Abhyankar's Conjecture and embedding problems

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Abstract: This paper proves results concerning the fundamental group of an affine curve over an algebraically closed field of characteristic p, particularly concerning embedding problems and inertia groups. It is shown that Galois covers of such curves can be modified so as to enlarge their Galois groups by quasi-p groups and to enlarge the p-part of inertia groups; that embedding problems for unramified covers with quasi-p group kernel can be solved with control on inertia; and that the analogous result holds for tamely ramified covers of affine curves. In addition, a tame analog of the geometric Shafarevich Conjecture is shown, and simplifications are given for the proofs of previous results including Abhyankar's Conjecture for general affine curves.

Section 1. Introduction.

Consider an affine curve U over an algebraically closed field k of characteristic $p > 0$. In this paper we consider the Galois covers of U, and how they fit together, with particular attention to solving embedding problems, enlarging inertia groups, and strengthening Abhyankar's Conjecture.

The étale covers of an affine curve U form an inverse system, whose automorphisms form the algebraic fundamental group $\pi_1(U)$. This fundamental group is known in characteristic 0 [Gr, XIII, Cor. 2.12] but not in characteristic $p > 0$. An explicit description of the set of finite quotients of $\pi_1(U)$ in characteristic p was conjectured in 1957 by Abhyankar [Ab], and this conjecture was proven in [Ra] and [Ha3]. This gives a necessary and sufficient condition for a finite group to be a Galois group of an étale cover of U. But it does not indicate which covers are dominated by other covers having specified Galois groups, nor which inertia groups can arise over points "at infinity". These questions are explored in the present paper.

In Section 2 below, it is shown that a characteristic p cover can be modified so as to enlarge its Galois group by a specified quasi-p group (Theorem 2.1). This is then used to obtain (Corollary 2.5) a quick proof of the (Strong) Abhyankar Conjecture. That result states that if U is obtained by deleting $r > 0$ points from a smooth projective k-curve

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of genus g, then G is the Galois group of an étale cover of U if and only if the maximal prime-to-p quotient of G can be generated by a set of $\leq 2g + r - 1$ elements; and in this case one can choose all but one infinite place to be tamely ramified (this last condition being the "strong" part). Here we give a short proof of existence; the converse follows from [Gr, XIII, Cor. 2.12].

Section 3 contains a result (Theorem 3.6) that allows a cover to be modified by enlarging its wild ramification. This result strengthens a previous result of the author [Ha2, Theorem 2], by permitting wild inertia groups whose orders need not be powers of p.

Theorems 2.1 and 3.6 are combined in Section 4, to obtain Theorem 4.1. This in turn provides a simplified proof and a generalization of a result of Pop [Po, Theorem B] on embedding problems with quasi-p kernel (Corollaries 4.2, 4.5, 4.6), as well as providing a related result on tame solutions to quasi-p embedding problems (Theorem 4.4). Theorem 4.1 also yields a proof of a further strengthening of Abhyankar's Conjecture (Corollary 4.7), in particular taking the unique wildly ramified fibre to have maximal possible wild inertia (i.e. containing a Sylow p-subgroup). In a related situation, in which extra tamely ramified points are allowed to be added, it is shown that all finite embedding problems can be solved (Corollary 4.8), thus extending results of [Ha4], [Po], and [Ha5]. The generalized tame fundamental group $\pi_1^{\mathrm{t}}(U, \Sigma)$ is also studied (where covers are permitted to be tamely ramified over $\Sigma \subset U$ and are required to be étale elsewhere over U). It is shown (Theorem 4.9) that this group is projective (i.e. has cohomological dimension ≤ 1), and that if $U - \Sigma$ is finite then this group is free. In particular, this gives the Galois group of the maximal extension of $k(x)$ that is tamely ramified away from infinity, as well as giving the structure of the analogous Galois group over $\mathbb{F}_p(x)$. This result can be regarded as a tame version of the geometric Shafarevich Conjecture ([Ha4], [Po, Corollary to Theorem A]).

Concerning methods, there have been three main techniques employed in studying covers of affine curves in characteristic p: patching, the p-embedding property, and semi-stable reduction.

Patching methods, in rigid or formal geometry, permit cut-and-paste constructions analogous to ones that can be performed on complex curves, and are useful in enlarging Galois groups of covers (e.g. in [Ra], [Ha2], [Ha3], [Po]). Here we use a version of formal patching that appeared in [HS], and which systematized the approach used in [Ha2].

The p-embedding property, in its simplest form, asserts that if H is a quotient of a finite group G by a p-group, then any H-Galois étale cover of U is dominated by a G-Galois étale cover. This was used in [Se2] to prove the solvable case of Abhyankar's Conjecture over the affine line, and stronger versions (imposing local conditions in rigid or formal geometry) were used in [Ra], [Ha3], and [Po]. The present paper uses a more general but more elementary "arithmetic" version of the p-embedding property with local conditions, which was proven in [Ha6]. It is used here in order to dominate a given cover by a cover that can be patched to a quasi-p cover without introducing new branch points.

The technique of semi-stable reduction permits one to pass from covers in characteristic 0 to covers in characteristic p, even when there is wild ramification. It was used in the proof of Abhyankar's Conjecture over the affine line [Ra], to handle the case in which the previous two methods do not apply. This method is used indirectly here, in that the main result of [Ra] is relied upon to obtain quasi-p covers of the affine line.

For further discussions of algebraic fundamental groups, patching methods, formal and rigid geometry, embedding problems, and semi-stable reduction, see [BLS], [MM], and [Vo].

Conventions and notation:

Throughout this paper, k denotes an algebraically closed field of characteristic $p > 0$. If R is a ring of characteristic p, there is an $\mathbb{F}_p$-linear map $\wp : R \to R$ given by $\wp(r) = r^p - r$. Its image $\wp(R)$ is an $\mathbb{F}_p$-subspace of R.

For a point ξ on a scheme X, the total ring of fractions of the complete local ring $\hat{\mathcal{O}}_{X,\xi}$ is denoted $\mathcal{K}_{X,\xi}$. If $Y \to X$ is a finite morphism , and $\xi \in X$, then $\mathcal{K}_{Y,\xi}$ will denote the direct sum of $\mathcal{K}_{Y,\eta}$, where η ranges over the (finitely many) points of Y over ξ. Thus $\operatorname{Spec} \mathcal{K}_{Y,\xi}$ is the fibre of $Y \to X$ over $\mathcal{K}_{X,\xi}$. If Z is a closed subset of an affine scheme $U = \operatorname{Spec} A$, defined by the ideal $I \subset A$, then $\hat{\mathcal{O}}_{U,Z}$ will denote the I-adic completion of A. If the affine scheme U is an open subset of a scheme V, then we will also write $\hat{\mathcal{O}}_{V,Z}$ for this completion (which depends only on V and Z, not on the choice of U).

Following [HS], a *cover* $Y \to X$ is a morphism of schemes that is finite and generically separable. In particular, étale covers are covers in this sense. If G is a finite group, then a *G-Galois cover* is a cover $Y \to X$ together with a homomorphism $G \to \operatorname{Aut}_X(Y)$ with respect to which G acts simply transitively on each generic geometric fibre. If H is a subgroup of G, and if $Y \to X$ is an H-Galois cover, then there is an induced G-Galois cover $\operatorname{Ind}_H^G Y \to X$, which consists of a disjoint union of $[G : H]$ copies of Y, indexed by the cosets of H in G. Reduced and irreducible covers $Y \to X$ and $Z \to X$ are *linearly disjoint* if the function fields of Y and Z are linearly disjoint over the function field of X.

If G is a finite group and H is a subgroup of G, then the normalizer of H in G is denoted $N_G(H)$. The *normal closure* of H in G is the smallest normal subgroup of G containing H. The subgroup of G generated by its p-subgroups will be denoted by $p(G)$. This is a characteristic subgroup of G, and $G/p(G)$ is the maximal prime-to-p quotient of G. A finite group G is a *quasi-p group* if $p(G) = G$. According to Abhyankar's Conjecture, these are precisely the Galois groups of finite étale covers of $\mathbb{A}^1_k$, while the finite Galois groups over an affine curve U are those G such that $G/p(G)$ is a Galois group over an "analogous complex curve" (i.e. a complex curve with the same genus and the same number of punctures). The *p-rank* of a profinite group G is the dimension of the $\mathbb{F}_p$-vector space $\operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$.

Given finite or profinite groups Π, Γ, G, an *embedding problem* $\mathcal{E}$ *for* Π consists of a pair of surjective group homomorphisms $(\alpha : \Pi \to G, f : \Gamma \to G)$. A *weak* [resp. *proper*] *solution* to $\mathcal{E}$ consists of a group homomorphism [resp. epimorphism] $\beta : \Pi \to \Gamma$ such that $f\beta = \alpha$. Here $\mathcal{E}$ is called *non-trivial* [resp. *finite*, resp. a *p-embedding problem*, resp. a *quasi-p embedding problem*] if the kernel of f is non-trivial [resp. finite, resp. a p-group, resp. a quasi-p group]. Also, $\mathcal{E}$ is called *split* if f has a section; such embedding problems automatically have weak solutions. A profinite group Π is *projective* if every finite embedding problem for Π has a weak solution. This is equivalent to $\mathrm{cd}(\Pi) \leq 1$ [Se1, I, 5.9, Proposition 45]. Similarly, if p is a prime, then the condition $\mathrm{cd}_p(\Pi) \leq 1$ is equivalent to every finite p-embedding problem for Π having a weak solution [Se1, I, 3.4, Proposition 16].

If Π is the fundamental group of a pointed connected scheme (X, ξ), and G is a finite group, then a surjection $\Pi \twoheadrightarrow G$ corresponds to a pointed connected G-Galois étale cover of (X, ξ). So giving an embedding problem $(\alpha : \Pi \to G, f : \Gamma \to G)$ for Π is equivalent to giving a pointed G-Galois cover $\phi : (Y, \eta) \to (X, \xi)$ together with a surjection $f : \Gamma \to G$. We call (ϕ, f) an *embedding problem* for (X, ξ); a *weak* [resp. *proper*] *solution* to such an embedding problem is a pointed Γ-Galois cover [resp. connected cover] that dominates ϕ. Such a solution corresponds to a solution to the embedding problem $(\alpha : \Pi \to G, f : \Gamma \to G)$ for Π. In discussing fundamental groups, reference to the base points will generally be suppressed, since the choice of base point does not affect the isomorphism class of the fundamental group.

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Section 2. Abhyankar's conjecture and enlarging Galois groups.

The main result of this section (Theorem 2.1) shows that a Galois group over an affine k-curve can be enlarged by a quasi-p group, provided that a group-theoretic normalization condition is satisfied. The enlarged cover can be taken to "look like" the original cover at all but one of the branch points, where wild inertia will in general be added. This result is proven using formal patching [HS], the p-embedding property with local conditions [Ha6], and Abhyankar's Conjecture over the affine line [Ra]. As a special case, we obtain a quick proof of Abhyankar's Conjecture over general affine k-curves (Corollary 2.5).

Theorem 2.1. *Let G be a finite group, let $Y \to X$ be a G-Galois cover of smooth connected projective k-curves, and let $\xi_0 \in X$. Let Γ be a finite group generated by G and a quasi-p group Q such that G normalizes a Sylow p-subgroup $P \subset Q$. Let N be the normal closure of Q in Γ. Then there is a smooth connected Γ-Galois cover $Z \to X$ such that*

(i) For every point $\xi \neq \xi_0$ in X, the inertia groups of $Y \to X$ over ξ are also inertia groups of $Z \to X$ over ξ;

(ii) For every inertia group I of $Y \to X$ over ξ_0, there is an inertia group H of $Z \to X$ over ξ_0 having the same maximal prime-to-p quotient as I, and satisfying $H \subset IP$; and

(iii) $Z/N \approx Y/(G \cap N)$ as Γ/N-covers of X.

In order to prove this theorem, it suffices to show that the analogous assertion holds for some overfield $K \supset k$. For then, the cover and its automorphisms are defined over some k-subalgebra $C \subset K$ of finite type over k. As a result, one obtains a family of covers of X, parametrized by $\operatorname{Spec} C$, such that the generic member satisfies the conclusion of 2.1. Specializing this family then gives a cover $Z \to X$ as desired. This strategy is carried out in detail below. Specifically, in Proposition 2.3 we prove the analogous assertion for the overfield $K = k((t))$, using formal geometry. After the proof of 2.3, we carry out the above specialization procedure, in order to complete the proof of Theorem 2.1.

Remark. (a) In fact, even more is true in Theorem 2.1: There are many choices of the Γ-Galois cover $Z \to X$ there, viz. as many as the cardinality of the base field k. The reason is that in the specialization argument just alluded to, there are that many points of $\operatorname{Spec} C$; and these points give that many distinct specializations. See Theorem 4.1 below (which subsumes Theorem 2.1) for details. The key step in the proof of this cardinality assertion appears in Lemma 3.5, which is used in Theorem 3.6 and thereby in Theorem 4.1. As a consequence of this cardinality assertion, we are afterwards able to determine the structure of certain fundamental groups, in Theorem 4.9.

(b) Theorem 2.1 allows one to "expand" a given Galois group by constructing a related cover whose group Γ is generated by the original group G together with elements of p-power order. As a special case, one can consider a split quasi-p embedding problem $\mathcal{E} = (\alpha : \Pi \to G,\ f : \Gamma \to G)$ for $\Pi = \pi_1(X - B)$, where $B \subset X$ contains ξ_0 and the branch locus of $Y \to X$, and where α corresponds to the cover $Y \to X$. With respect to a splitting of f, we may regard G as a subgroup of Γ. If this copy of G in Γ normalizes a Sylow-p subgroup of Γ, then Theorem 2.1 asserts that there is a proper solution to this embedding problem such that the corresponding Γ-Galois cover $Z \to X$ has the property that $Z \to Y$ is unramified away from ξ_0. Conversely, from an assertion on the solvability of quasi-p embedding problems (as in [Po]), one can deduce a result similar to Theorem 2.1; see Remark (b) after Corollary 4.2 below for a further discussion of this point. □

We begin by proving a lemma about branch loci of p-covers.

Lemma 2.2. *Let p be a prime number, and let P be a finite p-group. Let $\tilde{U} \to U$ be a P-Galois cover of schemes, with U regular and $\tilde{U}$ normal. Let V be an irreducible closed subset of U. Suppose the cover $\tilde{U} \to U$ is étale away from V and is totally ramified over some closed point $\nu \in V$. Then $\tilde{U} \to U$ is totally ramified over every point of V.*

Proof. Let $\tilde{V} \subset \tilde{U}$ be an irreducible component of the inverse image of V under $\tilde{U} \to U$. Let $I \subset P$ be the inertia group of $\tilde{U} \to U$ at the generic point of $\tilde{V}$.

Suppose that I is a proper subgroup of P. Since P is a p-group, it follows from [Sc,6.4.10] that I is contained in a proper normal subgroup J of P. The P/J-Galois cover $\tilde{U}/J \to U$ is then still totally ramified at ν, and étale away from V; and it is also étale over the generic point of V. Thus $\tilde{U}/J \to U$ is étale in codimension 1. Since $\tilde{U}$ is normal, so is $\tilde{U}/J$. Since also U is regular, Purity of Branch Locus [Na, 41.1] applies, and asserts that $\tilde{U} \to U$ is étale everywhere. But this P/J-Galois cover is totally ramified at ν, and P/J is non-trivial. This is a contradiction.

So in fact $I = P$. This shows that $\tilde{U} \to U$ is totally ramified over the generic point of V. Hence it is totally ramified over every point of V, since the set of totally ramified points is closed. □

Proposition 2.3. *Under the hypotheses of Theorem 2.1, there is an absolutely irreducible Γ-Galois cover $Z_t \to X_t := X \times_k k((t))$ that is normal and is $k((t))$-smooth away from $\xi_{0t} := \xi_0 \times k((t))$, such that the following conditions hold:*

(i) *For every point $\xi \neq \xi_0$ in X, the inertia groups of $Y \to X$ over ξ are also inertia groups of $Z_t \to X_t$ over $\xi_t := \xi \times k((t))$.*

(ii) *For every point η_0 of Y over ξ_0 with inertia group $I \subset G$, there is a point ζ_{0t} of Z_t over ξ_{0t} with inertia group $H \subset IP \subset \Gamma$ such that $\hat{\mathcal{O}}_{Z_t,\zeta_{0t}}$ contains $\hat{\mathcal{O}}_{\bar{Y}_t,\bar{\eta}_{0t}}$ as an $\hat{\mathcal{O}}_{X_t,\xi_{0t}}$-subalgebra, where $\bar{\eta}_0$ is the image of η_0 in $\bar{Y} = Y/(G \cap P)$, and where $\bar{Y}_t = \bar{Y} \times_k k((t))$ and $\bar{\eta}_{0t} = \bar{\eta}_0 \times_k k((t))$.*

(iii) *$Z_t/N \approx Y/(G \cap N) \times k((t))$ as Γ/N-covers of X_t.*

The proof of the proposition will proceed in several steps. In Step 1, we construct an auxiliary $\tilde{G}$-Galois cover of X defined over $k((s))$, where $\tilde{G}$ is the subgroup of Γ generated by P and G. This cover will be constructed so as to dominate the $\bar{G} := G/(G \cap P)$-Galois cover $Y/(G \cap P) \times_k k((s))$, using [Ha6] to guarantee a solution to this p-embedding problem (even with prescribed behavior over finitely many specified points δ — yielding a compatibility condition for Step 2). In Step 2, we construct the desired Γ-Galois cover of $X_t = X \times_k k((t))$. To do this, we take the blow up $\mathcal{X}_t$ of $X \times_k k[[t]]$ at some closed points δ; this has exceptional divisors S_δ which we view as copies of the s-line, and it contains a copy of X (viz. the proper transform of $(t = 0)$). Away from ξ_0 we identify the formal completion of $\mathcal{X}_t$ along X with $X \times_k k[[s]]$. A $\tilde{G}$-Galois cover of $\mathcal{X}_t$ is constructed by formal patching, so that its restriction to this formal completion consists of disjoint copies of the $\tilde{G}$-Galois cover from Step 1; and the restriction to each S_δ consists of copies of a Q-Galois cover W of the line, branched only at the point δ where S_δ meets X. (Here W exists by [Ra], and the patching is possible because local compatibility near δ was insured by Step 1.) The general fibre of this cover of $\mathcal{X}_t$ is then a Γ-Galois cover $Z_t \to X_t$. Finally, in Step 3, we verify that $Z_t \to X_t$ has all the required properties.

Proof of 2.3. We begin by fixing notation. Let $\tilde{G}$ be the subgroup of Γ generated by P and G, and let $\bar{G} = \tilde{G}/P = G/(G \cap P)$. Thus $\bar{Y} := Y/(G \cap P)$ is a $\bar{G}$-Galois cover. Let $X' = X - \{\xi_0\}$; this is an affine curve $\operatorname{Spec} R'$. For any scheme V equipped with a morphism to X, let V' denote $V \times_X X'$. Let s, t be transcendentals, and for any k-scheme V, let V_s denote $V \times_k k((s))$ and V_t denote $V \times_k k((t))$. Thus $Y'_s \to X'_s$ is a G-Galois cover of affine $k((s))$-curves that dominates the $\bar{G}$-Galois cover $\bar{Y}'_s = \bar{Y}' \times_k k((s)) \to X'_s$. By Riemann-Roch, we may choose a non-constant function $r \in R'$ on X' all of whose zeroes are simple, and such that the zero locus $D \subset X'$ of r is disjoint from the branch locus B of $Y \to X$. For each $\delta \in D$, let u_δ be a uniformizer for X at δ. Let $X'' = X - D$, and write $X'' = \operatorname{Spec} R''$; thus $\bar{r} := r^{-1} \in R''$, and $R'[\bar{r}] = R''[r] = \mathcal{O}(X' \cap X'')$.

Step 1: Construction of a $\tilde{G}$-Galois cover over $k((s))$.

Let S be a copy of the projective k-line, with parameter s. Since k is algebraically closed, the Abhyankar Conjecture for the affine k-line holds [Ra]. So there is a smooth connected Q-Galois cover $W \to S$ which is ramified only over $s = 0$. Moreover, by [Ra], this cover may be chosen so that its inertia groups over $(s = 0)$ are the Sylow p-subgroups of Q. In particular, one of the inertia groups, say at $\omega \in W$, is equal to P. The extension of local fields $k((s)) = \mathcal{K}_{\mathbb{P}^1,0} \subset \mathcal{K}_{W,\omega}$ over $(s = 0)$ is thus P-Galois and totally ramified.

We now apply [Ha6, Theorem 5.6] to the cover $\bar{Y}'_s \to X'_s$ in order to obtain a connected $\tilde{G}$-Galois cover $\tilde{Y}'_s \to X'_s$ that is étale over $\bar{Y}'_s$ and has specified behavior over finitely many local fields — viz. those at the points of B'_s and D_s. Specifically, $\tilde{Y}'_s \to X'_s$ can be taken to have the following properties: it is étale away from B'_s; it dominates the $\bar{G}$-Galois cover $\bar{Y}'_s \to X'_s$; the P-Galois cover $\tilde{Y}'_s \to \bar{Y}'_s$ is étale; for each $\beta \in B'$, its fibre over

$\mathcal{K}_{X'_s,\beta_s}$ is given by the $\tilde{G}$-Galois extension $\mathrm{Ind}_G^{\tilde{G}} \mathcal{K}_{Y'_s,\beta_s}$; and for each $\delta \in D$, its fibre over $\mathcal{K}_{X'_s,\delta_s} = k((s))((u_\delta))$ is given by the $\tilde{G}$-Galois extension $\mathrm{Ind}_P^{\tilde{G}} \mathcal{K}_{W,\omega}((u_\delta))$. (This last condition provides compatibility with the Q-Galois cover $W \to S$, and will allow $\tilde{Y}'_s$ to be patched to W in Step 2 below.) Thus for any $\xi \in X'$, the inertia groups of $Y' \to X'$ over ξ are also inertia groups of $\tilde{Y}'_s \to X'_s$ over ξ_s, and these groups are trivial for $\xi \notin B'$.

The above Galois covers $Y'_s, \bar{Y}'_s, \tilde{Y}'_s$ of X'_s are defined over $k((s))$, and have Galois groups $G, \bar{G}, \tilde{G}$ respectively. The normalizations $\mathcal{Y}'_s, \bar{\mathcal{Y}}'_s, \tilde{\mathcal{Y}}'_s$ of $\mathcal{X}'_s := X' \times k[[s]] = \mathrm{Spec}(R' \otimes_k k[[s]])$ in these three covers are Galois covers of $\mathcal{X}'_s$, defined over the ring $k[[s]]$; they have the same respective Galois groups as the above $k((s))$-covers, which are their generic fibres. Here $\mathcal{Y}'_s = Y' \times k[[s]]$ and $Y'_s = Y' \times k((s))$. So the cover $Y'_s \to X'_s$ splits completely over δ_s for $\delta \in D$, since the same holds for $Y' \to X'$ over δ. Also, $\bar{\mathcal{Y}}'_s = \mathcal{Y}'_s/(G \cap P)$, and its closed fibre is $\bar{Y}' = Y'/(G \cap P)$, which is irreducible. The normal $\tilde{G}$-Galois cover $\tilde{\mathcal{Y}}'_s \to \mathcal{X}'_s$ dominates $\bar{\mathcal{Y}}'_s \to \mathcal{X}'_s$, and it corresponds to a normal $\tilde{G}$-Galois extension $\tilde{A}'$ of the ring $\mathcal{O}(\mathcal{X}'_s) = R' \otimes_k k[[s]]$. The P-Galois cover $\tilde{\mathcal{Y}}'_s \to \bar{\mathcal{Y}}'_s$ is étale on its general fibre, and is totally ramified over each point $\delta \in D$ on the closed fibre, since over δ_s the cover agrees with the discrete valuation field $\mathcal{K}_{W,\omega}$ (which as a P-Galois extension of $k((s))$ is totally ramified). Hence $\tilde{\mathcal{Y}}'_s \to \bar{\mathcal{Y}}'_s$ is totally ramified all along its closed fibre, by Lemma 2.2. Thus the closed fibre of $\tilde{\mathcal{Y}}'_s$ is irreducible.

Step 2: Construction of a Γ-Galois cover over $k((t))$.

Let T be the affine k-line with parameter t, and let Σ be the blow-up of $X \times T$ with respect to the ideal $(r, t) \subset R'[t]$. Thus $\Sigma \to X \times T$ is an isomorphism away from $D \times (t = 0)$, and over each of those points the inverse image is an exceptional divisor S_δ, viz. a copy of the projective k-line S with parameter $s := t/r = \bar{r}t$. Here s is a well-defined morphism $\Sigma \to S$. Writing $\mathcal{X}_s = \Sigma \times_S k[[s]]$, where we identify $k[[s]]$ with the complete local ring of S at $(s = 0)$, we have that $\mathcal{X}'_s$ (from the end of Step 1) agrees with $\mathcal{X}_s \times_X X'$.

Let $\mathcal{X}_t = \Sigma \times_T k[[t]]$. This is a projective curve over $k[[t]]$ whose closed fibre consists of the proper transform of $X \times (t = 0)$ (which we identify with X) and the projective lines S_δ, where X and S_δ meet at the point δ on X. The general fibre of $\mathcal{X}_t$ is $X_t = X \times k((t))$. Observe that if $\xi \in X'$, viewed as a point on the closed fibre of Σ over $(t = 0)$, then $\hat{\mathcal{O}}_{\mathcal{X}_t,\xi} = \hat{\mathcal{O}}_{\Sigma,\xi} = \hat{\mathcal{O}}_{\mathcal{X}_s,\xi} \approx \hat{\mathcal{O}}_{X,\xi}[[s]]$.

The homomorphism $k[[s]] \to R''[[t]] = \hat{\mathcal{O}}_{\mathcal{X}_t,X''}$, given by $s \mapsto \bar{r}t$, induces a homomorphism from $\mathcal{O}(\mathcal{X}'_s) = R' \otimes_k k[[s]]$ to $R' \otimes_k R''[[t]]$ and hence to $R''[[t]][r]$ (using the map $R' \to R'[\bar{r}] = R''[r]$). So we may form the tensor product $\tilde{\mathcal{A}} := \tilde{A}' \otimes_{\mathcal{O}(\mathcal{X}'_s)} R''[[t]][r]$, which is $\tilde{G}$-Galois and flat over $R''[[t]][r]$ because $\tilde{A}'$ is $\tilde{G}$-Galois and flat over $\mathcal{O}(\mathcal{X}'_s)$. Thus $\hat{\mathcal{O}}_{\mathcal{X}_t,X''} = R''[[t]] \subset R''[[t]][r]$ is a subring of $\tilde{\mathcal{A}}$, and its integral closure $\tilde{A}_X$ in $\tilde{\mathcal{A}}$ is a $\tilde{G}$-Galois extension of $R''[[t]]$. For $\delta \in D$, viewed as a point on X in the closed fibre of $\mathcal{X}_t$, let $\tilde{A}_\delta = \tilde{A}' \otimes_{\mathcal{O}(\mathcal{X}'_s)} \hat{\mathcal{O}}_{\mathcal{X}_s,\delta} = \tilde{A}' \otimes_{\mathcal{O}(\mathcal{X}'_s)} \hat{\mathcal{O}}_{\mathcal{X}_t,\delta}$. Thus (in the notation of [HS, §1, p. 275]) $\tilde{A}_X$ and $\tilde{A}_\delta$ agree over the complete local ring $\hat{\mathcal{O}}_{\mathcal{X}_t,\delta,\mathfrak{p}_\delta} = \mathcal{K}_{X,\delta}[[s]] = \mathcal{K}_{X,\delta}[[t]]$, where $\mathfrak{p}_\delta = (s)$ is the height one prime of $\hat{\mathcal{O}}_{\mathcal{X}_t,\delta}$ whose closure in $\mathcal{X}_t$ is X. The corresponding statement then holds for the induced Γ-Galois algebras $E_X = \mathrm{Ind}_{\tilde G}^{\Gamma}\tilde A_X$ and $E_\delta = \mathrm{Ind}_{\tilde G}^{\Gamma}\tilde A_\delta$.

Let $S''_\delta = S_\delta - \{\delta\} \subset S_\delta$ and let W'' be the inverse image of S''_δ under $W \to S \xrightarrow{\sim} S_\delta$ (so W'' is independent of δ). Thus $\hat{\mathcal{O}}_{\mathcal{X}_t,S''_\delta} = k[\bar s][[t]]$, where $s\bar s = 1$. Define $\tilde A_{S_\delta} = \hat{\mathcal{O}}_{W''} \otimes_{\hat{\mathcal{O}}_{S''_\delta}} \hat{\mathcal{O}}_{\mathcal{X}_t,S''_\delta} = \hat{\mathcal{O}}_{W''}[[t]]$ and let $E_{S_\delta} = \mathrm{Ind}_Q^\Gamma \tilde A_{S_\delta}$. The fibre of $\tilde Y'_s \to X'_s$ was constructed to agree over δ_s with the extension $k((s)) \subset \mathcal{K}_{W,\omega}$, so E_{S_δ} and E_δ agree over δ_s, i.e. at the residue field of the height one prime $\mathfrak{q}_\delta = (u_\delta) \subset \hat{\mathcal{O}}_{\mathcal{X}_t,\delta}$ whose closure in $\mathcal{X}_t$ is S_δ. Since two étale extensions of a power series ring that agree over the residue field must themselves agree [Gr, I, Théorème 6.1], it follows that E_{S_δ} and E_δ agree over the complete local ring $\hat{\mathcal{O}}_{\mathcal{X}_t,\delta,\mathfrak{q}_\delta} = k((s))[[t]]$.

The singular locus of the closed fibre of $\mathcal{X}_t$ is the set D, whose complement in the closed fibre is $X'' \cup S''$, where $S'' = \bigcup_\delta S''_\delta$. By the above compatibilities at $\mathfrak{p}_\delta$ and $\mathfrak{q}_\delta$, we may apply the Corollary to the Patching Theorem [HS, Theorem 1] to the Γ-Galois extensions E_δ of $\hat{\mathcal{O}}_{\Sigma,\delta}$ and $E_X \times E_S$ of $\hat{\mathcal{O}}_{\mathcal{X}_t,X''\cup S''}$. The conclusion is that that there is a Γ-Galois cover $\mathcal{Z}_t \to \mathcal{X}_t$ whose restrictions to $\hat{\mathcal{O}}_{\mathcal{X}_t,X''}$, $\hat{\mathcal{O}}_{\mathcal{X}_t,S''_\delta}$, and $\hat{\mathcal{O}}_{\Sigma,\delta}$ agree with E_X, E_{S_δ}, and E_δ respectively, compatibly with the above identifications. In particular, $\mathcal{Z}_t$ is a normal variety. Its general fibre is a normal Γ-Galois cover of $k((t))$-curves, $Z_t \to X_t = X \times k((t)) = \mathcal{X}_t \times_{k[[t]]} k((t))$.

Step 3: Verification of the desired properties of the Γ*-Galois cover, over* $k((t))$.

Absolute irreducibility of Z_t: Pick $\delta \in D$ and let $\zeta \in \mathcal{Z}_t$ be the point over δ that corresponds to the closed point of the identity copy of $\mathrm{Spec}\,\tilde A_\delta$ in $\mathrm{Spec}\,E_\delta = \mathrm{Ind}_{\tilde G}^{\Gamma}\mathrm{Spec}\,\tilde A_\delta$. In the inverse image of $X' \subset \mathcal{X}_t$ under $\mathcal{Z}_t \to \mathcal{X}_t$, there is a unique irreducible component passing through ζ, viz. the closed fibre of the identity copy of $\tilde{\mathcal{Y}}'_s$ (which is indeed irreducible, as shown at the end of Step 1). The decomposition group of this component is $\tilde G$. Also, the inverse image of S_δ has a unique irreducible component passing through ζ, viz. the identity copy of W, with decomposition group Q. Since $\tilde G$ and Q generate Γ, it follows that Γ is the decomposition group of the connected component of $\mathcal{Z}_t$ containing ζ; i.e. that the Γ-Galois cover $\mathcal{Z}_t \to \mathcal{X}_t$ is connected. Since $\mathcal{Z}_t$ is normal, it is also irreducible, as is its generic fibre Z_t. The same argument shows that this irreducibility is preserved after passing to a finite extension of $k((t))$; i.e. Z_t is an absolutely irreducible $k((t))$-variety.

Smoothness and condition (i): If $\xi \in X' - D$, we may identify ξ_t in $\mathcal{X}_t$ with ξ_s in $\mathcal{X}_s$. Also, in this situation, the complete localizations of $\tilde A_X$ and $\tilde A'$ at ξ agree. Hence $\mathcal{Z}_t$, which is given near ξ by $E_X = \mathrm{Ind}_{\tilde G}^{\Gamma}\tilde A_X$, locally consists of copies of $\tilde{\mathcal{Y}}'_s = \mathrm{Spec}\,\tilde A'$.

In particular, in the case that $\xi = \beta \in B' \subset X' - D$, the generic fibre $\tilde Y'_s$ of $\tilde{\mathcal{Y}}'_s$ consists locally of copies of Y'_s near β_s. Hence the inertia groups of $Y \to X$ over ξ are also inertia groups of $\tilde Y'_s \to X'_s$ over $\xi_s = \xi_t$, and hence of $Z_t \to X_t$ over ξ_t, as desired. Moreover, in this case, Z_t is $k((t))$-smooth along the fibre over β_t since Y is smooth on the fibre over β.

On the other hand, if $\xi \in X' - B' - D$, then the inertia groups of $Y \to X$ over ξ are trivial, and hence so are those of $\bar Y'_s \to X'_s$ over $\xi_s = \xi_t$; those of $\mathrm{Spec}\,\tilde A_X \to \mathrm{Spec}\,\hat{\mathcal{O}}_{\mathcal{X}_t,X''}$

over ξ_t; and so also those of $Z_t \to X_t$ over ξ_t. Since X_t is $k((t))$-smooth at ξ_t, it follows that Z_t is $k((t))$-smooth over that point.

The remaining possibility is that $\xi = \delta \in D$. In this case, the cover $W'' \to S''_\delta$ is étale; hence so is the extension $\hat{\mathcal{O}}_{\mathcal{X}_t, S''_\delta} \subset \tilde{A}_{S_\delta}$, and so is $\mathcal{Z}_t \to \mathcal{X}_t$ over $\delta \times k[[t]]$. So the inertia groups of $Z_t \to X_t$ over δ_t, like those of $Y \to X$ over δ, are trivial. Again, it follows that Z_t is $k((t))$-smooth there.

Condition (ii): Since the $\tilde{G}$-Galois cover $\tilde{\mathcal{Y}}'_s \to \mathcal{X}'_s$ dominates the $\bar{G}$-Galois cover $\bar{\mathcal{Y}}'_s = \mathcal{Y}'_s/(G \cap P) \to \mathcal{X}'_s$, it follows that the $\tilde{G}$-Galois cover $\operatorname{Spec} \tilde{A}_X \to \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_t, X''}$ obtained above from $\tilde{\mathcal{Y}}'_s$ dominates the $\bar{G}$-Galois cover $\operatorname{Spec} \bar{A}_X \to \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_t, X''}$ that is similarly obtained from $\bar{\mathcal{Y}}'_s$. But $\operatorname{Spec} \bar{A}_X = \bar{Y} \times_X \hat{\mathcal{O}}_{\mathcal{X}_t, X''}$. So if I is the inertia group of $Y \to X$ at a point η_0 over ξ_0, then $\bar{I} := I/(I \cap P)$ is the inertia group of $\operatorname{Spec} \bar{A}_X$ at the induced point $\bar{\eta}_{0t}$; and the complete local ring of $\operatorname{Spec} \bar{A}_X$ at $\bar{\eta}_{0t}$ is an $\hat{\mathcal{O}}_{\mathcal{X}_t, X''}$-subalgebra of the complete local ring of $\operatorname{Spec} \tilde{A}_X$ at any point over $\bar{\eta}_{0t}$. Hence any inertia group H of $\operatorname{Spec} \tilde{A}_X \to \hat{\mathcal{O}}_{\mathcal{X}_t, X''}$ over $\bar{\eta}_{0t}$ is contained in IP, with $\bar{I} = IP/P$ as its quotient under $\tilde{G} \twoheadrightarrow \bar{G} = \tilde{G}/P$. The kernels of $I \twoheadrightarrow \bar{I}$ and $H \twoheadrightarrow \bar{I}$ are p-groups, so I and H have the same maximal prime-to-p quotient. Now the inertia groups and complete local rings of Z_t over $\xi_{0t} \in X_t$ are the same as those of $\mathcal{Z}_t$ over $\xi_{0t} \in \mathcal{X}_t$, which in turn are the same as those of $\operatorname{Spec} E_X = \operatorname{Ind}_{\tilde{G}}^{\Gamma} \operatorname{Spec} \tilde{A}_X$ over $\xi_{0t} \in \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_t, X''}$. Also, the complete local rings of $\bar{Y}_t$ over $\xi_{0t} \in X_t$ are the same as those of $\operatorname{Spec} \bar{A}_X$ over $\xi_{0t} \in \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_t, X''}$. So H is an inertia group of $Z_t \to X_t = X \times k((t))$ over ξ_{0t}, and the complete local rings of Z_t over $\xi_{0t} \in X_t$ dominate those of $\bar{Y}_t$ over ξ_{0t}.

Condition (iii): Observe that in the special case that Q is trivial, the cover Z_t is simply $Y \times k((t))$, since in that case W and P are trivial. More generally, for arbitrary Q, the above construction is compatible with taking quotients. In particular, $Z_t \to X \times k((t))$ has the property that its quotient $Z_t/N \to X \times k((t))$ is the Γ/N-Galois cover obtained by applying the construction to the $G/(G \cap N)$-cover $Y/(G \cap N) \to X$ and the trivial quasi-p group. (Here $G/(G \cap N) \approx \Gamma/N$.) So by the above observation, Z_t/N is isomorphic to $Y/(G \cap N) \times k((t))$ as Γ/N-covers of $X \times k((t))$. □

Remark. The above proof combined patching methods with a result on the existence of solutions to p-embedding problems with prescribed local behavior. As mentioned in the introduction, such a strategy has been employed previously in related results ([Ra], [Ha3], [Po]). In [Ra] (on Abhyankar's Conjecture for the affine line) and in [Po] (on solving embedding problems with quasi-p kernel), the notion of "local behavior" was in the context of rigid geometry. The p-embedding result used in those papers was proven in [Ra, §4] in the language of "Runge pairs", and the idea was to specify the behavior of the cover over given affinoid discs. In [Ha3] (on Abhyankar's Conjecture for affine curves), "local behavior" was in the sense of formal geometry, and referred to agreement over complete local rings of dimension 2. The existence of such solutions to p-embedding problems was proven in [Ha3] by means of results in [Ha1] on p-covers of affine curves. In the present paper, the notion of "local behavior" has a more arithmetic sense, in terms of agreement over finitely many points. Since the base field (here, a Laurent series field) is not algebraically closed, this type of local behavior provides non-trivial information; and it is sufficient to carry out the desired strategy without additional reliance on formal or rigid geometry. The p-embedding result used in the above proof was proven in [Ha6], which used a strategy that paralleled the proof of the p-embedding result in [Ra, §4]. The proof in [Ha6] also drew on related ideas that had appeared previously in Serre's proof of Abhyankar's Conjecture for solvable groups over the affine line [Se2], as well as in the proof of a result of Katz and Gabber [Ka, §2] on cyclic-by-p covers of curves (which in turn generalized the key result in [Ha1], though by a different proof). □

Using the above proposition and the fact that k is algebraically closed, we conclude the proof of Theorem 2.1:

Proof of 2.1. Consider the Γ-Galois cover $Z_t \to X \times k((t))$ given by Proposition 2.3. Since this cover is of finite type over its base, it descends to a Γ-Galois cover $\mathcal{Z} \to X \times_k C$ for some k-algebra $C \subset k((t))$ of finite type. Here we may assume that C is a integral domain, say with fraction field K. Since the asserted properties for $Z_t \to X \times k((t))$ are of finite type, we may choose this descended cover so that its generic fibre $\mathcal{Z}_K \to X_K$ has the same properties, though over K instead of $k((t))$. (The assertion about complete local rings in property (ii) is of finite type because it is equivalent to saying that the normalization of $\bar{Y}_t \times_{X_t} Z_t$ is étale over Z_t.)

More precisely, we may choose the descent so that $\mathcal{Z}_K$ is absolutely irreducible, normal, and K-smooth away from $\xi_0 \times K$, and so that properties (i)-(iii) of Proposition 2.3 hold with $k((t))$ replaced by K. Replacing $\operatorname{Spec} C$ by a Zariski dense open subset, we may assume that (i)-(iii) of Proposition 2.3 hold (with $k((t))$ replaced by k) for each fibre $\mathcal{Z}_\gamma \to X$, where γ ranges over the closed points of the k-variety $\operatorname{Spec} C$. Each such fibre is also smooth away from ξ_0, since $\mathcal{Z}$ is. By [Ha2, Proposition 5] (or by the Bertini-Noether theorem [FJ, Proposition 9.29]), there exists a k-point $\gamma \in \operatorname{Spec} C$ such that the fibre $\mathcal{Z}_\gamma \to X$ is irreducible. Its normalization $Z \to X$ is a smooth connected k-curve which agrees with $\mathcal{Z}_\gamma \to X$ over $X' = X - \{\xi_0\}$, by smoothness of $\mathcal{Z}_\gamma$ there. So condition (i) of Theorem 2.1 holds for $Z \to X$. Condition (iii) of Theorem 2.1 holds for $\mathcal{Z}_\gamma \to X$ and hence for $Z \to X$, since it holds generically and both sides are smooth. Finally, condition (ii) of Proposition 2.3 (over k) holds for $\mathcal{Z}_\gamma \to X$ and hence for $Z \to X$. So if I is any inertia group of $Y \to X$ over ξ_0, and $\bar{I}$ is the corresponding inertia group of $\bar{Y} \to X$, then $\bar{I}$ is a quotient of an inertia group H of Z over ξ_0. Thus I and H have the same maximal prime-to-p quotient. This shows that (ii) of Theorem 2.1 holds. □

Remarks. (a) The above proof of Theorem 2.1 actually shows more, viz. that the asserted cover $Z \to X$ can be chosen to have the property that its complete local rings over ξ_0 dominate those of $\bar{Y} = Y/(G \cap P) \to X$ over ξ_0, viewed as Galois $\hat{\mathcal{O}}_{X,\xi_o}$-algebras. With such a choice, we have that $\bar{I} = I/(I \cap P)$ is a quotient of H.

(b) The statement of Theorem 2.1 can be strengthened to assert that $H = IP$ in part (ii) of the theorem. This is shown in Theorem 4.1.

(c) In the formal patching results of [HS], cited above, the category $\mathcal{P}(T)$ of projective $\mathcal{O}_T$-modules is considered. More precisely, this consists of "sheaves of projective $\mathcal{O}_T$-modules" (i.e. $\mathcal{O}_T$-modules which assign, to each affine open set $U = \operatorname{Spec} A$, a projective A-module). Unless T itself is affine, this is not the same as the category of "projective sheaves of $\mathcal{O}_T$-modules" (i.e. projective objects in the category of all $\mathcal{O}_T$-modules). □

Recall the following elementary group-theoretic lemma from [Ha3]:

Lemma 2.4 [Ha3, Lemma 5.3] *Let Γ be a finite group, let $Q = p(\Gamma)$, and let $\pi : \Gamma \to \Gamma/Q$ be the natural quotient map. Let P be a Sylow p-subgroup of Γ, and let $\Gamma' = N_\Gamma(P)$. Then Γ' contains a subgroup F having order prime to p, such that $\pi(F) = \Gamma/Q$.*

Using the above, we obtain a proof of the Strong Abhyankar Conjecture [Ha3, Theorem 6.2]:

Corollary 2.5. (Strong Abhyankar Conjecture) *Let X be a smooth connected projective k-curve of genus $g \geq 0$, let $B \subset X$ be a set of $r > 0$ points, and let $\xi_0 \in B$. Let Γ be a finite group such that $\Gamma/p(\Gamma)$ has a generating set of at most $2g + r - 1$ elements. Then there is a smooth connected Γ-Galois cover $Z \to X$ that is unramified outside B and is tamely ramified away from ξ_0.*

Proof. Let $Q = p(\Gamma)$, which is a normal subgroup of Γ. Let P be a Sylow p-subgroup of Γ (or equivalently, of Q, since $Q = p(\Gamma)$). Let $a_1, \dots, a_n$ be generators of Γ/Q, with $n \leq 2g + r - 1$.

By Lemma 2.4, there is a prime-to-p subgroup $G \subset \Gamma$ that normalizes P and surjects onto Γ/Q (i.e. G and Q generate Γ). By this surjectivity, there are elements $g_1, \dots, g_n \in G$ that map to $a_1, \dots, a_n$ respectively, modulo Q. Replacing G by its subgroup generated by $g_1, \dots, g_n$, we may assume that G has a generating set of $n \leq 2g + r - 1$ elements.

Since G has order prime to p, by [Gr, XIII, Cor. 2.12] there is a smooth connected G-Galois cover $Y \to X$ that is unramified away from B, and is at most tamely ramified over B. Applying Theorem 2.1, we obtain a smooth connected Γ-Galois cover $Z \to X$ whose inertia groups away from ξ_0 agree with those of $Y \to X$. So $Z \to X$ is unramified away from B and is tamely ramified away from ξ_0. □

See Corollary 4.7 below for a stronger version of this result.

Section 3. Enlarging inertia groups.

This section considers the problem of modifying a cover by enlarging inertia groups, including the possibility of adding new branch points, and enlarging the Galois group by the new inertia. A result of this sort appeared at [Ha2, Theorem 2], using formal patching, and giving information toward Abhyankar's Conjecture. Here, in Theorem 3.6, more general modifications are obtained, viz. cyclic-by-p inertia groups can be enlarged by expanding the p-part (rather than just enlarging inertia groups that are p-groups, as in [Ha2]).

The proofs in this section, like those of Section 2, use formal patching [HS] and the p-embedding property [Ha6]. But this section can be read independently of Section 2. The key results of the two sections (Theorems 2.1 and 3.6) will be combined in Section 4, to prove Theorem 4.1.

Theorem 3.6 shows that it is possible to enlarge both Galois groups and the p-parts of inertia groups, with connectivity guaranteed provided that enough inertia is allowed. As in Section 2 above, the strategy is to constuct the desired cover over a Laurent series field $k((t))$; then to descend the cover to a finite-type k-subalgebra $A \subset k((t))$; and finally to specialize to a k-point of $\operatorname{Spec} A$ in order to obtain the desired cover of X defined over k itself. The cover defined over $k((t))$ is obtained as the generic fibre of a cover over $k[[t]]$ which is constructed using formal patching. Namely, this cover is constructed locally, near each branch point, in Proposition 3.4. These local covers are then patched to a disjoint union of copies of the original cover (base changed from k to $k[[t]]$) away from the branch locus. This last step is carried out in the proof of Theorem 3.6.

We begin with some preliminary results, used in the proofs of Proposition 3.4 and Theorem 3.6.

Lemma 3.1. *Let R be an integral domain of characteristic p, and suppose that $R/\wp(R)$ is infinite. Let P be a non-trivial normal p-subgroup of a finite group G, and let $Y \to X = \operatorname{Spec} R$ be a connected G/P-Galois étale cover. Then there are infinitely many connected G-Galois étale covers $Z \to X$ that dominate $Y \to X$ and are linearly disjoint over Y.*

Proof. Let $H = G/P$, let $\Pi = \pi_1(X)$, and let $\alpha : \Pi \twoheadrightarrow H$ correspond to the H-Galois cover $Y \to X$. The map α and the exact sequence $1 \to P \to G \to H \to 1$ define a finite p-embedding problem for $\Pi := \pi_1(X)$, whose proper solutions correspond to covers $Z \to X$ as above. Since $V := R/\wp(R)$ is infinite, and hence infinite dimensional as an $\mathbb{F}_p$-vector space, there are infinitely many non-isomorphic $\mathbb{Z}/p\mathbb{Z}$-Galois étale covers of $X := \operatorname{Spec} R$ (viz. given by $y^p - y = a$, where $a \in R$ ranges over a lift to R of a $\mathbb{Z}/p\mathbb{Z}$-basis of the vector space V). So $\Pi := \pi_1(\operatorname{Spec} R)$ has infinite p-rank. By [Ha6, Corollary 3.3(c)], $\operatorname{cd}_p(\Pi) \le 1$. So by [Ha6, Theorem 2.3], every finite p-embedding problem for Π has a proper solution. Thus such a Z exists.

To complete the proof, it suffices to show that for every $n > 0$ there are n such covers $Z_1, \dots, Z_n$ that are linearly disjoint as P-Galois covers of Y (and hence are non-isomorphic since P is non-trivial). For this, consider the n-fold fibre power of G over H, i.e. the group $G^n_H = \{(g_1, \dots, g_n) \in G^n \mid \bar{g}_1 = \dots = \bar{g}_n \in H\}$, where $\bar{g} \in H$ denotes the image of $g \in G$ under $G \twoheadrightarrow H$. We have an exact sequence $1 \to P^n \to G^n_H \to H \to 1$, and by the previous paragraph there is a connected étale G^n_H-Galois cover $\mathcal{Z} \to X$ that dominates $Y \to X$. Let $Z_i = \mathcal{Z}/N_i$, where N_i is the kernel of the i^{th} coordinate projection $G^n_H \twoheadrightarrow G$. Thus each $Z_i \to X$ is a connected étale G-Galois cover that dominates Y, and their fibre product over Y (viz. $\mathcal{Z}$) is irreducible. So their function fields $K(Z_i)$ are Galois over $K(Y)$, and their compositum over $K(Y)$ is equal to their tensor product over $K(Y)$. In particular the degree of this compositum over $K(Y)$ is the product of the degrees $[K(Z_i)/K(Y)]$. So by [FJ, Cor. 9.2], the covers $Z_1, \dots, Z_n$ are linearly disjoint over Y. □

Examples 3.2. (a) Lemma 3.1 holds for X an irreducible affine variety of finite type over a field of characteristic p, other than a point, since X has infinite p-rank by [Ha6, Cor. 3.7].

(b) Lemma 3.1 also holds for the case $R = F((x))$, with F any field of characteristic p, since the elements x^i (with $i > 0$, and i prime to p) are $\mathbb{F}_p$-linearly independent in $R/\wp(R)$ — which is thus infinite. □

Lemma 3.3. *Let $R = k[[x]]$, let S_0 be a normal integral affine k-scheme of finite type, and let $S = S_0 \times_k R$. Let P be a finite p-group, and let $T \to S$ be a normal P-Galois cover that is totally ramified over $(x = 0)$ and is unramified elsewhere. For $\sigma \in S_0$, let T_σ be the normalization of the fibre of $T \to S$ over $\sigma_R := \sigma \times_k R$.*

(a) Then for all σ in a dense open subset of S_0, the P-Galois cover $T_\sigma \to \operatorname{Spec} k[[x]]$ is totally ramified over the closed point.

(b) For any $\sigma \in S_0$, the pullback $\mathcal{T}_\sigma \to \operatorname{Spec} \mathcal{K}_{S_0,\sigma}[[x]]$ is a P-Galois cover which is totally ramified over the closed point.

Proof. (a) For $\sigma \in S_0$, let $I_\sigma \subset P$ be an inertia group of $T_\sigma \to \operatorname{Spec} k[[x]]$ over the closed point. Also, let Φ be the Frattini subgroup of P. Then I_σ and Φ generate P if and only if $I_\sigma = P$ (i.e. if and only if $T_\sigma \to \operatorname{Spec} k[[x]]$ is totally ramified), since Φ is the set of non-generators of P [Sc, 7.3.2]. But also, I_σ and Φ generate P if and only if the P/Φ-Galois cover $T_\sigma/\Phi \to \operatorname{Spec} k[[x]]$ is totally ramified. So it suffices to prove the result with P replaced by P/Φ; i.e. we may assume that P is an elementary abelian p-group $(\mathbb{Z}/p\mathbb{Z})^m$. Treating each factor separately, we are reduced to the case that $P = \mathbb{Z}/p\mathbb{Z}$, which we now assume.

The general fibre of $T \to S$ is an étale cover of $S_K = S \times_R K$, where $K = k((x))$. So writing $S_0 = \operatorname{Spec} A$, this fibre is given by an equation $y^p - y = \alpha$, where $\alpha \in A_K := A \otimes_k K$. Here α is uniquely determined modulo $\wp(A_K)$. Pick $h \geq 0$ such that $x^{p^h}\alpha \in A_R := A \otimes_k R$. Then modulo the subgroup $\wp(A_K) + A_R \subset A_K$, the element α may uniquely be written in the form $\sum_{i=p^{h-1}+1}^{p^h} a_i x^{-i}$, with $a_i \in A$ (because for $1 \leq i \leq p^{h-1}$, $ax^{-i} \equiv a^p x^{-ip}$ modulo

$\wp(A_K))$. Here not all $a_i = 0$, since α cannot be chosen in A_R (because the cover is not étale over X). Now if $\sigma \in S$, then the general fibre of the above cover over σ is given by $y^p - y = \alpha(\sigma)$, where $\alpha(\sigma) \equiv \sum_{i=p^{h-1}+1}^{p^h} a_i(\sigma)x^{-i}$ modulo $\wp\big(k((x))\big) + k[[x]]$. (Here $a_i(\sigma) \in k$ denotes the reduction of $a_i \in A$ modulo the maximal ideal corresponding to $\sigma \in S_0$.) So $T_\sigma \to \operatorname{Spec} k[[x]]$ is unramified if and only if $\alpha(\sigma) \in \wp\big(k((x))\big) + k[[x]]$. By the uniqueness of the above summation in its congruence class modulo $\wp\big(k((x))\big) + k[[x]]$, being unramified is thus equivalent to the condition that each $a_i(\sigma) = 0$. Since not all the elements $a_i \in A$ are zero, the simultaneous vanishing of the a_i's occurs only on a proper closed subset of S_0. Elsewhere, the normalized specialization to σ is ramified over the closed fibre, and hence totally ramified (since $P = \mathbb{Z}/p\mathbb{Z}$).

(b) Preserving the notation from part (a), we are again reduced to the case that $P = \mathbb{Z}/p\mathbb{Z}$, with the general fibre of $T \to S$ being given by $y^p - y = \alpha$, where $\alpha \equiv \sum_{i=p^{h-1}+1}^{p^h} a_i x^{-i}$ modulo $\wp\big(k((x))\big) + k[[x]]$, and where not all a_i equal 0 in A. Since $A \subset \mathcal{K}_\sigma$, some $a_i \neq 0 \in \mathcal{K}_\sigma$. Again by the uniqueness of the above summation representation of α in its congruence class (this time over $\mathcal{K}_\sigma$), it follows that $\alpha \notin \wp\big(\mathcal{K}_\sigma((x))\big) + \mathcal{K}_\sigma[[x]]$, and hence that the cover $\mathcal{T}_\sigma \to \operatorname{Spec} \mathcal{K}_\sigma[[x]]$ is not étale. Since the Galois group is $\mathbb{Z}/p\mathbb{Z}$, the cover is totally ramified over its closed point. □

The following result is a variant of the Lemma to Theorem 2 in [Ha2, §2]. The most important difference is that cyclic-by-p groups are allowed here, and not just p-groups, as in [Ha2, §2].

Proposition 3.4. *Let G' be a finite group of the form $P' \rtimes C$, where P' is a p-group and C is a cyclic group of order prime-to-p. Let G be a subgroup of G' of the form $P \rtimes C$, where P is a subgroup of P'. Let $N \subset P'$ be a non-trivial normal subgroup of G' such that N, P generate P'. Let L be a G-Galois field extension of $K = k((x))$. Then there is an irreducible normal G'-Galois cover $Z \to \mathbb{A}^1_{k[[x]]}$ of the t-line over $k[[x]]$ such that*

(i) the cover has branch locus $(x = 0)$, over which it is totally ramified;

(ii) on the locus of $(t = 0)$, the fibre over $\operatorname{Spec} K$ is isomorphic to $\operatorname{Ind}_G^{G'} \operatorname{Spec} L$;

(iii) $Z/N \times_{k[[x]]} k((x)) \approx (\operatorname{Spec} L)/(N \cap G) \times_k k[t]$ as a G'/N-Galois cover of $\mathbb{A}^1_{k((x))}$;

(iv) the pullback to $\operatorname{Spec} k[[x,t]][1/x]$ is not isomorphic to $\operatorname{Ind}_G^{G'} \operatorname{Spec} L \times_K k[[x,t]][1/x]$, as a G'-Galois cover.

Proof. First consider the special case that $N \cap P = 1$, i.e. $P' = N \rtimes P$. This is equivalent to supposing that $N \cap G = 1$, i.e. $G' = N \rtimes G$. In this case, the inclusion $G \hookrightarrow G'$ induces an isomorphism $G \xrightarrow{\sim} G'/N$.

Let n be the order of the cyclic group C, let $x' = x^{1/n}$ in an algebraic closure of K, and let $K' = K[x'] = k((x'))$. Note that K' is the only C-Galois field extension of K, since n is prime to p and since k is algebraically closed. Let $X = \mathbb{A}^1_{k[[x]]}$ and let $X' = \mathbb{A}^1_{k[[x']]}$. Consider the connected G-Galois étale cover $V = \operatorname{Spec} L[t] \to U := \mathbb{A}^1_K$. Let $U_\alpha \subset U$ be the locus of $(t = \alpha x)$, for $\alpha \in k$. Consider the (possibly disconnected) G'-Galois cover $W_0 = \mathrm{Ind}_G^{G'} \operatorname{Spec} L \to U_0$. By Lemma 3.1 and Example 3.2(b), there is a connected G'-Galois étale cover $W_1 \to U_1$ that dominates the G-Galois étale cover $\operatorname{Spec} L \to U_1$ and is linearly disjoint from $W_0 \to U_0$ over $\operatorname{Spec} L$ (and in particular is non-isomorphic to $W_0 \to U_0$). Thus W_1 is the spectrum of a G'-Galois field extension of K that dominates L. Applying [Ha6, Theorem 3.11], there is a connected G'-Galois étale cover $W \to U$ that dominates $V \to U$ and restricts to $W_i \to U_i$ for $i = 0, 1$. Let Z be the normalization of X in W. Thus $Z \to X$ is an irreducible normal G'-Galois cover which is étale away from the locus of $(x = 0)$. Moreover $Z/N \approx Y$ as a G-Galois cover of X, where Y is the normalization of X in V. So $Z/P' \approx Y/P \approx X'$ as a C-Galois cover of X, since its general fibre corresponds to the (unique) C-Galois field extension K'/K.

We claim that $Z \to X$ is totally ramified over $(x = 0)$. Since $Y/P \approx X' \to X$ is totally ramified there, it suffices to show that the P'-Galois cover $Z \to X' = \mathbb{A}^1_{k[[x']]}$ is totally ramified there. So let $I \subset P'$ be an inertia group over the general point of the closed fibre of X'. If I is a proper subgroup of P', then it is contained in a proper normal subgroup E of P' [Sc,6.4.10]. The P'/E-Galois cover $Z/E \to X'$ is then unramified over the general point of the closed fibre $(x = 0)$, as well as being unramified off of the closed fibre. By Purity of Branch Locus [Na,41.1], it follows that $Z/E \to X'$ is étale. But its restriction to the locus $(t = x)$ corresponds to a connected normal P'/E-Galois cover of $\operatorname{Spec} k[[x']]$ (viz. the normalization of $k[[x_i]]$ in W_1/E). Since $k[[x']]$ is a complete discrete valuation ring with an algebraically closed residue field, this P'/E-Galois cover must be totally ramified at its closed point — and hence $Z/E \to X'$ is also totally ramified over the closed point of $(t = x)$, i.e. at the point $(x = t = 0)$. This is a contradiction; showing that $I = P'$ and that $Z \to X$ is totally ramified along $(x = 0)$. This shows (i).

Assertion (ii) follows from the fact that on the locus of $(t = 0)$, the general fibre of $Z_0 \to \operatorname{Spec} k[[x]]$ is $W_0 = \mathrm{Ind}_G^{G'} \operatorname{Spec} L \to U_0$. Assertion (iii) is immediate from the fact that the G'-Galois cover $W \to U$ dominates the G-Galois cover $V \to U$. Assertion (iv) follows from the fact that the fibers W_0 and W_1 over $(t = 0)$ and $(t = x)$ are linearly disjoint over $\operatorname{Spec} L$. This completes the proof in the special case that $N \cap P$ is trivial.

In the general case, let $\tilde{G}' = N \rtimes G$, where the semidirect product is taken with respect to the conjugation action of G on N in G'. Let $\tilde{P}' = N \rtimes P \subset \tilde{G}'$. Thus $\tilde{G}' \approx \tilde{P}' \rtimes C$. There is a surjection $\tilde{G}' \twoheadrightarrow G'$ given by the identity inclusion on each factor, restricting to $\tilde{P}' \twoheadrightarrow P'$; let $H \lhd \tilde{G}'$ be the common kernel. We may regard N, G as subgroups of $\tilde{G}'$ that intersect trivially, and which each meet H trivially. Applying the above special case of the proposition, with $\tilde{G}', \tilde{P}'$ playing the roles of G', P', we obtain an irreducible normal $\tilde{G}'$-Galois cover $\tilde{Z} \to \mathbb{A}^1_{k[[x]]}$ satisfying the analogs of (i)-(iv). Let $Z = \tilde{Z}/H$, so that $\tilde{Z} \approx Z \times_{Z/N} \tilde{Z}/N$. Then the G'-Galois cover $Z \to \mathbb{A}^1_{k[[x]]}$ satisfies conditions (i)-(iii) because $\tilde{Z}$ does. Condition (iv) follows from the fact that the fibres of $\tilde{Z} \to \tilde{Z}/N$ over

U_0 and U_1 are linearly disjoint over their bases (i.e. W_0 and W_1 are linearly disjoint over $\operatorname{Spec} L$), and hence the fibres of $Z \to Z/N$ over U_0 and U_1 are linearly disjoint over their bases. □

If X is a k-variety, and K is a field containing k, then there is a natural morphism $\pi : X_K := X \times_k K \to X$. So for every k-point ξ of X, we may consider the fibre $\pi^{-1}(\xi)$ of X_K over ξ. Similarly, if $f : Y \to X$ is a morphism of k-schemes, we may consider the fibre of $f_K : Y_K \to X_K$ over a k-point ξ of X, viz. $(\pi f_K)^{-1}(\xi) \to \pi^{-1}(\xi)$.

Lemma 3.5. *Let X be an irreducible affine k-variety, let $X_K = X \times_k K$ where $K = k((x))$, let P be a p-group, and let $Y_K \to X_K$ be a connected P-Galois étale cover which is not isomorphic to $\operatorname{Spec} L \times_K X_K \to X_K$ for any P-Galois field extension L/K. Then the cardinality of the set of isomorphism classes of fibres of $Y_K \to X_K$ over k-points of X is equal to the cardinality of k.*

Proof. Let κ be the cardinality of k. Since the k-points of X form a set of cardinality κ, it suffices to show that there are at least κ distinct fibres, up to isomorphism.

Let $Z_K \to X_K$ be the maximal subcover of $Y_K \to X_K$ that is induced from a field extension of K; i.e. which is of the form $Z_K = \operatorname{Spec} M \times_K X_K \to X_K$ for some field extension M/K. Thus $Z_K = Y_K/N$, for some non-trivial normal subgroup $N \triangleleft P$, where $P/N = \operatorname{Gal}(M/K)$. Since k is algebraically closed, M is also a Laurent series field over k. So replacing K by M and X_K by Z_K, we may assume that no non-trivial subcover of $Y_K \to X_K$ is induced from a field extension of K.

Let E be a maximal (proper) subgroup of the p-group P; thus E is normal in P of index p [Sc, 6.4.9, 7.2.8]. Replacing P by P/E and Y_K by Y_K/E, we are reduced to the case that P is cyclic of order p. So writing $X = \operatorname{Spec} R$, we may assume that $Y_K \to X_K$ is given by an Artin-Schreier extension $y^p - y = \sum_{i=m}^n r_i x^{-i}$, where $0 \le m \le n$ and each $r_i \in R$. Replacing R by the étale extension given by $z^p - z = r_0$, we may assume that $r_0 = 0$, or equivalently that $m > 0$. Since the extension is not altered if a term $r_i x^{-i}$ is replaced by its pth power, we may assume for some $h > 0$ that $m = p^{h-1} + 1$ and $n = p^h$ (as in the proof of Lemma 3.3).

Since $Y_K \to X_K$ is not induced from a field extension of K, some r_i does not lie in $k \subset R$. This r_i thus defines a dominating map $X \to \mathbb{A}^1_k$, say with dense image $U \subset \mathbb{A}^1_k$. If ξ_1, ξ_2 are k-points of X that map to distinct points of U, then the corresponding Artin-Schreier equations of the corresponding fibres of $Y_K \to X_K$ are different. Because of the choice of m, n above, different Artin-Schreier equations give rise to non-isomorphic P-Galois covers (as in the proof of Lemma 3.3; i.e. since no two elements of the form $\sum_{i=m}^n c_i x^{-i}$, with $c_i \in k$, can differ by an element of the form $a^p - a$). Since the cardinality of U is equal to κ, it follows that there are at least this many non-isomorphic fibres, as desired. $\square$

The following result strengthens [Ha2, Theorem 2], by allowing more general inertia groups, and also by adding the more technical part (c) (which will be useful in Section 4).

Theorem 3.6. *Let G be a finite group, let H be a subgroup of G, and let $Y \to X$ be an H-Galois cover of smooth connected projective k-curves, unramified outside a non-empty finite set $B = \{\xi_1, \ldots, \xi_r\} \subset X$. For each i let $H_i \subset H$ be an inertia group over ξ_i, so that $H_i = P_i \rtimes C_i$ for some p-subgroup $P_i \subset H$ and some cyclic group C_i that is of order prime to p. For each i suppose that $H_i' = P_i' \rtimes C_i$ is a subgroup of G that contains H_i, where P_i' is a p-subgroup of G containing P_i.*

(a) Then there is a G-Galois cover $Z \to X$ of smooth k-curves that is unramified away from B, such that H_i' is an inertia group over ξ_i for all i.

(b) We may take Z to be connected provided that G is generated by $H, H_1', \ldots, H_n'$.

(c) If N is a normal subgroup of G such that $P_i' \subset NP_i$ for all i, then Z may be chosen so that $Z/N \approx Y/(N \cap H)$ as G/N-Galois covers of X. Moreover, if some $N \cap P_i' \neq 1$, then the cardinality of the set of isomorphism classes of such G-Galois covers $Z \to X$ is equal to the cardinality of k.

Proof. (a) Let x_i be a local uniformizer at $\xi_i \in X$, so that we may identify $R_i := \hat{\mathcal{O}}_{X,\xi_i}$ with $k[[x_i]]$ and $K_i := \mathcal{K}_{X,\xi_i}$ with $k((x_i))$. The affine curve $X_0 := X - B$ is of the form $X_0 = \operatorname{Spec} R_0$; let $X_0^* = \operatorname{Spec} R_0[[t]]$. For $1 \leq i \leq r$, let $X_i = \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_i}$ and let $X_i^* = \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi_i}[[t]] = \operatorname{Spec} \hat{\mathcal{O}}_{\mathbb{A}^1_R,(\xi_i,0)}$, the spectrum of the complete local ring at the point $(\xi_i, 0) \in \mathbb{A}^1_R$. For $i = 0, \ldots, r$, let $Y_i = Y \times_X X_i$ and $Y_i^* = Y \times_X X_i^*$. Also let $X^* = X \times_k k[[t]]$, $Y^* = Y \times_k k[[t]]$, and $B^* = B \times_k k[[t]]$; and let $X^\circ = X \times_k k((t))$ and $\xi_i^\circ = \xi_i \times_k k((t))$, for $i = 1, \ldots, r$.

For $i = 1, \ldots, r$, let $\eta_i \in Y$ be a point over $\xi_i \in X$ at which the inertia group of $Y \to X$ is H_i. Let $L_i = \mathcal{K}_{Y,\eta_i}$, an H_i-Galois field extension of K. Proposition 3.4 then yields an irreducible normal H_i'-Galois cover $W_i \to \mathbb{A}^1_{R_i}$ which is totally ramified over $\{\xi_i\} \times \mathbb{A}^1_k$ and is étale elsewhere, and which on $(t = 0)$ agrees with $\operatorname{Ind}_{H_i}^{H_i'} \operatorname{Spec} L_i$ over $\operatorname{Spec} K_i$. This lifts to an agreement of W_i with $\operatorname{Ind}_{H_1}^{H_1'} \operatorname{Spec} L_i[[t]]$ over $\operatorname{Spec} K_i[[t]]$, since a power series deformation of an étale cover is unique, by [Gr, I, Corollaire 6.2]. (Moreover, for any normal subgroup $N_i' \triangleleft P_i'$ which is a supplement of P_i, e.g. P_i' itself, the cover W_i may be chosen so as to satisfy the condition corresponding to (iii) of Proposition 3.4, and also the condition corresponding to (iv) of Proposition 3.4 if N_i' is non-trivial. We will return to this in the proof of (c).) By Lemma 3.3(b), for $i = 1, \ldots, r$ the pullback of the H_i'-Galois cover $W_i \to \mathbb{A}^1_{R_i}$ over $k((t))[[x_i]]$ is totally ramified over the closed point; and so the inertia group and decomposition groups there are equal to H_i'. Let $W_i^* = W_i \times_{\mathbb{A}^1_{R_i}} X_i^*$; this is an H_i'-Galois cover of X_i^*.

By [HS, Corollary to Theorem 1], there is a normal G-Galois cover $Z^* \to X^*$ that agrees with $\mathrm{Ind}_H^G Y_0^*$ over X_0^* and agrees with each $\mathrm{Ind}_{H_i'}^G W_i^*$ over X_i^* (for $1 \le i \le r$), compatibly with the above identification of $\mathrm{Ind}_{H_i}^{H_i'} \mathrm{Spec}\, L_i[[t]]$ with the pullback of W_i over $\mathrm{Spec}\, K_i[[t]]$. Thus this cover is unramified away from B^*, with H_i' equal to the inertia group and to the decomposition group at the $k((t))$-point $\eta_i^\circ := \eta_i \times_k k((t))$ over ξ_i°, for each $i > 1$. Let $Z^\circ \to X^\circ$ be the generic fibre of $Z^* \to X^*$, and let $Z_s \to X$ be the special fibre. By the uniqueness assertion of [HS, Corollary to Theorem 1] in the case $m = 0$ (i.e. no deformation variables), the normalization of Z_s is isomorphic to $\mathrm{Ind}_H^G Y$; i.e. it is a disjoint union of copies of Y, which are indexed by the cosets of H in G, and which respectively map to the irreducible components of Z_s. Moreover, the point η_i on the identity irreducible component $Z_s^1 \subset Z_s$, viewed as a point of Z_s, has inertia group $H_i' \subset G$. Hence the decomposition group of the connected component of Z_s containing Z_s^1 contains H and each H_i'; and thus Z_s is connected if G is generated by $H, H_1', \dots, H_r'$.

Since the cover $Z^* \to X^*$ and the above properties are of finite type, there is a k-subalgebra $A \subset k[[t]]$ of finite type and a normal G-Galois cover $Z_A \to X_A = X \times_k S$, where $S = \mathrm{Spec}\, A$, together with a $(\mathrm{frac}\, A)$-point $\eta_{A,i}$ over the general point of $\xi_i \times_k S$ for each i, such that $Z_A \to X_A$ induces $Z^* \to X^*$ over $k[[t]]$ and hence induces $Z^\circ \to X^\circ$ over $k((t))$; such that $\eta_{A,i}$ specializes to η_i°; such that the inertia group and the decomposition group at $\eta_{A,i}$ are equal to H_i'; such that over the point $s \in S$ mapping to the closed point of $\mathrm{Spec}\, k[[t]]$, the fibre of $Z_A \to X_A$ is isomorphic to $Z_s \to X$; and such that the cover $Z_A \to X_A$ is unramified away from $B_A := B \times_k S$. For each $i = 1, \dots, r$, the pullback $Z_{A,i} = Z_A \times_X X_i$ is a G-Galois cover of $X_i \times_k S$ with a section over $\{\xi_i\} \times S$, corresponding to $\eta_{A,i}$. Let $\bar{Z}_{A,i}$ be the connected component of $Z_{A,i}$ containing $\eta_{A,i}$. Thus $\bar{Z}_{A,i} \to X_i \times_k S$ is an H_i'-Galois cover that is totally ramified over $\{\xi_i\} \times S$, and we have an isomorphism $Z_{A,i} \approx \mathrm{Ind}_{H_i'}^G \bar{Z}_{A,i}$ of G-Galois covers of $X_i \times_k S$. Applying Lemma 3.3(a) to $\bar{Z}_{A,i} \to X_i \times_k S$, we have that the normalization of the fibre of $\bar{Z}_{A,i}$ over σ is totally ramified over the closed point of X_i, for all σ in some dense open subset $U_i \subset S$. Let U be the intersection of the subsets $U_1, \dots, U_r$. Then U is a dense open subset of S, and for every $\sigma \in U$, the normalization of the fibre of Z_A over σ has the property that the inertia group is H_i' at a point over ξ_i (viz. at the specialization of the point $\eta_{A,i}$ to σ). This normalized fibre is then the desired cover $Z \to X$.

(b) If G is generated by $H, H_1', \dots, H_r'$, then as observed above, the closed fibre Z_s of Z^* is connected. Hence so is Z^*, since each connected component of Z^* must meet the closed fibre, being finite over X^*. Since Z^* is unibranched over ξ_i^* (because W_i^* is totally ramified over the general point of $(x_i = 0)$), and is smooth elsewhere, it follows that Z^* is irreducible. Hence so is its dense open subset Z°. The same holds for the pullback of Z° by any finite field extension F of $k((t))$, by considering the integral closure of $k[[t]]$ in F and using that the normalized base change of W_i^* to F remains totally ramified over the general point of $(x_i = 0)$. That is, Z° is absolutely irreducible. Hence so is Z_A. By the Bertini-Noether theorem [FJ, Proposition 9.29], it follows that after shrinking U, the fibre of Z_A over any $\sigma \in U$ will be irreducible. So $Z \to X$ above will be irreducible, if the choice of σ is taken within this smaller dense open subset.

(c) Let $N_i = N \cap P_i$ and $N_i' = N \cap P_i'$. Thus N_i is a normal subgroup of P_i' which is a supplement to P_i in P_i'. So, as noted in the proof of (a), the cover W_i may be chosen so that $W_i/N_i' \times_{k[[x_i]]} k((x_i)) \approx (\operatorname{Spec} L_i)/N_i' \times_k k[t]$ as H_i'/N_i'-Galois covers of $\mathbb{A}^1_{k((x_i))}$ (condition (iii) of 3.4; here $H_i'/N_i' \approx H_i/N_i$). Let $\bar{Y}_i$ be the normalization of X_i in $\operatorname{Spec} L_i$; thus $Y_i = \operatorname{Ind}_{H_i}^H \bar{Y}_i$. Here W_i/N_i' and $\bar{Y}_i/N_i \times_k k[t]$ are normal H_i'/N_i'-Galois covers of $\mathbb{A}^1_{k[[x_i]]}$ whose general fibres agree; so they are isomorphic. Completing along $(t = 0)$, we have that Z^*/N agrees over X_i^* with $(\operatorname{Ind}_{H_i'}^G W_i^*)/N \approx \operatorname{Ind}_{H_i/N_i}^{G/N} \bar{Y}_i^*/N_i$, for $1 \le i \le r$, where $\bar{Y}_i^* = \bar{Y}_i \times_{k[[x_i]]} k[[x_i, t]]$; and similarly Z^*/N agrees over X_0^* with $\operatorname{Ind}_{H/(N\cap H)}^{G/N} Y_0^*/(N \cap H)$, since $Z^* \to X^*$ agrees with $\operatorname{Ind}_H^G Y_0^*$ over X_0^*. Moreover these agreements are compatible with the previous identifications. So by the uniqueness assertion in [HS, Corollary to Theorem 1], it follows that $Z^*/N \approx (\operatorname{Ind}_H^G Y^*)/N$ as G/N-Galois covers of X^*, and similarly over X°. So Z_A may be chosen so that $Z_A/N \approx (\operatorname{Ind}_H^G Y \times_k S)/N \approx Y/(N \cap H) \times_k S$ as G/N-Galois covers of X_A. Hence $Z \to X$, which is the normalized fibre of $Z_A \to X_A$ over σ, has the property that $Z/N \approx Y/(N \cap H)$ as G/N-Galois covers of X. This proves the first part of (c).

For the second part of (c), assume that some $N_i' \ne 1$. Since there are at most κ non-isomorphic covers of X, it suffices to show that there are at least κ covers with the desired properties — or in particular that there are κ non-isomorphic fibres $Z_\sigma \to X$ (with $\sigma \in U$) in the construction in the proof of (a) above. As noted in the proof of (a), by Proposition 3.4(iv) we may choose W_i so that its restriction to $X_i^* - \xi_i^*$ is not induced from its closed fibre. So then the same holds for the restriction of $Z^* \to X^*$ to $X_i^* - \xi_i^*$, since Z^* restricts to copies of W_i^* over X_i^*. Hence the general fibre of $Z_{A,i} \to X_{A,i} := X_i \times_k S$ is not isotrivial, i.e. is not induced by the fibre over a k-point of S. Now $Z_i^* \approx \operatorname{Ind}_{H_i'}^G W_i^*$ and $Z_{A,i} \approx \operatorname{Ind}_{H_i'}^G \bar{Z}_{A,i}$, with $\bar{Z}_{A,i}$ inducing W_i^* over X_i^*. Also, $W_i^*/N_i' \approx \bar{Y}_i^*/N_i$. So in (a), we may choose Z_A so that $\bar{Z}_{A,i}/N_i' \approx (\bar{Y}_i/N_i) \times_k S$. Since $N_i' \subset P_i'$ and $P_i' \cap H_i = P_i$, we also have $\bar{Z}_{A,i}/P_i' \approx (\bar{Y}_i/P_i) \times_k S$, and there is a P_i'-Galois cover $\bar{Z}_{A,i} \to (\bar{Y}_i/P_i) \times_k S$. Its general fibre $\bar{Z}_{A,i}^\circ \to L_i^{P_i} \times_k S$ is a P_i'-Galois étale cover of K-curves, and we may apply Lemma 3.5 to the restriction of this cover to $L_i^{P_i} \times_k U$. The conclusion is that the fibres over k-points σ of $U \subset S$ yield a set of isomorphism classes of P_i'-Galois covers $\bar{Z}_{\sigma,i}^\circ \to \operatorname{Spec}(L_i^{P_i})$ having cardinality κ. But for $\sigma \in U$, $Z_{\sigma,i}^\circ = \operatorname{Ind}_{H_i}^G \bar{Z}_{\sigma,i}^\circ$ is the restriction of $Z_\sigma \to X$ to the general fibre of X_i. Hence the fibres $Z_\sigma \to X$, as σ ranges over U, also form a set of cardinality at least (and hence exactly) κ. □

Remarks. (a) The above proof shows that when the hypotheses of (b) and (c) of Theorem 3.6 are simultaneously satisfied, we may choose $Z \to X$ so as to satisfy both extra conditions simultaneously.

(b) The hypotheses of part (c) of Theorem 3.6 are both necessary and sufficient. Namely, for the first part of (c), if $Z/N \approx Y/(N \cap H)$ then these two covers have the same inertia groups over ξ_i, and hence $P_i' \subset NP_i$. For the second part of (c), if all $N \cap P_i' = 1$, then $Z \to Z/N \approx Y/(N \cap H)$ is a tamely ramified N-Galois cover of the connected projective k-curve $Y/(N \cap H)$ with branch locus contained in the inverse image of $B \subset X$. But there are only finitely many such covers, by [Gr, XIII, Cor. 2.12]. (Note also that the hypothesis of the second part of (c) is satisfied if some $P_i' \neq P_i$.)

(c) As with Theorem 2.1, a key special case of Theorem 3.6 is that of split quasi-p embedding problems (taking $G = Q \rtimes H$ for some quasi-p group Q, generated by the p-subgroups P_i of G). And as in Theorem 2.1, there is a partial converse, in which part of Theorem 3.6 can be deduced from this special case. See Remark (b) after Corollary 4.2 below for a further discussion of this point. □

Section 4. Embedding problems and enlarging Galois groups.

By combining Theorems 2.1 and 3.6, we obtain Theorem 4.1 below, which allows enlarging the Galois group of a cover by a quasi-p group and also enlarging inertia groups. Using this, we obtain a number of consequences concerning covers of curves in characteristic p and how they fit together. In particular we strengthen a result of Pop [Po], proving the existence of solutions to quasi-p embedding problems (Corollary 4.6) which moreover can be chosen so as to preserve tameness over a given affine curve (Theorem 4.4). We also prove a strengthening of Abhyankar's Conjecture (Corollary 4.7), prove a tame analog of the geometric Shafarevich Conjecture (Theorem 4.9), and obtain the structure of the Galois group of the maximal extension of a function field over $\mathbb{F}_p$ that is tamely ramified at all but a specified finite set of places (Corollary 4.10).

Theorem 4.1. *In the situation of Theorem 2.1, let I be an inertia group of $Y \to X$ over ξ_0. Then $Z \to X$ may be chosen so that the subgroup of Γ generated by I and P is an inertia group of $Z \to X$ over ξ_0. Moreover, if Q is non-trivial, then up to isomorphism there are* $\operatorname{card}(k)$ *distinct choices of $Z \to X$.*

Proof. Let $Z_0 \to X$ be the Γ-Galois cover given by Theorem 2.1. Thus there is an inertia group H_0 of $Z_0 \to X$ over ξ_0 such that $H_0 \subset IP$ and H_0 has the same maximal prime-to-p quotient as I. So writing $I = P_I \rtimes C$ and $H_0 = P_0 \rtimes C$, where P_I and P_0 are the Sylow p-subgroups of I and H_0 respectively, we have that $P_0 \subset P_0' := PP_I$. Here P_0' is a p-subgroup of $\tilde{G}$, the subgroup of Γ generated by G and P (in which P is normal). Consider the cyclic-by-p subgroup of $\tilde{G}$ given by $H_0' := P_0' \rtimes C = IP$. Applying Theorem 3.6 with the roles of G, H, Y, N there played by $\Gamma, \Gamma, Z_0, \bar{Q}$, we obtain (in $\operatorname{card}(k)$ distinct ways, provided $Q \neq 1$) a smooth connected Γ-Galois cover $Z \to X$ whose inertia groups away from ξ_0 are the same as those of $Z_0 \to X$ over the respective points; H'_0 is an inertia group over ξ_0; and $Z/\bar{Q} \approx Z_0/\bar{Q}$. (Here, as in the statement of Theorem 2.1, $\bar{Q}$ is the normal closure of the quasi-p group Q.) Hence properties (i) - (iii) of Theorem 2.1, which held for Z_0, remain true for Z, and moreover (ii) is strengthened so that IP is an inertia group of $Z \to X$ over ξ_0. □

As a special case of Theorem 4.1, we obtain the following result of F. Pop [Po, second half of Theorem B]:

Corollary 4.2 *Let Γ be a finite group of the form $Q \rtimes G$, where Q is a quasi-p group. Suppose that the subgroup $G \subset \Gamma$ normalizes a Sylow p-subgroup of Q. Let $\pi : Y \to X$ be a G-Galois branched cover of smooth connected projective k-curves and let $\xi_0 \in X$. Then there is a smooth connected Γ-Galois cover $Z \to X$ such that $Z/Q \approx Y$ as G-Galois covers, such that $Z \to Y$ is unramified away from $\pi^{-1}(\xi_0)$, and such that the inertia groups of $Z \to Y$ over $\pi^{-1}(\xi_0)$ are the Sylow p-subgroups of Q.*

Proof. Theorem 4.1 provides a smooth connected Γ-Galois cover $Z \to X$ satisfying conditions (i)-(iii) of 2.1, with $H = IP$ in (ii). By (iii), $Z/Q \approx Y$ as a G-Galois cover of X, using that $\bar{Q} = Q$ since Q is normal in Γ. By (i), it follows that $Z \to Y$ is unramified away from $\pi^{-1}(\xi_0)$. By (ii) with $H = IP = P \rtimes I \subset Q \rtimes G = \Gamma$, one of the inertia groups over $\pi^{-1}(\xi_0)$ is equal to P, and the others are its conjugates — i.e. they range over the Sylow p-subgroups of Q. □

Remark. (a) As Theorem 4.1 shows, if $Q \neq 1$ then there are card(k) non-isomorphic choices of the asserted cover $Z \to X$ in Corollary 4.2.

(b) Much of Theorem 4.1 (and hence of its weaker form, Theorem 2.1) can be deduced formally from its special case, Corollary 4.2. Namely, in the situation of Theorem 4.1, let E be the fibre product of G and Γ over $F := \Gamma/N = G/(G \cap N)$. Then the exact sequence $1 \to N \to E \to G \to 1$ is split (via the diagonal). So the given G-Galois cover $Y \to X$, together with this exact sequence, gives a split quasi-p embedding problem for $\Pi := \pi_1(X - B)$, where B contains ξ_0 and the branch locus of $Y \to X$. By Corollary 4.2, there is a proper solution to this problem corresponding to a connected E-Galois cover $W \to X$ dominating $Y \to X$, such that the inertia groups of $W \to Y$ are trivial away from ξ_0 and are Sylow p-subgroups of E over ξ_0. The Γ-Galois intermediate cover $Z \to X$ then has the desired properties, except possibly for (i) of Theorem 2.1. Instead, we have a somewhat weaker condition on the inertia groups away from ξ_0, viz. that the inertia groups of $Z \to X$ over $\xi \neq \xi_0$ are isomorphic to those of $Y \to X$ there. The issue is that $Z \to Y$ is unramified there, so the inertia groups are isomorphic; but we do not know from 4.2 that the inertia groups of $Y \to X$ (viewed as subgroups of $G \subset \Gamma$) are also inertia groups of $Z \to X$. The proof of Corollary 4.2 in [Po] does show a bit more than what is asserted there: that there is a section s of $\Gamma \to G$ such that if I is an inertia group of $Y \to X$ over $\xi \neq \xi_0$, then $s(I)$ is an inertia group of $Z \to X$ over ξ. But this section s is generally *not* the given splitting of $\Gamma = Q \rtimes G$ corresponding to the given inclusion $G \hookrightarrow \Gamma$. So this still does not give the full strength of (i) of 2.1. But apart from this (and from the assertion on cardinality), the rest of Theorem 4.1 (and the corresponding parts of Theorem 2.1) can be deduced from Corollary 4.2, i.e. from [Po, Theorem B]. Similarly, in Theorem 3.6, one can obtain a G-Galois cover from the given data and from Corollary 4.2. But using this approach one cannot also obtain the desired inertia groups even up to isomorphism (since in [Po, Theorem B] the inertia groups of the "new part" of the cover are either trivial or Sylow p-subgroups, whereas Theorem 3.6 allows much more flexibility in controlling inertia). □

As another consequence of Theorem 4.1, we obtain Theorem 4.4 below. First we prove a variant on the group-theoretic Lemma 2.4:

Lemma 4.3. *Let Γ be a finite group, let Q be a normal quasi-p subgroup of Γ, and let $\pi : \Gamma \to \Gamma/Q$ be the natural quotient map. Let P be a Sylow p-subgroup of Q, and let $\Gamma' = N_\Gamma(P)$. Then $\pi(\Gamma') = \Gamma/Q$.*

Proof. Let $G = \Gamma/Q$. We want to show that $\pi(\Gamma') = G$. To see this, let $g \in G$; we will show that $\pi(\gamma') = g$ for some $\gamma' \in \Gamma'$. By definition of G, we know that there is a $\gamma \in \Gamma$ such that $\pi(\gamma) = g$. Since P is a Sylow p-subgroup of Q and since Q is normal, it follows that $\gamma P \gamma^{-1}$ is also a Sylow p-subgroup of Q. Since Sylow p-subgroups of Q are conjugate in Q, there must be an element $q \in Q \subset \Gamma$ such that $q(\gamma P \gamma^{-1})q^{-1} = P$. Let $\gamma' = q\gamma$. Thus $\gamma' P \gamma'^{-1} = P$, and so $\gamma' \in N_\Gamma(P) = \Gamma'$. Also, $\pi(\gamma') = \pi(q)\pi(\gamma) = g$. So γ' is as desired, proving the result. □

Theorem 4.4. *Let Γ be a finite group, Q a normal quasi-p subgroup of Γ, and $G = \Gamma/Q$. Let $\pi : Y \to X$ be a G-Galois cover of smooth connected projective k-curves that is étale away from $B \subset X$. Suppose that $Y \to X$ is tamely ramified away from a non-empty subset $S \subset B$. Then there is a smooth connected Γ-Galois cover $Z \to X$ which is étale away from B and tamely ramified away from S, such that $Y \approx Z/Q$ as G-Galois covers, and such that each inertia group over S contains a Sylow p-subgroup of Q. Moreover there are* card(k) *non-isomorphic choices of $Z \to X$ if $Q \neq 1$.*

Proof. We have a short exact sequence $1 \to Q \to \Gamma \to G \to 1$. Let P be a Sylow p-subgroup of Q, and let $Q' = N_Q(P)$ and $\Gamma' = N_\Gamma(P)$ be the normalizers of P in Q and in Γ. By Lemma 4.3, we have exact sequences $1 \to Q' \to \Gamma' \to G \to 1$ and

$1 \to Q'/P \to \Gamma'/P \to G \to 1$.

Let $U = X - S$, let $U' = X - B \subset U$, and let V, V' be the inverse images of U, U' under $Y \to X$. Thus $V \to U$ [resp. $V' \to U'$] is a tamely ramified [resp. unramified] G-Galois cover of smooth connected affine k-curves. The étale cover $V' \to U'$ corresponds to a surjective homomorphism $\alpha : \pi_1(U') \twoheadrightarrow G$. By [Se2, Prop. 1], $\mathrm{cd}(\pi_1(U')) \le 1$. So there is a homomorphism $\beta : \pi_1(U') \to \Gamma'/P$ such that the composition of β with the map $\Gamma'/P \twoheadrightarrow G$ is equal to α. Let $\bar{F} \subset \Gamma'/P$ be the image of β. The surjection $\beta : \pi_1(U') \twoheadrightarrow \bar{F}$ corresponds to a smooth connected $\bar{F}$-Galois étale cover $\bar{W}' \to U'$, dominating the G-Galois étale cover $V' \to U'$. Let $\bar{W}$ be the normalization of U in $\bar{W}'$. Thus $\bar{W} \to U$ is a smooth connected $\bar{F}$-Galois cover which dominates the G-Galois cover $V \to U$.

Now $\bar{W} \to V$ is a (connected) Galois cover whose group is $\bar{F} \cap (Q'/P) = \ker(\bar{F} \twoheadrightarrow G)$. Since P is a Sylow p-subgroup of Q and hence of Q', this intersection is of order prime to p. So $\bar{W} \to V$ is (at most) tamely ramified. Hence so is the $\bar{F}$-Galois cover $\bar{W} \to U$.

Let F be the inverse image of $\bar{F}$ under $\Gamma \twoheadrightarrow \Gamma/P$. So $F \subset \Gamma'$, and we have an exact sequence $1 \to P \to F \to \bar{F} \to 1$. By [Ha6, Theorem 5.14] (in the case $r = 0$), the tamely ramified smooth connected $\bar{F}$-Galois cover $\bar{W} \to U$ is dominated by a tamely ramified smooth connected F-Galois cover $W_0 \to U$ having the same branch locus. Thus $W_0 \to U$ dominates $V \to U$; i.e. $V \approx W_0/(F \cap Q)$ as G-Galois covers of U.

Let Z_0 be the normalization of X (or equivalently, of Y) in W_0. Thus $Z_0 \to X$ is an F-Galois cover of smooth connected k-curves such that $Y \approx Z_0/(F \cap Q)$ as G-Galois covers of X. Choose $\xi_0 \in S = X - U$. Since $F \to \bar{F}$ and $\bar{F} \to G$ are surjective, so is $F \to G = \Gamma/Q$. Thus F and the quasi-p group Q generate G. Also, F normalizes the Sylow p-subgroup P of Q, since $F \subset \Gamma'$. So by Theorem 4.1 (with F, Z_0 playing the role of G, Y in Theorem 2.1, and using that Q is normal in Γ here), there is a smooth connected Γ-Galois cover $Z \to X$ such that $Z/Q \approx Z_0/(F \cap Q) \approx Y$; such that for every $\xi \in U \subset X - \{\xi_0\}$, the inertia groups of $Y \to X$ over ξ are also inertia groups of $Z \to X$ over ξ; and such that the inertia groups over ξ_0 contain the conjugates of P. In particular, $Z \to X$ is tamely ramified away from S and is unramified away from B. So $Z \to X$ is as desired. If $Q \neq 1$ then by Theorem 4.1 there are $\mathrm{card}(k)$ distinct choices of $Z \to X$. □

Restricting to an affine open subset, we have:

Corollary 4.5. *Let Γ be a finite group, Q a normal quasi-p subgroup of Γ, and $G = \Gamma/Q$. Let $V \to U$ be a G-Galois tamely ramified cover of smooth connected affine k-curves. Then there is a smooth connected tamely ramified Γ-Galois cover $W \to U$ such that $V = W/Q$ and such that $W \to V$ is étale away from the ramification locus of $V \to U$. If $Q \neq 1$ there are $\mathrm{card}(k)$ non-isomorphic choices of $W \to U$.*

Proof. Let $B_0 \subset U$ be the branch locus of $V \to U$. Thus B_0 is a finite subset of U and

$V \to U$ is tamely ramified over the points of B_0. Let X be the smooth completion of U, let Y be the normalization of X in V, let $S = X - U$, and let $B = B_0 \cup S$. Theorem 4.4 yields a smooth connected Γ-Galois cover $Z \to X$ that dominates $Y \to X$, is unramified away from B, and is tamely ramified away from S. Let W be the inverse image of U under $Z \to X$. Then $W \to U$ is a smooth connected Γ-Galois étale cover that is tamely ramified, and is unramified away from the branch locus of $V \to U$. So $W \to U$ is as asserted, yielding the first assertion. The second assertion follows from the cardinality assertion of Theorem 4.4, since non-isomorphic covers $Z \to X$ yield non-isomorphic restrictions $W \to U$. □

Remark. (a) Abhyankar's Conjecture and the above result might lead one to suspect that if Γ is a quasi-p group and $G = \Gamma/N$ is a quotient (which is thus necessarily quasi-p), then any connected G-Galois étale cover $V \to U$ of affine curves should be dominated by a connected Γ-Galois étale cover $W \to U$. But actually, this is not the case, as is shown in [Se2]. In fact, N can be of order prime-to-p; and if moreover $U = \mathbb{A}^1$ and N is an elementary abelian ℓ-group (for some prime $\ell \neq p$), then [Se2, Prop. 2] provides a necessary and sufficient condition for such a W to exist. But when this condition fails, [Se2] shows that there is some other connected G-Galois étale cover of the line which *is* dominated by a W (proving Abhyankar's Conjecture in that case).

(b) The above corollary does not apply if U is replaced by a projective curve, as can be seen by taking the curve $\mathbb{P}^1$ and the groups $\Gamma = Q = \mathbb{Z}/p\mathbb{Z}$, $G = 1$. For then there are no connected tamely ramified Γ-Galois covers of the line. □

Corollary 4.6. *All finite quasi-p embedding problems over smooth connected affine k-curves may be solved properly, and in* $\mathrm{card}(k)$ *non-isomorphic ways if the embedding problem is non-trivial.*

Proof. To give such an embedding problem is to give the following data: a finite group Γ, a quasi-p subgroup $Q \subset \Gamma$, and a Γ-Galois étale cover $V \to U$ of smooth connected affine k-curves. A proper solution consists of a Γ-Galois étale cover $W \to U$ of smooth connected k-curves that dominates $V \to U$. This exists by Corollary 4.5 (and in $\mathrm{card}(k)$ non-isomorphic ways if the embedding problem is non-trivial), since the cover $W \to U$ there is étale because the ramification locus of $V \to U$ is empty. □

Remark. (a) The first half of [Po, Theorem B] is the existence part of Corollary 4.6 in the special case that the quasi-p embedding problem is split; i.e. that the corresponding exact sequence $1 \to Q \to \Gamma \to G \to 1$ splits. That half of [Po, Theorem B], unlike the other half (Corollary 4.2 above), does not require that the quotient group normalize a Sylow p-subgroup of the quasi-p part Q, but it also does not yield the conclusion concerning inertia. (In the restatement of [Po, Theorem B] appearing at [Ha5, Theorem 5.2], the normalization hypothesis was not included. So the inertia condition in the conclusion, which was not used in the sequel, should also have been omitted from that assertion.)

(b) The proof of Corollary 4.5, and hence that of Corollary 4.6, relies on the assumption of dimension 1, in order to have that $\mathrm{cd} \leq 1$. This raises the question of whether or not these assertions nevertheless hold for higher dimensional affine k-varieties U of finite type. In fact, they do not. Namely, such a generalization would imply the "higher dimensional Abhyankar Conjecture" asserting that a finite group Γ is the Galois group of a connected étale cover of U if and only if its maximal prime-to-p quotient $\Gamma/p(\Gamma)$ is. But by [HP], this conjecture fails in general, in dimension greater than 1 (even for the complement of the two axes in $\mathbb{A}^2$). Hence so do the higher dimensional generalizations of Corollaries 4.5 and 4.6. □

The above results give an improved version of the Strong Abhyankar Conjecture for curves (Corollary 2.5 above), in which the wild inertia is taken to be as large as possible, and in which the maximal prime-to-p subcover is specified in advance:

Corollary 4.7. *Let X be a smooth connected projective k-curve of genus $g \geq 0$, let $B \subset X$ be a set of $r > 0$ points, and let $\xi_0 \in B$. Let Γ be a finite group such that $\Gamma/p(\Gamma)$ has a generating set of at most $2g + r - 1$ elements.*

(a) Then there is a smooth connected Γ-Galois cover $Z \to X$ that is unramified outside B, is tamely ramified away from ξ_0, and whose inertia groups over ξ_0 contain Sylow p-subgroups of Γ.

(b) Moreover, the maximal prime-to-p subcover $Y = Z/p(\Gamma) \to X$ may be specified in advance, and (if p divides the order of Γ) there are $\mathrm{card}(k)$ *non-isomorphic choices of $Z \to X$ dominating each specified choice of $Y \to X$.*

Proof. (a) By [Gr, XIII, Cor. 2.12], there is a smooth connected $\Gamma/p(\Gamma)$-Galois étale cover $V \to U := X - B$. Let Y be the normalization of X in V. Thus $Y \to X$ is a smooth connected $\Gamma/p(\Gamma)$-Galois cover which is étale away from B. This cover is Galois of degree prime to p, so it is at most tamely ramified over B. Taking $S = \{\xi_0\}$ and applying Theorem 4.4, we obtain a smooth connected Γ-Galois cover $Z \to X$ having the desired properties, and such that $Z/p(\Gamma) \approx Y$ as a $\Gamma/p(\Gamma)$-Galois cover of X. (Here we use that Γ and $p(\Gamma)$ have the same Sylow p-subgroups.)

(b) In the proof of (a), we may choose arbitrarily the smooth connected $\Gamma/p(\Gamma)$-Galois cover $Y \to X$ which is unramified outside B. Doing so, we then obtain $Z \to X$ having the required properties and satisfying $Y \approx Z/p(\Gamma)$. So the first part of assertion (b) holds. The second part of (b) follows from the cardinality assertion of Theorem 4.4, since $p(\Gamma) \neq 1$ if and only if the order of Γ is divisible by p. □

As another consequence of Corollary 4.5, we are able to strengthen a result of [Ha4] and [Po] on solving embedding problems for branched covers. According to [Ha4, Theorem 3.6] and [Po, Theorem A], arbitrary finite embedding problems can be solved if additional branch points are allowed (and whose positions can be taken to avoid a given finite set of points). By using Corollary 4.5 above, this result can be strengthened to show that the new cover can be taken to be tamely ramified over the given cover:

Corollary 4.8. *Let Γ be a finite group, N a normal subgroup of Γ, and $G = \Gamma/N$. Let $V \to U$ be a G-Galois tamely ramified cover of smooth connected affine k-curves, and let $T \subset U$ be a finite set that is disjoint from the branch locus of $V \to U$. Then there is a smooth connected tamely ramified Γ-Galois cover $W \to U$ such that $V = W/N$ and such that $W \to V$ is étale over T. Moreover, if N is non-trivial, then the set of isomorphism classes of such covers has cardinality equal to that of the field k.*

Proof. Let κ be the cardinality of k. Let $Q = p(N)$ and let $\bar{N} = N/Q$, which has order prime to p. Since Q is a characteristic subgroup of $N \lhd \Gamma$, it is a normal subgroup of Γ. Let $\bar{N} = N/Q$ and $\bar{\Gamma} = \Gamma/Q$. Thus there are exact sequences $1 \to Q \to \Gamma \to \bar{\Gamma} \to 1$ and $1 \to \bar{N} \to \bar{\Gamma} \to G \to 1$.

By [Ha4, Theorem 3.6(a)] or [Po, Theorem A], there is a smooth connected $\bar{\Gamma}$-Galois cover $\bar{W} \to U$ that dominates the G-Galois cover $V \to U$ and is étale over T. Moreover, by [Ha4, Theorem 3.6(b)], there are κ such covers up to isomorphism if $\bar{N} \neq 1$. Since $V \to U$ is tamely ramified over U and since the order of $\bar{N} = \mathrm{Gal}(\bar{W}/V)$ is prime to p, it follows that each such $\bar{W} \to U$ is tamely ramified over U.

By Corollary 4.5, for each such choice of $\bar{W} \to U$, there is a smooth connected tamely ramified Γ-Galois cover $W \to U$ that dominates the $\bar{\Gamma}$-Galois cover $\bar{W} \to U$ and which is étale away from the ramification locus of $\bar{W} \to U$. Moreover, by that result, if $Q \neq 1$ then there are κ such covers up to isomorphism. Each choice of $W \to U$ dominates the given G-Galois cover $V \to U$ (since $\bar{W} \to U$ does), and each is étale over T since the branch locus of $\bar{W} \to U$ is disjoint from T.

Thus a desired cover $W \to U$ exists. If N is non-trivial, then at least one of Q and $\bar{N}$ is non-trivial, and so there are κ such covers $W \to U$ up to isomorphism. □

Following [Ha6], if U is a connected normal curve, and if Σ is a subset of U, then let $\pi_1^{\mathrm{t}}(U, \Sigma)$ denote the Galois group of the maximal extension of the function field of U that is at most tamely ramified over the places in Σ, and is étale over all places corresponding to other points of U. According to [Ha6, Corollary 5.16(a)], if U is affine over an arbitrary field of characteristic p (not necessarily algebraically closed), and if Σ is a proper closed subset of U, then $\mathrm{cd}_p(\pi_1^{\mathrm{t}}(U, \Sigma)) \leq 1$. In our situation here, where the base field k is algebraically closed, more is true, as is shown in the next result. Part (a) strengthens the result from

[Ha6] in the case that U is affine, to allow the full cohomological dimension, not just the p-cohomological dimension, and to allow arbitrary Σ. That it, all finite embedding problems for $\pi_1^{\mathrm{t}}(U,\Sigma)$ have a weak solution. Part (b) provides a tame version of the geometric Shafarevich Conjecture that the absolute Galois group of the function field of U is free, and more generally that the fundamental group $\pi_1(U_S)$ of the semi-localization at S is free for $S \subset U$ finite ([Ha4, Theorem 4.4], [Po, Cor. to Theorem A]).

Theorem 4.9. *Let U be a smooth connected affine k-curve and let $\Sigma \subset U$.*

(a) Then $\mathrm{cd}(\pi_1^{\mathrm{t}}(U,\Sigma)) \le 1$, or equivalently the group $\pi_1^{\mathrm{t}}(U,\Sigma)$ is projective. Again equivalently, every embedding problem for $\pi_1^{\mathrm{t}}(U,\Sigma)$ has a weak solution.

(b) If $U - \Sigma$ is finite then $\pi_1^{\mathrm{t}}(U,\Sigma)$ is a free profinite group on $\mathrm{card}(k)$ generators.

Proof. (a) Let $1 \to N \to \Gamma \to G \to 1$ be a short exact sequence of finite groups, and let $\alpha : \pi_1^{\mathrm{t}}(U,\Sigma) \twoheadrightarrow G$ be a surjective homomorphism, corresponding to a smooth connected G-Galois tamely ramified cover $V \to U$ that is étale away from Σ. We wish to find a (not necessarily surjective) homomorphism $\beta : \pi_1^{\mathrm{t}}(U,\Sigma) \to \Gamma$ that lifts α. This is equivalent to finding a (possibly disconnected) smooth Γ-Galois tamely ramified cover $W \to U$ that dominates $V \to U$ and is étale away from Σ.

Let $Q = p(N)$, let $\bar{N} = N/Q$, and let $\bar{\Gamma} = \Gamma/Q$. Let $B \subset \Sigma$ be the branch locus of $V \to U$. Let $U' = U - B$ and let $V' \subset V$ be the inverse image of U' under $V \to U$. Thus $V' \to U'$ is a smooth connected G-Galois étale cover, corresponding to a surjection $\alpha' : \pi_1(U') \twoheadrightarrow G$ that is compatible with α. Since U' is an affine curve over the algebraically closed field k, we have that $\mathrm{cd}(\pi_1(U')) \le 1$ [Se2, Proposition 1]. Thus there is homomorphism $\bar{\beta}' : \pi_1(U') \to \bar{\Gamma}$ that lifts α', say with image $\bar{\Gamma}_0 \subset \bar{\Gamma}$. The surjection $\bar{\beta}' : \pi_1(U') \to \bar{\Gamma}_0$ corresponds to a smooth connected $\bar{\Gamma}_0$-Galois étale cover $\bar{W}_0' \to U'$ that dominates $V' \to U'$. Taking the normalization of U in $\bar{W}_0'$, we obtain a smooth connected $\bar{\Gamma}_0$-Galois cover $\bar{W}_0 \to U$ that dominates $V \to U$ and is étale away from B. Here $\bar{W}_0 \to V$ is an $\bar{N}_0$-Galois cover, where $\bar{N}_0 = \bar{N} \cap \bar{\Gamma}_0 = \ker(\bar{\Gamma}_0 \twoheadrightarrow G)$. Since p is prime to the order of $\bar{N}$ and hence of $\bar{N}_0$, the cover $\bar{W}_0 \to V$ is tamely ramified. Since $V \to U$ is also tamely ramified, so is $\bar{W}_0 \to U$. Let Γ_0 be the inverse image of $\bar{\Gamma}_0$ under $\Gamma \twoheadrightarrow \bar{\Gamma}$. Thus $1 \to Q \to \Gamma_0 \to \bar{\Gamma}_0 \to 1$ is exact. By Corollary 4.5, there is a smooth connected tamely ramified Γ_0-Galois cover $W_0 \to U$ that dominates $\bar{W}_0 \to U$ and is étale away from B. Let $W = \mathrm{Ind}_{\Gamma_0}^{\Gamma} W_0$. Then $W \to U$ is as desired.

(b) By a result of O. Melnikov and Z. Chatzidakis [Ja, Lemma 2.1], a profinite group F is free of rank κ (for κ an infinite cardinal) if and only if every non-trivial finite embedding problem for F has exactly κ proper solutions. So it suffices to show that if $1 \to N \to \Gamma \to G \to 1$ is a short exact sequence of finite groups with $N \ne 1$, and if $V \to U$ is a smooth connected G-Galois tamely ramified cover that is étale away from Σ, then there are κ choices of a smooth connected Γ-Galois tamely ramified cover $W \to U$ that dominates

$V \to U$ and is étale away from Σ. This is just the assertion of Corollary 4.8, taking $T = U - \Sigma$. □

Corollary 4.10. *Let q be a power of a prime number p and let U be a smooth geometrically connected affine curve over the finite field $\mathbb{F}_q$.*

(a) If S is a finite subset of U, then the fundamental group $\pi_1(U_S)$ of the semi-localization of U at S is isomorphic to a semi-direct product $\hat{F}_\omega \rtimes \hat{\mathbb{Z}}$, where $\hat{F}_\omega$ is the free profinite group of countable rank.

(b) If Σ is a dense open subset of U, then $\pi_1^{\mathrm{t}}(U, \Sigma)$ is isomorphic to a semi-direct product $\hat{F}_\omega \rtimes \hat{\mathbb{Z}}$.

Proof. (a) The algebraic closure k of $\mathbb{F}_q$ is countably infinite, so by the geometric case of the Shafarevich Conjecture ([Ha4, Theorem 4.4], [Po, Cor. to Theorem A]), the fundamental group of $(U_S)_k := U_S \times_{\mathbb{F}_q} k$ is isomorphic to $\hat{F}_\omega$. Also, $G_{\mathbb{F}_q} \approx \hat{\mathbb{Z}}$, generated by Frobenius. So the fundamental exact sequence $1 \to \pi_1((U_S)_k) \to \pi_1(U_S) \to G_{\mathbb{F}_q} \to 1$ from Galois theory takes the form $1 \to \hat{F}_\omega \to \pi_1(U_S) \to \hat{\mathbb{Z}} \to 1$. Choosing any element of $\pi_1(U_S)$ lying over a topological generator of $\hat{\mathbb{Z}}$ provides a splitting, yielding the semi-direct product representation.

(b) By Galois theory and the definition of π_1^{t}, there is a quotient homomorphism $\pi_1^{\mathrm{t}}(U, \Sigma) \twoheadrightarrow G_{\mathbb{F}_q} \approx \hat{\mathbb{Z}}$ with kernel $\pi_1^{\mathrm{t}}(U_k, \Sigma)$. So by Theorem 4.9(b) we obtain the exact sequence $1 \to \hat{F}_\omega \to \pi_1^{\mathrm{t}}(U, \Sigma) \to \hat{\mathbb{Z}} \to 1$. Again, since the cokernel of the exact sequence is pro-cyclic, there is a splitting yielding the desired semi-direct product representation. □

In particular, taking $U = \Sigma = \mathbb{A}^1_{\mathbb{F}_q}$, $S = \emptyset$, we have that the absolute Galois group of $\mathbb{F}_q(t)$ is of the form $\hat{F}_\omega \rtimes \hat{\mathbb{Z}}$, as is the Galois group of the maximal extension of $\mathbb{F}_q(t)$ that is tamely ramified over every finite place. The first of these two facts may be regarded as evidence for the conjecture that every finite group is a Galois group over $\mathbb{F}_q(t)$, and the second fact may be regarded as suggesting:

Conjecture 4.11. *For every prime power q, and for every finite group G, there is a G-Galois field extension of $\mathbb{F}_q(t)$ that is at most tamely ramified over every finite place.*

This conjecture is a function field analog of a conjecture of B. Birch, that every finite group is the Galois group of a tamely ramified field extension of $\mathbb{Q}$.

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