Refinements to patching and applications to field invariants

David Harbater, Julia Hartmann, and Daniel Krashen

Abstract

We introduce a notion of refinements in the context of patching, in order to obtain new results about local-global principles and field invariants in the context of quadratic forms and central simple algebras. The fields we consider are finite extensions of the fraction fields of two-dimensional complete domains that need not be local. Our results in particular give the u-invariant and period-index bound for these fields, as consequences of more general abstract results.

1 Introduction

In this manuscript we introduce the notion of refinements in the context of patching, and use this to obtain results about quadratic forms and central simple algebras over fraction fields of two-dimensional complete domains. These provide strengthenings and analogs of results in earlier papers. Among our results here are local-global principles, which in the case of quadratic forms concern isotropy, the Witt group, the Witt index, and the u-invariant. In the case of central simple algebras they concern Brauer equivalence and the index. In addition, we obtain explicit results about the values of the u-invariant and the period-index bounds for these fraction fields.

Classically, one relates the u-invariant and period-index bound for complete discretely valued fields to those of their residue fields. Here, we consider the analogous situation of fraction fields of two-dimensional complete domains, which need not be local. We focus on these two situations:

(i) the fraction field of a two-dimensional Noetherian complete local domain R (e.g. $k((x,t))$);
(ii) a finite separable extension of the fraction field of the t-adic completion of $T[x]$, where T is a complete discrete valuation ring with uniformizer t.

In the context of central simple algebras, we obtain the following result. (Our use of the term "Brauer dimension" is explained before Theorem 4.21.)

Theorem 1.1. *In the above two situations, assume that the residue field k of R (resp. T) has Brauer dimension d away from $p := \mathrm{char}(k)$. Then $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^{d+1}$ for all $\alpha \in \mathrm{Br}(E)$ whose period is not divisible by p.*

See Theorem 4.23, which also treats Brauer classes $\alpha \in \mathrm{Br}(E)$ of *arbitrary* period in the mixed characteristic case. Using that, we obtain a local analog of [PS14, Theorem 1]:

Corollary 1.2. *Let L be the fraction field of $\mathbb{Z}_p[[x]]$ or of the p-adic completion of $\mathbb{Z}_p[x]$, and let E be a finite extension of L. Then $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^2$ for all $\alpha \in \mathrm{Br}(E)$.*

Theorem 4.23 also shows that the same conclusion holds if instead L is the fraction field of the p-adic completion of $\mathbb{Z}_p^{\mathrm{ur}}[[x]]$ or $\mathbb{Z}_p^{\mathrm{ur}}[x]$, with $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$ if $\mathrm{per}(\alpha)$ is prime to p.

For quadratic forms, we prove an analog of Theorem 1.1; see Theorem 4.11. This yields results about values of the u-invariant in mixed characteristic (Corollaries 4.13 and 4.14), and also the following result in equicharacteristic:

Theorem 1.3. *Let k be a field of characteristic unequal to two, and let E be a finite separable extension of the fraction field of $k[x][[t]]$ or of $k[[x,t]]$. Then*

- *$u(E) = 4$ if k is algebraically closed.*
- *$u(E) = 8$ if k is finite, or if $k = k_0((z))$ with k_0 algebraically closed.*
- *$u(E) = 16$ if $k = k_0((z))$ with k_0 finite, or if $k = \mathbb{Q}_p$ for some p (which can equal 2).*
- *$u(E) = 32$ if $k = \mathbb{Q}_p(z)$ or if $k = \mathbb{Q}_p((z))$ for some p.*

Note that the value of the u-invariant or the period-index bound for a given field does not in general give much information about the corresponding invariant for arbitrary finite separable extensions. So the above results would not follow simply from knowing the values of these invariants for the fraction fields of rings of the form $T[x]$ or $T[[x]]$.

Previous results about the u-invariant and period-index bounds for related fields appeared in such papers as [PS10], [HHK09], [Lee13], [Sal08], [deJ04], and [Lie11]. To obtain our present results, we build on the patching framework that was used in our previous manuscripts [HH10], [HHK09], [HHK15], and [HHK13].

As in those papers, the patching framework also enables us to obtain local-global principles. In particular, our Corollary 4.7 proves a local-global principle for isotropy; Corollary 4.8 relates the u-invariants of fields to those of their completions; and Corollary 4.17 provides a local-global principle for the period-index bound of a field. See also related results in [CPS12], [PS14], and [Hu15].

The key new ingredient in this paper is a refinement principle for patching. As in the patching framework, we consider a projective normal curve over a complete discrete valuation ring, and we choose a finite partition of the closed fiber. Criteria for patching and local-global principles are given in terms of intersection and factorization properties for a certain quadruple of rings arising from the partition. By enlarging the given partition or modifying the model (e.g. by blowing up), one can refine a quadruple, obtaining a new one with one part expanded. The refinement principle that we state in this manuscript relates the intersection and factorization properties of a given quadruple to that of two other quadruples: the refined one, and the quadruple arising from the part that was expanded. This principle is first stated in an abstract context (Proposition 2.14), and then used in a geometric context to establish "patching on patches" (Proposition 3.9) and patching on exceptional divisors of a blow-up (Proposition 3.10, answering a question of Yong Hu).

The manuscript is organized as follows. In Section 2 we present general results about patching and local-global principles for quadruples of groups or rings, and then state our abstract refinement principle. In Section 3 we turn to the geometric situation, generalizing the patching setup of [HHK09] in Section 3.1 to allow more general open subsets of the closed fiber, and then obtaining consequences of the refinement principle, in Sections 3.2 and 3.3. We then turn to quadratic forms and central simple algebras in Section 4, first proving local-global results in an abstract context (Theorems 4.1 and 4.15), and then specializing to the geometric situation to obtain results about numerical invariants, including those above.

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2 Patching and refinement

This section proves a refinement principle (Proposition 2.14) that will afterwards permit us to obtain results about patching and local-global principles over certain function fields, once such results are known for related fields. In this section the presentation is in a more general framework. We begin with a discussion of patching and local-global principles, and the related conditions of factorization and intersection for quadruples. That discussion draws heavily on prior papers of the present authors.

2.1 Diamonds of groups and rings

Patching is a method that mimics constructions from complex geometry to obtain global objects from more local ones. In our algebraic version, we work with objects in a category $\mathcal{C}$ consisting of sets, possibly with additional structure (e.g. groups or rings), and we will consider quadruples of objects $S_\bullet = (S, S_1, S_2, S_0)$ together with morphisms forming a commutative diagram

$$\begin{array}{ccccc} & & S_0 & & \\ & {\scriptstyle\beta_1}\nearrow & & \nwarrow{\scriptstyle\beta_2} & \\ S_1 & & & & S_2 \\ & {\scriptstyle\alpha_1}\nwarrow & & \nearrow{\scriptstyle\alpha_2} & \\ & & S & & \end{array} \qquad (*)$$

As a motivating example, one can think of these as being the collection of functions or other objects on a global space X, on two subsets U_1 and U_2 that cover X, and on their intersection U_0, as indicated in the following commutative diagram:

$$\begin{array}{ccccc} & & U_0 & & \\ & \swarrow & & \searrow & \\ U_1 & & & & U_2 \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

For patching, the two key properties of a quadruple are factorization and intersection (see Theorem 2.8). More precisely:

Definition 2.1. (a) A *diamond* $S_\bullet$ in $\mathcal{C}$ consists of a commutative diagram $(*)$ as above such that $(\alpha_1, \alpha_2) : S \to S_1 \times_{S_0} S_2$ is a monomorphism. For short we will often write $S_\bullet = (S, S_1, S_2, S_0)$ as a quadruple if the maps α_i, β_i are understood.

(b) A diamond $S_\bullet$ as above has the *intersection property* if the map $(\alpha_1, \alpha_2) : S \to S_1 \times_{S_0} S_2$ is an isomorphism. It is *injective* if each of the maps α_i, β_i is injective.

(c) Let $G_\bullet = (G, G_1, G_2, G_0)$ be a diamond of groups, together with maps α_i, β_i. We say that $G_\bullet$ has the *factorization property* if $G_0 = \beta_1(G_1)\beta_2(G_2)$, i.e. every element of G_0 is of the form $\beta_1(g_1)\beta_2(g_2)$ with $g_i \in G_i$.

In the injective case, we often regard the maps α_i, β_i as inclusions; and to emphasize this, we write the diamond as $(S \le S_1, S_2 \le S_0)$. With these identifications, the intersection property asserts that $S = S_1 \cap S_2$ in S_0; and the factorization property for diamonds of groups then asserts that each element of G_0 can be factored as $g_1 g_2$ with $g_i \in G_i$. By applying factorization to the inverse of each element of G_0, we obtain:

Lemma 2.2. *A diamond of groups (G, G_1, G_2, G_0) has the factorization property (respectively the intersection property) if and only if (G, G_2, G_1, G_0) does.*

We will often consider injective diamonds of rings $F_\bullet = (F \le F_1, F_2 \le F_0)$ with F is a field and each F_i a direct product of finitely many fields. In particular, we have:

Example 2.3. Let Γ be a bipartite connected (multi-)graph, with vertex set $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ and edge set $\mathcal{E}$. Suppose that we are given a Γ-field in the sense of [HHK14, Section 2.1.1]; i.e. a field F_v for each $v \in \mathcal{V}$ and a field F_e for each $e \in \mathcal{E}$, together with an inclusion $F_v \hookrightarrow F_e$ whenever v is a vertex of e. These fields and inclusions define an inverse system of fields; and if the inverse limit is a field F then this is called a "factorization inverse system" over F ([HHK15], Section 2), and the graph together with the associated fields is called a Γ/F-field ([HHK14], Section 2.1.1). In this situation, set $F_i = \prod_{v \in \mathcal{V}_i} F_v$ for $i = 1, 2$ and set $F_0 = \prod_{e \in \mathcal{E}} F_e$. Then the inclusions $F_v \hookrightarrow F_e$ induce inclusions $F_i \hookrightarrow F_0$ for $i = 1, 2$; and $F_\bullet = (F \le F_1, F_2 \le F_0)$ is an injective diamond of the above form.

In fact, every such diamond arises in this way:

Proposition 2.4. *Let $F_\bullet = (F \le F_1, F_2 \le F_0)$ be an injective diamond of rings having the intersection property, with F a field and each F_i a finite product of fields. Then $F_\bullet$ is induced from a factorization inverse system over F as in Example 2.3.*

Proof. Write $F_i = \prod_{\lambda \in \Lambda_i} F_\lambda$ with each F_λ a field, and let $\mathcal{E} = \Lambda_0, \mathcal{V}_1 = \Lambda_1, \mathcal{V}_2 = \Lambda_2$. To give the structure of a graph Γ, we will associate to every $e \in \mathcal{E}$ elements $v_i \in \mathcal{V}_i$ for $i = 1, 2$. To do this, choose $e' \in \mathcal{E}$ and for $i = 1, 2$ consider the composition $\prod_{v \in \mathcal{V}_i} F_v \to \prod_{e \in \mathcal{E}} F_e \to F_{e'}$. Since $F_{e'}$ is a field, the image of this homomorphism is a domain. So the kernel must be a prime and hence maximal ideal. Thus this composition factors through a unique projection $\prod_{v \in \mathcal{V}_i} F_v \to F_{v_i'}$. The assignment of e to (v_1', v_2') gives a graph Γ, and the above homomorphisms $F_{v_i'} \to F_{e'}$ give the structure of a Γ-field. The inverse limit of the fields F_v, F_e is the intersection $F_1 \cap F_2$, which is equal to F by the intersection property; and so these fields form a factorization inverse system, which induces the diamond. $\square$

2.2 Patching and local-global principles

To study patching and local-global principles in this framework, we will need to introduce the notion of diamonds of categories and tensor categories.

Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_0$ be categories, and suppose we are given functors $G_i : \mathcal{C}_i \to \mathcal{C}_0$. The (2-)fiber product $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ is defined to be the category whose objects are triples (C_1, C_2, ϕ) where $C_i \in \mathcal{C}_i$ and $\phi : G_1(C_1) \to G_2(C_2)$ is an isomorphism. A morphism $(C_1, C_2, \phi) \to (D_1, D_2, \psi)$ is defined to be a pair or morphisms $f_i : C_i \to D_i$ such that we have a commutative square

$$\begin{array}{ccc} G_1(C_1) & \xrightarrow{\phi} & G_2(C_2) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ G_1(D_1) & \xrightarrow[\psi]{} & G_2(D_2) \end{array}$$

Definition 2.5 (Patching Problems)**.** A *diamond* of (tensor) categories is a diagram

$$\begin{array}{ccccc} & & \mathcal{C}_0 & & \\ & {\scriptstyle G_1}\nearrow & & \nwarrow{\scriptstyle G_2} & \\ \mathcal{C}_1 & & & & \mathcal{C}_2 \\ & {\scriptstyle F_1}\nwarrow & & \nearrow{\scriptstyle F_2} & \\ & & \mathcal{C} & & \end{array}$$

of (tensor) categories and functors, together with a natural isomorphism of functors $\alpha : G_1F_1 \to G_2F_2$, such that the functor $\Phi : \mathcal{C} \to \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$, given by $\Phi(c) = (F_1(c), F_2(c), \alpha(c))$, is essentially injective. In this situation:

(a) A *patching problem* is an object $C_\bullet$ in the fiber product category $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$.
(b) A *solution* to a patching problem $C_\bullet$ is an object $C \in \mathcal{C}$ such that $\Phi(C) \cong C_\bullet$.
(c) *Patching holds* for the diamond if Φ is an equivalence of categories.

Example 2.6 (Patching for torsors)**.** If $R_\bullet = (R, R_1, R_2, R_0)$ is a diamond of rings, and G is an algebraic group over R, we obtain a diamond of categories of torsors $\mathrm{Tors}(G_{R_\bullet})$. In this case, we refer to patching problems (solutions, etc.) for $\mathrm{Tors}(G_{R_\bullet})$ as patching problems (solutions, etc.) for G-torsors.

Example 2.7 (Patching for free modules)**.** Similarly, if $R_\bullet = (R, R_1, R_2, R_0)$ is a diamond of rings, we obtain a diamond $\mathcal{F}(R_\bullet)$ of tensor categories of free modules of finite rank. If patching holds for $\mathcal{F}(R_\bullet)$, it follows that patching holds for other categories of structures which may be defined in terms of the category of vector spaces and its tensor structure (e.g. central simple algebras). The proof is as in [HH10], Theorems 7.1 and 7.5.

Theorem 2.8. *Let $R_\bullet = (R, R_1, R_2, R_0)$ be a diamond of rings.*

(a) The following conditions are equivalent:

(i) For every $n \geq 1$, the diamond of groups $\mathrm{GL}_n(R_\bullet)$ satisfies the intersection and factorization properties.

(ii) Patching holds for free modules; that is, for the diamond of categories $\mathcal{F}(R_\bullet)$.

(b) Under these conditions, the inverse to the equivalence $\Phi : \mathcal{F}(R) \to \mathcal{F}(R_1) \times_{\mathcal{F}(R_0)} \mathcal{F}(R_2)$ is given by taking the intersection of free modules.

(c) Suppose that R is a field, that each R_i is a finite product of fields, and the diamond is injective. Then the above two conditions are also equivalent to:

(iii) For every linear algebraic group G over R, patching holds for G-torsors, i.e. for $\mathrm{Tors}(G_{R_\bullet})$.

Proof. First observe that if Φ is an equivalence of categories, then necessarily R is the fiber product $R_1 \times_{R_0} R_2$. Namely, for any $c \in R_1 \times_{R_0} R_2$, consider the endomorphism of the rank one object (R_1, R_2, id) given by multiplication by c. Since Φ is an equivalence, this morphism is induced by an endomorphism of the free rank one R-module R; i.e. by multiplication by some element of R, which is necessarily equal to c. Thus $R_1 \times_{R_0} R_2 = R$. The first two parts of the assertion now follow from [Har84, Proposition 2.1], which says that if $R = R_1 \times_{R_0} R_2$, then Φ is an equivalence of categories if and only if factorization holds; and moreover that in this case the inverse of Φ is given by taking the fiber product of objects. The third part follows from [HHK14, Theorem 2.1.4], which applies here by Proposition 2.4 above. □

Definition 2.9. *Patching holds* for the diamond of rings $R_\bullet$ if either of the equivalent conditions of Theorem 2.8(a) holds.

Lemma 2.10. *Let $R_\bullet = (R, R_1, R_2, R_0)$ be a diamond of rings. Let $R \subseteq S$ be a finite extension of rings such that S is a free R-module. Set $S_i = S \otimes_R R_i$.*

(a) If the intersection property holds for $R_\bullet$, then it also holds for $S_\bullet$.

(b) If patching holds for $R_\bullet$, then it also holds for $S_\bullet$.

Proof. For the first part, suppose that $R_\bullet$ has the intersection property. We thus have a left exact sequence $0 \to R \to R_1 \times R_2 \to R_0$ of R-modules, where the map on the right is given by subtracting the image under $R_2 \to R_0$ from the image under $R_1 \to R_0$. Since S is free over R, the sequence $0 \to S \to S_1 \times S_2 \to S_0$ is also left exact. Consequently $S_\bullet$ also has the intersection property.

For the second statement, let (s_j) be a basis for the free R-module S and suppose that $s_j s_k = \sum a^\ell_{j,k} s_\ell$. Then the category of finitely generated free S_i-modules is equivalent to the category whose objects are finitely generated free R_i-modules together with R_i-endomorphisms $\widetilde{s}_j$ (corresponding to multiplication by s_j) such that $\widetilde{s}_j \widetilde{s}_k = \sum a^\ell_{j,k} \widetilde{s}_\ell$, and where the morphisms in the category are required to commute with each $\widetilde{s}_j$. In particular, since patching holds for the diamond of categories $\mathcal{F}(R_\bullet)$, it follows that patching also holds for $\mathcal{F}(S_\bullet)$. That is, patching holds for $S_\bullet$. □

Definition 2.11. Let $R_\bullet = (R, R_1, R_2, R_0)$ be a diamond of rings. Let $\mathcal{V}$ be a class of R-varieties. We say that the *local-global principle holds* for $\mathcal{V}$ with respect to $R_\bullet$ if for each $V \in \mathcal{V}$, the condition $V(R_i) \neq \emptyset$ for $i = 1, 2$ implies that $V(R) \neq \emptyset$.

This definition applies in particular to the key case considered above, where we are given an injective diamond ($F \le F_1, F_2 \le F_0$), with F is a field and each F_i a finite product of fields $\prod_j F_{ij}$, and where we take $\mathcal{V}$ to be the class of G-torsors over F, for some linear algebraic group G over F. The set of isomorphism classes of G-torsors over F is in natural bijection with the pointed Galois cohomology set $H^1(F, G)$, and we write $H^1(F_i, G)$ for $\prod_j H^1(F_{ij}, G)$. The local-global principle holds for (the class of) G-torsors if and only if the natural map $\phi : H^1(F, G) \to H^1(F_1, G) \times H^1(F_2, G)$ has trivial kernel.

Given a diamond $R_\bullet = (R, R_1, R_2, R_0)$ of rings, and a linear algebraic group G over R, there is an associated diamond of groups $G(R_\bullet) = (G(R), G(F_1), G(F_2), G(F_0))$ of rational points. By embedding G in $\mathrm{GL}_{n,R} \subset \mathbb{A}^N_R$ and considering coordinates, we immediately obtain the following lemma, which was implicitly used in [HHK09], Section 3.

Lemma 2.12. *Suppose that $R_\bullet$ is a diamond of rings with the intersection property and that G is a linear algebraic group over R. Then $G(R_\bullet)$ also has the intersection property.*

Theorem 2.13. *Suppose that $F_\bullet = (F \le F_1, F_2 \le F_0)$ is an injective diamond of rings with F a field and each F_i a finite direct product of fields. Assume moreover that patching holds for $F_\bullet$. Then the following statements are equivalent for a linear algebraic group G over F:*

(i) $G(F_\bullet)$ satisfies factorization.

(ii) The local-global principle holds for G-torsors with respect to $F_\bullet$.

(iii) The local-global principle holds, with respect to $F_\bullet$, for the class of F-varieties V equipped with a G-action such that $G(F_0)$ acts transitively on $V(F_0)$.

Proof. By Proposition 2.4, the diamond $F_\bullet$ arises from a factorization inverse system; and so [HHK15, Theorem 2.4] applies. Thus there is the following exact sequence of pointed sets:

$$H^0(F_1, G) \times H^0(F_2, G) \to H^0(F_0, G) \to H^1(F, G) \to H^1(F_1, G) \times H^1(F_2, G).$$

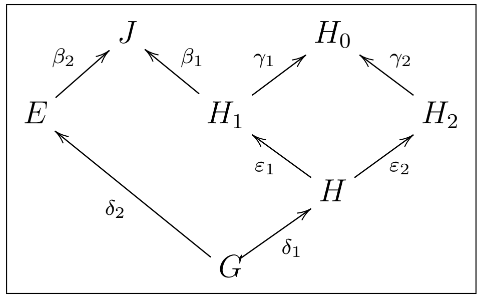
Condition (i) asserts surjectivity of the first arrow, which is equivalent to the third arrow having trivial kernel. That latter property is the same as condition (ii), as discussed above. Moreover the triviality of that kernel implies condition (iii) by [HHK15, Corollary 2.8]. Finally, condition (iii) trivially implies condition (ii), since G-torsors over F satisfy the transitivity hypothesis of condition (iii). □

2.3 A refinement principle

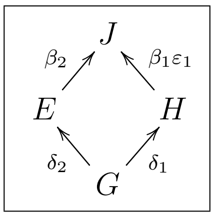
In studying factorization and intersection for a diamond $G_\bullet = (G, E, H, J)$ of groups, it will prove useful to consider the situation when H is itself the base for another diamond of groups $H_\bullet = (H, H_1, H_2, H_0)$. We would then like to combine the two diamonds into one new diamond, thereby refining the original situation. The natural question is to what extent such refinements preserve factorization and intersection.

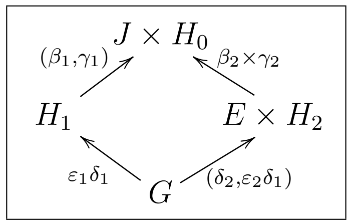
Below, given maps $f : A \to B$, $g : A \to B'$, and $f' : A' \to B'$, we write $(f, g) : A \to B \times B'$ for the map $a \mapsto (f(a), g(a))$, and write $f \times f' : A \times A' \to B \times B'$ for $(a, a') \mapsto (f(a), f'(a'))$.

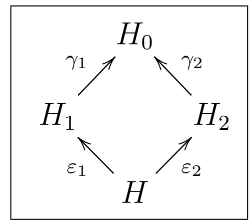
Proposition 2.14 (Refinement Principle)**.** *Suppose we are given a commutative diagram of groups and homomorphisms*

$\mathscr{R} =$ 

such that the following diagrams are diamonds of groups:

$G_\bullet =$ 

$G'_\bullet =$ 

$H_\bullet =$ 

Then

1. *If $G'_\bullet$ has the factorization property then so does $H_\bullet$.*
2. *If $G'_\bullet$ has the factorization property and $H_\bullet$ has the intersection property then $G_\bullet$ has the factorization property.*
3. *If $G'_\bullet$ has the intersection property then so does $G_\bullet$.*
4. *If $G'_\bullet$ has the intersection property and $G_\bullet$ has the factorization property then $H_\bullet$ has the intersection property.*
5. *If $G_\bullet$ and $H_\bullet$ have the intersection property then so does $G'_\bullet$.*
6. *If $G_\bullet$ and $H_\bullet$ have the factorization property then so does $G'_\bullet$.*

Proof.

Part 1: Assume that $G'_\bullet$ has the factorization property. Let $h_0 \in H_0$. By hypothesis, we may write $(1, h_0) \in J \times H_0$ as $(\beta_1, \gamma_1)(h_1) \cdot (\beta_2(e), \gamma_2(h_2))$ for some elements $h_1 \in H_1$ and $(e, h_2) \in E \times H_2$. In particular, $\gamma_1(h_1)\gamma_2(h_2) = h_0$, hence $H_\bullet$ has the factorization property.

Part 2: We now additionally assume that $H_\bullet$ has the intersection property. To see that $G_\bullet$ has the factorization property, suppose that $j \in J$ and consider $(j, 1) \in J \times H_0$. Using factorization for $G'_\bullet$, we may find $h_1 \in H_1$ and $(e, h_2) \in E \times H_2$ such that $j = \beta_1(h_1)\beta_2(e)$ and $1 = \gamma_1(h_1)\gamma_2(h_2)$. Since by the latter equality, h_1 and h_2^{-1} have the same image in H_0, the intersection hypothesis for $H_\bullet$ implies that there exists $h \in H$ with $\varepsilon_1(h) = h_1$ and $\varepsilon_2(h) = h_2^{-1}$. Hence $j = \beta_1\varepsilon_1(h)\beta_2(e)$, which proves factorization for $G_\bullet$ (see Lemma 2.2).

Part 3: Suppose that $G'_\bullet$ has the intersection property, and let $(e,h) \in E \times H$ satisfy $\beta_2(e) = \beta_1\varepsilon_1(h)$. We wish to show that e and h are the images of a single element $g \in G$. To see this, we first note that the elements $\varepsilon_1(h) \in H_1$ and $(e, \varepsilon_2(h)) \in E \times H_2$ have the same image in $J \times H_0$. Since $G'_\bullet$ has the intersection property, we may find $g \in G$ so that $\varepsilon_1\delta_1(g) = \varepsilon_1(h)$, $\delta_2(g) = e$, and $\varepsilon_2\delta_1(g) = \varepsilon_2(h)$. But $(\varepsilon_1, \varepsilon_2)$ is injective, since $H_\bullet$ is assumed to be a diamond. It follows that $\delta_1(g) = h$, and g is as desired.

Part 4: Suppose that $G'_\bullet$ has the intersection property and $G_\bullet$ has the factorization property. Assume that $h_0 = \gamma_1(h_1) = \gamma_2(h_2)$ with $h_i \in H_i$. We would like to show that h_1, h_2 are the images in H_1, H_2 of some element of H. Using the factorization property for $G_\bullet$, we may write $\beta_1(h_1) \in J$ as $\beta_2(e) \cdot \beta_1\varepsilon_1(h)$ with $e \in E$, $h \in H$. Let $h'_1 = h_1\varepsilon_1(h)^{-1} \in H_1$ and $h'_2 = h_2\varepsilon_2(h)^{-1} \in H_2$. Thus h'_1, h'_2 have the same image in H_0 under γ_1, γ_2 respectively, viz. the element $h'_0 := h_0\,\gamma_1\varepsilon_1(h)^{-1} = h_0\,\gamma_2\varepsilon_2(h)^{-1}$. Moreover $\beta_1(h'_1) = \beta_2(e)$. Consider the images of $h'_1 \in H_1$ and of $(e, h'_2) \in E \times H_2$ in $J \times H_0$. These are $(\beta_1, \gamma_1)(h'_1)$ and $(\beta_2(e), \gamma_2(h'_2))$, which by the previous considerations are equal.

Since $G'_\bullet$ has the intersection property, there exists an element $g \in G$ for which $\varepsilon_1\delta_1(g) = h'_1$ and $\varepsilon_2\delta_1(g) = h'_2$. So $\delta_1(g) \in H$ maps to $h'_1 \in H_1$ and $h'_2 \in H_2$ under $\varepsilon_1, \varepsilon_2$. Thus $h_1 \in H_1$ and $h_2 \in H_2$ are the images of the common element $\delta_1(g)h \in H$.

Part 5: Suppose $h_1 \in H_1$ and $(e, h_2) \in E \times H_2$ satisfy $(\beta_1(h_1), \gamma_1(h_1)) = (\beta_2(e), \gamma_2(h_2))$. The intersection property for $H_\bullet$ yields an element $h \in H$ such that $\varepsilon_i(h) = h_i$ for $i = 1, 2$. The intersection property for $G_\bullet$ then yields an element $g \in G$ such that $\delta_2(g) = e$ and $\delta_1(g) = h$. Thus $\varepsilon_1\delta_1(g) = \varepsilon_1(h) = h_1$ and $(\delta_2, \varepsilon_2\delta_1)(g) = (e, h_2)$; i.e. h_1 and (e, h_2) are the images of the common element $g \in G$.

Part 6: Let $(j, h_0) \in J \times H_0$. By factorization for $H_\bullet$, there exist $h_i \in H_i$ for $i = 1, 2$, such that $h_0 = \gamma_1(h_1)\gamma_2(h_2)$. By factorization for $G_\bullet$ and Lemma 2.2, there exist $h \in H$ and $e \in E$ such that $\beta_1(h_1)^{-1}j \in J$ equals $\beta_1\varepsilon_1(h) \cdot \beta_2(e)$; i.e. $j = \beta_1(h_1\varepsilon_1(h)) \cdot \beta_2(e) \in J$. Moreover $h_0 = \gamma_1(h_1)\gamma_2(h_2) = \gamma_1(h_1)\gamma_1\varepsilon_1(h)\gamma_2\varepsilon_2(h)^{-1}\gamma_2(h_2) = \gamma_1(h_1\varepsilon_1(h))\gamma_2(\varepsilon_2(h)^{-1}h_2)$. Thus the elements $h_1\varepsilon_1(h) \in H_1$ and $(e, \varepsilon_2(h)^{-1}h_2) \in E \times H_2$ provide a factorization of $(j, h_0) \in J \times H_0$. □

The following result will be useful in conjunction with the above proposition.

Lemma 2.15. *(a) Let $H^{(j)}_\bullet = (H^{(j)}, H^{(j)}_1, H^{(j)}_2, H^{(j)}_0)$ be a diamond of groups for each j. Write $H = \prod H^{(j)}$ and $H_i = \prod H^{(j)}_i$ for $i = 0, 1, 2$, and let $H_\bullet = (H, H_1, H_2, H_0)$, together with the products of the maps defining the diamonds $H^{(j)}_\bullet$. Then $H_\bullet$ is a diamond, and it satisfies intersection (resp. factorization) if and only if each $H^{(j)}_\bullet$ does.*

(b) Let $\widetilde{H}_\bullet = (\widetilde{H}, \widetilde{H}_1, \widetilde{H}_2, \widetilde{H}_0)$ be a diamond of groups, with associated maps $\alpha_i : \widetilde{H} \to \widetilde{H}_i$ and $\beta_i : \widetilde{H}_i \to \widetilde{H}_0$ for $i = 1, 2$. Let A be a group and write $H = \widetilde{H} \times A$, $H_1 = \widetilde{H}_1 \times A$, and $H_i = \widetilde{H}_i$ for $i = 0, 2$. Then $H_\bullet := (H, H_1, H_2, H_0)$ is a diamond with respect to the maps $\alpha_1 \times \mathrm{id}, \alpha_2 \circ \mathrm{pr}_1, \beta_1 \circ \mathrm{pr}_1, \beta_2$, where pr_1 is the first projection map. Moreover $H_\bullet$ satisfies intersection (resp. factorization) if and only if $\widetilde{H}_\bullet$ does.

Here part (a) is immediate, and part (b) then follows by applying part (a) to the two diamonds $\widetilde{H}_\bullet$ and $(A, A, 1, 1)$.

3 Patches and their fields

In this section, we apply the refinement principle, Proposition 2.14, to fields arising from curves over complete discretely valued fields. In that situation, it was shown in [HHK09] that patching holds for diamonds arising from a partition of the closed fiber of a normal projective model of the curve into finitely many closed points and irreducible open sets. Here we show the same holds for more general partitions of the closed fiber (Proposition 3.7); for partitions of a connected open subset of the closed fiber (Proposition 3.9); and for partitions of the exceptional divisor of a blow-up (Proposition 3.10). The second of these can be regarded as an assertion about "patching on patches." Related results for factorization with respect to algebraic groups, which will be used in Section 4, appear in Section 3.3.

For the sake of the applications in Section 4, we will need to consider reducible open sets in our partitions, in order to be able to treat branched covers that have reducible closed fibers over a given open subset of the base. This will require us first to generalize somewhat the framework that was considered in [HH10] and [HHK09], where the open sets had been assumed to be irreducible. (See also [Cuo13], where a similar generalization is considered.)

3.1 Setup

Consider a complete discrete valuation ring T with uniformizer t, residue field k, and fraction field K. Let F be a one-variable function field over K, and let $\widehat{X}$ be a *normal model* of F, i.e. a normal connected projective T-curve with function field F. Let X be the reduced closed fiber of $\widehat{X}$.

The following definition generalizes the notation in [HH10, Section 6] and [HHK09], where the open sets of X were required to be irreducible.

Definition 3.1. For a point $P \in X$ we let $\mathcal{O}_{\widehat{X},P}$ be its local ring, consisting of the elements of F that are regular at P. For a nonempty strict open subset $W \subset X$, we define $R_W = \bigcap_{P \in W} \mathcal{O}_{\widehat{X},P}$, and we let $\widehat{R}_W$ be the t-adic completion of R_W. If $\widehat{R}_W$ is a domain we also define F_W to be its fraction field.

Thus R_W is the subring of F consisting of the rational functions on $\widehat{X}$ that are regular at each point of W. The above definition agrees with those in [HH10] and [HHK09], which considered R_W and $\widehat{R}_W$ only when W meets just one irreducible component of X. For W an affine open subset of X, we view $\mathrm{Spec}(\widehat{R}_W)$ as a "thickening" of W, just as $\widehat{X}$ is a "thickening" of its closed fiber X. See Lemma 3.3(a) below. By convention, we also write $F_X := F$. If we have a second curve $\widehat{X}'$ with function field F', we will write R'_W, $\widehat{R}'_W$, and F'_W for the analogously defined rings (where W is now a nonempty strict open subset of the closed fiber X' of $\widehat{X}'$).

Remark 3.2. (a) For W a non-empty open subset of X as above, and for any point $P \in W$, the completion of $\widehat{R}_W$ at the ideal defined by $P \in \widehat{X}$ is the complete local ring of $\widehat{X}$ at P, which is a normal domain. Moreover, since R_W is normal and reduced (i.e. has no nilpotents), the same conditions hold for its completion $\widehat{R}_W$ (by [Mat80], 33.I, 34.A); this uses that T and hence R_W is excellent (by [Mat80], 34.B, 34.A).

(b) When studying rings of the form $\widehat{R}_W$ on normal models of F, it suffices to restrict attention to affine open sets W. This is for the following reason: Given a collection of some but not all irreducible components of X, by [BLR90, Sect. 6.7, Proposition 4] there is an associated contraction $\pi : \widehat{X} \to \widehat{Y}$. This is a projective birational T-morphism π to a normal model $\widehat{Y}$ of F over K that is an isomorphism away from these components, and which sends each of these components to a point. By the description of $\widehat{Y}$ given in [BLR90, Sect. 6.7, Theorem 1], if U is an open set strictly contained in the closed fiber Y of $\widehat{Y}$, and $W = \pi^{-1}(U)$, then the T-morphism π induces a T-algebra isomorphism between R_W (taken on $\widehat{X}$) and R_U (taken on $\widehat{Y}$). This in turn induces an isomorphism between the t-adic completions $\widehat{R}_W$ and $\widehat{R}_U$. In particular, let W be any connected open set strictly contained in X, and let π be chosen to contract precisely those components of X that are contained in W. Then $\widehat{R}_W = \widehat{R}_U$ for $U = \pi(W)$, and moreover U is affine.

With $W \subset X$ as above, the reduced closed fiber of $\mathrm{Spec}(R_W)$ is $\mathrm{Spec}(R_W/J)$, where J is the Jacobson radical of R_W (i.e. the radical of the ideal tR_W). The corresponding statement holds for $\mathrm{Spec}(\widehat{R}_W)$.

Lemma 3.3. *In the above situation, let W be a non-empty open subset of X.*

(a) If W is an affine open subset of X, then the reduced closed fibers of $\mathrm{Spec}(R_W)$ *and* $\mathrm{Spec}(\widehat{R}_W)$ *are each isomorphic to W.*

(b) More generally, if $W \neq X$, then the reduced closed fibers of $\mathrm{Spec}(R_W)$ *and* $\mathrm{Spec}(\widehat{R}_W)$ *are each isomorphic to $\pi(W)$, where π is the contraction of $\widehat{X}$ with respect to the irreducible components of X that are contained in W.*

Proof. (a) The assertion for R_W is clear, and it then follows for $\widehat{R}_W$ because R_W and $\widehat{R}_W$ have the same reduction modulo (t).

(b) Note that $\pi(\tilde{W})$ is open because the only components of W that are contracted by π are those contained in W. Also $R_W = R_{\pi(W)}$, as observed in Remark 3.2(b). So the assertion follows. □

Proposition 3.4. *Let W be a nonempty proper open subset of X, the closed fiber of $\widehat{X}$. Let $W_1, \dots, W_n$ be the connected components of W.*

(a) The ring $\widehat{R}_W$ is a domain (and so F_W is defined) if and only if W is connected.

(b) The ring $\widehat{R}_W$ is isomorphic to $\prod_{i=1}^n \widehat{R}_{W_i}$.

(c) *Let F' be a finite extension of F, and let $\widehat{X}'$ be the normalization of $\widehat{X}$ in F', with associated morphism $\pi : \widehat{X}' \to \widehat{X}$ and closed fiber X'. Let $W' = \pi^{-1}(W) \subset X'$. Then $\widehat{R}_W \otimes_{R_W} R'_{W'}$ is isomorphic to $\widehat{R}'_{W'} = \prod_{j=1}^n \widehat{R}'_{W'_j}$, where $W'_1, \dots, W'_n$ are the connected components of W'. If W is connected, $F_W \otimes_F F'$ is isomorphic to $\prod_{j=1}^n F'_{W'_j}$.*

Proof. We begin with part (b). By Lemma 3.3(b), the reduced closed fiber of $\mathrm{Spec}(R_W)$ is the disjoint union of the reduced closed fibers of $\mathrm{Spec}(R_{W_i})$, for $i = 1, \dots, r$. Thus the ideal $J \subset R_W$ defining the former closed fiber is the product of the relatively prime ideals $J_i \subset R_W$ defining the latter; and more generally $J^n = \prod_{i=1}^r J_i^n$ for all n. The Chinese Remainder Theorem implies that $R_W/J^n = \prod_{i=1}^r R_W/J_i^n = \prod_{i=1}^r R_{W_i}/J_i^n R_{W_i}$ for all n. Since J is the radical of the ideal $(t) \subset R_W$, and similarly for J_i and R_{W_i}, the asserted isomorphism follows by passing to the inverse limit.

The forward direction of part (a) is now immediate. For the reverse implication of (a), note that the condition that $\widehat{R}_W$ is a domain is equivalent to its spectrum being reduced and irreducible. It was observed in Remark 3.2(a) that $\widehat{R}_W$ (or equivalently its spectrum) is reduced and normal. Moreover a normal scheme is irreducible if and only if it is connected. Thus it suffices to show that $\mathrm{Spec}(\widehat{R}_W)$ is connected.

So suppose that $\mathrm{Spec}(\widehat{R}_W)$ is the disjoint union of two Zariski open subsets Y_1 and Y_2. We wish to show that Y_1 or Y_2 is empty. First note that $\mathrm{Spec}(\widehat{R}_W/(t))$ is the disjoint union of the two Zariski open subsets $\bar{Y}_1$ and $\bar{Y}_2$, the restrictions of Y_1 and Y_2 to $\mathrm{Spec}(\widehat{R}_W/(t))$. Also, $\mathrm{Spec}(\widehat{R}_W/(t))$ is connected since W and hence $\pi(W)$ is, using Lemma 3.3(b). So either $\bar{Y}_1$ or $\bar{Y}_2$ is empty. But any maximal ideal of $\widehat{R}_W$ contains t, and so any closed point of $\mathrm{Spec}(\widehat{R}_W)$ (including any closed point of Y_i) lies on $\mathrm{Spec}(\widehat{R}_W/(t))$. Hence if Y_i is non-empty, then so is $\bar{Y}_i$. Thus either Y_1 or Y_2 is empty, concluding the proof of (a).

Finally, we prove part (c). The map $\widehat{R}_W \otimes_{R_W} R'_{W'} \to \widehat{R}'_{W'}$ is an isomorphism by [Bou72, III, §3.4, Theorem 3(ii)]. For the second assertion, where W is connected, write $S = \widehat{R}_W^\times$. Then the localization $S^{-1}\widehat{R}'_{W'_j} = F_W \otimes_{\widehat{R}_W} \widehat{R}'_{W'_j}$ is a domain that is a finite extension of F_W. Thus it is a field, and is equal to its fraction field $F'_{W'_j}$. So $F_W \otimes_F F' = F_W \otimes_{R_W} R'_{W'} = F_W \otimes_{\widehat{R}_W} \widehat{R}_W \otimes_{R_W} R'_{W'} = F_W \otimes_{\widehat{R}_W} \widehat{R}'_{W'} = F_W \otimes_{\widehat{R}_W} \prod \widehat{R}'_{W'_j} = \prod S^{-1}\widehat{R}'_{W'_j} = \prod F'_{W'_j}$. □

The above definition of $\widehat{R}_W$ requires that W is non-empty. But if the closed fiber X of $\widehat{X}$ is irreducible with generic point η, then we define $\widehat{R}_\varnothing$ to be $\widehat{R}_\eta$. In this situation note that the equivalence in Proposition 3.4 still holds.

The fields F_W arise in particular when considering a finite morphism $f : \widehat{X} \to \widehat{X}'$ of projective normal T-curves, corresponding to a finite field extension F/F'. If U is a non-empty connected open subset of the closed fiber X' of $\widehat{X}'$, then by Proposition 3.4(c) the tensor product $F'_U \otimes_{F'} F$ is a product of finitely many fields, each of them of the form F_W for some open subset $W \subseteq X$. Namely, these sets W are the connected components of $f^{-1}(U) \subseteq X$. Here the sets W can each meet more than one irreducible component of X, even if U meets just one irreducible component of X'.

In the other direction, consider a finite separable extension E_U of the field F'_U, where $F' = K(x)$ is the function field of the projective line $\widehat{X}' = \mathbb{P}^1_T$; where $U = \mathbb{A}^1_k$; and where F'_U is the patching field associated to U on the closed fiber X' of $\widehat{X}'$. Then according to the second part of the next result, there is a finite field extension F/F', corresponding to a finite morphism $f : \widehat{X} \to \widehat{X}'$ of projective normal T-curves, such that $F \otimes_{F'} F'_U$ is F'_U-isomorphic to E_U. Hence $W := f^{-1}(U)$ is a connected affine open subset of the closed fiber X of $\widehat{X}$, and F_W is F'_U-isomorphic to the given field E_U.

Before stating the proposition, recall some notation. Let $\widehat{X}$ be a normal model of a one-variable function field F over the complete discretely valued field K, and X the closed fiber of $\widehat{X}$. For each point $P \in X$, let R_P be the local ring of $\widehat{X}$ at P. Its completion $\widehat{R}_P$ is a domain ([HH10, page 88]), with fraction field denoted by F_P. Each height one prime $\wp$ in $\widehat{R}_P$ that contains the uniformizer t of K defines a *branch* of X at P, lying on some irreducible component of X. The t-adic completion $\widehat{R}_\wp$ of the local ring $R_\wp$ of $\widehat{R}_P$ at $\wp$ is a complete discrete valuation ring; its fraction field is denoted by $F_\wp$. Hence $F_\wp$ contains F_P, and is its completion. The field $F_\wp$ also contains F_U if U is an irreducible open subset of X such that $P \in \bar{U} \smallsetminus U$. If $\widehat{X}'$ is another curve with function field F' we will write $\widehat{R}'_\wp$, $\widehat{R}'_P$, $F'_\wp$ etc. for the analogously defined objects.

Proposition 3.5. *Let $U = \mathbb{A}^1_k$, let P be the point $(x = \infty)$ on $\mathbb{P}^1_k$, and let $\wp$ be the unique branch of $\mathbb{P}^1_k$ at P, where $X = \mathbb{P}^1_k$ is viewed as the closed fiber of $\widehat{X} = \mathbb{P}^1_T$.*

(a) For every finite separable field extension $E_\wp$ of $F_\wp$, there is a finite separable field extension E_P of F_P such that $E_P \otimes_{F_P} F_\wp \cong E_\wp$ over $F_\wp$.

(b) For every finite separable field extension E_U of F_U, there is a finite separable field extension F' of $F := K(x)$ such that $F' \otimes_F F_U \cong E_U$ over F_U. Moreover if $\widehat{X}'$ is the normalization of $\widehat{X}$ in F', with closed fiber X' and associated morphism $\pi : \widehat{X}' \to \widehat{X}$, then $F' \otimes_F F_U \cong F'_{U'}$ over F_U, where $U' = \pi^{-1}(U) \subset X'$ is connected.

Proof. (a) Since $F_\wp$ is the $\wp$-adic completion of F_P, this follows from Krasner's Lemma ([Lan70], Prop. II.2.3).

(b) The tensor product $E_U \otimes_{F_U} F_\wp$ is a finite direct product $\prod_i E_{\wp,i}$ of finite separable field extensions $E_{\wp,i}$ of $F_\wp$. By part (a), for each i there is a finite separable field extension $E_{P,i}$ of F_P such that $E_{P,i} \otimes_{F_P} F_\wp$ is isomorphic to $E_{\wp,i}$ over $F_\wp$. We thus have an isomorphism of separable $F_\wp$-algebras

$$E_U \otimes_{F_U} F_\wp \to \Big(\prod_i E_{P,i}\Big) \otimes_{F_P} F_\wp.$$

Applying the patching assertion Theorem 7.1 of [HH10] (in the context of Theorem 5.9 of that paper), we obtain a finite separable F-algebra F' that induces E_U over F_U and induces $\prod_i E_{P,i}$ over F_P, compatibly with this isomorphism. Since E_U is a field, so is F'.

For the last part, $F' \otimes_F F_U \cong \prod_{j=1}^n F'_{U'_j}$ by Proposition 3.4(c), where $U'_1, \dots, U'_n$ are the connected components of U'. But $F' \otimes_F F_U \cong E_U$ is a field. So $n = 1$ and the assertion follows. □

As the above proof shows, Proposition 3.5 holds more generally for non-empty affine open subsets U of the closed fiber X of a smooth projective T-curve $\widehat{X}$, together with the set of points $P \in X$ in the complement of U. For this, one cites Theorem 7.1 of [HH10] in the context of Theorem 5.10 of that paper.

In the case that T is an *equal characteristic* complete discrete valuation ring, an analog of Proposition 3.5(b) for a finite extension of F_P appeared in [HHK13, Lemma 3.8]. For a more general choice of T, there is the following weaker result, which nevertheless will suffice for our purposes below (in Corollary 3.12):

Proposition 3.6. *Let $\widehat{X}$ be a projective normal T-curve, let P be a closed point on the closed fiber X, and let E be a finite separable extension of F_P. Let S be the integral closure of $\widehat{R}_P$ in E, and let $\widehat{V}^* \to \mathrm{Spec}(S)$ be a birational projective morphism, with $\widehat{V}^*$ normal. Then there exist normal schemes $\widehat{V}$, $\widehat{Z}$, and $\widehat{Y}$ and a commutative diagram*

$$\begin{array}{ccccc} \widehat{V} & \longrightarrow & \widehat{V}^* & \longrightarrow & \mathrm{Spec}(S) \\ \downarrow & & & & \downarrow \\ \widehat{Z} & & \longrightarrow & & \mathrm{Spec}(\widehat{R}_P) \\ \downarrow & & & & \downarrow \\ \widehat{Y} & & \longrightarrow & & \widehat{X} \end{array}$$

of T-schemes, where the horizontal maps are birational projective morphisms that are isomorphisms away from (the inverse image of) P; the bottom half of the diagram is a pullback square; and the morphism $\widehat{V} \to \widehat{Z}$ is finite.

Proof. Let L be the Galois closure of E over F_P, let $G = \mathrm{Gal}(L/F_P)$, and let R be the integral closure of $\widehat{R}_P$ in L. Let $H = \mathrm{Gal}(L/E)$ and let $\widehat{W}^*$ be the normalization of $\widehat{V}^* \times_S R$. It suffices to prove the assertion with E, S, and $\widehat{V}$ replaced by L, R, and $\widehat{W}^*$, provided we also show that the G-action on $\mathrm{Spec}(R)$ lifts to a G-action on the asserted space $\widehat{W}$. Namely, we can then take $\widehat{V} = \widehat{W}/H$. So we now assume that E is Galois over F_P.

The Galois group $G = \mathrm{Gal}(E/F_P)$ acts on the (isomorphism classes of) birational projective morphisms to $\mathrm{Spec}(S)$. Consider the fiber product of the morphisms in the orbit of $\widehat{V}^* \to \mathrm{Spec}(S)$, and let $\widehat{V}$ be the irreducible component that dominates $\mathrm{Spec}(S)$. Then $\widehat{V}$ is normal since each G-conjugate of $\widehat{V}^*$ is; and $\widehat{V} \to \mathrm{Spec}(S)$ is a G-stable birational projective morphism that factors through $\widehat{V}^* \to \mathrm{Spec}(S)$. Thus the action of G on $\mathrm{Spec}(S)$ lifts to an action of G on $\widehat{V}$. Let $\widehat{Z}$ be the quotient of $\widehat{V}$ by G. Then the birational projective morphism $\widehat{V} \to \mathrm{Spec}(S)$ descends to a birational projective morphism $\widehat{Z} \to \mathrm{Spec}(\widehat{R}_P)$. That is, we obtain a commutative diagram

$$\begin{array}{ccccc} \widehat{V} & \longrightarrow & \widehat{V}^* & \longrightarrow & \mathrm{Spec}(S) \\ \downarrow & & & & \downarrow \\ \widehat{Z} & & \longrightarrow & & \mathrm{Spec}(\widehat{R}_P) \end{array}$$

whose vertical arrows are each finite and G-Galois, with generic fiber corresponding to the G-Galois field extension E/F_P.

The bottom horizontal morphism is a composition of blowups and blowdowns, centered at P and at points on exceptional divisors lying over P. We may perform the corresponding blowups and blowdowns on $\widehat{X}$, observing inductively that at each step the spaces mapping to $\mathrm{Spec}(\widehat{R}_P)$ and to $\widehat{X}$ have the same exceptional divisors (fibers over P), and that generators of the local ring at a closed point over $P \in \widehat{X}$ also generate the local ring at the corresponding closed point over $P \in \mathrm{Spec}(\widehat{R}_P)$. This process yields a pullback diagram

$$\begin{array}{ccc} \widehat{Z} & \longrightarrow & \mathrm{Spec}(\widehat{R}_P) \\ \downarrow & & \downarrow \\ \widehat{Y} & \longrightarrow & \widehat{X} \end{array}$$

where the bottom horizontal map is a birational projective morphism which is an isomorphism away from $P \in \widehat{X}$. This gives the desired conclusion. □

3.2 Patching

Below, $\widehat{X}$ is a (projective) normal model of a one-variable function field F over the complete discretely valued field K, and X is the closed fiber of $\widehat{X}$. As in Section 3.1, for each point P on the closed fiber X of $\widehat{X}$ we have an associated complete local domain $\widehat{R}_P$ with fraction field F_P; and for each non-empty connected Zariski strict open subset U of X we may consider the domain $\widehat{R}_U$ and its fraction field F_U. For $P \in \mathcal{P}$ and $\wp$ a branch of X at P, we also have the complete discrete valuation ring $\widehat{R}_\wp$ and its fraction field $F_\wp$.

Consider a non-empty finite subset $\mathcal{P} \subset X$. Let $\mathcal{W}$ be the set of connected components of the complement of $\mathcal{P}$; each of these connected components $U \in \mathcal{W}$ is a strict open subset of X. The set of all the branches of X at points of $\mathcal{P}$ is denoted by $\mathcal{B}$.

If $U \subset U'$ are connected strict open subsets of X, then $F_{U'} \subset F_U$. For $P \in U$, there is also an inclusion $F_U \subset F_P$; and if $\wp$ is a branch at P lying on the closure of U, then there are inclusions $F_P, F_U \subset F_\wp$. These containments are compatible, as U, U' vary.

The next result generalizes [HH10, Theorem 6.4] and [HHK13, Proposition 2.3(a)].

Proposition 3.7. *Let $\mathcal{P}$ be a non-empty finite set of closed points of $\widehat{X}$, let $\mathcal{W}$ be the set of connected components of the complement V of $\mathcal{P}$ in the closed fiber X, and let $\mathcal{B}$ be the set of branches of X at the points of $\mathcal{P}$. For $Q \in \mathcal{P}$, let $\widehat{R}_Q^\circ$ be the subring of F_Q that consists of the elements that lie in $\widehat{R}_\wp$ for each branch $\wp$ of X at Q. Then patching holds for the following injective diamonds.*

(a) $F_\bullet = (F \leq \prod_{U \in \mathcal{W}} F_U,\ \prod_{Q \in \mathcal{P}} F_Q \leq \prod_{\wp \in \mathcal{B}} F_\wp)$.

(b) $R_\bullet = (R_V \leq \prod_{U \in \mathcal{W}} \widehat{R}_U,\ \prod_{Q \in \mathcal{P}} \widehat{R}_Q^\circ \leq \prod_{\wp \in \mathcal{B}} \widehat{R}_\wp)$.

Proof. For short write $F_1 = \prod_{U \in \mathcal{W}} F_U$, $F_2 = \prod_{Q \in \mathcal{P}} F_Q$, $F_0 = \prod_{\wp \in \mathcal{B}} F_\wp$, $R_1 = \prod_{U \in \mathcal{W}} \widehat{R}_U$, $R_2^\circ = \prod_{Q \in \mathcal{P}} \widehat{R}_Q^\circ$, and $R_0 = \prod_{\wp \in \mathcal{B}} \widehat{R}_\wp$. Thus $F_\bullet = (F \leq F_1, F_2 \leq F_0)$ and $R_\bullet = (R_V \leq R_1, R_2^\circ \leq R_0)$. The proofs for the two diamonds are similar. We begin with $F_\bullet$.

First consider the special case that the set $\mathcal{P}$ meets each irreducible component of X non-trivially. By [HHK15, Proposition 3.3], there is then a finite morphism $f : \widehat{X} \to \mathbb{P}^1_T$ such that $\mathcal{P}$ is the fiber over the point ∞ on the closed fiber $\mathbb{P}^1_k$ of $\mathbb{P}^1_T$. Thus $\mathcal{W}$ is the set of connected components of $f^{-1}(U')$, where $U' = \mathbb{A}^1_k$. Let F' be the function field of $\mathbb{P}^1_T$, so that the map f gives a finite field extension F/F'. Let $F'_\bullet = (F' \leq F'_1, F'_2 \leq F'_0)$ be defined analogously as above for the curve $\mathbb{P}^1_T$, with $\mathcal{P}' = \{\infty\}$ and $\mathcal{W}' = \{U'\}$. By [HH10, Theorem 5.9], patching holds for $F'_\bullet$. Since F/F' is a finite field extension, Lemma 2.10 implies that patching holds for $(F, F'_1 \otimes_{F'} F, F'_2 \otimes_{F'} F, F'_0 \otimes_{F'} F)$. The proposition now follows from the assertion that $F_i = F'_i \otimes_{F'} F$, which holds for $i = 1$ by Proposition 3.4(c) and for $i = 0, 2$ by [HH10, Lemma 6.2] (enlarging the set S' there if necessary).

Now consider the case that $\mathcal{P}$ does not meet each irreducible component of X. Since $\mathcal{P}$ is non-empty, not every irreducible component of X is disjoint from $\mathcal{P}$. So by Remark 3.2(b), we may contract the components that are disjoint from $\mathcal{P}$, via a proper birational morphism $\pi : \widehat{X} \to \widehat{Y}$. The set $\mathcal{P}$ maps bijectively to its image in $\widehat{Y}$, with π inducing an isomorphism between F_Q (taken on $\widehat{X}$) and $F_{\pi(Q)}$ (taken on $\widehat{Y}$), for $Q \in \mathcal{P}$. Similarly, π induces an isomorphism between $F_\wp$ and $F_{\pi(\wp)}$ for $\wp \in \mathcal{B}$. Moreover for $U \in \mathcal{W}$, the morphism π induces an isomorphism between F_U and $F_{\pi(U)}$, by Remark 3.2(b), since these are the fraction fields of $\widehat{R}_U$ and $\widehat{R}_{\pi(U)}$ (where the rings are taken on $\widehat{X}$ and $\widehat{Y}$ respectively). Thus the assertion for $\widehat{X}$ is equivalent to the assertion for $\widehat{Y}$, which holds by the first case. This completes the proof of patching for $F_\bullet$.

Next we turn to patching for the diamond $R_\bullet$. As above, we are reduced to the case that the set $\mathcal{P}$ meets each irreducible component of X non-trivially, so that there is a finite morphism $f : \widehat{X} \to \mathbb{P}^1_T$ such that $\mathcal{P} = f^{-1}(\infty)$. With notation as above, consider the analogous diamond $R'_\bullet = (R'_{U'} \leq R'_1, R'^\circ_2 \leq R'_0)$ taken with respect to $\mathcal{P}' = \{\infty\}$, where $R'_1 = \widehat{R}'_{U'}$. Note that $R_1 = \widehat{R}_V = R'_1 \otimes_{R'_{U'}} R_V$ by Proposition 3.4(b,c). Together with [HH10, Lemma 6.2], this implies that $R_\bullet = R'_\bullet \otimes_{R'_{U'}} R_V$.

Also, R_V is a finitely generated free module over $R'_{U'}$ by [Bou72, Proposition II.3.2.5(ii)], using that the finitely generated module R_V/tR_V over the principal ideal domain $R'_{U'}/(t) = k[x]$ is torsion-free and hence free and also that (t) is the Jacobson radical of $R'_{U'}$. Now intersection holds for $R'_\bullet$ because $R'_{U'} \subseteq R'_1 \cap R'^\circ_2 \subseteq R'_1 \cap F'_1 \cap F'_2 = R'_1 \cap F' = R'_{U'}$, and factorization holds for $\mathrm{GL}_n(R'_\bullet)$ by [HHK13, Proposition 2.3(a)]. That is, patching holds for $R'_\bullet$. Hence it also holds for $R_\bullet = R'_\bullet \otimes_{R'_{U'}} R_V$, by Lemma 2.10. □

The next result generalizes [HHK13, Corollary 2.4, Theorem 3.1(a), and Corollary 3.3(a)], the second of which is a form of the Weierstrass Preparation Theorem. It follows easily from Proposition 3.7(b) in the same way that those three previous results followed from [HHK13, Proposition 2.3(a)]. See also [Cuo13, Theorem 3.6].

Corollary 3.8. *Let $\mathcal{W}$ be as in Proposition 3.7.*

(a) *Suppose that for every* $U \in \mathcal{W}$ *we are given an element* $a_U \in F_U^\times$. *Then there exist* $b \in F$ *and elements* $c_U \in \widehat{R}_U^\times$ *such that* $a_U = bc_U \in F_U^\times$ *for all* $U \in \mathcal{W}$.

(b) *If* $U \in \mathcal{W}$ *and* $a \in F_U$ *then there exist* $b \in F$ *and* $c \in \widehat{R}_U^\times$ *such that* $a = bc \in F_U$. *More generally, if* $a \in F_U$ *and* n *is a positive integer that is not divisible by the characteristic of the residue field* k *of* T, *then there exist* $b \in F$ *and* $c \in \widehat{R}_U^\times$ *such that* $a = bc^n \in F_U$.

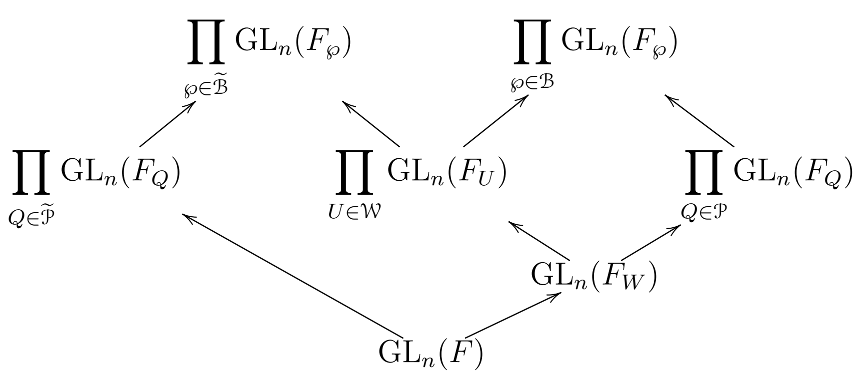
The next result is analogous to Proposition 3.7(a), with F replaced by F_W, for $W \subset X$.

Proposition 3.9. *Let* $W \subseteq X$ *be a connected open subset of* X. *Let* $\mathcal{P}$ *be a non-empty finite set of closed points of* W*; let* $\mathcal{W}$ *be the set of connected components of the complement of* $\mathcal{P}$ *in* W*; and let* $\mathcal{B}$ *be the set of branches of* W *at the points of* $\mathcal{P}$. *Then patching holds for the injective diamond* $F_{W\bullet} = \left(F_W \le \prod_{U\in\mathcal{W}} F_U, \prod_{Q\in\mathcal{P}} F_Q \le \prod_{\wp\in\mathcal{B}} F_\wp\right)$.

Proof. If $W = X$ then the assertion is given by Proposition 3.7(a). So assume that W is strictly contained in X. After blowing down all irreducible components of X that do not intersect W as in Remark 3.2(b), we may assume that the closure of W is X. Let $\widetilde{\mathcal{P}}$ be the complement of W in its closure X. Also let $\widetilde{\mathcal{B}}$ be the set of branches at the points of $\widetilde{\mathcal{P}}$. By Proposition 3.7(a), patching holds for the diamond $\widetilde{F}_\bullet = (F \le F_W, \prod_{Q\in\widetilde{\mathcal{P}}} F_Q \le \prod_{\wp\in\widetilde{\mathcal{B}}} F_\wp)$.

Let $\widehat{\mathcal{P}}$ be the disjoint union $\widetilde{\mathcal{P}} \sqcup \mathcal{P}$ and let $\widehat{\mathcal{B}}$ be the set of branches at the points of $\widehat{\mathcal{P}}$. Thus $\widehat{\mathcal{B}} = \widetilde{\mathcal{B}} \sqcup \mathcal{B}$. Notice that the set of connected components of the complement of $\widehat{\mathcal{P}}$ in X is the set of connected components of the complement of $\mathcal{P}$ in W, i.e., $\mathcal{W}$. Again, patching holds for the diamond $\widehat{F}_\bullet = (F \le \prod_{U\in\mathcal{W}} F_U, \prod_{Q\in\widehat{\mathcal{P}}} F_Q \le \prod_{\wp\in\widehat{\mathcal{B}}} F_\wp)$ by Proposition 3.7(a).

The general linear groups for the various products of fields form the following diagram:



As noted above, patching holds for the diamonds $\widetilde{F}_\bullet$ and $\widehat{F}_\bullet$; and so by Theorem 2.8(a), factorization and intersection hold for the diamonds of groups

$$G_\bullet := \mathrm{GL}_n(\widetilde{F}_\bullet) = \big(\mathrm{GL}_n(F) \le \mathrm{GL}_n(F_W), \prod_{Q\in\widetilde{\mathcal{P}}} \mathrm{GL}_n(F_Q) \le \prod_{\wp\in\widetilde{\mathcal{B}}} \mathrm{GL}_n(F_\wp)\big),$$

$$G'_\bullet := \mathrm{GL}_n(\widehat{F}_\bullet) = \big(\mathrm{GL}_n(F) \le \prod_{U\in\mathcal{W}} \mathrm{GL}_n(F_U), \prod_{Q\in\widehat{\mathcal{P}}} \mathrm{GL}_n(F_Q) \le \prod_{\wp\in\widehat{\mathcal{B}}} \mathrm{GL}_n(F_\wp)\big).$$

Proposition 2.14 (parts 1 and 4) and Lemma 2.2 yield factorization and intersection for

$$\mathrm{GL}_n(F_{W^\bullet}) = \big(\mathrm{GL}_n(F_W) \le \prod_{U\in\mathcal{W}} \mathrm{GL}_n(F_U), \prod_{Q\in\mathcal{P}} \mathrm{GL}_n(F_Q) \le \prod_{\wp\in\mathcal{B}} \mathrm{GL}_n(F_\wp)\big).$$

That is, patching holds for the diamond $F_{W^\bullet}$, as desired. □

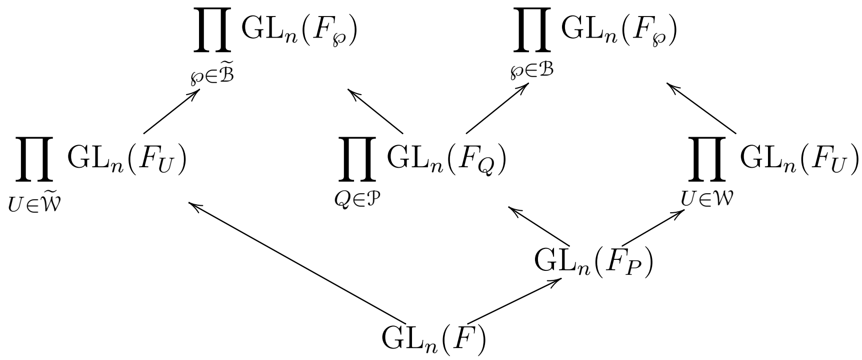
The next result, which answers a question posed by Yong Hu, permits patching on the exceptional divisor of a blow-up $f : \widehat{X} \to \widehat{Y}$. Alternatively, we can view f as a blow-down, in which a non-empty connected union V of some but not all of the irreducible components of the closed fiber $X \subset \widehat{X}$ are contracted to a point $P \in Y \subset \widehat{Y}$ (cf. Remark 3.2(b)).

Proposition 3.10. *Let $f : \widehat{X} \to \widehat{Y}$ be a proper birational morphism of projective normal T-curves, having closed fibers X, Y respectively. Let $P \in Y$ be a closed point, let $V \subset X$ be the inverse image of P in X, and let $\widetilde{Y} \subseteq X$ be the proper transform of Y. Suppose that $\dim(V) = 1$, and that f restricts to an isomorphism $\widehat{X} \smallsetminus V \to \widehat{Y} \smallsetminus \{P\}$. Choose a finite collection of closed points $\mathcal{P}$ in V that includes all the points of $V \cap \widetilde{Y}$. Let $\mathcal{W}$ be the set of connected components of $V \smallsetminus \mathcal{P}$, and let $\mathcal{B}$ be the set of branches at the points in $\mathcal{P}$ along the components of V. Then patching holds for the injective diamond $F_{P^\bullet} = \left(F_P \le \prod_{Q\in\mathcal{P}} F_Q, \prod_{U\in\mathcal{W}} F_U \le \prod_{\wp\in\mathcal{B}} F_\wp\right)$, with respect to the natural inclusions.*

Proof. First observe that there are natural inclusions of F_P into F_U and F_Q, for $U \in \mathcal{W}$ and $Q \in \mathcal{P}$. Namely, the natural morphism $\mathrm{Spec}(\widehat{R}_U) \to \widehat{X}$ factors through $\widehat{X}_P := f^{-1}(\mathrm{Spec}(\widehat{R}_P))$, the pullback of $\widehat{X} \to \widehat{Y}$ via $\mathrm{Spec}(\widehat{R}_P) \to \widehat{Y}$. Since $\widehat{X} \to \widehat{Y}$ is birational, so is $\widehat{X}_P \to \mathrm{Spec}(\widehat{R}_P)$, and hence the function field of $\widehat{X}_P$ is F_P. The morphism $\mathrm{Spec}(\widehat{R}_U) \to \widehat{X}_P$ induces a homomorphism of function fields in the other direction, $F_P \to F_U$, which is necessarily an inclusion. The case of $F_P \to F_Q$ is similar.

Let $\widetilde{\mathcal{W}}$ be the set of connected components of $Y \smallsetminus \{P\}$. Let $\widetilde{\mathcal{B}}$ be the set of branches of Y at P. Via f, we may identify $X \smallsetminus V$ with its isomorphic image $Y \smallsetminus \{P\}$. We may thus regard the elements of $\widetilde{\mathcal{W}}$ as open subsets of X, and the elements of $\widetilde{\mathcal{B}}$ as branches of $\widetilde{Y}$. Write $\widehat{\mathcal{W}}$ for the disjoint union $\widetilde{\mathcal{W}} \sqcup \mathcal{W}$. The set of points of X that lie in no element of $\widehat{\mathcal{W}}$ is exactly $\mathcal{P}$; let $\widehat{\mathcal{B}}$ be the set of branches of X at points of $\mathcal{P}$. Thus $\widehat{\mathcal{B}}$ equals the disjoint union $\widetilde{\mathcal{B}} \sqcup \mathcal{B}$. Note that at the points of $\mathcal{P}$, some of the branches of X are elements of $\mathcal{B}$ and some are in $\widetilde{\mathcal{B}}$, depending on whether the branches lie on a component of V or of $\widetilde{Y}$.

We may now consider the associated diagram of groups:



By Proposition 3.7(a) and Lemma 2.2, patching holds for the diamonds

$$\widetilde{F}_\bullet = (F \le \prod_{U\in\widetilde{\mathcal{W}}} F_U, F_P \le \prod_{\wp\in\widetilde{\mathcal{B}}} F_\wp), \quad \widehat{F}_\bullet = (F \le \prod_{Q\in\mathcal{P}} F_Q, \prod_{U\in\widehat{\mathcal{W}}} F_U \le \prod_{\wp\in\widehat{\mathcal{B}}} F_\wp).$$

That is, factorization and intersection hold for the diamonds of groups $G_\bullet := \mathrm{GL}_n(\widetilde{F}_\bullet)$ and $G'_\bullet := \mathrm{GL}_n(\widehat{F}_\bullet)$. Parts (1) and (4) of Proposition 2.14 imply that the diamond $\mathrm{GL}_n(F_{P\bullet})$ satisfies factorization and intersection; i.e. patching holds for $F_{P\bullet}$, as desired. □

Example 3.11. Let $T = k[[t]]$ and let $\widehat{Y}$ be the projective y-line $\mathbb{P}^1_T$, with closed fiber $Y = \mathbb{P}^1_k$. Let P be the point $y = 0$ on Y, with complete local ring $\widehat{R}_P = k[[y,t]]$. Consider the blow-up $\widehat{X} \to \widehat{Y}$ of $\widehat{Y}$ at P. The exceptional divisor V is a copy of the x-line over k, with $x = 0$ at the point P' of $\widehat{X}$ where V meets the proper transform of Y. The complete local ring $\widehat{R}_{P'}$ is $k[[y,t,x]]/(t-xy) = k[[x,y]]$. Writing W for the complement of P' in V, the ring $\widehat{R}_W$ is the t-adic completion of $k[[y,t]][x^{-1}]/(x^{-1}t-y)$, which is naturally isomorphic to $k[x^{-1}][[t]]$ (with $y = x^{-1}t$). The unique branch $\wp$ of V at P' has associated ring $\widehat{R}_\wp = k((x))[[y]]$, which contains $\widehat{R}_{P'}$ and $\widehat{R}_W$. The intersection of these two rings in $\widehat{R}_\wp$ is $\widehat{R}_P$. The respective fraction fields are $F_P = k((y,t))$, $F_{P'} = k((x,y))$, $F_W = \mathrm{frac}\big(k[x^{-1}][[t]]\big)$, and $F_\wp = k((x))((y))$. They satisfy the intersection condition $F_P = F_{P'} \cap F_W \subset F_\wp$, and they also satisfy factorization for GL_n. This example is a twisted form of the example given in [HH10] after Theorem 5.9 there. It is also related to the situations discussed in [PN10, Section 1] and [BT13].

The next corollary, which will be useful in Section 4.2, is a variant of the previous proposition. Here we blow up $\mathrm{Spec}(S)$ for some two-dimensional complete ring S that need not be of the form $\widehat{R}_P$, but instead can be a finite extension of some $\widehat{R}_P$ or some $\widehat{R}_W$. In this situation, we can again consider fields of the form F_Q, F_U, and $F_\wp$, associated to this blowup; the previous definitions carry over mutatis mutandis to the case of any two-dimensional normal scheme whose closed points all lie on a connected curve.

Corollary 3.12. *Let $\widehat{X}$ be a projective normal T-curve with closed fiber X, and let ξ be either a closed point $P \in X$ or a connected affine open subset $W \subset X$. Let E be a finite separable extension of F_ξ, let S be the integral closure of $\widehat{R}_\xi$ in E, and let $\widetilde{\xi}$ be the inverse* *image of* ξ *under* $\mathrm{Spec}(S) \to \mathrm{Spec}(\widehat{R}_\xi)$. *Let* D *be a divisor on* $\mathrm{Spec}(S)$. *Then there exist a birational projective morphism* $\pi : \widehat{V} \to \mathrm{Spec}(S)$ *and a finite set* $\mathcal{P}$ *of closed points of* $V := \pi^{-1}(\widetilde{\xi})$ *such that the following hold:*

(i) $\widehat{V}$ *is a normal scheme.*

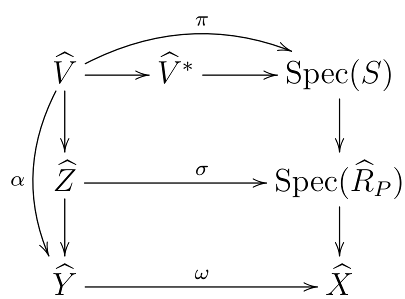
(ii) $D' := \pi^{-1}(D)$ *is a normal crossing divisor on* $\widehat{V}$.

(iii) $\mathcal{P}$ *contains all the points of* V *where* $V \cup D'$ *is not regular, and it meets every connected component of the exceptional locus of* π.

(iv) Let $\mathcal{W}$ *be the set of connected components of* $V \smallsetminus \mathcal{P}$, *and let* $\mathcal{B}$ *be the collection of branches of* V *at the points of* $\mathcal{P}$. *If* $\mathcal{P}, \mathcal{W}$ *are non-empty, then patching holds for the injective diamond* $E_\bullet = (E \le \prod_{Q \in \mathcal{P}} F_Q, \prod_{U \in \mathcal{W}} F_U \le \prod_{\wp \in \mathcal{B}} F_\wp)$.

Proof. Case I: $\xi = P \in X$.

Since $\mathrm{Spec}(S)$ is two-dimensional, excellent, and normal, by [Abh69] and [Lip75] there is a birational projective morphism (viz. a composition of blowups) $\pi^* : \widehat{V}^* \to \mathrm{Spec}(S)$ such that $\widehat{V}^*$ is regular and $D^* := (\pi^*)^{-1}(D) \subset \widehat{V}^*$ is a normal crossing divisor. By Proposition 3.6, we obtain a diagram



with the properties asserted there. Here $D' := \pi^{-1}(D)$ is a normal crossing divisor on $\widehat{V}$ because D^* is, and since the map $\widehat{V} \to \widehat{V}^*$ is a blow-up.

Recall that $V := \pi^{-1}(\widetilde{P})$, where $\widetilde{P} \in \mathrm{Spec}(S)$ is the inverse image of $P \in \mathrm{Spec}(\widehat{R}_P)$. Let $Y = \omega^{-1}(X)$ be the closed fiber of $\widehat{Y}$, and $V_0 := \alpha(V) = \omega^{-1}(P) \subset Y$. Choose a non-empty finite set $\mathcal{P}_0$ of closed points of V_0 that contains the image under α of the locus where $V \cup D'$ is not regular, and also contains the points where V_0 meets the proper transform of X in Y. Let $\mathcal{P} = \alpha^{-1}(\mathcal{P}_0) \subseteq V$. Thus the above properties (i), (ii), (iii) hold. Let $\mathcal{W}$ be as in (iv).

If π is an isomorphism, then $\mathcal{W}$ is empty and we are done. Otherwise, V is a curve, $\mathcal{W}$ is non-empty, and it remains to show that patching holds for the diamond $E_\bullet$.

Let $\mathcal{W}_0$ be the set of connected components of $V_0 \smallsetminus \mathcal{P}_0$ and let $\mathcal{B}_0$ be the set of branches of V_0 at the points of $\mathcal{P}_0$. Thus the elements of $\mathcal{W}$ are the inverse images of the elements of $\mathcal{W}_0$, and similarly for $\mathcal{B}$ and $\mathcal{B}_0$. By Proposition 3.10, patching holds for the diamond $F_{P\bullet} = \big(F_P \le \prod_{Q \in \mathcal{P}_0} F_Q, \prod_{U \in \mathcal{W}_0} F_U \le \prod_{\wp \in \mathcal{B}_0} F_\wp\big)$, taking F_P with respect to $\widehat{X}$ and taking the other fields with respect to $\widehat{Y}$.

By Proposition 3.6, the bottom half of the above diagram is a pullback square, with $Z := \sigma^{-1}(P) \to V_0$ an isomorphism. For each $U \in \mathcal{W}_0$, with inverse image $U' \subseteq Z$, the natural map $F_U \to F_{U'}$ is an isomorphism; and similarly for $\mathcal{P}_0$ and $\mathcal{B}_0$. The diamond $F_{P\bullet}$ may thus be considered to be taken with respect to $\widehat{Z}$.

The morphism $\widehat{V} \to \widehat{Z}$ in Proposition 3.6 is finite. Thus $\prod_{U\in\mathcal{W}} F_U = \prod_{U\in\mathcal{W}_0} F_U \otimes_{F_P} E$, where F_U on the left is taken with respect to $\widehat{V}$ and F_U on the right is taken with respect to $\widehat{Z}$ (or $\widehat{Y}$); and the analogous isomorphisms hold for the fields F_Q and $F_\wp$. (These isomorphisms are as in Proposition 3.4(c) and [HH10, Lemma 6.2], whose statements and proofs carry over to this somewhat more general situation.) Applying Lemma 2.10(b) with respect to the field extension E/F_P, we obtain the desired conclusion.

Case II: ξ is a connected affine open subset $W \subset X$.

Recall that $\widetilde{W} \subset \mathrm{Spec}(S)$ is the inverse image of $W \subset \mathrm{Spec}(\widehat{R}_W)$. Choose a non-empty finite subset $\mathcal{P}^*$ of closed points of W that contains the image under $f : \mathrm{Spec}(S) \to \mathrm{Spec}(\widehat{R}_W)$ of the points where $\widetilde{W} \cup D$ is not regular, and let $\widetilde{\mathcal{P}} = f^{-1}(\mathcal{P}^*) \subset \widetilde{W} \subset \mathrm{Spec}(S)$. Write $\widetilde{\mathcal{W}}$ for the set of connected components of the complement of $\widetilde{\mathcal{P}}$ in $\widetilde{W}$, and $\widetilde{\mathcal{B}}$ for the set of branches of $\widetilde{W}$ at the points of $\widetilde{\mathcal{P}}$.

Let $\mathcal{W}^*$ be the set of connected components of $W \smallsetminus \mathcal{P}^*$ and let $\mathcal{B}^*$ be the set of branches of W at the points of $\mathcal{P}^*$. Thus the elements of $\widetilde{\mathcal{W}}$ are the inverse images under f of the elements of $\mathcal{W}^*$, and similarly for $\widetilde{\mathcal{B}}$ and $\mathcal{B}^*$. By Proposition 3.9, patching holds for the diamond $F_{W\bullet} = \big(F_W \leq \prod\limits_{U\in\mathcal{W}^*} F_U, \prod\limits_{Q\in\mathcal{P}^*} F_Q \leq \prod\limits_{\wp\in\mathcal{B}^*} F_\wp\big)$. Since f is finite, $\prod_{U\in\widetilde{\mathcal{W}}} F_U = \prod_{U\in\mathcal{W}^*} F_U \otimes_{F_W} E$, and similarly for the fields F_Q and $F_\wp$, as at the end of the proof of Case I. As in that proof, patching holds for the diamond $\widetilde{E}_\bullet := (E \leq \prod_{U\in\widetilde{\mathcal{W}}} F_U, \prod_{Q\in\widetilde{\mathcal{P}}} F_Q \leq \prod_{\wp\in\widetilde{\mathcal{B}}} F_\wp)$, by Lemma 2.10(b). That is, factorization and intersection hold for $G_\bullet := \mathrm{GL}_n(\widetilde{E}_\bullet)$ for all n (see Definition 2.9).

Let $\widetilde{\mathcal{P}}' \subseteq \widetilde{\mathcal{P}}$ consist of the closed points of $\widetilde{W}$ where D is not a normal crossing divisor, and write $\widetilde{\overline{\mathcal{P}}} = \widetilde{\mathcal{P}} \smallsetminus \widetilde{\mathcal{P}}'$. Thus $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}' \sqcup \widetilde{\overline{\mathcal{P}}}$. Our strategy will be to blow up $\mathrm{Spec}(S)$ at the points of $\widetilde{\mathcal{P}}'$, obtaining a refined diamond $E_\bullet$; and then to use that patching holds for $\widetilde{E}_\bullet$ and the diamond arising from the exceptional locus to obtain the same for $E_\bullet$ via Proposition 2.14.

For each $Q \in \widetilde{\mathcal{P}}'$, consider the complete local ring $\widehat{R}_Q$ of $\mathrm{Spec}(S)$ at the point Q, with fraction field F_Q. Thus $f(Q) \in \mathcal{P}^* \subset W \subset \widehat{W} = \mathrm{Spec}(\widehat{R}_W)$, and F_Q is a finite separable extension of $F_{f(Q)}$, viz. a factor of $F_{f(Q)} \otimes_{F_W} E$. Let D_Q be the restriction of $\widetilde{W} \cup D$ to $\mathrm{Spec}(\widehat{R}_Q)$.

By Case I, for each $Q \in \widetilde{\mathcal{P}}'$, there is a birational projective morphism (viz. a composition of blowups) $\pi'_Q : \widehat{V}'_Q \to \mathrm{Spec}(\widehat{R}_Q)$ for which conditions (i)-(iv) are satisfied, with respect to the divisor D_Q, some finite subset $\mathcal{P}'_Q$ of $V'_Q := \pi_Q'^{-1}(Q)$, the associated sets $\mathcal{W}'_Q$ and $\mathcal{B}'_Q$, and the diamond $F'_{Q\bullet} := (F_Q \leq \prod_{Q'\in\mathcal{P}'_Q} F_{Q'}, \prod_{U\in\mathcal{W}'_Q} F_U \leq \prod_{\wp\in\mathcal{B}'_Q} F_\wp)$. In particular, patching holds for $F'_{Q\bullet}$ by (iv), where $\mathcal{P}'_Q, \mathcal{W}'_Q$ are non-empty since $Q \in \widetilde{\mathcal{P}}'$. That is, intersection and factorization hold for $\mathrm{GL}_n(F'_{Q\bullet})$ for all n. By Lemma 2.15(a), these properties also hold for $\mathrm{GL}_n(F'_\bullet)$, where $F'_\bullet = \prod_{Q\in\widetilde{\mathcal{P}}'} F'_{Q\bullet}$, with the product of diamonds being taken entry by entry.

Since blowing up is local, we may take the corresponding blowups of $\mathrm{Spec}(S)$ at ideals respectively supported at the points $Q \in \widetilde{\mathcal{P}}'$. We thus obtain a projective birational morphism $\pi : \widehat{V} \to \mathrm{Spec}(S)$ which is an isomorphism away from $\widetilde{\mathcal{P}}'$, and whose pullback under $\mathrm{Spec}(\widehat{R}_Q) \to \mathrm{Spec}(S)$ may be identified with $\widehat{V}'_Q$, for $Q \in \widetilde{\mathcal{P}}'$. We may similarly regard $\widetilde{W} \smallsetminus \widetilde{\mathcal{P}}'$ as contained in $\widehat{V}$. With respect to these identifications, let $\mathcal{P}' \subset V := \pi^{-1}(\widetilde{W})$ be the disjoint union of the sets $\mathcal{P}'_Q$ for $Q \in \widetilde{\mathcal{P}}'$, and similarly define $\mathcal{W}'$ and $\mathcal{B}'$. Note that $\mathcal{P} := \mathcal{P}' \sqcup \widehat{\mathcal{P}}$ contains all the points of V at which $V \cup D'$ is not regular, where $D' = \pi^{-1}(D)$.

Now properties (i), (ii), (iii) hold for π with respect to the divisor D and the set $\mathcal{P}$, where (ii) uses that $D \subset \mathrm{Spec}(S)$ is a normal crossing divisor away from $\widetilde{\mathcal{P}}'$. Let $\mathcal{W}$ be the set of connected components of $V \smallsetminus \mathcal{P}$, and let $\mathcal{B}$ be the set of branches of V at the points of $\mathcal{P}$. Thus $\mathcal{W} = \widetilde{\mathcal{W}} \sqcup \mathcal{W}'$ and $\mathcal{B} = \widetilde{\mathcal{B}} \sqcup \mathcal{B}'$. It remains to show that property (iv) is satisfied for the diamond $E_\bullet := (E \le \prod_{Q\in\mathcal{P}} F_Q, \prod_{U\in\mathcal{W}} F_U \le \prod_{\wp\in\mathcal{B}} F_\wp)$.

It was shown above that for any n, intersection and factorization hold for

$$\widetilde{H}_\bullet := \mathrm{GL}_n(F'_\bullet) = \Big(\prod_{Q\in\widetilde{\mathcal{P}}'} \mathrm{GL}_n(F_Q) \le \prod_{Q'\in\mathcal{P}'} \mathrm{GL}_n(F_{Q'}), \prod_{U\in\mathcal{W}'} \mathrm{GL}_n(F_U) \le \prod_{\wp\in\mathcal{B}'} \mathrm{GL}_n(F_\wp)\Big).$$

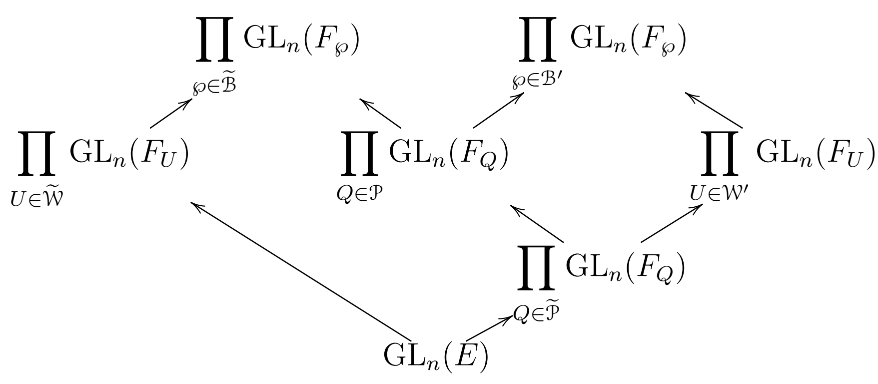
Applying Lemma 2.15(b) with $A = \mathrm{GL}_n(\prod_{Q\in\widehat{\mathcal{P}}} F_Q)$, it follows that these two properties also hold for the (non-injective) diamond

$$H_\bullet := \Big(\prod_{Q\in\widetilde{\mathcal{P}}} \mathrm{GL}_n(F_Q), \prod_{Q\in\mathcal{P}} \mathrm{GL}_n(F_Q), \prod_{U\in\mathcal{W}'} \mathrm{GL}_n(F_U), \prod_{\wp\in\mathcal{B}'} \mathrm{GL}_n(F_\wp)\Big).$$

(Here we use that $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}' \sqcup \widehat{\mathcal{P}}$ and that $\mathcal{P} = \mathcal{P}' \sqcup \widehat{\mathcal{P}}$.) Consider the injective diamond

$$G'_\bullet := \mathrm{GL}_n(E_\bullet) = \Big(\mathrm{GL}_n(E) \le \prod_{Q\in\mathcal{P}} \mathrm{GL}_n(F_Q), \prod_{U\in\mathcal{W}} \mathrm{GL}_n(F_U) \le \prod_{\wp\in\mathcal{B}} \mathrm{GL}_n(F_\wp)\Big).$$

Since $\mathcal{W} = \widetilde{\mathcal{W}} \sqcup \mathcal{W}'$ and $\mathcal{B} = \widetilde{\mathcal{B}} \sqcup \mathcal{B}'$, Proposition 2.14 applies to the diamonds $G_\bullet$, $G'_\bullet$, and $H_\bullet$, with respect to the following diagram:



By parts (5) and (6) of that proposition, it follows that intersection and factorization hold for $G'_\bullet = \mathrm{GL}_n(E_\bullet)$ for all n; i.e. patching holds for $E_\bullet = (E \le \prod_{Q\in\mathcal{P}} F_Q, \prod_{U\in\mathcal{W}} F_U \le \prod_{\wp\in\mathcal{B}} F_\wp)$. □

Remark 3.13. As the proof of Corollary 3.12 shows, if the conclusion holds for a given choice of $\widehat{V}$ and of $\mathcal{P} \subset V$, and if $\mathcal{P}' \subset V$ is any other finite subset of V, then one can enlarge $\mathcal{P}$ so as to contain $\mathcal{P}'$ and still satisfy the conclusion of the corollary.

3.3 Factorization for diamonds of groups

In the situation of Section 3.2, we can obtain results about factorization for diamonds that arise from algebraic groups other than just GL_n. This is useful for obtaining local-global principles for torsors; see Theorem 2.13.

Proposition 3.14. *Let $\widehat{X}$ be a normal model of a one-variable function field F over K, and let G be an algebraic group over F. In the notation of Proposition 3.7, 3.9, or 3.10, let $\mathcal{P}'$ be a finite set of closed points of X that contains $\mathcal{P}$. In the context of Proposition 3.9, assume also that $\mathcal{P}'$ contains $\overline{W} \smallsetminus W$. Let $\mathcal{W}'$ be the set of components of $X \smallsetminus \mathcal{P}'$, and let $\mathcal{B}'$ be the set of branches of X at the points of $\mathcal{P}'$. Set $F'_\bullet = (F \le \prod_{U \in \mathcal{W}'} F_U,\ \prod_{Q \in \mathcal{P}'} F_Q \le \prod_{\wp \in B'} F_\wp)$. If factorization holds for the diamond $G(F'_\bullet)$ then it holds for $G(F_\bullet)$, $G(F_{W\bullet})$, or $G(F_{P\bullet})$, respectively.*

Proof. For short write $F'_\bullet = (F \le F'_1, F'_2 \le F'_0)$. We consider each of the three cases in turn.

Case of Proposition 3.7. Let n be the number of points in $\mathcal{P}'$ that are not in $\mathcal{P}$. By induction we are reduced to the case that $n = 1$, since each set $\mathcal{P}''$ with $\mathcal{P} \subset \mathcal{P}'' \subset \mathcal{P}'$ is also an allowable finite subset of X in the notation of Proposition 3.7. Write $\mathcal{P}' = \mathcal{P} \sqcup \{P\}$.

Let W be the unique element of $\mathcal{W}$ that contains P, and let $W' = W \smallsetminus \{P\}$. Consider the associated diamond $F_{W\bullet}$, defined as in Proposition 3.9 with respect to the set $\{P\} \subset W$. Thus $F_{W\bullet} = (F_W \le \prod_{U \in \mathcal{W}'_P} F_U,\ F_P \le \prod_{\wp \in \mathcal{B}_P} F_\wp)$, where $\mathcal{W}'$ be the set of connected components of W', and where $\mathcal{B}_P$ is the set of branches of W at P. Patching holds for $F_{W\bullet}$ by Proposition 3.9; and in particular the intersection property holds for $F_{W\bullet}$ (see Definition 2.9).

Let

$$G_\bullet = G(F_\bullet) = (G(F) \le \prod_{U \in \mathcal{W}} G(F_U),\ \prod_{Q \in \mathcal{P}} G(F_Q) \le \prod_{\wp \in \mathcal{B}} G(F_\wp)),$$

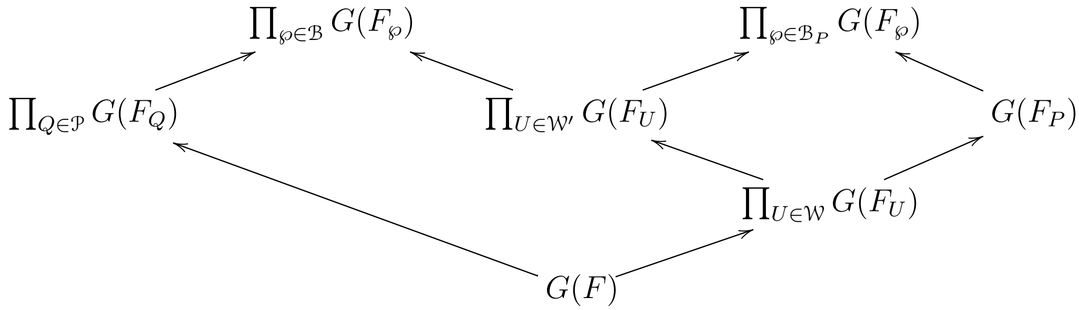
$$G'_\bullet = G(F'_\bullet) = (G(F) \le \prod_{U \in \mathcal{W}'} G(F_U),\ \prod_{Q \in \mathcal{P}'} G(F_Q) \le \prod_{\wp \in \mathcal{B}'} G(F_\wp)),$$

$$\widetilde{H}_\bullet = G(F_{W\bullet}) = (G(F_W) \le \prod_{U \in \mathcal{W}'} G(F_U),\ G(F_P) \le \prod_{\wp \in \mathcal{B}_P} G(F_\wp)).$$

By Lemma 2.12, intersection holds for $\widetilde{H}_\bullet$ since it holds for $F_{W\bullet}$. Let $A = \prod_{U \in \mathcal{W} \smallsetminus \{W\}} G(F_U)$. Let $H_\bullet$ be the coordinate-wise product of diamonds $\widetilde{H}_\bullet \times (A, A, 1, 1)$. That is,

$$H_\bullet = (\prod_{U \in \mathcal{W}} G(F_U) \le \prod_{U \in \mathcal{W}'} G(F_U),\ G(F_P) \le \prod_{\wp \in \mathcal{B}_P} G(F_\wp)).$$

Then intersection holds for $H_\bullet$ by Lemma 2.15(b). Using Lemma 2.2, the desired conclusion then follows from Proposition 2.14(2), with $G_\bullet$, $G'_\bullet$, and $H_\bullet$ as above, with respect to the following refinement diagram:



Case of Proposition 3.9. Let $\mathcal{P}''$ be the subset of $\mathcal{P}'$ obtained by deleting those points that lie in W but not in $\mathcal{P}$, and consider the corresponding diamond $F''_\bullet$. By the case of Proposition 3.7, it follows that factorization holds for $G(F''_\bullet)$. So after replacing $\mathcal{P}'$ by $\mathcal{P}''$, we may assume that $\mathcal{P}' \cap W = \mathcal{P}$.

Write $\mathcal{P}'$ as a disjoint union $\mathcal{P} \sqcup \tilde{\mathcal{P}}$. Thus $\tilde{\mathcal{P}}$ is disjoint from W. Let $\tilde{\mathcal{W}}$ be the set of connected components of the complement of $\tilde{\mathcal{P}}$ in X. Thus $W \in \tilde{\mathcal{W}}$, using the hypothesis that $\mathcal{P}'$ contains $\overline{W} \smallsetminus W$. Let $\tilde{\mathcal{B}}$ be the set of branches of X at the points of $\tilde{\mathcal{P}}$. Consider the diamond $\tilde{F}_\bullet = (F \le \tilde{F}_1, \tilde{F}_2 \le \tilde{F}_0)$, where $\tilde{F}_1, \tilde{F}_2, \tilde{F}_0$ are defined analogously to F'_1, F'_2, F'_0. Write $G_\bullet = G(\tilde{F}_\bullet)$, $G'_\bullet = G(F'_\bullet)$, and $\widetilde{H}_\bullet = G(F_{W\bullet})$. Letting $A = \prod_{U\in\tilde{\mathcal{W}}\smallsetminus\{W\}} G(F_U)$, and setting $H_\bullet = \widetilde{H}_\bullet \times (A, A, 1, 1)$ as in the first case, we then obtain a refinement diagram as in that case. Using Lemma 2.2, it follows from Proposition 2.14(1) that $H_\bullet$ satisfies factorization. By Lemma 2.15(b), so does $\widetilde{H}_\bullet = G(F_{W\bullet})$.

Case of Proposition 3.10. We proceed analogously to the case of Proposition 3.9. As in that case, we may assume that $\mathcal{P}' \cap V = \mathcal{P}$, via the case of Proposition 3.7. Write $\mathcal{P}' = \mathcal{P} \sqcup \mathcal{P}''$, so that $\mathcal{P}'' \subset X$ is disjoint from V. Identifying $X \smallsetminus V$ with $Y \smallsetminus \{P\}$, we may view $\mathcal{P}''$ as a subset of $Y \smallsetminus \{P\}$. Let $\tilde{\mathcal{P}} = \mathcal{P}'' \sqcup \{P\} \subset Y$. Let $\tilde{\mathcal{W}}$ be the set of connected components of the complement of $\tilde{\mathcal{P}}$ in Y, and $\tilde{\mathcal{B}}$ the set of branches of Y at the points of $\tilde{\mathcal{P}}$. Consider the diamond $\tilde{F}_\bullet = (F \le \tilde{F}_1, \tilde{F}_2 \le \tilde{F}_0)$, where $\tilde{F}_1, \tilde{F}_2, \tilde{F}_0$ are defined as in the previous case (though with respect to $\widehat{Y}$, rather than $\widehat{X}$; note that F is also the function field of $\widehat{Y}$). Write $G_\bullet = G(\tilde{F}_\bullet)$, $G'_\bullet = G(F'_\bullet)$, and $\widetilde{H}_\bullet = G(F_{P\bullet})$. Setting $A = \prod_{Q\in\widetilde{P}\smallsetminus\{P\}} G(F_Q)$ and $H_\bullet = \widetilde{H}_\bullet \times (A, A, 1, 1)$ as before, we again obtain a refinement diagram. As in the previous case, using Lemma 2.2, factorization for $H_\bullet$ follows from Proposition 2.14(1). Lemma 2.15(b) then implies that factorization also holds for $\widetilde{H}_\bullet = G(F_{P\bullet})$. □

Corollary 3.15. *Under the hypothesis of Proposition 3.7, 3.9, or 3.10, let G be a rational connected linear algebraic group over F. Then factorization holds for the diamond $G(F_\bullet)$, $G(F_{W\bullet})$, or $G(F_{P\bullet})$, respectively.*

Proof. Choose a finite set $\mathcal{P}'$ of closed points of X that contains $\mathcal{P}$ and all the points where irreducible components of X meet. In the context of Proposition 3.9, assume also that $\mathcal{P}'$ contains $\overline{W} \smallsetminus W$. With notation as in the statement of Proposition 3.14, factorization holds for the diamond $G(F'_\bullet)$ by [HHK09, Theorem 3.6] (using that the simultaneous factorization condition in [HHK09] is equivalent to the factorization assertion for diamonds; see [HHK14, Section 2.1.3]). Hence the conclusion follows by Proposition 3.14. □

In the situation of Corollary 3.15, it then follows from Theorem 2.13 that if $\mathcal{V}$ is the class of F-varieties V with a G-action such that $G(\prod_{\wp\in\mathcal{B}} F_\wp)$ acts transitively on $V(\prod_{\wp\in\mathcal{B}} F_\wp)$, then the local-global principle holds for $\mathcal{V}$ with respect to $F_\bullet$. The corresponding local-global assertions also follow for varieties over F_W or F_P (satisfying transitivity over $\prod_{\wp\in\mathcal{B}} F_\wp$), with respect to $F_{W^\bullet}$ or $F_{P^\bullet}$, respectively.

Remark 3.16. Proposition 3.14 similarly implies the conclusion of Corollary 3.15 for any linear algebraic group G over F such that factorization holds for G with respect to any choice of a non-empty finite subset $\mathcal{P}'$ of X that includes all the points where distinct components meet. This includes all connected retract rational groups G, by [Kra10]. As shown in [HHK15, Corollary 6.5], if the reduction graph associated to F is a tree then this property also holds for all linear algebraic groups G over F that are rational but not necessarily connected (i.e. each connected component is F-rational).

4 Applications to local-global principles and field invariants

We now apply the above results in order to obtain applications in the contexts of quadratic forms and central simple algebras. These applications, which concern local-global principles and invariants of fields, extend and build on results that appeared in [HHK09], [HHK15], and [HHK13], as well as in [Lee13], [Hu13], [Hu12], [Hu15], and [PS14]. The fields we consider will be finite extensions of fields F_P and F_U, for P a closed point and U a connected open subset of the closed fiber of a curve over a complete discrete valuation ring, in the notation of Section 3.

4.1 Applications to quadratic forms

Here we present applications to quadratic forms, concerning local-global principles and invariants of fields, especially the u-invariant. We focus on the fields arising in Section 3 and finite separable extensions of these, in particular proving results that generalize and extend assertions in [HHK15, Section 9.2], and [HHK13, Section 4.1] regarding the Witt ring, Witt index, and u-invariant. As a consequence, we obtain a local-global result for the value of the u-invariant (Corollary 4.8). Due to [PS14], this latter result also applies in the case of mixed characteristic $(0, 2)$, which is often avoided in quadratic form theory. Afterwards we obtain a result (Theorem 4.11) concerning the value of the u-invariant for finite separable extensions of fields such as $k((x, t))$ and the fraction field of $k[x][[t]]$, as well as mixed characteristic analogs of such fields.

We begin with local-global principles, starting with a more general result in the abstract context of diamonds. The reader is referred to [Lam05] for basic notions such as isotropic and hyperbolic forms, the Witt ring and fundamental ideal, the Witt index, and the u-invariant.

If F_v are fields (for v in some index set), we define $W(\prod F_v) := \prod W(F_v)$ and $I(\prod F_v) := \prod I(F_v)$. Recall that for any field E and quadratic form q over E, $H^1(E, \mathrm{SO}(q))$ classifies quadratic forms of the same dimension and discriminant as q, with q corresponding to the distinguished element of the Galois cohomology set; see [KMRT98, 29.29]. (In part (b) below we write μ_2 rather than $\mathbb{Z}/2\mathbb{Z}$ as in [HHK15]; but these are equivalent since the characteristic is not two.)

Theorem 4.1. *Suppose that $F_\bullet = (F \le F_1, F_2 \le F_0)$ is a diamond of rings with F a field of characteristic unequal to two, and each F_i a finite direct product of fields. Write $F_i = \prod_{v \in \mathcal{V}_i} F_v$ where each F_v is a field. Assume moreover that patching holds for $F_\bullet$ and that we have factorization for the diamond $\mathrm{SO}(nh)(F_\bullet)$ for each $n > 0$, where h is a hyperbolic plane $\langle 1, -1\rangle$. Then:*

(a) We have an exact sequence

$$0 \to P_{F_\bullet} \to \mathrm{W}(F) \to \mathrm{W}(F_1) \times \mathrm{W}(F_2) \to \mathrm{W}(F_0),$$

where the map on the right is given by taking the difference of the restrictions of the two Witt classes, and where $P_{F_\bullet}$ is the subgroup of $\mathrm{W}(F)$ consisting of classes of locally hyperbolic binary Pfister forms; i.e. forms $\langle 1, -d\rangle$ where $d \in F^\times$ becomes a square in each F_i.

(b) The group $P_{F_\bullet}$ is naturally isomorphic to the kernel of the local-global map

$$H^1(F, \mu_2) \to \prod_v H^1(F_v, \mu_2),$$

i.e. the group of square classes in $F^\times$ that become the trivial square class in each $F_v^\times$.

(c) For every quadratic form q over F, factorization holds for the diamond of groups $\mathrm{SO}(q)(F_\bullet)$.

(d) The local-global principle for isotropy holds for quadratic forms over F that are of dimension unequal to two, and for binary forms that do not lie in $P_{F_\bullet}$. That is, if such a form q becomes isotropic over F_v for each $v \in \mathcal{V}_1 \cup \mathcal{V}_2$, then q is isotropic.

(e) If q is a regular quadratic form over F then $\mathrm{i_W}(q) = \min\{\mathrm{i_W}(q_v) \mid v \in \mathcal{V}_1 \cup \mathcal{V}_2\} - \varepsilon$, where $\varepsilon = 1$ if q represents a non-trivial class in $P_{F_\bullet}$ and otherwise $\varepsilon = 0$.

Proof. Proof of part (a): Let $P_{F_\bullet}$ be the kernel of the diagonal map $\mathrm{W}(F) \to \mathrm{W}(F_1) \times \mathrm{W}(F_2)$, and let $\alpha \in P_{F_\bullet} \subseteq \mathrm{W}(F)$. Thus $\alpha_{F_v} = 0$ for each v. Here α is the class of a quadratic form q such that q_{F_v} is hyperbolic; hence q is of even dimension $2n$. Let d be the discriminant of q and let $b = \langle 1, -d\rangle$. Since q_{F_v} has trivial discriminant for each v, it follows that b_{F_v} is hyperbolic for each v; i.e. $d \in (F_v^\times)^2$. Hence the form $q' = q \perp -b$ has trivial discriminant and is hyperbolic for each v. Thus q' corresponds to a class in $H^1(F, \mathrm{SO}((n+1)h))$.

Let H be the $\mathrm{SO}((n+1)h)$-torsor corresponding to q'. Since q'_{F_v} is hyperbolic for each v, we see that $H(F_i) \neq \varnothing$ for $i = 1, 2$. But by Theorem 2.13, since factorization holds for $\mathrm{SO}((n+1)h)(F_\bullet)$, it follows that $H(F)$ is non-empty. Hence the torsor H is split, and so $q' = q \perp -b$ is hyperbolic. But this implies that q is equivalent to the binary Pfister form b.

For exactness on the right, suppose that we have Witt classes $\alpha_i \in \mathrm{W}(F_i)$ such that $(\alpha_1)_{F_0} = (\alpha_2)_{F_0}$. We wish to show that there is a class $\alpha \in \mathrm{W}(F)$ such that $\alpha_{F_i} = \alpha_i$ for $i = 1, 2$. To begin, we choose representative forms q_i over F_i with class α_i of the same dimension n. Since these forms become Witt equivalent over F_0, and since they have the same dimension, they necessarily become isometric over F_0. But the category of quadratic forms of dimension n under isometry is equivalent to the category of $\mathrm{O}(n\langle 1\rangle)$-torsors; so by Theorem 2.8, there is a quadratic form q over F such that $q_{F_i} \cong q_i$, as desired.

Proof of part (b): Since elements of $P_{F_\bullet}$ are represented by binary forms, they lie in the fundamental ideal $I(F)$. So the exact sequence in part (a) restricts to an exact sequence

$$0 \to P_{F_\bullet} \to I(F) \to I(F_1) \times I(F_2) \to I(F_0).$$

We claim that the induced map $P_{F_\bullet} \to I(F)/I^2(F)$ is injective. To see this, observe that if q is a quadratic form whose Witt class is in $P_{F_\bullet} \cap I^2(F)$, then q has trivial discriminant and even dimension $2n$, and thus corresponds to a class $\alpha \in H^1(F, \mathrm{SO}(nh))$ with α_{F_v} trivial for each v. But since factorization holds for $\mathrm{SO}(nh)(F_\bullet)$, it follows from Theorem 2.13 that α is split and hence q is hyperbolic. Consequently, the above map is injective.

Consider the composition

$$P_{F_\bullet} \to I(F) \to I(F)/I^2(F) \xrightarrow{\sim} F^\times/(F^\times)^2 \xrightarrow{\sim} H^1(F, \mu_2),$$

which is thus also injective. Its image is contained in the kernel of the map $H^1(F, \mu_2) \to \prod_v H^1(F_v, \mu_2)$, by the definition of $P_{F_\bullet}$ and the functoriality the isomorphism $I(F)/I^2(F) \xrightarrow{\sim} H^1(F, \mu_2)$. The reverse containment follows from the description of $P_{F_\bullet}$ in part (a), together with the fact that the map $I(F) \to F^\times/(F^\times)^2$ takes the class of the quadratic form $\langle 1, -d\rangle$ to the square class of d. Hence we obtain the asserted isomorphism.

Proof of part (c): Let $n = \dim q$. Suppose that q' is a quadratic form class with $[q'] \in H^1(F, \mathrm{SO}(q))$ such that $[q']_{F_v}$ is trivial in $H^1(F_v, \mathrm{SO}(q))$ for each v. Then $q \perp -q'$ is a quadratic form over F of even dimension and trivial discriminant that is trivial over each F_v; and hence its Witt class lies in $P_{F_\bullet}$. Since none of the nontrivial elements of $P_{F_\bullet}$ have trivial discriminant, it follows that $q \perp -q'$ is hyperbolic, and that q and q' are isometric (being of the same dimension). Hence $[q']$ is trivial in $H^1(F, \mathrm{SO}(q))$. This shows that the local-global principle holds for $\mathrm{SO}(q)$-torsors. By Theorem 2.13, it follows that factorization holds for $\mathrm{SO}(q)$.

Proof of part (d): We prove the contrapositive; i.e. if q is anisotropic then so is some q_{F_v}. This is clear if q is binary but not in the kernel $P_{F_\bullet}$ of the local-global map on Witt rings, since a binary form is hyperbolic if and only if it is isotropic. So now assume that the anisotropic form q is not binary. The group $\mathrm{O}(q)$ acts on the projective quadric hypersurface Q defined by q, and the action of $\mathrm{O}(q)(F_0)$ is transitive on $Q(F_0)$ by the Witt Extension Theorem (see the proof of [HHK09, Theorem 4.2]). Since $\dim(q) > 2$, it follows that Q is connected; and hence $\mathrm{SO}(q)(F_0)$ also acts transitively on $Q(F_0)$. By part (c) above, factorization holds for $\mathrm{SO}(q)(F_\bullet)$. Hence Theorem 2.13 implies that the local-global principle holds for Q. If q is anisotropic, then $Q(F)$ is empty and thus some $Q(F_v)$ must be empty; i.e. q_{F_v} is anisotropic.

Proof of part (e): By Witt decomposition, we are reduced to the case that q is anisotropic. The desired assertion is clear if the class of q is in $P_{F_\bullet}$, so assume otherwise. Then some q_{F_v} is anisotropic by part (d). Hence $\mathrm{i_W}(q) = \min\{\mathrm{i_W}(q_v)\} = 0$, and thus $\varepsilon = 0$ as asserted. $\square$

This abstract result can in particular be applied to the concrete situation of Section 3.

We recall the standing hypotheses: T is a complete discrete valuation ring with fraction field K and residue field k; and $\widehat{X}$ is a projective normal T-curve with fraction field F and closed fiber X. For the remainder of this section on quadratic forms, we *additionally assume* that K (or equivalently, F) has characteristic unequal to two.

Theorem 4.1 yields local-global principles for quadratic forms in this context:

Example 4.2. In the situation of Proposition 3.7, 3.9, or 3.10, write L for the field F, F_W, or F_P, respectively. Recall that we assume $\mathrm{char}(L) \neq 2$. Theorem 4.1 applies because those three propositions say that patching holds for the given diamond, and because factorization holds for $\mathrm{SO}(nh)(F_\bullet)$ by Corollary 3.15 (since $\mathrm{SO}(nh)$ is a rational connected linear algebraic F-group by the Cayley parametrization; e.g. see [HHK09, Remark 4.1]).

The local-global principles given in Example 4.2 can be carried over to the situation of points on the closed fiber. First we prove a lemma, with notation as above.

Lemma 4.3. *Let X_0 be an irreducible component of X, with generic point η. Let U_0 be a non-empty affine open subset of X_0 that meets no other irreducible component of X, and let q be a quadratic form over F_{U_0}. If q becomes isotropic over F_η then q is isotropic over F_U for some non-empty affine open subset $U \subseteq U_0$.*

Proof. We may assume that q is a diagonal form $\langle a_1, \ldots, a_n \rangle$, with each $a_i \in F_{U_0} \subset F_\eta$. By [HHK13, Corollary 3.3(a)], there exist elements $b_i \in F$ and $c_i \in F_{U_0}^\times$ such that $a_i = b_i c_i^2$. So q is isometric over F_{U_0} to the form $q' := \langle b_1, \ldots, b_n \rangle$ that is defined over F. The projective quadric hypersurface Q over F that is defined by q' has an F_η-point, since q' is isotropic over F_η. Hence by [HHK15, Proposition 5.8], Q has an F_U-point for some non-empty affine open subset $U \subseteq X_0$. After shrinking U, we may assume that $U \subseteq U_0$. Thus $q'_U = q_U$, and q' is isotropic over F_U. Hence q is isotropic over F_U. $\square$

Proposition 4.4. *Let $\widehat{X}$ be a projective normal curve with closed fiber X over a complete discrete valuation ring T of characteristic not two. Let S equal X, or a non-empty connected open proper subset $W \subset X$, or a non-empty connected proper subset $V \subset X$ consisting of a union of irreducible components of X. In these three cases, let L respectively equal F or F_W or F_P, where in the third case we consider the model $\widehat{Y}$ of F obtained by contracting V and where the point $P \in \widehat{Y}$ is the image of V. Let q be a quadratic form over L.*

(a) If $\dim(q) \neq 2$ and q_{F_Q} is isotropic for each point $Q \in S$, then q is isotropic.

(b) If q is a regular quadratic form, then $\mathrm{i_W}(q) \in \{\min(\mathrm{i_W}(q_{F_Q})), \min(\mathrm{i_W}(q_{F_Q})) - 1\}$, where the minimum is taken over all $Q \in S$. The second case occurs precisely when q is Witt equivalent to an anisotropic binary Pfister form that becomes isotropic over each F_Q.

(c) *The kernel of the map* $\pi : \mathrm{W}(L) \to \prod_{Q \in S} \mathrm{W}(F_Q)$ *is equal to the kernel* $Ш_S(L, \mu_2)$ *of the local-global map* $H^1(L, \mu_2) \to \prod_{Q \in S} H^1(F_Q, \mu_2)$ *in Galois cohomology.*

Proof. We begin with the observation that if q_{F_Q} is isotropic for each point $Q \in S$, then there is a finite set $\mathcal{P}$ of closed points of S such that q_{F_U} is isotropic for each connected component U of $S \smallsetminus \mathcal{P}$. To see this, note that for each irreducible component S_0 of S, the form q_{F_η} is isotropic, where η is the generic point of S_0. Hence q is isotropic over F_U for some non-empty open subset $U \subseteq S_0$, by Lemma 4.3. We may now take $\mathcal{U}$ to be the collection of these sets U (one for each irreducible component of S), and take $\mathcal{P}$ to be the complement in S of the union of the sets $U \in \mathcal{U}$. This proves the observation.

We now prove part (a). By the above observation, there are sets $\mathcal{P}$ and $\mathcal{U}$ as above such that q_{F_ξ} is isotropic for each $\xi \in \mathcal{P} \cup \mathcal{U}$. Consider Example 4.2 in the situation of Proposition 3.7, 3.9, or 3.10, if S is equal to X, W, or V respectively. By Theorem 4.1(d) in the context of this example, q is isotropic over L.

For part (b), take the Witt decomposition $q = q_a \perp ih$, where q_a is anisotropic, h is a hyperbolic plane, and $i \geq 0$. The assertion is trivial if q is itself hyperbolic, and so we may assume that q_a is a non-trivial form. If q_a remains anisotropic over F_{Q_0} for some $Q_0 \in S$, then $\mathrm{i_W}(q) = i = \mathrm{i_W}(q_{Q_0})$, and $\mathrm{i_W}(q_Q) \geq i$ for all other $Q \in S$. Thus $\mathrm{i_W}(q) = \min(\mathrm{i_W}(q_{F_Q}))$ in this case.

The other case is that q_a becomes isotropic over each F_Q. Then by the above claim, q_a becomes isotropic over F_ξ for each $\xi \in \mathcal{P} \cup \mathcal{U}$ as above. So by Theorem 4.1(e) in the context of Example 4.2, q_a is an anisotropic binary Pfister form that becomes isotropic (or equivalently, hyperbolic) over each F_ξ, and hence over each F_Q. Thus q is Witt equivalent to such a form and $\mathrm{i_W}(q) = i = \mathrm{i_W}(q_{F_Q}) - 1$ for all $Q \in S$.

For part (c), observe that by Theorem 4.1(a,b) in the context of Example 4.2, the kernel of the map $\pi_{\mathcal{P}} : \mathrm{W}(L) \to \prod_\xi \mathrm{W}(F_\xi)$ is equal to the kernel $Ш_{\mathcal{P}}(L, \mu_2)$ of the local-global map $H^1(L, \mu_2) \to \prod_\xi H^1(F_\xi, \mu_2)$, where ξ ranges over the elements of $\mathcal{P} \cup \mathcal{W}$ in each product. So it suffices to show that $\ker(\pi) = \bigcup \ker(\pi_{\mathcal{P}})$ and $Ш_S(L, \mu_2) = \bigcup Ш_{\mathcal{P}}(L, \mu_2)$, where in each case $\mathcal{P}$ ranges over the non-empty sets of closed points of S. For any choice of $\mathcal{P}$ (and hence of $\mathcal{W}$), $F_U \subset F_Q$ for all $Q \in U \in \mathcal{W}$. Thus $\ker(\pi_{\mathcal{P}}) \subseteq \ker(\pi)$ and $Ш_{\mathcal{P}}(L, \mu_2) \subseteq Ш_S(L, \mu_2)$ for all $\mathcal{P}$. It therefore remains to show that $\ker(\pi) \subseteq \bigcup \ker(\pi_{\mathcal{P}})$ and $Ш_S(L, \mu_2) \subseteq \bigcup Ш_{\mathcal{P}}(L, \mu_2)$.

We begin with the first of these inclusions. By Witt decomposition, every non-trivial class in $\ker(\pi)$ is represented by a non-trivial anisotropic form q. Such a q becomes hyperbolic and hence isotropic over each F_Q. So by the observation at the beginning of the proof, q becomes isotropic over F_ξ for every $\xi \in \mathcal{P} \cup \mathcal{W}$, for some choice of $\mathcal{P}$. Also, since q is not hyperbolic over F but becomes hyperbolic over each F_Q, it follows from part (b) above that q is a binary form. Thus the isotropic forms q_{F_ξ} are also binary and hence hyperbolic, and so the class of q lies in $\ker(\pi_{\mathcal{P}})$ as desired.

To prove the second inclusion, note that a non-trivial element of $Ш_S(L, \mu_2)$ is given by a quadratic field extension of L, of the form $L[a^{1/2}]$ for some non-square $a \in L^\times$. By definition of $Ш_S(L, \mu_2)$, the element a is a square in $L_\eta = F_\eta$ for every generic point η of S. For each η, choose an irreducible connected open neighborhood $U_0 \subset S$. By Corollary 3.8(b), $a = bc^2$ for some $b \in F^\times$ and $c \in \widehat{R}_{U_0}^\times$. Thus b is a square in F_η; i.e. the μ_2-torsor given by

$F[b^{1/2}]$ over F has an F_η-point. By [HHK15, Proposition 5.8], this torsor has an F_U-point for some open neighborhood $U \subseteq U_0$ of η; i.e. b and hence a is a square in F_U. Let $\mathcal{P}$ be the complement of the union of the sets U as η varies. Then the given element of $Ш_S(L, \mu_2)$ lies in $Ш_{\mathcal{P}}(L, \mu_2)$. □

In the above result, the first case (where $S = X$ and $L = F$) was previously shown in Theorem 9.3, Corollary 9.5, and Theorem 9.6 of [HHK15]; but here a uniform argument proves all three cases. Analogous local-global assertions have been also proven with respect to discrete valuations on F rather than with respect to points on X; see [CPS12, Theorem 3.1] and [HHK15, Theorem 9.11]. By combining Proposition 4.4 with the strategy used in [HHK15, Proposition 9.10], we obtain a local-global principle for isotropy over F_P:

Proposition 4.5. *Let $\widehat{X}$ be a normal projective T-curve, let P be a closed point of $\widehat{X}$, and let q be a quadratic form on F_P of dimension $\neq 2$. Assume* $\mathrm{char}(k) \neq 2$. *Then q is isotropic over F_P if and only if it is isotropic over $(F_P)_v$ for every discrete valuation v on F_P.*

Even more is true: q is isotropic over F_P provided that it is isotropic over $(F_P)_v$ for each discrete valuation v on F_P whose restriction to F is induced by a codimension one point on a regular projective model of F over T.

Proof. The forward direction of the first assertion is trivial. For the reverse direction, consider a quadratic form q on F_P that is isotropic on $(F_P)_v$ for every discrete valuation v on F_P. We may assume that q is a diagonal form $\langle a_1, \dots, a_n\rangle$, with $a_i \in \widehat{R}_P$.

By resolution of singularities for surfaces ([Abh69], [Lip75]) and Weierstrass Preparation ([HHK13], Corollary 3.7), there is a birational projective morphism $\pi : \widehat{X}' \to \widehat{X}$ such that $\widehat{X}'$ is regular, and such that on the pullback $\pi' : \widehat{X}'_P \to \mathrm{Spec}(\widehat{R}_P)$ of π with respect to $\mathrm{Spec}(\widehat{R}_P) \to \widehat{X}$, the support of q is a normal crossing divisor at every point of $\pi'^{-1}(P) \subset \widehat{X}'_P$ (which we may identify with $V := \pi^{-1}(P) \subset \widehat{X}'$; here the support of q is defined to be the union of the supports of the divisors of the elements a_i.) By the third case of Proposition 4.4(a), in order to show that q is isotropic over F_P it suffices to show that q is isotropic over F_Q for every point $Q \in V$.

First note that by [HHK15, Proposition 7.5], for any $Q \in V$ and any discrete valuation v on F_Q, the restriction of v to F is a (non-trivial) discrete valuation. Since $F \subseteq F_P \subseteq F_Q$, it follows that the same holds for the restriction of v to F_P.

Consider a closed point Q of V. By the condition on the support of q at Q, there exists a generating set $\{x, y\}$ for the maximal ideal of $\widehat{R}_Q$ whose support contains that of q in $\mathrm{Spec}(\widehat{R}_Q)$. Let v be the y-adic valuation on F_Q, and let $v_0 = v|_{F_P}$. Thus q is isotropic over the completion $(F_P)_{v_0}$, by the previous paragraph and by the hypothesis of this direction of the proposition. Hence q is also isotropic over $(F_Q)_v$, which contains $(F_P)_{v_0}$. Since $(F_Q)_v$ has residue characteristic unequal to two, it follows from [HHK15, Lemma 9.9] that q is isotopic over F_Q.

The other case is that Q is a generic point of V. Thus F_Q is a complete discretely valued field, say with valuation v. Again, the restriction v_0 of v to F_P is a discrete valuation such that q is isotropic over $(F_P)_{v_0}$. Hence q is also isotropic over F_Q, which contains $(F_P)_{v_0}$. This completes the proof of the reverse implication.

For the last assertion, note that we may assume in the above argument that the model $\widehat{X}'$ has the property that distinct branches of the closed fiber X' at any closed point must lie on distinct irreducible components of X'. With respect to this choice of model $\widehat{X}'$, it suffices to check that the valuations used in the above argument are induced by codimension one points on $\widehat{X}'$.

If Q is a generic point of V, then this condition is trivial, since Q is itself a codimension one point on $\widehat{X}'$, and the restriction of v_0 to F is the valuation associated to that point on that model. So consider the case that Q is a closed point of V, and take the valuation $v_0 = v|_{F_P}$ considered in the argument above. By the above condition on branches, the hypothesis of [HHK13, Theorem 3.1(c)] is automatically satisfied; and so there exist $b \in F$ and $c \in \widehat{R}_Q^\times$ such that $y = bc$. Here $b = yc^{-1} \in \widehat{R}_Q$, and $\{x, b\}$ is a generating set for the maximal ideal of $\widehat{R}_Q$. So there is a unique irreducible component D of the zero locus of b on $\widehat{X}'$ that passes through Q. The y-adic valuation v on F_Q is equal to the b-adic valuation on F_Q, and the valuation v_0 on F_P thus restricts to the b-adic valuation on F. That is, $v_0|_F$ is the discrete valuation associated to the generic point of D, which is of codimension one on $\widehat{X}'$, as desired. □

Lemma 4.6. *Let E be the fraction field of a two-dimensional Noetherian complete local domain R. Then E is isomorphic to a finite separable extension of a field of the form F_P. Moreover if R is regular or equicharacteristic, then E is itself of the form F_P.*

Proof. If R is regular, then by [Coh46, Theorem 15] it is of the form $T[[x]]$ for some complete discrete valuation ring T. Thus $E = F_P$ with respect to a point on the projective T-line.

In the general case, by [Coh46, Theorem 16], R is a finite extension of a two-dimensional regular complete local domain having residue field k. So by the previous paragraph, E is a finite extension of a field of the form F_P. This extension is automatically separable if $\mathrm{char}(E) = 0$; and by [GaOr08, Théorème 7.1], it can be chosen to be separable if $\mathrm{char}(E) > 0$. Hence E is isomorphic to a finite separable extension of F_P.

Finally, if R is equicharacteristic, then by the above it is a finite generically separable extension of some $T[[x]]$, where $T = k[[t]]$ since it is equicharacteristic. So E is a finite separable extension of $k((t, x))$, and thus of the form F_P by [HHK13], Lemma 3.8. □

Proposition 4.5 and Lemma 4.6 then yield the following strengthening of [Hu12, Theorem 1.2]):

Corollary 4.7. *Let E be the fraction field of a regular or equicharacteristic two-dimensional complete local ring whose residue field k has characteristic unequal to two. Then a quadratic form q over E of dimension $\neq 2$ is isotropic if and only if it becomes isotropic over E_v for every discrete valuation v on E.*

Corollary 4.8. *In the situation of Proposition 3.7 (resp. 3.9 or 3.10), with* $\mathrm{char}(K) \neq 2$, *let L be the field F (resp. F_W or F_P) and let S be the set X (resp. W or V).*

(a) *Then $u(L) \leq \max_{\xi \in \mathcal{P} \cup \mathcal{W}} u(F_\xi) \leq \sup_{Q \in S} u(F_Q)$.*

(b) *If the residue field k of T has characteristic unequal to two, then* $u(L) = \max_{\xi\in\mathcal{P}\cup\mathcal{W}} u(F_\xi) = \sup_{Q\in S} u(F_Q) = \sup_{v\in\Omega} u(L_v)$*, where Ω is the set of discrete valuations on L whose restriction to F is a discrete valuation that is induced by a codimension one point on a regular model of F.*

(c) *If k is perfect of characteristic two, and* $\mathrm{char}(K) = 0$*, then the four quantities* $u(L)$, $\max_{\xi\in\mathcal{P}\cup\mathcal{W}} u(F_\xi)$, $\sup_{Q\in S} u(F_Q)$, $\sup_{v\in\Omega} u(L_v)$ *are each less than or equal to* 8.

Proof. For $\xi \in \mathcal{P}\cup\mathcal{W}$, the field F_ξ is not quadratically closed, since the integrally closed ring $\widehat{R}_\xi$ is not. Thus $u(F_\xi) \geq 2$ by [Lam05], Chapter XI, Example 6.2(1).

If q is a quadratic form over L having dimension greater than $\max_{\xi\in\mathcal{P}\cup\mathcal{W}} u(F_\xi)$, then q_ξ is isotropic for all $\xi \in \mathcal{P}\cup\mathcal{W}$. Hence q is isotropic over L by Theorem 4.1(d) in the context of Example 4.2, using that $\dim(q) > 2$. This shows that $u(L) \leq \max_{\xi\in\mathcal{P}\cup\mathcal{W}} u(F_\xi)$.

Next, if $U \in \mathcal{W}$ and q is a quadratic form over F_U of dimension greater than $\sup_{Q\in S} u(F_Q)$, then q_{F_Q} is isotropic for every point Q of U. Hence q is isotropic over F_U by Proposition 4.4(a) for F_U, using that $\dim(q) > 2$. Thus $\max_{\xi\in\mathcal{P}\cup\mathcal{W}} u(F_\xi) \leq \sup_{Q\in S} u(F_Q)$. This proves part (a).

Next, we show part (b), assuming that $\mathrm{char}(k) \neq 2$. By part (a), it suffices to show the two inequalities $\sup_{Q\in S} u(F_Q) \leq \sup_{v\in\Omega} u(L_v) \leq u(L)$.

For the first of these inequalities, we may consider just closed points Q. Let $\pi : \widehat{Y} \to \widehat{X}$ be a birational projective morphism such that $\widehat{Y}$ is smooth, and let $\Sigma = \pi^{-1}(S)$. Applying Proposition 4.4(a) to F_Q for every $Q \in S$ at which π is not an isomorphism, we see that $\sup_{Q\in S} u(F_Q) \leq \sup_{Q\in\Sigma} u(F_Q)$. So after replacing $\widehat{X}$ by $\widehat{Y}$, we may assume that $\widehat{X}$ is regular. Next, since $\mathrm{char}(k) \neq 2$, there is a split cover $\omega : \widehat{X}' \to \widehat{X}$, say with function field extension F'/F and closed fiber X', such that for each $Q' \in S' := \omega^{-1}(S)$ and each $a \in F'_{Q'}$, there exist $b \in F'$ and $c \in F'^{\times}_{Q'}$ such that $a = bc^2$. (See Corollaries 3.3(c) and 3.7 of [HHK13], the latter applied to the set of non-unibranched points of S.) By Proposition 5.1 of [HHK15], the set of isomorphism classes of fields $F'_{Q'}$, for $Q' \in X'$, is the same as the set of isomorphism classes of fields F_Q, for $Q \in X$. Also, since $\widehat{X}$ is regular, the analogous assertion is true for the fields F_v, by Proposition 7.6 of [HHK15]. So we may replace $\widehat{X}$ by $\widehat{X}'$, and therefore assume that $\widehat{X}$ satisfies the above factorization condition on elements $a \in F_Q$.

Now let q be a quadratic form over F_Q for some $Q \in S$, and assume that $n := \dim(q) > u(L_v)$ for all $v \in \Omega$. To prove the first inequality we wish to show that q is isotropic over F_Q. We may assume that $q = \langle a_1, \ldots, a_n\rangle$ with $a_i \in F_Q$. By the above condition, we may write $a_i = b_i c_i^2$ with $b_i \in F$ and $c \in F_Q^\times$. Replacing q by the F_Q-equivalent form $\langle b_1, \ldots, b_n\rangle$, we may assume that q is defined over F. Now let w be any discrete valuation on F_Q whose restriction to F is a discrete valuation induced by a codimension one point on a regular projective model of F. Thus $v := w|_L \in \Omega$, and L_v is contained in $(F_Q)_w$. But q is isotropic over L_v by the dimension assumption on q, since $v \in \Omega$ and q is a quadratic form over L_v. Thus q is isotropic over each $(F_Q)_w$. By Proposition 4.5, it follows that q is isotropic over F_Q. This completes the proof of the first inequality.

For the second inequality, let v be a discrete valuation on L. Since $\mathrm{char}(k) \neq 2$, the residue field κ_v of L_v also has characteristic unequal to two. So $u(L_v) = 2u(\kappa_v)$ for each

$v \in \Omega$ by Springer's theorem [Spr55, Proposition 2], and $2u(\kappa_v) \leq u(L)$ by the first part of [HHK09, Lemma 4.9]. Hence $u(L_v) \leq u(L)$, concluding the proof of part (b).

For part (c), we first show under the given hypotheses that $u(F_Q) \leq 8$ for all $Q \in S$. Let q be a quadratic form of dimension 9 over F_Q; we wish to show that q is isotropic. Since $\mathrm{char}(L) = \mathrm{char}(K) = 0$, we may assume that q is a diagonal form $\langle a_1, \dots, a_9 \rangle$, with $a_i \in \widehat{R}_P$. We may assume that each a_i is non-zero. Let D be the union of the supports of the divisors $(a_1), \dots, (a_9), (2)$ on $\mathrm{Spec}(\widehat{R}_Q)$. In the special case that $\widehat{X}$ is regular at Q and D has at most a normal crossing at Q, [PS14, Proposition 4.6] asserts that q is isotropic. More generally, let $\pi : \widehat{X}' \to \widehat{X}$ be a blow-up such that $\widehat{X}'$ is regular and D has only normal crossings on $\mathrm{Spec}(\widehat{R}_Q) \times_{\widehat{X}} \widehat{X}'$, the corresponding blow-up of $\mathrm{Spec}(\widehat{R}_Q)$. Then q is isotropic over $F_{Q'}$ for every $Q' \in \pi^{-1}(Q)$, by the special case just shown. By Proposition 4.4(a), q is isotropic over F_Q, as desired. Thus $u(F_Q) \leq 8$.

So by part (a), $u(L) \leq \max_{\xi \in \mathcal{P} \cup \mathcal{W}} u(F_\xi) \leq \sup_{Q \in S} u(F_Q) \leq 8$ in this case. To complete the proof of part (c), it suffices to show that $u(L_v) \leq 8$ for each $v \in \Omega$. If the residue characteristic of v is zero (and thus not two), then $u(L_v) \leq u(L)$ by the same argument as at the end of the proof of part (b); and so $u(L_v) \leq 8$. Otherwise, v is the valuation associated to the generic point of a component of the special fiber of some model of L, and $u(L_v) \leq 8$ as in the first part of the proof of [PS14, Theorem 4.7]. □

Remark 4.9. (a) Corollary 4.8(b) remains valid if one takes the supremum over *all* the discrete valuations on L, since the above argument that $u(L_v) \leq u(L)$ does not use that $v \in \Omega$.

(b) Corollary 4.8(b) is related to Theorem 4.9 in [Hu15]. The proof in [Hu15] used that the function field there was assumed to be purely transcendental. Note that $\mathrm{char}(k) \neq 2$ in that assertion, by a standing hypothesis there.

(c) Part (c) of the corollary extends Theorem 4.7 of [PS14], which asserted that $u(F) \leq 8$ under these hypotheses.

The next result generalizes an assertion given in [HHK13, Theorem 4.1] concerning the value of $u(F_U)$ for an open subset U of X. Here, as in [HHK09] and [HHK13], given a field E, $u_s(E)$ denotes the smallest n such that $u(L) \leq 2^i n$ for every finitely generated field extension L/E of transcendence degree $i \leq 1$. The proof is as for [HHK13, Theorem 4.1], but using Corollary 3.8 in place of the less general [HHK13, Corollary 3.3(a)]. As in that result, we need to assume that the residue field k of the complete discrete valuation ring T has characteristic unequal to two (not just that its fraction field K has characteristic $\neq 2$).

Proposition 4.10. *Let $\widehat{X}$ be a normal projective T-curve, and let U be a non-empty connected open subset of the closed fiber X. Assume that* $\mathrm{char}(k) \neq 2$.

(a) Then $u(F_U) \leq 4u_s(k)$.

(b) Let $\widetilde{X}$ be the normalization of X, and let $\widetilde{Q} \in \widetilde{X}$ be a closed point lying over some point $Q \in U$. Then $u(F_U) \geq 4u(\kappa(\widetilde{Q}))$.

See also [Cuo13, Corollary 3.7].

Theorem 4.11. *Let E be one of the following:*

(i) the fraction field of a two-dimensional Noetherian complete local domain R that is regular or equicharacteristic; or

(ii) a finite separable extension of the fraction field of the t-adic completion of $T[x]$, where T is a complete discrete valuation ring with uniformizer t.

Assume that the residue field k of R (resp. T) does not have characteristic two.

(a) Then $u(E) \le 4u_s(k)$.

(b) If $u(k) = u_s(k)$, and if $u(k') = u(k)$ for every finite extension k' of k, then $u(E) = 4u(k)$.

Proof. In case (i), E is of the form F_P by Lemma 4.6. The assertion then follows in this case from [HHK13], Theorem 4.1, by choosing a finite set of points $\mathcal{P}$ on the closed fiber X of the model $\widehat{X}$, such that $\mathcal{P}$ contains P and the points where distinct components of X meet.

In case (ii), E is a finite separable extension of F_U, where $U = \mathbb{A}^1_k$, viewed as an open subset of the closed fiber of $\mathbb{P}^1_T$. By Proposition 3.5(b), E is isomorphic to a field $F'_{U'}$ for some finite extension F' of F, where F is the fraction field of $T[x]$, and for some non-empty connected open subset $U' \subset X'$. (Here X' is the closed fiber of a projective normal model $\widehat{X}'$ of F'.) By Proposition 4.10, $u(E) = u(F'_{U'}) \le 4u_s(k)$, proving part (a). For part (b), $u(E) \le 4u_s(k) = 4u(k)$ by part (a); and the reverse inequality follows from Proposition 4.10(b), using that $u(\kappa(\widetilde{Q})) = u(k)$ by hypothesis, for any closed point $Q \in U'$. □

Example 4.12. Theorem 4.11 applies in particular to the broad class of fields k that satisfy Leep's $A_n(2)$ property. Recall that k is called an $A_n(2)$ field if for every $r > 0$, every system of r homogeneous forms of degree two over k in more than $r \cdot 2^n$ variables has a non-trivial zero over a finite extension of k having odd degree. (see [Lee13, Section 2]). Every C_n field is an $A_n(2)$ field ([Sha72], Lemma IV.3.7), but not conversely. Although p-adic fields are not C_n for any n, they are $A_2(2)$ field for all primes p, including $p = 2$ (see [Lee13], Corollary 2.7). Moreover if k is an $A_n(2)$ field then so is every finite extension of k ([Lee13], Theorem 2.5); and the fields $k(z)$ and $k((z))$ are $A_{n+1}(2)$ fields ([Lee13], Theorem 2.3). Since $u(k) \le 2^n$ for an $A_n(2)$ field ([Lee13], Theorem 2.2), it follows from the above properties that $u_s(k) \le 2^n$, and thus $u(E) \le 2^{n+2}$ in the notation of the corollary, provided $\mathrm{char}(k) \ne 2$. In the special case that $u(k') = 2^n$ for every finite extension k'/k, it follows from [Lee13, Lemma 3.2] that $u(k) = u_s(k) = 2^n$. Thus the hypotheses of Theorem 4.11(b) hold.

As a consequence, we obtain Theorem 1.3:

Proof of Theorem 1.3. By Example 4.12, the fields k are respectively $A_0(2)$, $A_1(2)$, $A_2(2)$, $A_3(2)$, as are their finite extensions; and moreover the hypotheses of Theorem 4.11(b) hold (using also [Lee13, Theorem 3.4] in the last two cases). We conclude by that theorem. □

As pointed out to us by David Leep, the case of the theorem for $u\big(k((x,t))\big)$, where $k = \mathbb{Q}_p$ or $\mathbb{Q}_p(z)$ or $\mathbb{Q}_p((z))$, can be deduced directly from results in [Lee13]. Namely, if k is an $A_n(2)$-field of characteristic unequal to two, then $k((t))(x)$ is an $A_{n+2}(2)$ field by [Lee13, Theorem 2.3] and so $u\big(k((t))(x)\big) \le 2^{n+2}$. If $u(k) = 2^n$ (as in the case of $k = \mathbb{Q}_p$ or $\mathbb{Q}_p(z)$ or $\mathbb{Q}_p((z))$), it then follows that $u\big(k((x,t))\big) = 2^{n+2}$ by [Lee13, Proposition 5.1].

Theorem 4.11 also provides explicit values for the u-invariant in mixed characteristic, when the residue characteristic is odd:

Corollary 4.13. *Let p be an odd prime, and $\mathbb{Z}_p^{\mathrm{ur}}$ the maximal unramified extension of $\mathbb{Z}_p$. Let R be $\mathbb{Z}_p[[x]]$ or the p-adic completion of $\mathbb{Z}_p[x]$ (resp. the p-adic completion of $\mathbb{Z}_p^{\mathrm{ur}}[[x]]$ or of $\mathbb{Z}_p^{\mathrm{ur}}[x]$). Let E be the fraction field of a finite extension S of R; and if $R = \mathbb{Z}_p[[x]]$ or the p-adic completion of $\mathbb{Z}_p^{\mathrm{ur}}[[x]]$, assume that S is regular. Then $u(E) = 8$ (resp. 4).*

Proof. In the first two cases, let $T = \mathbb{Z}_p$ and apply Theorem 4.11, using that the hypotheses of part (b) hold with $u_s(\mathbb{F}_p) = 2$, by [HHK09, Theorem 4.10]. In the other cases, let T be the p-adic completion of $\mathbb{Z}_p^{\mathrm{ur}}$; this is the ring of Witt vectors of $\overline{\mathbb{F}}_p$. Again the hypotheses of Theorem 4.11(b) hold, this time with $u_s(\overline{\mathbb{F}}_p) = 1$. □

In the case of mixed characteristic with residue characteristic two, we obtain the following, by combining [PS14] with Theorem 4.11:

Corollary 4.14. *Let k be a complete discretely valued field of characteristic zero whose residue field κ is perfect of characteristic two. If E is a finite extension of $k((x,t))$ or of the fraction field of $k[x][[t]]$, then $u(E) \le 16$.*

Proof. Every finite extension λ of κ is perfect, so $u(\lambda) \le 2$ by [MMW91, Corollary 1]. Thus $u(\ell) \le 4$ for every finite extension ℓ of k, by a theorem of Springer ([Spr55, Proposition 2]). By [PS14, Theorem 4], $u(F) \le 8$ for every finitely generated extension F of k of transcendence degree one. Thus $u_s(k) \le 4$. We conclude by Theorem 4.11, where separability holds since $\mathrm{char}(k) = 0$. □

4.2 Applications to central simple algebras

Finally, we turn to applications of our results to central simple algebras, especially concerning the period and index of elements of the Brauer group of fields of the sort considered in Section 3, and their finite extensions. In particular, for a finite separable extension E of a field of the form F_P or F_U, we find an integer d such that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^d$ for $\alpha \in \mathrm{Br}(E)$. See Theorems 4.21 and 4.22, as well as the associated corollaries, for the precise statements, which strengthen and extend results in [HHK13, Section 4]. For example, for two-dimensional p-adic cases, we obtain a sharp bound for the period-index bound d regardless of the period of α; see Theorems 4.23 and 1.2.

As in Section 4.1, we begin with an abstract local-global result (Theorem 4.15) that applies to diamonds that satisfy patching. But unlike the analogous Theorem 4.1, Theorem 4.15 does not require a factorization hypothesis. This makes it more applicable, permitting its use in conjunction with Corollary 3.12, which in turn makes possible the period-index applications mentioned above for finite separable extensions of fields F_P and F_U. Theorem 4.15 also yields local-global results about the value of the period-index bound for the fields F_P and F_U; see Example 4.16 and Corollary 4.17.

Below we use that if E is a product of fields F_v, then $\mathrm{Br}(E) = \prod \mathrm{Br}(F_v)$ because an Azumaya algebra over E is the same as a product of central simple F_v-algebras.

Theorem 4.15. *Suppose that $F_\bullet = (F \le F_1, F_2 \le F_0)$ is a diamond of rings with F a field and each F_i a finite direct product of fields. Write $F_i = \prod_{v \in \mathcal{V}_i} F_v$ where each F_v is a field. Assume moreover that patching holds for $F_\bullet$. Then:*

(a) We have a short exact sequence $0 \to \mathrm{Br}(F) \to \mathrm{Br}(F_1) \times \mathrm{Br}(F_2) \to \mathrm{Br}(F_0)$, where the map on the right is given by taking the difference of the restrictions of the two Brauer classes.

(b) For a class $\alpha \in \mathrm{Br}(F)$, we have $\mathrm{ind}(\alpha) = \mathrm{lcm}\{\mathrm{ind}(\alpha_v) \mid v \in \mathcal{V}_1 \cup \mathcal{V}_2\}$.

Proof. For any field L, the natural map $\mathrm{GL}_n(L) \to \mathrm{PGL}_n(L)$ is surjective, by Hilbert's Theorem 90 and the long exact cohomology sequence associated to the short exact sequence of algebraic groups $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$. Now factorization for GL_n holds for $F_\bullet$, since $F_\bullet$ has the patching property (see Theorem 2.8(a)). The above surjectivity then implies that factorization for PGL_n also holds for $F_\bullet$. Theorem 2.13 then in turn implies that the map of pointed sets $H^1(F, \mathrm{PGL}_n) \to \prod_v H^1(F_v, \mathrm{PGL}_n)$ has trivial kernel.

Recall that $H^1(F, \mathrm{PGL}_n)$ classifies isomorphism classes of central simple F-algebras of degree n ([KMRT98, p. 396]). So if A is a central simple F-algebra such that A_{F_v} is split for each v, then A itself is split. Thus the homomorphism $\mathrm{Br}(F) \to \prod_v \mathrm{Br}(F_v)$ is injective.

Now, suppose that we have classes $\alpha_i \in \mathrm{Br}(F_i) = \prod_{v \in \mathcal{V}_i} \mathrm{Br}(F_v)$ for $i = 1, 2$ such that $(\alpha_1)_{F_0} = (\alpha_2)_{F_0}$. We wish to show that there is an $\alpha \in \mathrm{Br}(F)$ with $\alpha_{F_i} = \alpha_i$. Choose a positive integer n that is divisible by $\mathrm{ind}(\alpha_v)$ for each v. We may then choose central simple algebras A_v over F_v of degree n such that the class of $A_i = \prod_{v \in \mathcal{V}_i} A_v$ in $\mathrm{Br}(F_i) = \prod_{v \in \mathcal{V}_i} \mathrm{Br}(F_v)$ is α_i. Since $(\alpha_1)_{F_0} = (\alpha_2)_{F_0}$, the algebras $(A_1)_{F_0}$ and $(A_2)_{F_0}$ are Brauer equivalent, and thus isomorphic, being of the same degree. Using patching for central simple algebras (see Example 2.7), there is a central simple F-algebra A such that $A_{F_i} \cong A_i$, compatibly. This gives exactness of the given sequence, proving part (a).

We now turn to part (b). By [Pie82, Proposition 13.4(iv)], $\mathrm{ind}(\alpha_{F_v})$ divides $\mathrm{ind}(\alpha)$. It thus suffices to show that if each $\mathrm{ind}(\alpha_{F_v})$ divides an integer i then so does $\mathrm{ind}(\alpha)$. Choose a central simple algebra A with Brauer class α and with some degree $n > i$. Let SB_i be the i-th generalized Severi-Brauer variety, parametrizing ni-dimensional right ideals of A (see [VdB88, p. 334], [See99, Theorem 3.6], or [HHK09, p. 255]). For any field extension L/F, the group $\mathrm{PGL}_1(A)(L)$ acts transitively on the L-points of SB_i, via [KMRT98, Proposition 1.12, Definition 1.9] (see also [HHK09, p. 255]). Also, $\mathrm{ind}(A_L)$ divides i if and only if $\mathrm{SB}_i(L) \neq \varnothing$, by [KMRT98, Proposition 1.17]. In particular, $\mathrm{SB}_i(F_1)$ and $\mathrm{SB}_i(F_2)$ are non-empty. We claim that factorization holds for $\mathrm{PGL}_1(A)(F_\bullet)$. Assuming this for the moment, it follows from Theorem 2.13 that $\mathrm{SB}_i(F) \neq \varnothing$. Therefore $\mathrm{ind}(\alpha)$ divides i, as desired.

To complete the proof of part (b), it remains to prove the above claim. By Theorem 2.13, it suffices to prove that the map $H^1(F, \mathrm{PGL}_1(A)) \to \prod_v H^1(F_v, \mathrm{PGL}_1(A))$ has trivial kernel. Let β be in the kernel of this map. Now for any field E, the cohomology set $H^1(E, \mathrm{PGL}_1(A))$ parametrizes the set of isomorphism classes of central simple E-algebras of degree n, with the trivial element corresponding to the class of A (see [KMRT98, Proposition 29.1 and p. 396]). Let B be a central simple F-algebra B of degree n whose isomorphism class corresponds to β. Thus $B \otimes A^{\mathrm{op}}$ induces the trivial element in $\mathrm{Br}(F_v)$ for all v; and hence $B \otimes A^{\mathrm{op}} \in \mathrm{Br}(F)$ is itself trivial by part (a). Equivalently, β is trivial in $H^1(F, \mathrm{PGL}_1(A))$. This proves the claim and hence the result. □

Example 4.16. Under the hypotheses of Proposition 3.7 (resp. Proposition 3.9 or Proposition 3.10 or Corollary 3.12), the conclusions of Theorem 4.15 hold, since its hypotheses hold by those propositions. In particular, a central simple algebra A over F (resp. over F_W or F_P or E) is split if and only if it is split over each F_ξ for $\xi \in \mathcal{P} \cup \mathcal{W}$. Moreover the index of A is the least common multiple of the indices of the algebras A_{F_ξ}. See also [HH10, Theorem 7.2], [HHK09, Theorem 5.1], and [RS13, Theorem 2] for related results.

Corollary 4.17. *Let L be the field F (resp. F_W or F_P or E) in the situation of Proposition 3.7 (resp. Proposition 3.9 or Proposition 3.10 or Corollary 3.12). Let d be a positive integer and let $\alpha \in \mathrm{Br}(L)$. For $\xi \in \mathcal{P} \cup \mathcal{W}$, let α_ξ be the image of α in $\mathrm{Br}(F_\xi)$. If $\mathrm{ind}(\alpha_\xi)$ divides $\mathrm{per}(\alpha_\xi)^d$ for all $\xi \in \mathcal{P} \cup \mathcal{W}$, then $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^d$.*

Proof. Since $\mathrm{per}(\alpha_\xi)$ divides $\mathrm{per}(\alpha)$ for each $\xi \in \mathcal{P} \cup \mathcal{W}$, we have that $\mathrm{ind}(\alpha_\xi)$ divides $\mathrm{per}(\alpha)^d$ for each ξ. Since $\mathrm{ind}(\alpha) = \mathrm{lcm}\big(\mathrm{ind}(\alpha_\xi)\big)_\xi$ by Example 4.16, it follows that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^d$. □

We now turn to results that provide period-index bounds for finite separable extensions of fields of the form F_P and F_W, in terms of such bounds for k. Even in the special case of the fields F_P and F_W themselves, the results strengthen [HHK09, Corollary 5.10] and [HHK13, Theorem 4.6] by improving the exponent on the period and also considering more general open subsets. First we prove some lemmas.

Lemma 4.18. *Suppose that $\widehat{R}$ is a excellent complete regular local ring of dimension 2 with residue field k and fraction field F. Let n, d be positive integers such that $\mu_n \subset k$ and such that $\mathrm{ind}\,\alpha | (\mathrm{per}\,\alpha)^d$ for every n-torsion Brauer class $\alpha \in {}_n\mathrm{Br}(k)$. Let $\beta \in {}_n\mathrm{Br}(F)$ be a Brauer class ramified only along a regular sequence for $\widehat{R}$. Then $\mathrm{ind}\,\beta \,|\, (\mathrm{per}\,\beta)^{d+2}$.*

Proof. Let $\beta \in {}_n\mathrm{Br}(F)$ be as above. Thus $m := \mathrm{per}(\beta)$ divides n. By [Sal97], Proposition 1.2, we may write $\beta = \beta_0 + \gamma_1 + \gamma_2$, where $\beta_0 \in \mathrm{Br}(\widehat{R})$, where γ_i are the classes of cyclic algebras of degree m and hence have index dividing m. Thus γ_1, γ_2 have periods dividing m and hence the same is true of β_0. By [Mil80, Corollary IV.2.13], we may identify $\mathrm{Br}(\widehat{R}) = \mathrm{Br}(k)$ via specialization. Moreover the index of the class $\beta_0 \in \mathrm{Br}(\widehat{R}) \subset \mathrm{Br}(F)$ divides that of its image in $\mathrm{Br}(k)$, since specialization induces a natural bijection between étale extensions of k and of $\widehat{R}$, and hence of étale splittings of associated central simple algebras. Thus $\mathrm{ind}(\beta) \,|\, \mathrm{ind}(\beta_0)\,\mathrm{ind}(\gamma_1)\,\mathrm{ind}(\gamma_2) \,|\, \mathrm{per}(\beta_0)^d m^2 \,|\, m^{d+2} = \mathrm{per}(\beta)^{d+2}$. □

Lemma 4.19. *Suppose $\widehat{R}$ is a 2-dimensional excellent ring with fraction field F, $t \in \widehat{R}$, and $\widehat{R}$ is complete with respect to the t-adic topology. Suppose that $\widehat{R}/t\widehat{R} \cong k[U]$ is the coordinate ring of a regular affine k-curve U with $\mathrm{char}(k) \nmid m$, and that $\mathrm{ind}\,\alpha \,|\, (\mathrm{per}\,\alpha)^d$ for all $\alpha \in {}_m\mathrm{Br}(k[U])$. If $\beta \in {}_m\mathrm{Br}(F)$ is ramified only at the support of $t\widehat{R}$, then $\mathrm{ind}\,\beta \,|\, (\mathrm{per}\,\beta)^{d+1}$.*

Proof. The ramification of β defines a finite connected cover $V \to U$ of curves of degree $n := \mathrm{per}(\beta)$, together with a generator σ of its cyclic Galois group C_n. Applying [Sal08, Theorem 1.1] at each closed point of U, and using that β is ramified only at $t\widehat{R}$, it follows that the cover $V \to U$ is actually unramified and hence étale. By [AGV72, VII.5.5], specialization induces an equivalence of categories between the étale covers of U and of $\mathrm{Spec}(\widehat{R})$; and hence there is an étale algebra $S/\widehat{R}$ that lifts $V \to U$ and which is Galois with generator $\hat{\sigma}$ lifting σ. Let L be the fraction field of S, and consider the cyclic algebra $C = (L/F, \hat{\sigma}, t)$. By [Sal99, Lemma 10.2], C and β define the same cyclic cover $V \to U$ and the same Galois generator σ. Thus the Brauer class $\beta - [C] \in \mathrm{Br}(F)$ is unramified over $\widehat{R}$; i.e. it lies in $\mathrm{Br}(\widehat{R})$, and is represented by an Azumaya algebra B over $\widehat{R}$. Now $\mathrm{per}([C]) \mid \mathrm{ind}([C]) \mid n = \mathrm{per}(\beta)$, and so $\mathrm{per}(B) = \mathrm{per}(\beta - [C]) \mid \mathrm{per}(\beta)$.

By reducing modulo (t), the $\widehat{R}$-algebra B induces an Azumaya algebra B_0 over $k[U]$, hence in turn a class in the Brauer group of $k(U)$. We claim that $\mathrm{ind}(B)$ divides $i := \mathrm{ind}(B_0)$, over F and $k(U)$ respectively. To see this, consider the i-th generalized Severi-Brauer variety SB_i associated to B over $\widehat{R}$. Its fiber $(\mathrm{SB}_i)_0$ modulo (t) is the i-th generalized Severi-Brauer variety associated to B_0 over $k[U]$. Since the index of B_0 over $k(U)$ is i, there is a $k(U)$-point on $(\mathrm{SB}_i)_0$, by the key property of generalized Severi-Brauer varieties (recalled in the proof of Theorem 4.15 above). Now $(\mathrm{SB}_i)_0$ is smooth and projective over $k[U]$, and U is a regular curve; so the valuative criterion for properness implies that this $k(U)$-point extends to a $k[U]$-point, viz. a section of $(\mathrm{SB}_i)_0 \to U = \mathrm{Spec}(k[U]) = \mathrm{Spec}(\widehat{R}/t\widehat{R})$. By Lemma 4.5 of [HHK09] and the comment after that, this section over U lifts to a section of $\mathrm{SB}_i \to \mathrm{Spec}(\widehat{R})$. The generic point of the image of this section is an F-point of SB_i. Thus $\mathrm{SB}_i(F)$ is non-empty, and so the index of $[B] \in \mathrm{Br}(F)$ divides i, proving the claim.

Now $\mathrm{per}(B_0) \mid \mathrm{per}(B)$, since $\mathrm{Br}(\widehat{R}) \to \mathrm{Br}(k[U])$ is a group homomorphism taking $[B]$ to $[B_0]$. Thus $\mathrm{per}(B_0) \mid \mathrm{per}(\beta) = n \mid m$. The above claim and the hypothesis on $\mathrm{Br}(k[U])$ then yield that $\mathrm{ind}(B) \mid \mathrm{ind}(B_0) \mid (\mathrm{per}\ B_0)^d \mid (\mathrm{per}\ B)^d$. But $\beta = [B] + [C]$. Hence we have that $\mathrm{ind}(\beta) \mid \mathrm{ind}(B)\,\mathrm{ind}(C) \mid (\mathrm{per}\ B)^d\,\mathrm{per}(\beta) \mid (\mathrm{per}\ \beta)^{d+1}$, as asserted. □

Lemma 4.20. *For a (general) field L and an integer n, the following are equivalent:*

1. *For every finite field extension L'/L, and $\alpha \in {}_n\mathrm{Br}(L')$, we have* $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^d$.
2. *For every prime q dividing n, every finite field extension L'/L and every Brauer class $\alpha \in \mathrm{Br}(L')$ of period q, we have* $\mathrm{ind}(\alpha) | q^d$.

Moreover if $\mathrm{char}(L)$ *does not divide n then the same assertion holds with respect to the class of finite* separable *extensions L'/L in conditions 1 and 2.*

Proof. The forward implication is trivial. For (2) ⇒ (1), by considering primary parts we may assume that $\mathrm{per}(\alpha)$ is a prime power q^r. The implication is then given by [PS14, Lemma 1.1]. In the case that $\mathrm{char}(L)$ does not divide n, the corresponding assertion for separable extensions L'/L holds because the proof of [PS14, Lemma 1.1] involves only extensions of q-power degree. □

Following [Lie11] and [HHK09], we define the *Brauer dimension* of a field k *away from* a prime p as follows: The value is 0 if the absolute Galois group of k is a pro-p group (e.g. if k is separably closed). Otherwise, it is the infimum of the positive integers d such that for every finite generated field extension E/k of transcendence degree $i \le 1$, every $\alpha \in \mathrm{Br}(E)$ of period prime to p satisfies $\mathrm{ind}\,\alpha \,|\, (\mathrm{per}\ \alpha)^{d+i-1}$. (We note that the term "Brauer dimension" was used in somewhat different senses in the manuscripts [ABGV11] and [PS14].)

Theorem 4.21. *Let T be a complete discrete valuation ring whose residue field k has Brauer dimension d away from $p := \mathrm{char}(k)$. Let $\widehat{X}$ be a normal projective T-curve with function field F and closed fiber X. Let ξ be either a closed point $P \in X$ or a connected Zariski open subset $W \subset X$, and let E be a finite separable extension of F_ξ. Then $\mathrm{ind}(\alpha) \,|\, \mathrm{per}(\alpha)^{d+1}$ for all α in $\mathrm{Br}(E)$ of period not divisible by p.*

Proof. By hypothesis, $\mathrm{per}(\alpha)$ is not divisible by $\mathrm{char}(k)$ and hence also not by the characteristic of K, the fraction field of T. By Lemma 4.20, we may assume that $\mathrm{per}(\alpha)$ is a prime number $q \neq \mathrm{char}(K), \mathrm{char}(k)$. Since the degree $[K(\zeta_q) : K]$ is prime to q, we may also assume that K, k each contain μ_q. Namely, let $K' = K(\zeta_q)$, $k' = k(\zeta_q)$, and $E' := E(\zeta_q)$. Then the period of the induced element $\alpha' \in \mathrm{Br}(E')$ is equal to the period of $\alpha \in \mathrm{Br}(E)$, by [Pie82, Proposition 14.4.b(v)]. Also, the index of α (which is a power of q) divides $[E' : E]\,\mathrm{ind}(\alpha')$ by [Pie82, Proposition 13.4(v)]; and hence it divides the index of α' since $[E' : E]$ is relatively prime to q. So we may henceforth assume that K, k each contain μ_q.

In the case that $\xi = W$, we may assume that W is affine, since $F_W = F_{W'}$ for some connected affine open set W' on a normal model of T (see Remark 3.2(b)). In both cases $\xi = P, W$, let S be the integral closure of $\widehat{R}_\xi$ in E, let D be the ramification divisor of α on $\mathrm{Spec}(S)$, and let $\widetilde{\xi}$ be the inverse image of ξ under $\mathrm{Spec}(S) \to \mathrm{Spec}(\widehat{R}_\xi)$. Applying Corollary 3.12, we obtain a birational projective morphism $\pi : \widehat{V} \to \mathrm{Spec}(S)$ and a non-empty set $\mathcal{P}$ of closed points of $V := \pi^{-1}(\widetilde{\xi})$ satisfying the four conditions there.

Observe that the assertion holds in the special case that π is an isomorphism. Namely, if $\xi = P$, then $\mathrm{ind}\,\alpha | (\mathrm{per}\,\alpha)^{d+1}$ for all $\alpha \in {}_q\mathrm{Br}(E)$ by Lemma 4.18, since $\mathrm{ind}\,\gamma | (\mathrm{per}\,\gamma)^{d-1}$ for all $\gamma \in {}_q\mathrm{Br}(k_{\widetilde{P}})$ by the assumption on the Brauer dimension of k. (Here $k_{\widetilde{P}}$ is the residue field at $\widetilde{P}$.) Similarly, if $\xi = W$, then $\mathrm{ind}\,\alpha | (\mathrm{per}\,\alpha)^{d+1}$ for $\alpha \in {}_q\mathrm{Br}(E)$ by Lemma 4.19.

So we may now assume that π is not an isomorphism, and therefore that the patching assertion in the last part of Corollary 3.12 holds. Let $D', \mathcal{W}, \mathcal{B}$ be as in that result. By properties (ii) and (iii) of Corollary 3.12, $\alpha_U \in \mathrm{Br}(F_U)$ is unramified away from the closed fiber $U \subset \mathrm{Spec}(\widehat{R}_U)$ for each $U \in \mathcal{W}$.

Now $\mathrm{ind}\,\alpha_Q | (\mathrm{per}\,\alpha_Q)^{d+1}$ in $\mathrm{Br}(F_Q)$ for each $Q \in \mathcal{P}$ by Lemma 4.18, and therefore $\mathrm{ind}\,\alpha_Q | (\mathrm{per}\,\alpha)^{d+1}$, since $\mathrm{per}\,\alpha_Q | \mathrm{per}\,\alpha$. Similarly, $\mathrm{ind}\,\alpha_U | (\mathrm{per}\,\alpha)^{d+1}$ in $\mathrm{Br}(F_U)$ for all $U \in \mathcal{W}$ by Lemma 4.19. But the index of $\alpha \in \mathrm{Br}(E)$ is the least common multiple of the indices of all the induced elements α_Q, α_U, for $Q \in \mathcal{P}$ and $U \in \mathcal{W}$, by Example 4.16 in the context of Corollary 3.12. So the desired conclusion follows. □

Above, we restricted attention to elements of the Brauer group whose period is prime to the residue characteristic p. But in [PS14], a result was shown about elements whose period is equal to p. Namely, suppose that $\operatorname{char}(K) = 0$ and $\operatorname{char}(k) = p > 0$. If F is a finitely generated K-algebra of transcendence degree one, and if $\alpha \in \operatorname{Br}(F)$ has period p, then $\operatorname{ind}(\alpha)$ divides p^{2n+2}, where n is the p-rank of the residue field k of T. (See [PS14, Theorem 3.6]. Recall that the *p-rank*, or *imperfect exponent*, of a field k of characteristic p is the integer n such that $[k : k^p] = p^n$.) Combining the ideas there with the ideas above, we obtain the following:

Theorem 4.22. *In the situation of Theorem 4.21, suppose that T has mixed characteristic $(0, p)$. If the period of $\alpha \in \operatorname{Br}(E)$ is a power of p, then $\operatorname{ind}(\alpha)$ divides $(\operatorname{per} \alpha)^{2n+2}$, where n is the p-rank of k.*

Proof. Case I: $\operatorname{per}(\alpha) = p$.

As in the proof of Theorem 4.21, we may assume that the fraction field K (though not the residue field k) contains a primitive p-th root of unity. As in the proof of Theorem 4.21, we obtain a birational projective morphism $\pi : \widehat{V} \to \operatorname{Spec}(S)$ and an associated non-empty finite set $\mathcal{P} \subset V$.

In the case that π is not an isomorphism, we proceed as in the proof of Theorem 4.21 but use [PS14, Proposition 3.5 and Theorem 2.4] instead of using Lemmas 4.18 and 4.19. Namely, as before we obtain a set $\mathcal{W}$ consisting of the components of $V \smallsetminus \mathcal{P}$. If $U \in \mathcal{W}$, then U is irreducible and we may consider its generic point η. The p-rank of the field F_η is $n + 1$. Applying [PS14, Theorem 2.4] to F_η, we thus obtain that $\operatorname{ind}(\alpha_{F_\eta})$ divides p^{2n+2}. By [HHK15, Proposition 5.8] and [KMRT98, Proposition 1.17], after shrinking U (and correspondingly enlarging $\mathcal{P}$), we have that $\operatorname{ind}(\alpha_{F_U}) = \operatorname{ind}(\alpha_{F_\eta})$, which divides p^{2n+2}. Meanwhile, by [PS14, Proposition 3.5], $\operatorname{ind}(\alpha_{F_P})$ divides p^{2n+2} for every $P \in \mathcal{P}$. As in the proof of Theorem 4.21, we conclude via Example 4.16.

In the case that π is an isomorphism, we similarly use those two results in [PS14] instead of Lemmas 4.18 and 4.19. In the case that $\xi = W$, we apply [PS14, Theorem 2.4] at each generic point of W, and as above obtain a finite collection of points and open sets that partition W. We then conclude via Example 4.16 as in the above case.

Case II: General case.

Let E' be a finite extension of E. Recall that the p-rank of E' is also equal to n, i.e. $[E' : E'^p] = [E : E^p]$, because $[E' : E] = [E'^p : E^p]$ via the Frobenius isomorphism. Thus Case I applies to elements of $\operatorname{Br}(E')$ having period p. The result now follows from Lemma 4.20 applied to the field E and the integer $\operatorname{per}(\alpha)$. □

Theorem 4.23. *Let E be one of the following:*

(i) the fraction field of a two-dimensional Noetherian complete local domain R; or

(ii) a finite separable extension of the fraction field of the t-adic completion of $T[x]$, where T is a complete discrete valuation ring with uniformizer t.

Assume that the residue field k of R (resp. T) has Brauer dimension d away from $p := \operatorname{char}(k)$, and has p-rank n. Let $\alpha \in \operatorname{Br}(E)$. Then $\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^{d+1}$ if p does not divide $\operatorname{per}(\alpha)$; and $\operatorname{ind}(\alpha) \mid \operatorname{per}(\alpha)^{\max(d+1, 2n+2)}$ with no restriction on $\operatorname{per}(\alpha)$ if $\operatorname{char}(E) = 0$.

Proof. In case (i), Lemma 4.6 says that E is a finite separable extension of a field of the form F_P. In case (ii), E is a finite separable extension of F_U, where $U = \mathbb{A}^1_k \subset \mathbb{P}^1_T$. So in both cases, E is of the form considered in Theorems 4.21 and 4.22.

Those two theorems therefore yield this result if the period of α is either prime to p or a power of p. The general case follows by decomposing α into its primary parts. □

For example, taking $T = k[[t]]$, this theorem applies to finite separable extensions of the fraction field of $k[x][[t]]$, as well as of $k((x,t))$. This strengthens [HHK13, Corollary 4.7].

Corollary 4.24. *In the situation of Theorem 4.23, suppose that k is finite (resp. algebraically closed). If* $\mathrm{char}(E)$ *does not divide* $\mathrm{per}(\alpha)$ *then* $\mathrm{ind}(\alpha)$ *divides* $\mathrm{per}(\alpha)^2$. *Moreover* $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$ *in the algebraically closed case if* $\mathrm{char}(k)$ *does not divide* $\mathrm{per}(\alpha)$.

Proof. If k is algebraically closed, then $d = 0$ and $n = 0$ in the notation of Theorem 4.23. Since the period always divides the index, the assertion in this case follows from the theorem.

If k is finite, then $d = 1$ by Wedderburn's Theorem and [Rei75, Theorem 32.19]; and $n = 0$ since k is perfect. So again the result follows from the theorem. □

In particular, in the p-adic case this yields Corollary 1.2 and the assertion after it. See also [Hu13, Theorem 3.4] for a related result in the local case.

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Author Information:

David Harbater
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA
email: harbater@math.upenn.edu

Julia Hartmann
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA
email: hartmann@math.upenn.edu

Daniel Krashen
Department of Mathematics, University of Georgia, Athens, GA 30602, USA
email: dkrashen@math.uga.edu

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