Shafarevich Conjecture – The Shafarevich Conjecture in inverse Galois theory asserts that the absolute Galois group $G_{\mathbb{Q}^{ab}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}^{ab})$ of $\mathbb{Q}^{ab}$ is a free profinite group of countable rank. Here $\mathbb{Q}^{ab}$ is the maximal abelian extension of $\mathbb{Q}$, or equivalently (by the Kronecker-Weber Theorem) the maximal cyclotomic extension of $\mathbb{Q}$.

I.R. Shafarevich posed this assertion as an important problem during a 1964 series of talks at Oberwolfach on the solution to the class field tower problem. The conjecture would imply an affirmative answer to the inverse Galois problem over $\mathbb{Q}^{ab}$, i.e. that every finite group is a Galois group over $\mathbb{Q}^{ab}$. By a theorem of Iwasawa [Iw, p.567] (see also [FJ, Cor. 24.2]), a profinite group Π of countable rank is free (as a profinite group) if and only if every finite embedding problem for Π has a proper solution. Thus the Shafarevich Conjecture is equivalent to the assertion that if H is a quotient of a finite group G, then every H-Galois field extension of $\mathbb{Q}^{ab}$ is dominated by a G-Galois field extension of $\mathbb{Q}^{ab}$.

A weakening of this assertion is known: that the profinite group $G_{\mathbb{Q}^{ab}}$ is projective, i.e. every finite embedding problem for $G_{\mathbb{Q}^{ab}}$ has a weak solution. Projectivity is equivalent to the condition of cohomological dimension ≤ 1 [Se, Chap. 1, Props. 16 and 45], and this holds for $G_{\mathbb{Q}^{ab}}$ by [Se, Chap. 2, Prop. 9]. On the other hand, the absolute Galois group $G_{\mathbb{Q}}$ is not projective, since the surjection $G_{\mathbb{Q}} \to \mathbb{Z}/2\mathbb{Z}$ corresponding to the extension $\mathbb{Q}(i)/\mathbb{Q}$ does not factor through $\mathbb{Z}/4\mathbb{Z}$. Thus the analog of the Shafarevich Conjecture does not hold for $\mathbb{Q}$.

Evidence for the conjecture. Many finite groups, including "most" simple groups, have been realized as Galois groups over $\mathbb{Q}^{ab}$ [MM, Chapter II, §10]. These realizations provide evidence for the inverse Galois problem over $\mathbb{Q}^{ab}$ and hence for the Shafarevich Conjecture. Typically these realizations have been achieved by constructing Galois branched covers of the projective line over $\mathbb{Q}^{ab}$. Since $\mathbb{Q}^{ab}$ is Hilbertian [Vo, Cor. 1.28], such a realization implies that the covering group is a Galois group of a field extension of $\mathbb{Q}^{ab}$. Most of these branched covers have been constructed by means of rigidity; cf. [MM] and [Vo] for a discussion of this approach. (Some of these covers are actually defined over the $\mathbb{Q}$-line, and their covering groups are thus Galois groups over $\mathbb{Q}$.)

The rigidity approach also suggests a possible way of proving the Shafarevich Conjecture. Matzat introduced the notion of GAR-realizability of a group, this being realizability as the Galois group of a branched cover with certain additional properties (cf. [MM, Chapter 4, §3.1]). Many simple groups have been GAR-realized over $\mathbb{Q}^{ab}$, and the Shafarevich Conjecture would follow if it were shown that *every* finite simple group has a GAR-realization over $\mathbb{Q}^{ab}$. See [MM, Chapter 4, §§3,4].

The solvable case of the Shafarevich Conjecture has been proven: Iwasawa [Iw] showed that the maximal pro-solvable quotient of $G_{\mathbb{Q}^{ab}}$ is a free pro-solvable group of countable rank. In particular, every finite solvable group is a Galois group over $\mathbb{Q}^{ab}$, and every embedding problem for $G_{\mathbb{Q}^{ab}}$ with finite solvable kernel has a proper solution. Iwasawa's result also holds for the maximal abelian extension K^{ab} of any global field K, and for the maximal cyclotomic extension K^{cycl} of any global field K [Iw, Theorems 6 and 7].

Generalizations of the conjecture. The Shafarevich Conjecture can be posed with $\mathbb{Q}$ replaced by any global field K. In this generalized form, it asserts that the absolute Galois group of K^{cycl} is free of countable rank (as a profinite group). This conjecture remains open in the number field case, but has been proven by Harbater [Ha, Cor. 4.2] and Pop

[Po1] in the case that K is the function field of a curve over a finite field k. (See also [HV, Cor. 4.7] and [MM, §V.2.4].) Since $k^{\mathrm{cycl}} = \bar{\mathbb{F}}_p$ if k is a finite field of characteristic p, this assertion is equivalent to stating that the absolute Galois group of K is free of countable rank if K is the function field of a curve over $\bar{\mathbb{F}}_p$. This result is shown by using patching methods involving formal schemes or rigid analytic spaces, in order to show that all finite embedding problems for G_K have a proper solution — i.e. that every connected H-Galois branched cover of the curve is dominated by a connected G-Galois branched cover, if H is a quotient of the finite group G. By Iwasawa's theorem [Iw, p.567], the result follows. The proof also shows that if C is a curve over an arbitrary algebraically closed field of cardinality κ, then every finite embedding problem for G_K has exactly κ proper solutions. By a theorem of O. Melnikov and Z. Chatzidakis [Ja, Lemma 2.1], it follows that G_K is free profinite of rank κ, generalizing the geometric case of the Shafarevich Conjecture (see [Ha, Thm. 4.4], [Po1, Cor. to Thm. A]).

As another proposed generalization of the Shafarevich Conjecture (which would subsume the above case of global fields), Fried and Völklein conjectured [FV, p.470] that if K is a countable Hilbertian field whose absolute Galois group G_K is projective, then G_K is free of countable rank. They proved a special case of this [FV, Theorem A], viz. that G_K is free of countable rank if K is a countable Hilbertian pseudo-algebraically closed (PAC) field of characteristic 0. For example, this applies to the field $K = \mathbb{Q}^{\mathrm{tr}}(\sqrt{-1})$, where $\mathbb{Q}^{\mathrm{tr}}$ is the field of totally real algebraic numbers, by results of Weissauer and Pop; see [Vo, p.151], [MM, p.286]. Later Pop [Po2, Theorem 1] removed the characteristic 0 hypothesis from the above result. This solves [FJ, Problem 24.41]. (See also [HJ].) Since $\mathbb{Q}^{\mathrm{ab}}$ is not PAC (as proven by Frey [FJ, Cor. 10.15]), this result does not prove the Shafarevich Conjecture itself. But it does imply that $G_{\mathbb{Q}}$ has a free normal subgroup of countable rank for which the quotient is of the form $\prod_{n=2}^{\infty} S_n$ [FV] (instead of the form $\hat{\mathbb{Z}}^* = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ in the Shafarevich Conjecture). The Fried-Völklein conjecture holds if K is Galois over $k(x)$, for k an algebraically closed field ([Ja, Prop. 4.4], using the geometric case of the Shafarevich Conjecture [Ha], [Po1]). More generally, it holds if K is *large* in the sense of Pop [Po2, Theorem 2.1]; cf. also [MM, §V.4]. A solvable case of the conjecture holds, extending Iwasawa's result: For K Hilbertian with G_K projective, every embedding problem for G_K with finite solvable kernel has a proper solution [Vo, Cor. 8.25].

References.

[Fr] M. Fried, ed. "Recent Developments in the Inverse Galois Problem". AMS Contemporary Math. Series, vol. 186, 1995.

[FJ] M. Fried and M. Jarden. "Field Arithmetic". Springer-Verlag, Berlin-Heidelberg, 1986.

[FV] M. Fried and H. Völklein. The embedding problem over a Hilbertian PAC field. Annals of Math. **135** (1992), 469-481.

[HJ] D. Haran and M. Jarden. Regular split embedding problems over complete valued fields. Forum Math. **10** (1998), 329-351.

[HV] D. Haran and H. Völklein. Galois groups over complete valued fields. Israel J. Math. **93** (1996), 9-27.

[Ha] D. Harbater. Fundamental groups and embedding problems in characteristic p. In

[Fr], pp. 353-370.
[Iw] K. Iwasawa. On solvable extensions of algebraic number fields. Annals of Math. **58** (1953), 548-572.
[Ja] M. Jarden. On free profinite groups of uncountable rank. In [Fr], pp. 371-383.
[MM] G. Malle and B.H. Matzat. "Inverse Galois Theory". Springer-Verlag, Berlin-Heidelberg-New York, 1999.
[Po1] F. Pop. Étale Galois covers over smooth affine curves. Invent. Math. **120** (1995), 555-578.
[Po2] F. Pop. Embedding problems over large fields. Ann. of Math. **144** (1996), 1-34.
[Se1] J.-P. Serre. "Cohomologie Galoisienne". Lect. Notes in Math. vol. 5, Springer-Verlag, Berlin-Heidelberg-New York, 1964.
[Vo] H. Völklein. "Groups as Galois Groups". Cambridge Studies in Adv. Math. vol. 53, Cambridge Univ. Press, Cambridge-New York-Melbourne, 1996.