colon G \to H_G^{\mathbb{N}}$ is surjective. Lemma 2.5 implies the existence of $\aleph_0$ independent proper solutions, and thus F is semi-free. □

We extend [11, Theorem 2.1]:

Theorem 3.6 *Let F be a profinite group of infinite rank m. The following conditions are equivalent:*

a. F is free.
b. F is semi-free and projective.
c. F is quasi-free and projective.

Proof We show that (a) ⇒ (b). Let $\mathcal{E} = (\varphi\colon F \to G, \alpha\colon H \to G)$ be a nontrivial finite embedding problem for F. Fix a set I of cardinality m. Let H_G^I be the corresponding fiber product; let $\mathrm{pr}_i\colon H_G^I \to H$ be the projection on the i-th coordinate, for each $i \in I$; and let $\alpha^I = \alpha \circ \pi_i\colon H_G^I \to G$ be the canonical epimorphism.

Since F is free of rank m and since $\mathrm{rank}(H_G^I) \le m$, we have a proper solution $\psi\colon F \to H_G^I$ of the embedding problem $(\varphi\colon F \to G, \bar{\alpha}\colon H_G^I \to G)$ [22, Theorem 3.5.9]. Put $\psi_i = \pi_i \circ \psi$ for each $i \in I$. Then, by Lemma 2.5, solutions $\{\psi_i\}_{i\in I}$ of $\mathcal{E}$ are independent and proper. As $\mathcal{E}$ is nontrivial, they are distinct.

Implication (b) ⇒ (c) is trivial and (c) ⇒ (a) is [11, Theorem 2.1]. □

From technical point of view, it is preferable to work with a set of *pairwise* proper solutions of a FSEP instead of independent set of solutions. The following result shows that it is possible.

Proposition 3.7 *Let $\mathcal{M}$ be an infinite family of pairwise F-independent open normal subgroups of a profinite group F. Then $\mathcal{M}$ contains an F-independent subfamily $\mathcal{M}_0$ of cardinality $|\mathcal{M}|$.*

Proof By Zorn's Lemma there is a maximal F-independent subfamily $\mathcal{M}_0$ of $\mathcal{M}$. We have to show that $|\mathcal{M}_0| = |\mathcal{M}|$. Assume the contrary; that is, $|\mathcal{M}_0| < |\mathcal{M}|$.

Let $\mathcal{M}_1$ be the family of all finite intersections of the elements of $\mathcal{M}_0$. If $\mathcal{M}_0$ is finite, then so is $\mathcal{M}_1$; if $\mathcal{M}_0$ is infinite, then $|\mathcal{M}_1| = |\mathcal{M}_0|$. In particular, $|\mathcal{M}_1| < |\mathcal{M}|$. The groups in $\mathcal{M}_1$ are open in F. Let $\mathcal{M}_2$ be the family of all open subgroups of F containing a group in $\mathcal{M}_1$. Again, if $\mathcal{M}_1$ is finite, then so is $\mathcal{M}_2$; if $\mathcal{M}_1$ is infinite, then $|\mathcal{M}_2| = |\mathcal{M}_1|$. In particular, $|\mathcal{M}_2| < |\mathcal{M}|$.

For every proper subgroup N of F there exists at most one $M \in \mathcal{M}$ such that $M \le N$. Indeed, if $M_1, M_2 \in \mathcal{M}$ are distinct, then $M_1 M_2 = F$, by Proposition 2.2(e), and hence we cannot have $M_1, M_2 \leq N < F$. Since $|\mathcal{M}_2| < |\mathcal{M}|$, there exists $M \in \mathcal{M}$ such that

$$M \leq N \in \mathcal{M}_2 \ \text{ only for } \ N = F. \tag{*}$$

We claim that $\mathcal{M}_0 \cup \{M\}$ is F-independent. (This will produce the desired contradiction to the maximality of $\mathcal{M}_0$.) Thus we have to show, for distinct $M_1, \ldots, M_n \in \mathcal{M}_0$, that $M_1, \ldots, M_n, M$ are F-independent.

Put $N = \bigcap_{i=1}^n M_i$. By Proposition 2.2(c) it suffices to show that M, N are F-independent. By construction, $N \in \mathcal{M}_1$. Hence $MN \in \mathcal{M}_2$. Since $M \leq MN$, by (*), $MN = F$. Hence, by Proposition 2.2(e), M, N are F-independent. □

Corollary 3.8 *Let m be an infinite cardinal and let F be a profinite group of rank at most m. Then F is semi-free of rank m if and only if every nontrivial FSEP has m pairwise independent proper solutions.*

4 Finite split embedding problems and twisted wreath products

We follow [5] and establish the connection between FSEPs and twisted wreath products.

Definition 4.1 (Twisted wreath product) Let A, $G_0 \leq G$ be finite groups with a (right) action of G_0 on A. Write $\mathrm{Ind}_{G_0}^G(A)$ for all functions $f\colon G \to A$ such that $f(\sigma\tau) = f(\sigma)^\tau$ for all $\sigma \in G$ and $\tau \in G_0$ with component wise multiplication. Then $\mathrm{Ind}_{G_0}^G(A) \cong A^{(G:G_0)}$ and G acts on $\mathrm{Ind}_{G_0}^G(A)$ by

$$f^\sigma(\rho) = f(\sigma\rho), \qquad \sigma, \rho \in G,\ f \in \mathrm{Ind}_{G_0}^G(A).$$

The **twisted wreath product**, $A \,\mathrm{wr}_{G_0}\, G$, is defined to be the semidirect product of $\mathrm{Ind}_{G_0}^G(A)$ and G, i.e. $A \,\mathrm{wr}_{G_0}\, G = \mathrm{Ind}_{G_0}^G(A) \rtimes G$. Here and below, $\alpha : A \,\mathrm{wr}_{G_0}\, G \to G$ denotes the canonical projection $f\sigma \mapsto \sigma$ (see [4, Definition 13.7.1]). Similarly, $\alpha_0 : A \rtimes G_0 \to G_0$ denotes the canonical projection $a\sigma \mapsto \sigma$ of the semidirect product.

There is an epimorphism $\pi_0\colon \mathrm{Ind}_{G_0}^G(A) \to A$ defined by $\pi_0(f) = f(1)$. It extends to an epimorphism $\pi\colon \mathrm{Ind}_{G_0}^G(A) \rtimes G_0 \to A \rtimes G_0$ defined by $f\tau \mapsto f(1)\tau$ for $f \in \mathrm{Ind}_{G_0}^G(A)$ and $\tau \in G_0$, since $\pi_0(f^\tau) = f^\tau(1) = f(\tau) = f(1)^\tau = \pi_0(f)^\tau$ for all $f \in \mathrm{Ind}_{G_0}^G(A)$ and $\tau \in G_0$. We call π the **Shapiro map** of $A \,\mathrm{wr}_{G_0}\, G$.

Remark 4.2 a. If $G = G_0$ in Definition 4.1, then $A \,\mathrm{wr}_{G_0}\, G = A \rtimes G$.

b. See [21], where a related notion, known as a permutational wreath product, is used in a similar context.

The following technical result will be needed later.

Lemma 4.3 *Under the above notation, let $B = \pi^{-1}(G_0)$. Then B is a subgroup of $A \,\mathrm{wr}_{G_0}\, G$ of index $(G : G_0)|A|$. If $A \neq 1$, then B does not contain $\mathrm{Ind}_{G_0}^G(A)$.*

Proof As the Shapiro map π is surjective, $(\mathrm{Ind}_{G_0}^G(A) \rtimes G_0 : B) = |A|$. Thus the index of B in $A \,\mathrm{wr}_{G_0}\, G$ is $(G : G_0)|A|$.

If $A \neq 1$, there is $f \in \mathrm{Ind}_{G_0}^G(A)$ such that $f(1) \neq 1$; then $\pi(f) \notin G_0$, and hence $f \notin B$. □

Lemma 4.4 *Consider groups $H_i = A_i \,\mathrm{wr}_{G_0}\, G$, for $i = 1, \ldots, n$. Then G_0 acts on $\prod A_i$ componentwise and $\bigtimes_G H_i \cong (\prod A_i) \,\mathrm{wr}_{G_0}\, G$.*

Proof We have

$$\bigtimes\nolimits_G H_i = \{((f_1\sigma), \ldots, (f_n\sigma)) \mid \ f_i \in \mathrm{Ind}_{G_0}^G(A_i),\ \sigma \in G\},$$
$$\Big(\prod A_i\Big) \,\mathrm{wr}_{G_0}\, G = \{(f_1, \ldots, f_n)\sigma \mid \ f_i \in \mathrm{Ind}_{G_0}^G(A_i),\ \sigma \in G\},$$

and the isomorphism is given by $((f_1\sigma), \ldots, (f_n\sigma)) \mapsto (f_1, \ldots, f_n)\sigma$. □

Lemma 4.5 *Let $\varphi\colon F \to G$ be an epimorphism of a profinite group F onto a finite group G. Let M be a closed subgroup of F, let $G_0 = \varphi(M) \leq G$, and assume that G_0 acts on a finite group A. Consider the FSEP*

$$\mathcal{E}_0(A) = (\varphi|_M\colon M \to G_0, \alpha_0\colon A \rtimes G_0 \to G_0),$$

and let ψ be a solution of the corresponding FSEP

$$\mathcal{E}(A) = (\varphi\colon F \to G, \alpha\colon A \,\mathrm{wr}_{G_0}\, G \to G),$$

with notation as in Definition 4.1. Let π be the Shapiro map of $A \,\mathrm{wr}_{G_0}\, G$. Then $\psi(M) \leq \mathrm{Ind}_{G_0}^G(A) \rtimes G_0$ and $\pi \circ \psi|_M$ is a solution of $\mathcal{E}_0(A)$.

Proof We have $\psi(M) \leq \alpha^{-1}(G_0) = \mathrm{Ind}_{G_0}^G(A) \rtimes G_0$. Thus $\pi \circ \psi|_M$ is defined. Let $\alpha'\colon \mathrm{Ind}_{G_0}^G(A) \rtimes G_0 \to G_0$ be the restriction of α. From the commutativity of

$$\begin{array}{ccccc} & & & & M \\ & & {\scriptstyle \psi|_M}\swarrow & & \downarrow{\scriptstyle \varphi|_M} \\ \mathrm{Ind}_{G_0}^G(A) \rtimes G_0 & & \xrightarrow{\ \alpha'\ } & & G_0 \\ & {\scriptstyle \pi}\searrow & & \nearrow{\scriptstyle \alpha_0} & \\ & & A \rtimes G_0 & & \end{array}$$

we have $\alpha_0 \circ \pi \circ \psi|_M = \varphi|_M$, i.e. $\pi \circ \psi|_M$ is a solution. □

Although the solution $\pi \circ \psi|_M$ in the preceding lemma need not be proper, even if ψ is proper, the proof of [4, Proposition 25.4.1] shows that, under some assumptions on M, the properness of ψ does imply the properness of $\pi \circ \psi|_M$. Moreover, if F is a free profinite group of infinite rank m, that proof produces a family of m distinct proper solutions of $\mathcal{E}_0(A)$. We generalize this in part b of the following proposition, where we consider proper solutions that are not just distinct, but in fact independent.

Proposition 4.6 *Let $M \leq F$ be profinite groups, let A, G_1 be finite groups together with an action of G_1 on A, and let*

$$\mathcal{E}_1(A) = (\mu\colon M \to G_1, \alpha_1\colon A \rtimes G_1 \to G_1)$$

be a FSEP for M. Let D, F_0, L be subgroups of F such that

(2a) *D is an open normal subgroup of F with $M \cap D \leq \mathrm{Ker}\mu$,*
(2b) *F_0 is an open subgroup of F with $M \leq F_0 \leq MD$,*
(2c) *L is an open normal subgroup of F with $L \leq F_0 \cap D$.*

Put $G = F/L$, $G_0 = F_0/L \leq G$, and let $\varphi\colon F \to G$ be the quotient map.

a. Then there is an epimorphism $\bar{\varphi}_1\colon G_0 \to G_1$, through which an action of G_0 on A is defined, such that every weak solution ψ of the FSEP

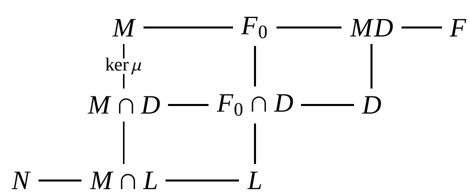
$$\mathcal{E}(A) = (\varphi\colon F \to G, \alpha\colon A \operatorname{wr}_{G_0} G \to G)$$

induces a weak solution $\nu = \rho \circ \pi \circ \psi|_M$ of $\mathcal{E}_1(A)$. Here π is the Shapiro map of $A \operatorname{wr}_{G_0} G$ and $\rho\colon A \rtimes G_0 \to A \rtimes G_1$ is the extension of $\bar{\varphi}_1$ by the identity of A.

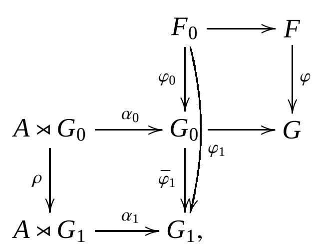
b. Let $n \in \mathbb{N}$. Assume that there is a closed normal subgroup N of F with $N \leq M \cap L$ such that there is no nontrivial quotient $\bar{A}$ of A^n through which the action of G_0 on A^n descends and for which the FSEP

$$(\bar{\varphi}\colon F/N \to G, \bar{\alpha}\colon \bar{A} \operatorname{wr}_{G_0} G \to G), \tag{3}$$

where $\bar{\varphi}$ is the quotient map, is properly solvable. Then any n independent proper solutions ψ of $\mathcal{E}(A)$ induce n independent proper solutions ν of $\mathcal{E}_1(A)$.



Proof (a) We can extend μ to a map $MD \to G_1$ by $md \mapsto \mu(m)$ for all $m \in M$ and $d \in D$. Its restriction to F_0 is an epimorphism $\varphi_1\colon F_0 \to G_1$. It decomposes as $\varphi_1 = \bar{\varphi}_1 \circ \varphi_0$, where $\varphi_0\colon F_0 \to G_0$ is the restriction of φ to F_0 and $\bar{\varphi}_1\colon G_0 \to G_1$ is an epimorphism. (Here we use that $\mathrm{Ker}\varphi|_{F_0} = L \leq D \leq \mathrm{Ker}\varphi_1$ to obtain $\bar{\varphi}_1$.) Let G_0 act on A via $\bar{\varphi}_1$. Then we have the following commutative diagram



where ρ is given by $\rho|_{G_0} = \bar{\varphi}_1$ and $\rho|_A = \mathrm{id}_A$. By Lemma 4.5, $\pi \circ \psi|_M$ is a (not necessarily proper) solution of $\mathcal{E}_0(A) : (\varphi_0|_M\colon M \to G_0, \alpha_0\colon A \rtimes G_0 \to G_0)$. Hence $\nu = \rho \circ \pi \circ \psi|_M$ is a solution of $\mathcal{E}_1(A)$.

(b) Let $\{\psi_i\}_{i=1}^n$ be a family of independent proper solutions of $\mathcal{E}(A)$. Let $1 \leq i \leq n$, and let $\nu_i = \rho \circ \pi \circ \psi_i|_M$ be the induced solution of $\mathcal{E}_1(A)$, as in (a). It suffices to show that each ν_i is proper and the family $\{\nu_i\}_{i=1}^n$ is independent.

By Lemma 4.4, $(A \operatorname{wr}_{G_0} G)^n_G = A^n \operatorname{wr}_{G_0} G$. So by Lemma 2.5, $\psi_1, \dots, \psi_n$ define a proper solution, $\psi\colon F \to A^n \operatorname{wr}_{G_0} G$, of

$$\mathcal{E}(A^n) = (\varphi\colon F \to G, \alpha\colon A^n \operatorname{wr}_{G_0} G \to G).$$

Applying Lemma 4.5, with A^n playing the role of A there, we get that $\nu = \rho' \circ \pi' \circ \psi$ is a solution of

$$\mathcal{E}_1(A^n) = (\mu\colon M \to G_1, \alpha_1\colon A^n \rtimes G_1 \to G_1).$$

(Here ρ' and π' are defined as ρ and π with A^n replacing A.) By Part C of [4, Proposition 25.4.1] (again, with A^n replacing A), $\pi'(\psi(N)) = A^n$. But $\nu(N) = \rho'(\pi'(\psi(N))) = \rho'(A^n) = A^n$. Therefore $A^n \leq \nu(M)$, and thus ν is a proper solution of $\mathcal{E}_1(A^n)$. As $\psi = \prod \psi_i$, we get that $\nu = \prod \nu_i$. Consequently, $\nu_1, \dots, \nu_n$ are independent proper solutions (Lemma 2.5). ◻

Corollary 4.7 (cf. [4, Proposition 25.4.1]) *Let F be a semi-free profinite group of infinite rank m and let M be a closed subgroup of F. Assume that for every open normal subgroup D of F there exist L and F_0 as in* (2b),(2c) *of Proposition 4.6, and there exists $N \triangleleft F$ with $N \leq M \cap L$ such that no FSEP*

$$(\varphi\colon F/N \to F/L, \alpha\colon A \operatorname{wr}_{F_0/L} F/L \to F/L),$$

where A is a nontrivial finite group on which F_0/L acts and where φ is the quotient map, is properly solvable.

Then M is semi-free of rank m.

Proof By [4, Corollary 17.1.4], $\operatorname{rank}(M) \leq \operatorname{rank}(F) = m$. Let $\mathcal{E}_1(A)$ be a FSEP as in Proposition 4.6. Choose D as in (2a) of Proposition 4.6. With F_0, L, N be as above, let $\mathcal{E}(A)$ be as in Proposition 4.6. Since F is quasi-free of rank m, there exists a family Ψ of independent proper solution of $\mathcal{E}(A)$ of cardinality m. This in turn induces a family $\mathcal{N}$ of solutions of $\mathcal{E}_1(A)$ (Lemma 4.5). The hypotheses of Proposition 4.6 hold by the assumptions of the present corollary. Therefore for every positive integer n and for every non-trivial quotient $\bar{A}$ of A^n, the embedding problem (3) of Proposition 4.6 has no proper solution. Hence $\psi_1, \dots, \psi_n \in \Psi$ induce $\nu_1, \dots, \nu_n \in \mathcal{N}$ which are independent and proper. Therefore $\mathcal{N}$ is a family of independent proper solutions of cardinality m. ◻

5 Semi-free subgroups

5.1 Proof of Main Theorem

Let F be semi-free of rank m and let $M \leq F$.

5.1.1 Case I

Assume that M is open in F. We apply Corollary 4.7. Given an open $D \triangleleft F$, we take an open $L \triangleleft F$ with $L \leq M \cap D$. Then for $F_0 = M$ and $N = L$, there are no proper solutions of the embedding problem appearing in Corollary 4.7, since φ is an isomorphism and α is not. Therefore, M is semi-free.

5.1.2 Case II

Assume that $F/\hat{M}$ is finitely generated, where $\hat{M} = \bigcap_{\sigma\in F} M^\sigma$ is the normal core of M in F.

We apply Proposition 4.6. Let $\mathcal{E}_1(A) = (\mu\colon M \to G_1, \alpha_1\colon A \rtimes G_1 \to G_1)$ be a nontrivial FSEP for M. Let D be an open normal subgroup of F with $M \cap D \le \mathrm{Ker}\mu$. Let $F_0 = MD$ and $N = \hat{M} \cap D$. Then F/N is finitely generated (as an open subgroup of $F/\hat{M} \times F/D$). Thus, F has only finitely many open subgroups containing N of index at most $r = (F : D)|A|^2$. Their intersection, L, is an open normal subgroup of F containing N and contained in D.

Now, for $n = 2$, the embedding problem (3), i.e.

$$(\bar{\varphi}\colon F/N \to F/L, \bar{\alpha}\colon \bar{A}\,\mathrm{wr}_{F_0/L}\, F/L \to F/L),$$

for any nontrivial quotient $\bar{A}$ of A^2, has no proper solution. Indeed, assume there exists a proper solution $\bar{\psi}\colon F/N \to \bar{A}\,\mathrm{wr}_{F_0/L}\, F/L$ of (3). By Lemma 4.3 there is a subgroup B of $H = \bar{A}\,\mathrm{wr}_{F_0/L}\, F/L$ of index $(H : B) = (F : F_0)|\bar{A}| \le r$ that does not contain $\mathrm{Ker}\bar{\alpha}$. In particular, $(H : B) > (H : B\,\mathrm{Ker}\bar{\alpha}) = (F/L : \bar{\alpha}(B))$. Write $\bar{\psi}^{-1}(B)$ as K/N, for some $N \le K \le F$. Then $(F : K) = (F/N : K/N) = (H : B) \le r$, and hence $L \le K$. As $\bar{\varphi} = \bar{\alpha} \circ \bar{\psi}$, we have $K/L = \bar{\varphi}(K/N) = \bar{\alpha}(\bar{\psi}(K/N)) = \bar{\alpha}(B)$. Therefore

$$(H : B) = (F : K) = (F/L : K/L) = (F/L : \bar{\alpha}(B)) < (H : B),$$

a contradiction.

Since F is semi-free, there exists a family Ψ of independent, and in particular pairwise independent, proper solutions of the nontrivial FSEP $\mathcal{E}(A) = (\varphi\colon F \to F/L, \alpha\colon A\,\mathrm{wr}_{F_0/L}\, F/L \to F/L)$ such that $|\Psi| = m$. By Proposition 4.6(b) with $n = 2$, Ψ induces a family $\mathcal{N}$ of pairwise independent proper solutions of $\mathcal{E}_1$ and $|\mathcal{N}| = |\Psi| = m$. By Corollary 3.8 we get that M is semi-free of rank m.

5.1.3 Cases IV, VI, and VII

The proof of Case VI is verbally identical with the proof of the Diamond Theorem, [4, Theorem 25.4.3], provided that we replace [4, Proposition 25.4.1] by our Corollary 4.7.

Case IV immediately follows from Case VI. So does Case VII: Since $(F : M) = (F/N : M/N)$ is divisible by two primes and the Sylow subgroups are normal in F/N, there are two (Sylow) normal subgroups P_1, P_2 of F/N such that $P_1 \cap P_2 = 1$ and $P_1, P_2 \not\subseteq M/N$. The preimages M_1, M_2 of P_1, P_2 are normal in F and satisfy $M_1 \cap M_2 = N \le M$, but $M_1 \not\le M$ and $M_2 \not\le M$.

5.1.4 Case V

Assume that $M \triangleleft F$ and F/M is abelian. It follows that M is also semi-free either by Cases II and VI or directly from Corollary 4.7. We show the former. If F/M is cyclic, then, by Case II, M is semi-free. Otherwise, there exists a pro-p subgroup of rank 2 in F/M, say H. It factors as $H = C_1 \times C_2$, where C_1, C_2 are nontrivial cyclic pro-p group. Then $C_1 \cap C_2 = 1$ and $C_1, C_2 \triangleleft F/M$ (since F/M is abelian). The preimages M_1, M_2 of C_1, C_2 are normal in F and satisfy $M_1 \cap M_2 = M$, but $M_1 \not\le M$ and $M_2 \not\le M$.

5.1.5 Cases III, VIII, and IX

The proofs of these three cases are based on Case I and on more elementary arguments than the other cases.

Recall that weight$(F/M) = 1$ if M is open, and weight(F/M) is the cardinality of the set of open subgroups of F that contain M if $(F : M) = \infty$ ([4, Section 25.2]).

Proof (Proof of Case III) Let $\mathcal{E}(M) = (\varphi\colon M \to G, \alpha\colon H \to G)$ be a FSEP for M and let $M_0 = \mathrm{Ker}\varphi$. There is an open $D \triangleleft F$ such that $D \cap M \le M_0$. By Case I we may replace F by its open subgroup DM to assume that $DM = F$. Then $dm \mapsto \varphi(m)$, for $d \in D,\ m \in M$, extends φ to an epimorphism $\varphi\colon F \to G$. Let F_0 be its kernel. It contains D, hence $F_0 M = F$ and $F_0 \cap M = M_0$. Thus $(M : M_0) = (F : F_0)$ and we have the FSEP $\mathcal{E}(F) = (\varphi\colon F \to G, \alpha\colon H \to G)$.

Let Ψ be a family of independent proper solutions of $\mathcal{E}(F)$ of cardinality m. Each $\psi \in \Psi$ defines a solution $\psi' := \psi|_M$ of $\mathcal{E}(M)$. Let $\Psi' = \{\psi' \mid \psi \in \Psi\}$ and let $\mathcal{X} \subseteq \Psi'$ be a maximal subset of independent proper solutions (Zorn's Lemma). We claim that $\mathcal{X}$ has cardinality m.

Assume differently, that is to say, assume $|\mathcal{X}| < m$. Let $N = \bigcap_{\psi' \in \mathcal{X}} \mathrm{Ker}\psi'$ if $\mathcal{X} \neq \emptyset$ and $N = M_0$ if $\mathcal{X} = \emptyset$. In both cases $N \le M_0$.

It suffices to find $\psi \in \Psi$ such that $N\mathrm{Ker}\psi = F_0$. Indeed, then for every open subgroup N_0 of M_0 containing N we have $(N_0 : N_0 \cap \mathrm{Ker}\psi) = (F_0 : \mathrm{Ker}\psi)$,

$$
\begin{array}{ccccccc}
 & & & & M & \text{———} & F \\
 & & & & | & & | \\
N & \text{———} & N_0 & \text{———} & M_0 & \text{———} & F_0 \\
| & & | & & | & & | \\
N \cap \mathrm{Ker}\psi & \text{———} & N_0 \cap \mathrm{Ker}\psi & \text{———} & M \cap \mathrm{Ker}\psi = \mathrm{Ker}\psi' & \text{———} & \mathrm{Ker}\psi
\end{array}
$$

i.e., N_0 and $\mathrm{Ker}\psi'$ are M_0-independent. In particular, taking $N_0 = M_0$, we have $(M_0 : \mathrm{Ker}\psi') = (M_0 : M \cap \mathrm{Ker}\psi) = (F_0 : \mathrm{Ker}\psi)$, and hence ψ' is surjective. Furthermore, for any finite subset $\mathcal{X}'$ of $\mathcal{X}$, taking $N_0 = \bigcap_{\psi' \in \mathcal{X}'} \mathrm{Ker}\psi'$ we get by Proposition 2.2(c) that $\mathcal{X}' \cup \{\psi'\}$ is an independent set of solutions. Therefore so is $\mathcal{X} \cup \{\psi'\}$, which contradicts the maximality of $\mathcal{X}$.

To complete the proof, for each $\psi \in \Psi$ let $L_\psi = N\mathrm{Ker}\psi$ and assume that $L_\psi \neq F_0$. Since $\{\mathrm{Ker}\psi \mid \psi \in \Psi\}$ is F_0-independent, the set $\{L_\psi \mid \psi \in \Psi\}$ is also independent by Proposition 2.2(**??**). Since $L_\psi \neq F_0$ for all $\psi \in \Psi$, this implies in particular $L_{\psi_1} \neq L_{\psi_2}$ for all distinct $\psi_1, \psi_2 \in \Psi$. Hence weight$(F_0/N) \ge m$. But weight$(F_0/M) < m$ by the hypothesis of Case III and the fact that F_0 is an open subgroup of F. Moreover weight$(M/N) < m$, by [4, Lemma 25.2.1(b)]. Hence weight$(F_0/N) < m$ by [4, Lemma 25.2.1(d)], a contradiction. □

Definition 5.1 A closed subgroup M of a profinite group F of infinite index is called **sparse** if for all $n \in \mathbb{N}$ there exists an open subgroup K of F containing M such that for every proper open subgroup L of K containing M we have $(K : L) \ge n$.

The following lemma shows that this definition is equivalent to [2, Definition 2.1]:

Lemma 5.2 *If M is sparse in F, then for every $\ell, n \in \mathbb{N}$ there exists K as in Definition 5.1 of index at least ℓ in F.*

Proof Let $\ell, n \in \mathbb{N}$. Choose an open subgroup K_0 of index $\ell_0 \geq \ell$ in F such that $M \leq K_0$. By the definition there exists K_1 with $M \leq K_1 \leq F$ such that $(K_1 : L) \geq n\ell_0$ for all proper open subgroups L of K_1 that contain M. Then the assertion follows with $K = K_0 \cap K_1$, since $(K_1 : K) \leq \ell_0$. □

Proof (Proof of Case VIII) Let M be a sparse subgroup of F. Let $\mathcal{E}_0(A) = (\mu\colon M \to G, \alpha\colon A \rtimes G \to G)$ be a nontrivial FSEP for M.

Choose an open normal subgroup E_0 of F such that $E_0 \cap M \leq \mathrm{Ker}\mu$ and let $F_0 = ME_0$. Since M is sparse in F_0 [2, Corollary 2.3], there is an open subgroup K of F_0 containing M such that $(K : L) > |A|^2|G|$ for each proper open subgroup L of M that contains M. Extend μ to an epimorphism $\varphi\colon K \to G$ by $\varphi(re) = \mu(r)$, $r \in M$, $e \in E_0$. By Case I, K is semi-free of rank m; hence it suffices to show that two independent proper solutions ψ_1, ψ_2 of $\mathcal{E}(A) = (\varphi\colon K \to G, \alpha\colon A \rtimes G \to G)$ induce two independent proper solutions $\psi_1|_M, \psi_2|_M$ (Corollary 3.8).

By Lemma 4.4, $A^2 \rtimes G$ is the fiber product of $A \rtimes G \to G$ with itself. Thus ψ_1, ψ_2 induce a proper solution ψ of $\mathcal{E}(A^2) = (\varphi\colon K \to G, \alpha\colon A^2 \rtimes G \to G)$ (Lemma 2.5). Let $L = \mathrm{Ker}\psi$. Then $(K : ML) = (A^2 \rtimes G : \psi(M)) \leq |A|^2|G|$. Hence, by the choice of K, we get that $ML = K$. Therefore, $\psi|_M$ is a proper solution of $\mathcal{E}_0(A^2) = (\varphi\colon M \to G, \alpha\colon A^2 \rtimes G \to G)$. But $\psi|_M = \psi_1|_M \times \psi_2|_M$. Consequently, $\psi_1|_M, \psi_2|_M$ are independent proper solutions of $\mathcal{E}_0(A)$, as claimed. □

The following corollary of Case VIII extends [2, Lemma 2.4] to free groups of uncountable infinite rank.

Corollary 5.3 *If M is a sparse subgroup of a free profinite group F of rank $m \geq 2$, then M is a free profinite group of* $\mathrm{rank}(M) = \max\{\aleph_0, \mathrm{rank}(F)\}$.

Proof The case where $\mathrm{rank}(F) \leq \aleph_0$ is proven in [2]. Assume $m = \mathrm{rank}(F)$ is infinite. By Theorem 3.6, F is semi-free of rank m. By Case VIII of the Main Theorem, M is semi-free of rank m. Also, M is projective, being a closed subgroup of a free profinite group. Consequently M is free of rank m (Theorem 3.6). □

Case IX is, in fact, a special case of Case VIII:

Lemma 5.4 *Let M be a closed subgroup of a profinite group F of infinite index. Assume $(F : M) = \prod_p p^{\alpha(p)}$ with all $\alpha(p)$ finite. Then M is sparse in F.*

Proof For $n \in \mathbb{N}$ take K to be an open subgroup of F containing M such that $p^{\alpha(p)} \mid (F : K)$ for all $p \leq n$. Then for each $M \leq L \lneqq K$ only primes $p > n$ can divide $(K : L)$. Therefore, $(K : L) > n$. □

As a consequence of Corollary 5.3 and Lemma 5.4, we get [15, Proposition 5.1]:

Corollary 5.5 *Let M be a closed subgroup of a free profinite group F of rank $m \geq 2$. Assume $(F : M) = \prod_p p^{\alpha(p)}$ with all $\alpha(p)$ finite. If $(F : M)$ is infinite, then M is free profinite group of rank* $\max\{\aleph_0, \mathrm{rank}(F)\}$.

6 Quasi-freeness vs. semi-freeness

We now construct an example of a quasi-free group that is not semi-free.

For a profinite group C and an infinite set X denote by $\mathop{\ast}\!\!\prod_X C$ the free product of copies $\{C_x\}_{x\in X}$ of C in the sense of [1]. That is, $\mathop{\ast}\!\!\prod_X C$ contains a copy C_x of C for each $x \in X$; and every family of homomorphisms $\psi_x : C_x \to A$ into a finite group A, such that $\psi_x(C_x) = 1$ for all but finitely many $x \in X$, uniquely extends to a homomorphism $\psi : \mathop{\ast}\!\!\prod_X C \to A$. As usual let $\hat{F}_\omega$ denote the free profinite group of countable rank.

Proposition 6.1 *Let X be a set of infinite cardinality m. Let $C = \prod_p \mathbb{Z}/p\mathbb{Z}$ be the direct product of all prime cyclic groups. Let $F = (\mathop{\ast}\!\!\prod_X C) * \hat{F}_\omega$. Then*

a. *F is quasi-free of rank m, and*
b. *the FSEP*

$$(F \to 1, \mathbb{Z}/4\mathbb{Z} \to 1) \tag{4}$$

has at most countably many independent proper solutions.

In particular, for $m > \aleph_0$, F is quasi-free but not semi-free.

Proof (a) The rank of $\mathop{\ast}\!\!\prod_X C$ is m and the rank of $\hat{F}_\omega$ is $\aleph_0 \le m$. Hence the rank of F is m. In particular, every FSEP for F has at most m proper solutions. Let

$$(\varphi : F \to G, \alpha : H \to G) \tag{5}$$

be a nontrivial FSEP. Let $\beta : G \to H$ be its splitting. We need two auxiliary maps: Firstly, there exists a nontrivial homomorphism $\pi : C \to \mathrm{Ker}\alpha$; namely, an epimorphism of C onto a subgroup of $\mathrm{Ker}\alpha$ of prime order. Secondly, since $\hat{F}_\omega$ is free of infinite rank, there exists an epimorphism $\psi' : \hat{F}_\omega \to \alpha^{-1}(\varphi(\hat{F}_\omega))$ such that $\alpha \circ \psi'$ is the restriction of φ to $\hat{F}_\omega$. In particular, $\psi'(\hat{F}_\omega)$ contains $\mathrm{Ker}\alpha$. Since φ is continuous, there is a $Y \subseteq X$ such that $X \smallsetminus Y$ is finite and $\varphi(C_y) = 1$ for every $y \in Y$.

For every $y \in Y$ define a homomorphism $\psi_y : F \to H$ in the following manner: Its restriction to $C_y \cong C$ coincides with π; if $y \neq x \in Y$, the restriction of ψ_y to C_x is trivial; if $x \in X \smallsetminus Y$, the restriction of ψ_y to C_x is $\beta \circ \varphi$; and, finally, the restriction of ψ_y to $\hat{F}_\omega$ is ψ'. Thus $\alpha \circ \psi_y = \varphi$. As $\psi_y(F) \supseteq \psi'(\hat{F}_\omega) \supseteq \mathrm{Ker}\alpha$, the map ψ_y is a proper solution of (5).

As $\psi_{y_1} \neq \psi_{y_2}$ for distinct $y_1, y_2 \in Y$, (5) has at least $|Y| = m$ distinct proper solutions.

(b) Let Ψ be an independent set of proper solutions of (4). The map $\alpha : \mathbb{Z}/4\mathbb{Z} \to 1$ decomposes as $\alpha = \beta\gamma$, where $\gamma : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and $\beta : \mathbb{Z}/2\mathbb{Z} \to 1$. If $\psi_1, \psi_2 \in \Psi$ are independent, then $\gamma \circ \psi_1, \gamma \circ \psi_2$ are independent proper solutions of $(\beta : \mathbb{Z}/2\mathbb{Z} \to 1, \varphi : F \to 1)$ (Proposition 2.2(d)). In particular, $\gamma \circ \psi_1 \neq \gamma \circ \psi_2$. Thus $\{\gamma \circ \psi \mid \psi \in \Psi\}$ has at least the cardinality of Ψ.

On the other hand, $\mathbb{Z}/4\mathbb{Z}$ is a 2-group and the 2-Sylow subgroup of C is of order 2. Hence every $\psi \in \Psi$ maps each $C_x \cong C$ into $\mathrm{Ker}\gamma$, the unique subgroup of $\mathbb{Z}/4\mathbb{Z}$ of order 2, and hence $\gamma \circ \psi$ is trivial on C_x. Therefore $\gamma \circ \psi$ is trivial on $\mathop{\ast}\!\!\prod_X C$. It follows that $\gamma \circ \psi$ is determined by its restriction to $\hat{F}_\omega$. But there are $\aleph_0$ (continuous) homomorphisms $\hat{F}_\omega \to \mathbb{Z}/4\mathbb{Z}$. Thus $|\Psi| \le \aleph_0$. □

Remark 6.2 One can modify the construction in the proposition to get an absolute Galois group F which is quasi-free but not semi-free. E.g., let $F = F_X(\prod_{p\neq 2}\mathbb{Z}_p) * D * \hat{F}_\omega$, where D is the free product of the constant sheaf of copies of $\mathbb{Z}/2\mathbb{Z}$ over some profinite space of weight m. One can show along the lines of the proof of Proposition 6.1 that F is quasi-free but not semi-free. Moreover, F is real projective in the sense of [6, p. 472] and hence isomorphic to an absolute Galois group by [6, Theorem 10.4]. We leave out the details, since the assertion is outside the scope of this work.

Remark 6.3 In order to complete the picture we show that being semi-free is strictly weaker than being free. In fact, if F is semi-free of infinite rank m and G is of rank $\leq m$, then $F * G$ is semi-free. This leads to many examples of semi-free but not free profinite groups; e.g., take G to be finite and recall that a free group has no torsion. Furthermore, we can construct a semi-free group of arbitrary cohomological dimension d, by taking F free and G of cohomological d. If $d > 1$ then the group is not free, or even projective, since its cohomological dimension is greater than one. Another example is the absolute Galois group given in Theorem 7.1 below, which is semi-free but is not projective (and hence not free) because its cohomological dimension is greater than one.

The condition $m > \aleph_0$ in the above proposition is essential:

Remark 6.4 If $\mathrm{rank}(F) = \aleph_0$, then F is semi-free if and only if it is quasi-free.

Indeed, assume F is quasi-free. Then every FSEP is solvable. By Lemma 3.5 F is semi-free. The opposite direction is immediate.

We now show that Case III of our Main Theorem does not carry over to quasi-free subgroups of quasi-free groups.

Example 6.5 Let X be a set of cardinality $m > \aleph_0$ and let $F = (\mathop{\ast}\limits_X C) * \hat{F}_\omega$ be the group of Proposition 6.1. Let M be the kernel of the map $F \to \hat{F}_\omega$. Then F is quasi-free of rank m, $\mathrm{weight}(F/M) < m$, but M is not quasi-free.

Indeed, by Proposition 6.1, F is quasi-free of rank m. We have

$$\mathrm{weight}(F/M) = \mathrm{rank}(\hat{F}_\omega) = \aleph_0$$

since $F/M = \hat{F}_\omega$. It is easy to see that M is generated by the conjugates of $\mathop{\ast}\limits_X C$ in F. Since $\mathop{\ast}\limits_X C$ is generated by copies of C and $C = \prod_p \mathbb{Z}/p\mathbb{Z}$ is generated by elements of prime order, also M is generated by elements of prime order. Hence $\mathbb{Z}/q^2\mathbb{Z}$ is not an image of M. In particular, M is not quasi-free.

Remark 6.6 It is interesting to ask which of the cases of the Main Theorem holds for quasi-free groups. As we have seen, Case III does not hold. In [23] Case I is proved. Case V is proved in [10] for $M = [F, F]$. Combining the methods of this paper together with [10], one can extend the result to any M such that F/M is abelian but not a pro-p group. The proof of Case VIII (and hence of (IX)) can be carried over to quasi-free groups. However, we do not know if the diamond theorem, i.e. Case VI, which is the central result of this paper, holds for quasi-free groups. All other cases are open in the quasi-free case.

In order to use our method, i.e. using wreath products, for quasi-free groups for M of infinite index in F, one needs to come up with a new idea, as explained at the end of Section 1.

7 Fields with semi-free absolute Galois groups

The main result in [11] (Theorem 5.1 there) was that for any field k, the absolute Galois group of $K := k((x, y))$ is quasi-free. In fact more is true:

Theorem 7.1 *Let k be a field. Then the absolute Galois group of the field $K := k((x, t))$ is semi-free of rank* $\operatorname{card} K$.

The proof of this stronger result is essentially contained in the proof of the original theorem in [11]. We explain below what additional observations need to be made to complete the argument, and how these observations also yield stronger forms of other results in [11]. See also [12, Theorem 5.1] for more details.

First we recall the strategy used to prove [11, Theorem 5.1]. The proof of that theorem relied on a related geometric assertion, [11, Proposition 5.3]. That proposition asserted that given a split short exact sequence $1 \to N \to \Gamma \xrightarrow{f} G \to 1$ of finite groups with non-trivial kernel, any G-Galois connected normal branched cover $Y^* \to X^* = \operatorname{Spec} k[[x, t]]$ can be dominated by a Γ-Galois connected normal branched cover $Z^* \to X^*$. Moreover it said that this cover may be chosen such that $Z^* \to Y^*$ satisfied a splitting condition (that $Z^* \to Y^*$ is totally split at the generic points of the ramification locus of $Y^* \to X^*$), and that the set of isomorphism classes of such covers $Z^* \to X^*$ has cardinality equal to $m := \operatorname{card} k((x, t))$.

The proof of [11, Proposition 5.3] relied on [11, Theorem 4.1], which was a more global version of that assertion. Namely, it considered a smooth connected curve X over a field $\hat{k} := k((t))$, and then considered a finite split embedding problem for the absolute Galois group of the function field K of X (this field K being a global analog of the more local field K considered in [11, Proposition 5.3]). The conclusion was similar: that any G-Galois branched cover $Y \to X$ of normal curves can be dominated by a Γ-Galois branched cover $Z \to X$; that this cover can be chosen with a splitting property; and that there are $m := \operatorname{card} K$ distinct such choices of corresponding normal branched covers $Z \to X$. (The splitting property is that $Z \to Y$ is totally split over a given finite set $D \subset Y$ of closed points, and the decomposition groups of $Z \to X$ at the points of Z over $\delta \in D$ are the conjugates of $\sigma(G_\delta)$, where G_δ is the decomposition group of $Y \to X$ at δ and where σ is a section of f.)

Moreover, for the sake of [11, Proposition 5.3], more was shown in [11, Theorem 4.1], to enable passage from a global solution to a more local solution. Let $\bar{X}$ be a smooth projective model for X over $k[[t]]$; and with Y, Z as above, let $\bar{Y}, \bar{Z}$ be the corresponding normal branched covers. Let P be a closed point of $\bar{X}$ whose residue field is separable over k, let X^* be the spectrum of the complete local ring of $\bar{X}$ at P, and suppose that the pullback $Y^* \to X^*$ of $\bar{Y} \to \bar{X}$ is connected. Then among the pullbacks $Z^* \to X^*$ of the above solutions $\bar{Z} \to \bar{X}$ there are m distinct proper solutions of the corresponding local embedding problem. This additional condition was applied in the case of the x-line over $\hat{k}$ in order to obtain [11, Proposition 5.3].

More specifically, the relationship between the local assertion [11, Proposition 5.3] and the more global assertion [11, Theorem 4.1] is based on viewing $k((x,t))$ as the fraction field of the complete local ring of $\bar{X} := \mathbb{P}^1_{k[[t]]}$ at the point $x = t = 0$. In order to apply [11, Theorem 4.1] to the proof of [11, Proposition 5.3], a change of variables can be made to reduce to the case in which the prime (t) is unramified in $Y^* \to X^*$. The reduction of this cover modulo (t) is then induced from a branched cover of the projective k-line, by the Katz-Gabber theorem [17, Theorem 1.4.1]. A patching argument then shows that this cover of $\mathbb{P}^1_k$ is in turn the closed fiber of a cover of $\mathbb{P}^1_{k[[t]]}$ that restricts to $Y^* \to X^*$. This enables [11, Theorem 4.1] to be cited; and by the extra conditions in the paragraph above, the proper solutions to the embedding problem over the function field of $\mathbb{P}^1_{k[[t]]}$ yield distinct proper solutions to the embedding problem over $k((x,t))$.

Theorem 4.1 of [11] was a variant on results of Pop [20, Main Theorem A] and of Haran and Jarden [7, Theorem 6.4], showing that finite split embedding problems over the function fields of curves over complete discretely valued (or more generally large) fields have proper regular solutions (and that some additional conditions can also be satisfied, e.g. the existence of an unramified rational point). Like those earlier results, [11, Theorem 4.1] was proven using patching. Generators were chosen for the kernel N of the given finite split embedding problem; and cyclic covers were constructed with groups generated by each of those elements in turn. These were then patched together to form a global solution; in doing so, a compatibility condition (agreement on overlaps) had to be satisfied by the cyclic covers on the "patches". Such a construction was carried out in [11, Proposition 3.5]. But the construction there assumed that branch points of $Z \to \bar{Y}$ that correspond to distinct generators of Z had the property that their closures in $\bar{Y}$ are disjoint. In order to apply this to the proof of [11, Theorem 4.1] (where the branch points all coalesce on the closed fiber at P, in order to preserve the solutions over X^*), it was necessary to blow up the closed fiber to separate the branch points.

We can now describe the proof of Theorem 7.1:

Proof As discussed above, this theorem is a strong form of [11, Theorem 5.1], and to prove this result it suffices to prove a corresponding strong form of [11, Proposition 5.3]: that among the covers $Z^* \to X^*$ whose existence is asserted in that proposition, there is a subset having cardinality m, and which is linearly disjoint as a set of covers of Y^*. To prove this, we need to see that in the situation of [11, Theorem 4.1], an additional property holds: that there are m choices of $Z \to X$ that are linearly disjoint over Y, that properly solve the given global embedding problem, and that induce proper solutions over X^* that are linearly disjoint over $Y^* = Y \times_X X^*$.

To show this stronger version of [11, Theorem 4.1], the key point is that the branch points associated to the generators of N can be chosen in m different (and even disjoint) ways. As shown in the original proof, given any choices of these points on X (which correspond to curves on $\bar{X}$ that are finite over $k[[x]]$), any other choice of points that is congruent to the original choice modulo a sufficiently high power of t will also work. (Indeed, this is how it was shown that there are m distinct solutions, both over X and over X^*.) What needs to be shown here is that by varying the branch points we can obtain m solutions that are linearly disjoint over Y. Since Galois branched covers with no common subcover are linearly disjoint, it suffices to show that the set of m solutions $Z \to X$, such that the covers $Z \to Y$ have pairwise disjoint branch loci, can be chosen such that each $Z \to Y$ has no non-trivial étale subcover $W \to Y$.

In the above situation, if $Z \to Y$ has a non-trivial étale subcover $W \to Y$, then the Galois group $\mathrm{Gal}(Z/W)$, which is a subgroup of $N = \mathrm{Gal}(Z/Y)$, must contain all the inertia groups of $Z \to Y$. But this is ruled out by the explicit construction in the proof of [11, Proposition 3.5]. Namely, that result asserts that the closed fiber $\bar{Z} \to \bar{Y}$ of $Z \to Y$ is an N-Galois mock cover; i.e., each irreducible component of $\bar{Z}$ maps isomorphically onto $\bar{Y}$, with the irreducible components being indexed by the cosets of N in Γ. The construction in the proof there shows that for each generator n of N, there is a closed point $Q_n \in \bar{Z}$ lying in the ramification locus of $\bar{Z} \to \bar{Y}$, such that n generates the inertia group of $\bar{Z} \to \bar{Y}$ at Q_n and also the inertia groups at the generic points of the ramification components passing through Q_n. Since the elements n together generate N, this shows that the N-Galois cover $Z \to Y$ has no non-trivial étale subcovers, as desired.

Thus the above strong form of [11, Theorem 4.1] indeed holds. Hence so does the strong form of [11, Proposition 5.3]; and thus also Theorem 7.1 above, the strong form of [11, Theorem 5.1]. □

Another key result of [11], viz. Corollary 4.4 there, asserted that if K is the function field of a smooth projective curve over a very large field k, then the absolute Galois group of K is quasi-free. This can also be strengthened, as follows:

Theorem 7.2 *If K is the function field of a smooth projective curve X_0 over a large field k, then the absolute Galois group of K is semi-free.*

Proof By a recent result of Pop (see [10, Proposition 3.3]), every large field is very large. So the assumption on k in [11, Corollary 4.4] can be (a priori) weakened from very large to large. Concerning the strengthening of the conclusion, this can be done in a similar way to what was done above for Theorem 7.1. Namely, [11, Corollary 4.4] followed from [11, Theorem 4.3], which was a variant of [11, Theorem 4.1] in which the field $\hat{k} = k((t))$ was replaced by a more general large field F. As in the case of Theorem 7.1, to prove 7.2 it suffices to show that the proper solutions $Z_0 \to X_0$ in [11, Theorem 4.3] can be chosen so as to be linearly disjoint over Y_0; and for this it suffices to show that they can be chosen so that each $Z_0 \to Y_0$ has no non-trivial étale subcovers.

Theorem 4.3 of [11] was proven using [11, Theorem 4.1], by taking $k = F$; obtaining a proper solution for the function field of the induced curve $\bar{X} := X_0 \times_F R$ over $R = k[[t]]$; descending from R to a k-algebra A of finite type, corresponding to a k-variety V; considering the descended Γ-Galois cover $Z_A \to X_A$ as a family of Γ-Galois covers of X_0 parametrized by V; and then specializing to k-points of V (thereby obtaining solutions over X_0) using that k is (very) large. To prove the desired strong form of [11, Theorem 4.3], observe that in the context of the above use of [11, Theorem 4.1], the branch points (which can be varied arbitrarily modulo some sufficiently high power of t) can be chosen so as not to be constant; i.e. not of the form $P' \times_k \hat{k}$ with P' a point of X_0. As a result, the the varying branch locus of the family of Γ-Galois covers of X_0 parametrized by V is base-point free. So as in the proof of the strong form of [11, Theorem 4.1], the specialized covers can be chosen to have no non-trivial étale subcovers; and hence they are linearly disjoint. This shows that [11, Theorem 4.3] can be strengthened as claimed to include the desired linear disjointness assertion; and hence Theorem 7.2, the strong form of [11, Corollary 4.4], also holds. □

8 Fields with free absolute Galois groups

We present two families of fields having free absolute Galois groups. For each we use Theorem 3.6 to reduce the proof of freeness to proving that the group is semi-free and projective.

The semi-freeness follows from the Diamond Theorem (Main Theorem, Case VI) together with the semi-freeness of the absolute Galois group of the base field, which was established in the previous section. The projectivity is achieved by different means (here we just quote it).

8.1 Fields containing the maximal abelian extension of $k((x,t))$

We follow [10] to find fields with free absolute Galois group. Let us start with a general fact and then give some concrete examples.

Corollary 8.1 *Let $K = k((x,y))$, where k is separably closed and let L be a separable extension of K. If L contains the maximal abelian extension of K, and its absolute Galois group* $\mathrm{Gal}(L)$ *satisfies one of the cases of the Main Theorem as a subgroup of* $\mathrm{Gal}(K)$, *then* $\mathrm{Gal}(L)$ *is a free profinite group.*

Proof The group $\mathrm{Gal}(K)$ is semi-free of rank m by Theorem 7.1. Hence so is $\mathrm{Gal}(L)$. Also, $\mathrm{Gal}(L)$ is projective [10, Theorem 4.4] (see also [3]). Thus, Theorem 3.6 yields that $\mathrm{Gal}(L)$ is free. □

Example 8.2 Let $K = k((x,y))$, where k is separably closed. Let E be a Galois extension of K not containing the maximal abelian extension K^{ab} of K. Let L be any subextension of EK^{ab}/K^{ab}. We claim that $\mathrm{Gal}(L)$ is free of rank equal to the cardinality of L.

To see this, first note that $\mathrm{Gal}(K)$ is semi-free (Theorem 7.1). If $L = K^{\mathrm{ab}}$, then by [10, Theorem 4.6(b)] it follows that $\mathrm{Gal}(L)$ is free. (Equivalently, this follows from Main Theorem Case V together with Corollary 8.1.)

Now consider the case $L \neq K^{\mathrm{ab}}$. Since $K^{\mathrm{ab}} \not\subseteq E$ and $K^{\mathrm{ab}} \subseteq L$, it follows that $L \not\subseteq E$. Furthermore, E/K and K^{ab}/K are Galois. Hence by the Galois correspondence, $M = \mathrm{Gal}(L)$ satisfies Case VI of the Main Theorem with $F = \mathrm{Gal}(K)$, $M_1 = \mathrm{Gal}(E)$, and $M_2 = \mathrm{Gal}(K^{\mathrm{ab}})$. By Corollary 8.1, $\mathrm{Gal}(L)$ is free.

$$\begin{array}{ccc} \mathrm{Gal}(K^{\mathrm{ab}}) & \text{---} & \mathrm{Gal}(K) \\ \Big| {\scriptstyle \mathrm{Gal}(L)} & & \Big| \\ \mathrm{Gal}(E) \cap \mathrm{Gal}(K^{\mathrm{ab}}) & \text{---} & \mathrm{Gal}(E) \end{array}$$

8.2 Jarden's example – extension of roots

This example is adapted from [14]. Let k be a PAC field of characteristic $p \geq 0$ and $K = k(x)$. Let $\mathcal{F} \subseteq k[x] \subseteq K$ be the set of all monic irreducible polynomials. For each $f \in \mathcal{F}$ choose a set of compatible roots

$$\{f^{\frac{1}{n}} \mid p \nmid n\} \subseteq K_s.$$

(Here compatible means that $(f^{\frac{1}{nn'}})^n = f^{\frac{1}{n'}}$ for all n, n' prime to p.) Let

$$L = K(f^{\frac{1}{n}} \mid f \in \mathcal{F} \text{ and } p \nmid n).$$

Note that L/K is Galois if and only if K contains all roots of unity. Thus in general L/K is not Galois. In what follows we show that $\mathrm{Gal}(L)$ is free of rank equal to the cardinality of L.

Fact 1 $\mathrm{Gal}(L)$ *is projective.*

This fact follows from a theorem of Pop (see Theorems 10.4.9 and 11.6.4 in [14]).

Lemma 8.3 *There exist Galois extensions L_1, L_2 of K such that $L \subseteq L_1L_2$, but $L \nsubseteq L_i$, $i = 1, 2$.*

Proof Let L_0 denote the extension of K generated by all roots of unity. Let

$$L_1 = L_0(x^{\frac{1}{n}} \mid p \nmid n) \text{ and } L_2 = L_0(f^{\frac{1}{n}} \mid f \in \mathcal{F} \smallsetminus \{x\} \text{ and } p \nmid n).$$

Clearly L_1, L_2 are Galois extensions of K. It is obvious that $L \subseteq L_1L_2$. Choose an integer $m > 1$ that is not divisible by p. Since $(x+1)^{\frac{1}{m}} \notin L_1$ we get that $L \nsubseteq L_1$; and similarly $x^{\frac{1}{m}} \notin L_2$ implies that $L \nsubseteq L_2$. □

Theorem 8.4 $\mathrm{Gal}(L)$ *is free of rank equal to the cardinality of L.*

Proof By Theorem 3.6 it suffices to show that $\mathrm{Gal}(L)$ is both projective and semi-free of rank equal to the cardinality of L. We already mentioned that $\mathrm{Gal}(L)$ is projective (Fact 1).

Theorem 7.2 implies that $\mathrm{Gal}(K)$ is semi-free of rank $m := |K| = |L|$. (Recall that k is PAC, and in particular large.) Taking absolute Galois groups of the fields L_1, L_2 in the above lemma establishes the condition of Case VI of the Main Theorem, thus $\mathrm{Gal}(L)$ is semi-free of rank m. □

In fact, even more is true. Namely, we have learned from Pop that the proof of his theorem (referred to above) applies more broadly. In particular, it applies in the case that $k = F((t))$ for some separably closed field F (using that this field k, like a PAC field, has projective absolute Galois group and "satisfies a universal local-global principle"). Following the same construction as above, we again deduce that the resulting field L has free absolute Galois group of rank $|L|$. Note that by Corollary 25.4.8 of [4], this also implies that the absolute Galois group of $F((t))(x)^{\mathrm{ab}}$ is free for F separably closed.

Moreover, if k' is the field obtained from k by adjoining a set of compatible n^{th} roots to all the non-zero elements of k, then Pop's argument also shows that $L' := Lk'$ has projective absolute Galois group in the case that k is a local field such as $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$. (Here the adjunction of additional roots is to deal with the fact that $\mathrm{Gal}(k)$ is no longer projective.) Since Lemma 8.3 then holds with L replaced by L' (and with L_i in the proof replaced by its compositum with k'), the above proof of Theorem 8.4 then shows that $\mathrm{Gal}(L')$ is a free profinite group.

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