Section 4: Rigid patching

This section, like Section 3, discusses an approach to carrying over the ideas of Section 2 from complex curves to more general curves. The approach here is due to Tate, who introduced the notion of rigid analytic spaces. The idea here is to consider power series that converge on metric neighborhoods on curves over a valued field, and to "rigidify" the structure to obtain a notion of "analytic continuation". Tate's original point of view, which is presented in Section 4.1, is rather intuitive. But the details of carrying it out become somewhat complicated, as the reader will see (particularly with regard to the precise method of rigidifying "wobbly spaces"). A simplified approach, due to Grauert, Remmert, and Gerritzen, is discussed later in Section 4.1, including their approach to a rigid analog of GAGA. Section 4.2 then discusses a later reinterpretation of rigid geometry that is due to Raynaud, and which establishes a kind of "dictionary" between the formal and rigid set-ups (and allows rigid GAGA to be deduced from formal GAGA). Applications to the construction of Galois covers of curves are then presented in Section 4.3, including a version of the (geometric) regular inverse Galois problem, and Pop's Half Riemann Existence Theorem. Additional applications of both rigid and formal geometry to Galois theory appear afterwards, in Section 5.

Section 4.1. Tate's rigid analytic spaces.

Another approach to generalizing complex analytic notions to spaces over other fields is provided by Tate's rigid analytic spaces. As in the formal approach discussed in Section 3, the rigid approach allows "small neighborhoods" of points, and permits objects (spaces, maps, sheaves, covers) to be constructed by giving them locally and giving agreement on overlaps (i.e. "patching"). Here the small neighborhoods are metric discs, rather than formal neighborhoods of subvarieties, as in the formal patching approach.

This approach was introduced by Tate in [Ta], a 1962 manuscript which he never submitted for publication. The manuscript was circulated in the 1960's by IHES, with the notation that it consisted of "private notes of J. Tate reproduced with(out) his permission". Later, the paper was published in Inventiones Mathematicae on the initiative of the journal's editors, who said in a footnote that they "believe that it is in the general interests of the mathematical community to make these notes available to everyone".

Tate's approach was motivated by the problem of studying bad reduction of elliptic curves (what we now know as the study of Tate curves; see e.g. [BGR, 9.7]). The idea is to work over a field K that is complete with respect to a non-trivial non-archimedean valuation — e.g. the p-adics, or the Laurent series over a coefficient field k. On spaces defined over such a field K, one can consider discs defined with respect to the metric on K; and one can consider "holomorphic functions" on those discs, viz. functions given by power series that are convergent there. One then wants to work more globally by means of analytic continuation, and to carry over the classical results over $\mathbb{C}$ (e.g. those of Section 2 above) to this context. As a result, one hopes to obtain a GAGA-type result, a version of Riemann Existence Theorem, the realization of all finite groups as Galois groups over $K(x)$, etc.

There are difficulties, however, that are caused by the fact that the topology on K is totally disconnected. For example, on the affine K-line, consider the characteristic function f_D of the open unit disc $|x| < 1$; i.e. $f(x) = 1$ for $|x| < 1$, and $f(x) = 0$ for $|x| \geq 1$. Then this function is continuous, and in a neighborhood of each point $x = x_0$ it is given by a power series. (Namely, on the open disc of radius 1 about $x = x_0$, it is identically 1 or identically 0, depending on whether or not $|x_0|$ is less than 1.) This is quite contrary to the situation over $\mathbb{C}$, where a holomorphic function is "rigid", in the sense that it is determined by its values on any open disc. Thus, if one proceeds in the obvious way, objects will have a strictly local character, and there will be no meaningful "patching".

Tate used two ideas to deal with this problem. The first of these is to consider functions that are locally given on *closed* discs, rather than on open discs, and to require agreement on overlapping boundaries. Note, though, that because the metric is non-archimedean, closed discs are in fact open sets. The second idea is to restrict the set of allowable maps between spaces, by choosing a class of maps that fulfills certain properties and creates a "rigid" situation.

Concerning the first of these ideas, let $K\{x\}$ denote the subring of $K[[x]]$ consisting of power series that converge on the closed unit disc $|x| \leq 1$. Because the metric is non-archimedean, this ring consists precisely of those series $\sum_{i=0}^{\infty} a_i x^i$ for which $a_i \to 0$ as $i \to \infty$. Similarly, the power series in $K[[x_1, \ldots, x_n]]$ that converge on the closed polydisc where each $|x_i| \leq 1$ form the ring $K\{x_1, \ldots, x_n\}$ of series $\sum a_{\underline{i}} x^{\underline{i}}$, where $\underline{i}$ ranges over n-tuples of non-negative integers, and where $a_{\underline{i}} \to 0$ as $\underline{i} \to \infty$. As an example, if $K = k((t))$ for some field k, then $K\{x\} = k[x][[t]][t^{-1}]$. (Verification of this equality is an exercise left to the reader.)

If $0 < r_1 \leq r_2$, then we may also consider the closed annulus $\{x \,|\, r_1 \leq |x| \leq r_2\}$. Since the metric is non-archimedean, this is an open subset, which we may consider even when $r_1 = r_2$. In particular, in the case $r_1 = r_2 = 1$, we may consider the ring $K\{x, x^{-1}\} = K\{x, y\}/(xy - 1)$ of functions converging on the annulus; this consists of doubly infinite series $\sum_{i=-\infty}^{\infty} a_i x^i$ such that $a_i \to 0$ as $|i| \to \infty$. Similarly, we may consider the ring $K\{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\} = K\{x_1, \ldots, x_n, y_1, \ldots, y_n\}/(x_i y_i - 1)$ of functions on the "poly-annulus" $|x_i| = 1$ (with $i = 1, \ldots, n$). In the case that $K = k((t))$, we have that $K\{x, x^{-1}\} = k[x, x^{-1}][[t]][t^{-1}]$. (Verification of this is again left to the reader. In this situation, the one-dimensional rings $K\{x\}$ and $K\{x, x^{-1}\}$ are obtained by inverting t in the two-dimensional rings $k[x][[t]]$ and $k[x, x^{-1}][[t]]$; cf. Figure 3.1.4 above.)

In order to consider more general analytic "varieties" over K, Tate considered quotients of the rings $K\{x_1, \dots, x_n\}$ by ideals. He referred to such quotients by saying that they were of *topologically finite type*; these are also now referred to as *affinoid algebras* [BGR] or as *Tate algebras* [Ra1] (though the latter term is sometimes used only for the ring $K\{x_1, \dots, x_n\}$ itself [BGR]). Tate showed that a complete K-algebra A is an affinoid algebra if and only if it is a finite extension of some $K\{x_1, \dots, x_n\}$ [Ta, Theorem 4.4]; and in this case A is Noetherian, every ideal is closed, and the residue field of every maximal ideal is finite over K [Ta, Theorem 4.5]. The association $A \mapsto \operatorname{Max} A$ is a contravariant functor from affinoid algebras to sets, where $\operatorname{Max} A$ is the maximal spectrum of A. (The map $\operatorname{Max} B \to \operatorname{Max} A$ associated to $\phi : A \to B$ is denoted by ϕ°, and is called *rigid*.) Since A/ξ is a finite extension L of K for any $\xi \in \operatorname{Max} A$, we may consider $f(\xi) \in L$ and $|f(\xi)| \in \mathbb{R}$ for any $f \in A$ (and thus regard A as a ring of functions on $\operatorname{Max} A$). By an *affinoid variety*, we then mean a pair $\operatorname{Sp} A := (\operatorname{Max} A, A)$, where A is an affinoid algebra.

Tate defined an *affine subset* $Y \subset \operatorname{Max} A$ to be a subset for which there is an affinoid algebra A_Y that represents the functor $h_Y : B \mapsto \{\phi : A \to B \mid \phi^\circ(\operatorname{Max} B) \subset Y\}$; i.e. such that $h_Y(B) = \operatorname{Hom}(A_Y, B)$. (This is called an *affinoid subdomain* in [BGR].) A *special affine subset* $Y \subset \operatorname{Max} A$ is a subset of the form

$$Y = \{\xi \in \operatorname{Max} A \,:\, |f_i(\xi)| \leq 1\,(\forall i),\ |g_j(\xi)| \geq 1\,(\forall j)\},$$

where $(f_i), (g_j)$ are finite families of elements of A. (These are called *Laurent domains* in [BGR].) Tate showed [Ta, Proposition 7.2] that every special affine subset is affine, viz. that if Y is given by $(f_i), (g_j)$ as above, then $A_Y = A\{f_i; g_j^{-1}\} := A\{x_i; y_j\}/(f_i - x_i, 1 - g_j y_j)$. Moreover if Y is an affine subset of $\operatorname{Max} A$, then the canonical map $\operatorname{Max} A_Y \to Y$ is a bijection [Ta, Proposition 7.3]. In fact, it is a homeomorphism [Ta, Cor. 2 to Prop. 9.1], if we give $\operatorname{Max} A$ the topology in which a fundamental system of neighborhoods of a point ξ_0 is given by sets of the form $U_\varepsilon(g_1, \dots, g_n) = \{\xi \in \operatorname{Max} A \,:\, |g_i(\xi)| < \varepsilon \text{ for } 1 \leq i \leq n\}$, where $\varepsilon > 0$ and where $g_1, \dots, g_n \in A$ satisfy $g_i(\xi_0) = 0$.

Tate defined Čech cohomology for coverings of affinoid varieties $V = (\operatorname{Max} A, A)$ by finitely many affine subsets, and proved his Acyclicity Theorem [Ta, Theorem 8.2], that $H^i(\mathfrak{V}, \mathcal{O}) = 0$ for $i > 0$; here $\mathcal{O}$ is the presheaf that associates to any affine subset its affinoid algebra, and $\mathfrak{V}$ is a finite covering of V by special affine subsets. (In fact, this holds even with a finite covering of V by affine subsets; see [BGR, §8.2, Theorem 1].) As a consequence, for such a covering $\mathfrak{V}$ of V and any A-module M of finite type, $H^0(\mathfrak{V}, \tilde{M})$ is isomorphic to M, and $H^i(\mathfrak{V}, \tilde{M}) = 0$ for $i > 0$ [Ta, Theorem 8.7]; here $\tilde{M}$ is the presheaf $Y \to M \otimes_A A_Y$ for Y an affine subset of V. These are analogs of the usual facts for the cohomology of affine varieties. Moreover, they imply that $\mathcal{O}$ and $\tilde{M}$ are sheaves. In particular [BGR, §8.2, Corollary 2], if $f, g \in A$ agree on each member U_i of a finite affine covering of V, then they are equal; and if for every i we are given a function f_i on U_i, with agreements on the overlaps, then they may be "patched" — i.e. there is a function $f \in A$ which restricts to each f_i.

As might be expected, if U is an affine open subset of an affinoid variety V, then the map $A_U \to \Gamma(U, \mathcal{O})$ is injective. Unfortunately, it is not surjective, e.g. because of characteristic functions like f_D, mentioned at the beginning of this section. Moreover, the functor $A \mapsto \operatorname{Max} A$ is faithful, but not fully faithful [Ta, Corollary 2 to Proposition 9.3]; i.e. not every K-ringed space morphism between two affinoid varieties is induced by a homomorphism between the corresponding rings of functions. Because of this phenomenon, if one defines more global analytic K-spaces simply by considering ringed K-spaces that are locally isomorphic to affinoid varieties, then one instead obtains a theory of "wobbly analytic spaces", rather than rigid ones.

In order to "rigidify" these wobbly spaces, Tate introduced the second of the two ideas mentioned earlier — viz. shrinking the class of allowable morphisms between such spaces, in such a way that in the case of affinoid varieties, the allowable morphisms are precisely the rigid ones (i.e. those induced by homomorphisms of the underlying algebras). He did this in a series of steps, which he said followed "fully and faithfully a plan furnished by Grothendieck" [Ta, §10]. First, he defined [Ta, Definition 10.1] an *h-structure* θ on a wobbly analytic space V to be a choice of a subset $V^\theta(A) \subset \operatorname{Hom}(\operatorname{Max} A, V)$ (of *structural* maps) for every affinoid K-algebra A, such that every point of V is in the image of some open structural immersion, and such that the composition of a rigid map of affinoids with a structural map is structural. An *h-space* is a wobbly analytic space together with an h-structure, and a *morphism* of h-spaces $(V, \theta) \to (V', \theta')$ is a ringed space morphism $V \to V'$ which pulls back structural maps to structural maps. If V, V' are affinoid, then a morphism of h-spaces between them is the same as a rigid morphism between them [Ta, Corollary to Prop. 10.4].

Next, Tate defined a *special covering* of an h-space [Ta, Def. 10.9] to be one that is obtained by taking a finite covering by special affine subsets, then repeating this process on each of those subsets, a finite number of times. An h-space V is then said to be *special* [Ta, Def. 10.12] if it has the property that a ringed space morphism $\operatorname{Max} B \to V$ is structural if and only if its restriction to each member of any special covering of $\operatorname{Max} B$ is structural. An open covering of an h-space V is *admissible* if its pullback by any structural morphism has a refinement that is a special covering. A *semi-rigid* analytic space V over K is a special h-space that has an admissible covering by affine open h-spaces. Finally, a *rigid* analytic space is a semi-rigid space V such that the above admissible covering has the property that the intersection of any two members is semi-rigid [Ta, Definition 10.16].

This rather cumbersome approach to rigidifying "wobbly spaces" was simplified and extended in a number of papers in the 1960's and 1970's, particularly in [GrRe1], [GrRe2], [GG]. From this point of view, the key idea is that analytic continuation on rigid spaces is permitted only with respect to "admissible" coverings by affinoid varieties, and where the only morphisms permitted between affinoid varieties are the rigid ones (i.e. those induced by homomorphisms between the corresponding affinoid algebras). To make sense of "admissibility", the notion of Grothendieck topology was used.

Recall (e.g. from [Ar1] or [Mi]) that a Grothendieck topology is a generalization of a classical topology on a space X, in which one replaces the collection of open sets $U \subset X$ by a collection of (admissible) maps $U \to X$, and in which certain families of such maps $\{V_i \to U\}_{i \in I}$ are declared to be *(admissible) coverings* (of U). This notion was originally introduced in order to provide a framework for the étale topology and for étale cohomology, which for algebraic varieties behaves much like classical singular cohomology in algebraic topology (unlike Zariski Čech cohomology).

In the case of rigid analytic spaces, a less general notion of Grothendieck topology is needed, in which the maps $U \to X$ are just inclusions of (certain) subsets of X, so that one speaks of "admissible subsets" of X [GuRo, §9.1]. According to the definition of a Grothendieck topology, the admissible subsets U and the admissible coverings of the U's satisfy several properties:

- the intersection of two admissible subsets is admissible;
- the singleton $\{U\}$ is an admissible covering of a set U;
- choosing an admissible covering of each member of an admissible covering together gives an admissible covering; and
- the intersection of an admissible covering of U with an admissible subset $V \subset U$ is an admissible covering of V.

Here, though, several additional conditions are imposed [BGR, p.339]:

- the empty set and X are admissible subsets of X;
- if V is a subset of an admissible $U \subset X$ and if the restriction to V of every member of some admissible covering of U is an admissible subset of X, then V is an admissible subset of X; and
- a family of admissible subsets $\{U_i\}_{i \in I}$ whose union is an admissible subset U, and which admits a refinement that is an admissible covering of U, is itself an admissible covering.

In this framework, a *rigid analytic space* is a locally ringed space $(V, \mathcal{O}_V)$ under a Grothendieck topology as above, with respect to which V has an admissible covering $\{V_i\}_{i \in I}$ where each $(V_i, \mathcal{O}_V|_{V_i})$ is an affinoid variety $\operatorname{Sp} A_i = (\operatorname{Max} A_i, A_i)$. (Here $A_i = \mathcal{O}_V|_{V_i}$.) A *morphism* of rigid analytic spaces $(V, \mathcal{O}_V) \to (W, \mathcal{O}_W)$ is a morphism (f, f^*) as locally ringed spaces. Thus morphisms between affinoid spaces are required to be rigid (i.e. of the form (ϕ°, ϕ), for some algebra homomorphism ϕ), and global morphisms are locally rigid with respect to an admissible covering. Analogously to the classical and formal cases, a *coherent* sheaf $\mathcal{F}$ (of $\mathcal{O}_V$-modules) is an $\mathcal{O}_V$-module that is locally (with respect to an admissible covering) of the form $\mathcal{O}_V^r \to \mathcal{O}_V^s \to \mathcal{F} \to 0$. In the case of an affinoid variety $\operatorname{Sp} A = (\operatorname{Max} A, A)$, coherent sheaves are precisely those of the form $\tilde{M}$, where M is a finite

A-module [FP, III, 6.2].

Rigid analogs of key results in the classical and formal situations (cf. Sections 2.2 and 3.2 above) have been proven in this context. A rigid version of Cartan's Lemma on matrix factorization [FP, III, 6.3] asserts that if $V = \mathrm{Sp}\, A$ is an affinoid variety and $f \in A$, and if we let V_1 [resp. V_2] be the set where $|f| \leq 1$ [resp. $|f| \geq 1$], then every invertible matrix in $\mathrm{GL}_n(\mathcal{O}(V_1 \cap V_2))$ that is sufficiently close to the identity can be factored as the product of invertible matrices over $\mathcal{O}(V_1)$ and $\mathcal{O}(V_2)$. There are also rigid analogs of Cartan's Theorems A and B, proven by Kiehl [Ki2]; they assert that a coherent sheaf $\mathcal{F}$ is generated by its global sections, and that $H^i(V, \mathcal{F}) = 0$ for $i > 0$, for "quasi-Stein" rigid analytic spaces V. (These are rigid spaces V that can be written as an increasing union of affinoid open subsets U_i that form an admissible covering of V, and such that $\mathcal{O}(U_{i+1})$ is dense in $\mathcal{O}(U_i)$. Compare Cartan's original version for complex Stein spaces [Ca2] discussed in §2.2 above.) Kiehl also proved [Ki1] a rigid analog of Zariski's Theorem on Formal Functions [Hrt2, III, Thm. 11.1], which together with Cartan's Theorem B (or Theorem A) was used to obtain GAGA classically. And indeed, there is a rigid analog of GAGA (or in this case, a "GRGA": géométrie rigide et géométrie algébrique) [Köp], asserting the equivalence between coherent rigid sheaves and coherent algebraic sheaves of modules over a projective algebraic K-variety. Thus, to give a coherent sheaf over such a variety, it suffices to give it over the members of an admissible covering (viewing the variety as a rigid analytic space), and giving the patching data on the overlaps.

As in Sections 2 and 3 above, it would be desirable to use these results in order to obtain a version of Riemann's Existence Theorem, which would classify covers. Ideally, this should be precise enough to give an explicit description of the tower of Galois groups of covers of a given space; and that description should be analogous to Corollary 2.1.2, the explicit form of the classical Riemann's Existence Theorem given at the beginning of Section 2.1. Unfortunately, to give such an explicit description, one needs to have a notion of a "topological fundamental group", and one needs to be able to compute that group explicitly. But unlike the complex case, one does not have such a notion, or computation, over more general fields K (in particular, because we cannot speak of "loops"). Thus, in this context, one does not have a full analog of Riemann's Existence Theorem 2.1.1, because one cannot assert an equivalence between finite rigid analytic covering maps and finite topological covering spaces. Still, one can ask for an analog of the first part of Theorem, 2.1.1 viz. an equivalence between finite étale covers of an algebraic curve V over K, and finite analytic covering maps of V (viewed as a rigid analytic space).

Such a result has been obtained (with some restrictions) by Lütkebohmert [Lü2]. As in the proof of the complex version (see Section 2.2), the proof proceeds using GAGA (here, the rigid version discussed above). Namely, as in the complex case, once one has the equivalence of categories that GAGA provides for sheaves of modules, one also obtains an equivalence (as a purely formal consequence) for sheaves of algebras, and hence for branched covers. But as in the complex case, GAGA applies to projective curves, but not to affine curves. So GAGA shows that there is an equivalence between branched (algebraic) covers of a projective K-curve X, and rigid analytic branched covers of the curve. Then to prove the desired portion of Riemann's Existence Theorem, it remains to show (both in the algebraic and rigid analytic settings) that covers of X branched only at a finite set B are equivalent to unramified covers of $V = X - B$ (i.e. that every unramified cover of V extends uniquely to a branched cover of X). In Section 2.2, we saw that this is immediate in the algebraic context, and follows easily from complex analysis in the analytic setting. But in the rigid analytic setting, this extension result for rigid analytic covers is harder, and moreover requires that the characteristic of K is 0.

Specifically, if $\mathrm{char}\, K = 0$, then unramified rigid covers of an affine K-curve $V = X-B$ do extend (uniquely) to rigid branched covers of the projective curve X; and so finite étale covers of V are equivalent to finite unramified rigid analytic covers of V. Moreover this generalizes to higher dimensions, where V is any K-scheme that is locally of finite type over K [Lü2, Theorem 3.1]. But there are counterexamples, even for curves, if $\mathrm{char}\, K = p$. For example, let $K = k((t))$, let V be the affine x-line over K, and consider the rigid unramified covering map $W \to V$ given by $y^p - y = \sum_{i=1}^{\infty} t^{(p+1)^i} x^{p^i}$. Then this map does not extend to a finite (branched) cover of the projective line, and so is not induced by any algebraic cover of V [Lü2, Example 2.10]. On the other hand, if one restricts attention to *tamely* ramified covers, then the desired equivalence between rigid and algebraic unramified covers does hold [Lü2, Theorem 4.1]. (Note that the above wildly ramified example does not contradict rigid GAGA, since that result applies in the projective case, whereas this example is affine.)

Still, we do not have an explicit description of the rigid analytic covers of a given curve (even apart from the difficulty with wildly ramified covers); so this result does not give explicit information about Galois groups and fundamental groups for K-curves (as a full rigid analog of Corollary 2.1.2 would). We return to this issue in Section 4.3, after considering another approach to rigid analytic spaces in Section 4.2.

Section 4.2. Rigid geometry via formal geometry.

Tate's rigid analytic spaces can be reinterpreted in terms of Grothendieck's formal schemes. This reinterpretation was outlined by Raynaud in [Ra1], and worked out in greater detail by Bosch, Lütkebohmert, and Raynaud in [Lü1], [BLü1], [BLü2], [BLüR1], [BLüR2]. (See also [Ra2, §3]; and Chapters 1 and 2, by M. Garuti [Ga] and Y. Henrio [He], in [BLoR].) As Tate said in [Ta], his approach was motivated by a suggestion of Grothendieck; and according to the introduction to [BLü1], Grothendieck's goal was to associate a generic fibre to a formal scheme of finite type. So this approach may actually be closer to Grothendieck's original intent than the more analytic framework discussed above.

The basic idea of this approach can be seen by revisiting examples from Sections 3.2 and 4.1. In Example 3.2.3, it was seen that $k[x][[t]]$ is the ring of formal functions along the affine x-line in the x, t-plane over a field k, or equivalently that its spectrum is a formal thickening of the affine x-line. The corresponding ring for the affine x^{-1}-line (i.e. the formal thickening of the projective x-line minus $x = 0$) is $k[x^{-1}][[t]]$, and the ring corresponding to the overlap (i.e. the formal thickening of $\mathbb{P}^1 - \{0, \infty\}$) is $k[x, x^{-1}][[t]]$. On the other hand, as seen in Section 4.1, if t is inverted in each of these three rings, one obtains the rings of functions on three affinoids over $K = k((t))$: the disc $|x| \leq 1$; the disc $|x^{-1}| \leq 1$ (i.e. $|x| \geq 1$ together with the point at infinity); and the "annulus" $|x| = 1$. In each of these two contexts (formal and rigid), the first two sets cover the projective line (over $R := k[[t]]$ and $K = \operatorname{frac} R$, respectively), and the third set is their "overlap". The ring of holomorphic functions on an affinoid set over K can (at least in this example) be viewed as the localization, with respect to t, of the ring of formal functions on an affine open subset of the closed fibre on an R-scheme. Correspondingly, an affinoid can be viewed as the generic fibre of the spectrum of the ring of formal functions (in the above example, a curve being the general fibre of a surface). Intuitively, then, a rigid analytic space over K is the general fibre of a (formal) scheme over R. (See Figure 4.2.1.)

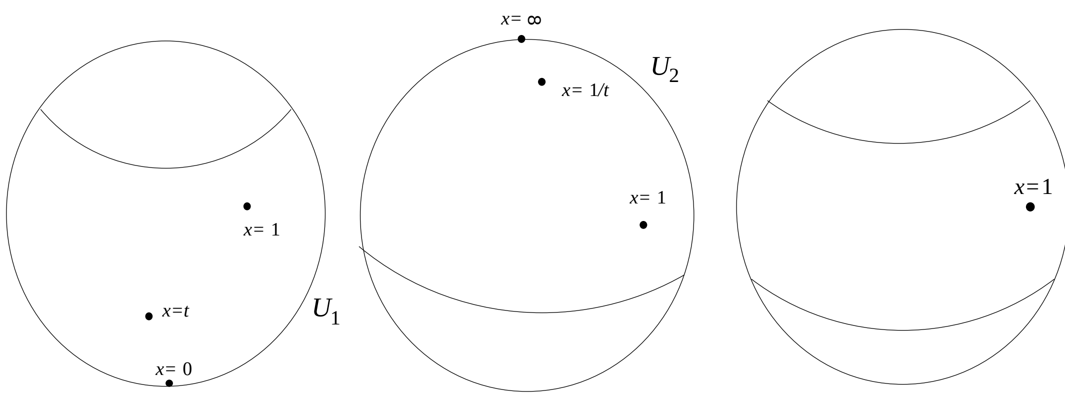

Figure 4.2.1: *A rigid covering of $\mathbb{P}^1_K$ (viewed as a sphere, in analogy with the complex case). The patches U_1, U_2 are discs around 0 and ∞, with rings of functions $k[x][[t]][1/t]$ and $k[1/x][[t]][1/t]$ (see §4.1). The overlap U_0 is an annulus containing the point $x = 1$, with ring of functions $k[x, 1/x][[t]][1/t]$. Compare Fig. 3.1.4 and see Example 4.2.3 below.*

The actual correspondence between formal schemes and rigid analytic spaces is a bit more complicated, because of several issues. The first concerns which base rings and fields are involved. Formal schemes are defined over complete local rings R, while rigid analytic spaces are defined over complete valuation fields K. The fraction field of a complete discrete valuation ring R is a discrete valuation field K, and every such K arises from such an R. But general valuation fields are not fraction fields of complete local rings, and the fraction fields of general complete local rings are not valuation fields. So in stating the correspondence, we restrict here to the case of a complete discrete valuation ring R, say with maximal ideal $\mathfrak{m}$ (though one can consider, somewhat more generally, a complete height 1 valuation ring R).

Secondly, in order for a formal space to induce a rigid space, it must locally induce affinoid K-algebras, i.e. K-algebras that are of topologically finite type. Correspondingly, we say that an R-algebra A is of *topologically finite type* if it is a quotient of the $\mathfrak{m}$-adic completion of some $R[x_1, \ldots, x_n]$. Observe that this $\mathfrak{m}$-adic completion is a subring of $R[[x_1, \ldots, x_n]]$, and in fact consists precisely of those power series $\sum_{\underline{i} \in \mathbb{N}^n} a_{\underline{i}} x^{\underline{i}}$, where $a_{\underline{i}} \to 0$ as $\underline{i} \to \infty$. It is then easy to verify that $A \otimes_R K$ is an affinoid K-algebra, for any R-algebra A that is of topologically finite type. (This is in contrast to the full rings of power series, where $K[[x_1, \ldots, x_n]]$ is much larger than $R[[x_1, \ldots, x_n]] \otimes_R K$.) A formal R-scheme $\mathcal{V}$ is *locally of topologically finite type* if in a neighborhood of every point, the structure sheaf $\mathcal{O}_{\mathcal{V}}$ is given by an R-algebra that is of topologically finite type. Such a formal scheme is said it be of *topologically finite type* if in addition it is quasi-compact. Thus formal schemes that are of topologically finite type induce quasi-compact rigid spaces.

The condition of a formal R-scheme $\mathcal{V}$ being locally of topologically finite type in turn implies that the corresponding $R/\mathfrak{m}^n$-schemes V_n are locally Noetherian (since the structure sheaf is locally a quotient of some $(R/\mathfrak{m}^n)[x_1, \ldots, x_n]$). Thus each V_n is quasi-separated, by [Gr4, IV, Cor. 1.2.8]; and hence so is $\mathcal{V}$ and so is the induced rigid space. On the other hand, not every rigid space is necessarily quasi-separated; so in order to get an equivalence between formal and rigid spaces, we will need to restrict attention to rigid spaces that are quasi-separated (this being a very mild finiteness condition).

A third issue concerns the fact that non-isomorphic R-schemes can have K-isomorphic general fibres. For example, let V be a proper R-scheme, where R is a complete discrete valuation ring. Let V_0 be the closed fibre of V, and let W be a closed subset of V_0. Let $\tilde{V}$ be the blow up of V along W (as a scheme). Then V and $\tilde{V}$ have the same general fibre. But they are not isomorphic as R-schemes (if the codimension of W in V is at least 2), since $\tilde{V}$ has an exceptional divisor over the blown up points. Hence they do not correspond to isomorphic formal schemes.

In order to deal with this third issue, the strategy is to regard two R-schemes as equivalent if they have a common *admissible* blow-up (i.e. a blow up at a closed subset of the closed fibre). Thus given two R-schemes V, V', to give a morphism from the equivalence class of V to that of V' is to give an admissible blow up $\tilde{V} \to V$ together with a morphism of R-schemes $\tilde{V} \to V'$. Here $V, \tilde{V}, V'$ induce formal R-schemes $\mathcal{V}, \tilde{\mathcal{V}}, \mathcal{V}'$ (given by the direct limit of the fibres $V_n, \tilde{V}_n, V'_n$ over $\mathfrak{m}^n$), and we regard the induced pair $(\tilde{\mathcal{V}} \to \mathcal{V}, \tilde{\mathcal{V}} \to \mathcal{V}')$ as a morphism between the equivalence classes of $\mathcal{V}, \mathcal{V}'$. Equivalently, we are considering morphisms from the class of $\mathcal{V}$ to the class of $\mathcal{V}'$, in the localization of the category of formal R-schemes with respect to the class of admissible formal blow-ups $\tilde{\mathcal{V}} \to \mathcal{V}$. (The *localization* is the category in which those blow-ups are formally inverted. Such a localization automatically exists, according to [Hrt1]; though to be set-theoretically precise, one may wish to work within a larger "universe" [We, Remark 10.3.3].)

Here, for a formal scheme $\mathcal{V}$ induced by a proper R-scheme V, one can correspondingly define admissible blow-ups of $\mathcal{V}$ as the morphisms of formal schemes induced by admissible blow-ups of V. Alternatively, and for a more general formal R-scheme $\mathcal{V}$, admissible blow-ups can be defined directly, despite the fact that the topological space underlying $\mathcal{V}$ is just the closed fibre of the associated R-scheme (if there is one). Namely, the blow-up can be defined algebraically, analogously to the usual definition for schemes. First, observe that if A is a complete R-algebra, then the closed subsets of the closed fibre of Spec A correspond to ideals of A that are *open* in the topology induced by that of R. Now recall [Hrt2, Chap. II, p.163] that if V is a Noetherian scheme, and $\mathcal{I}$ is a coherent sheaf of ideals on V, then the blow-up of V at $\mathcal{I}$ is $\operatorname{Proj} \mathcal{J}$, where $\mathcal{J}$ is the sheaf of graded algebras $\mathcal{J} = \bigoplus_{d \geq 0} \mathcal{I}^d$. So given a formal R-scheme $\mathcal{V}$ and a sheaf $\mathcal{I}$ of open ideals of $\mathcal{O}_{\mathcal{V}}$, define the blow-up of $\mathcal{V}$ along $\mathcal{I}$ to be the formal scheme associated to the direct system of $R/\mathfrak{m}^n$-schemes $\operatorname{Proj} \mathcal{J}_n$, where $\mathcal{J}_n = \bigoplus_{d \geq 0}(\mathcal{I}^d \otimes_{\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{\mathcal{V}}/\mathfrak{m}^n)$. We call such a blow-up of the formal scheme $\mathcal{V}$ *admissible*. This agrees with the previous definition, for formal schemes $\mathcal{V}$ induced by R-schemes V.

A fourth issue, which is similar to the third, is that an R-scheme V may have an irreducible component that is contained in the closed fibre V_0. In that case, the general fibre of V "does not see" that component, and so cannot determine V (or the induced formal scheme). So we avoid this case, by requiring that the formal scheme $\mathcal{V}$ have the property that its structure sheaf $\mathcal{O}_{\mathcal{V}}$ has no $\mathfrak{m}$-torsion. We call the formal scheme $\mathcal{V}$ *admissible* if it has this property and is of locally of topologically finite type. (So quasi-compact admissible is the same as $\mathfrak{m}$-torsion-free plus topologically finite type.)

With these restrictions and adjustments, the equivalence between formal and rigid spaces takes place. Consider an admissible formal R-scheme $\mathcal{V}$, whose underlying topological space is a k-scheme V_0 (where $k = R/\mathfrak{m}$). For any affine open subset $U \subset V_0$, let A be the ring of formal functions along U. So A is topologically of finite type, and has no $\mathfrak{m}$-torsion; and $A \otimes_R K$ is an affinoid K-algebra. In the notation of Section 4.1, $\operatorname{Sp} A = (\operatorname{Max} A, A)$ is an affinoid variety. This construction is compatible with shrinking U, and so from $\mathcal{V}$ we obtain a rigid analytic space, which we denote by $\mathcal{V}^{\mathrm{rig}}$. There is then the following key theorem of Raynaud [Ra1] (see also [BLü1, Theorem 4.1], for details):

Theorem 4.2.2. (Raynaud) *Let R be a complete valuation ring of height* 1 *with fraction field K. Let* For_R *be the category of quasi-compact admissible formal R-schemes, and let* For'_R *be the localization of* For_R *with respect to admissible formal blow-ups. Let* Rig_K *be* *the category of quasi-compact quasi-separated rigid analytic K-spaces. Then the functor* $\mathrm{rig} : \mathrm{For}_R \to \mathrm{Rig}_K$ *given by* $\mathcal{V} \mapsto \mathcal{V}^{\mathrm{rig}}$ *induces an equivalence of categories* $\mathrm{For}'_R \to \mathrm{Rig}_K$.

(Alternatively, the conclusion of the theorem could be stated by saying that $\mathrm{rig} : \mathrm{For}_R \to \mathrm{Rig}_K$ is a localizing functor with respect to all admissible blow-ups, rather than speaking in terms of For'_R.)

In particular, if V is a proper R-scheme, and if $\mathcal{V}$ is the associated formal scheme, then $\mathcal{V}^{\mathrm{rig}}$ is the rigid analytic space corresponding to the generic fibre V_K of V.

More generally, one can turn the above result around and make it a *definition*, to make sense of rigid analytic spaces over the fraction field K of a Noetherian complete local ring R which is not necessarily a valuation ring (e.g. $k[[x_1, \ldots, x_n]]$, where k is a field and $n > 1$). That is, for such a ring R and fraction field K, one can simply *define* the category Rig_K of rigid analytic K-spaces to be the category For'_R, obtained by localizing the category For_R of formal R-schemes with respect to admissible blow-ups [Ra1], [BLü1], [Ga]. The point is that formal schemes make sense in this context, and thus the notion of rigid spaces can be extended to this situation as well. (Of course, by Raynaud's theorem, the two definitions are equivalent in the case that R is a complete discrete valuation ring.)

The advantage to Raynaud's approach to rigid analytic spaces is it permits them to be studied using Grothendieck's results on formal schemes in EGA [Gr4]. It also permits the use of results in EGA on proper schemes over complete local rings, because of the equivalence of those schemes with formal schemes via by Grothendieck's Existence Theorem ([Gr2], [Gr4, III, Cor. 5.1.6]; see also Section 3.2 above). In particular, Grothendieck's Existence Theorem and Raynaud's theorem above together imply the rigid GAGA result (for projective spaces) discussed in Section 4.1 above. Moreover, Raynaud's approach permits the use of the rigid point of view over more general fields than Tate's original approach did, though with some loss of analytic flavor. Indeed, from this point of view, the rigid and formal contexts are not so different, though there is a difference in terms of intuition. Another difference is that in the formal context one works on a fixed R-model of a space, whereas in the rigid context one works just over K (and thus blow-ups are already included in the geometry).

We conclude this discussion by giving two examples comparing formal and rigid GAGA on the line, beginning with the motivating situation discussed earlier:

Example 4.2.3. Let k be a field; $R = k[[t]]$; $K = k((t))$; and $V = \mathbb{P}^1_R$. Let x be a parameter on V, and $y = x^{-1}$. So V is covered by two copies of the affine line over R, the x-line and the y-line, intersecting where $x, y \neq 0$. Letting $\mathcal{V}$ be the formal scheme associated to V, there is the induced rigid analytic space $V^{\mathrm{rig}} := \mathcal{V}^{\mathrm{rig}}$, viz. $\mathbb{P}^1_K$. According to rigid GAGA, giving a coherent sheaf on V^{rig} is equivalent to giving finite modules over (the rings of functions on) the admissible sets $U_1 : |x| \leq 1$ and $U_2 : |y| \leq 1$, with agreement on the overlap $U_0 : |x| = |y| = 1$. Here $U_1 = \mathrm{Sp}\, K\{x\}$, $U_2 = \mathrm{Sp}\, K\{y\}$, and

$U_0 = \mathrm{Sp}\, K\{x,y\}/(xy-1)$. Geometrically (and intuitively), U_1 and U_2 are discs centered around $x = 0, \infty$ respectively (the "south and north poles"), and U_0 is an annulus (a band around the "equator", if $\mathbb{P}^1_K$ is viewed as a "sphere"; see Figure 4.2.1 above).

On the formal level, U_1 is the general fibre of $S_1 = \mathrm{Spec}\, k[x][[t]]$, the formal thickening of the affine x-line (which pinches down near $x = \infty$). Similarly, U_2 is the general fibre of $S_2 = \mathrm{Spec}\, k[y][[t]]$, the formal thickening of the affine y-line (which pinches down near $x = 0$). And U_0 is the general fibre of $S_0 = \mathrm{Spec}\, k[x, x^{-1}][[t]]$, the formal thickening of the line with both 0 and ∞ deleted (and which pinches down near both points — cf. Figure 3.1.4). According to formal GAGA (i.e. Grothendieck's Existence Theorem; cf. Theorems 3.2.1 and 3.2.4), giving a coherent sheaf on V is equivalent to giving finite modules over S_1 and S_2 with agreement on the "overlap" S_0.

In the formal context, even less data is needed in order to construct a coherent sheaf on V — and this permits more general constructions to be performed (e.g. see [Ha6]). Namely, let $\hat{S}_1 = \mathrm{Spec}\, k[[x,t]]$, the complete local neighborhood of $x = t = 0$. Let $\hat{S}_0 = \mathrm{Spec}\, k((x))[[t]]$, the "overlap" of $\hat{S}_1$ with S_2. (See Figure 3.2.9, where $\hat{S}_1, S_2, \hat{S}_0$ are denoted by W^*, U^*, W'^*, respectively.) Then according to Theorem 3.2.8, giving a coherent sheaf on V is equivalent to giving finite modules over $\hat{S}_1$ and S_2 together with agreement over $\hat{S}_0$. On the rigid level, the generic fibres of $\hat{S}_1$ and S_2 are $\hat{U}_1 : |x| < 1$ and $U_2 : |x| \geq 1$. Those subsets of V^{rig} do not intersect, and moreover $\hat{U}_1$ is not an affinoid set. The result in the formal situation suggests that the generic fibre of $\hat{S}_0$, corresponding to $k((x))((t))$, forms a "glue" that connects $\hat{U}_1$ and U_2; but this cannot be formulated within the rigid framework. □

Example 4.2.4. With notation as in Example 4.2.3, rigid GAGA says that to give a coherent sheaf on $V^{\mathrm{rig}} = \mathbb{P}^1_K$ is equivalent to giving finite modules over the two discs $|x| \leq 1$ and $|y| \leq c^{-1}$, and over the annulus $c \leq |x| \leq 1$; here $0 < c = |t| < 1$, and the annulus is the overlap of the two discs. Writing $z = ty = t/x$, the rings of functions on these three sets are $K\{x\}$, $K\{z\}$, and $K\{x,z\}/(xz-t)$.

To consider the corresponding formal situation, let $\tilde{V}$ be the blow-up of V at the closed point $x = t = 0$. Writing $xz = t$, the closed fibre of $\tilde{V}$ consists of the projective x-line over k (the proper transform of the closed fibre of V) and the projective z-line over k (the exceptional divisor), meeting at the "origin" $O : x = z = t = 0$. The three affinoid open sets above are then the generic fibres associated to the formal schemes obtained by respectively deleting from the closed fibre of $\tilde{V}$ the point $x = \infty$ (which is where $z = 0$); the point $z = \infty$ (where $x = 0$); and both of these points. And by Grothendieck's Existence Theorem, giving compatible formal coherent modules over each of these sets is equivalent to giving a coherent module over V.

But as in Example 4.2.3, less is needed in the formal context. Namely, let X' and Z' be the projective x- and z-lines over k, with the points $(x = 0)$ and $(z = 0)$ respectively deleted. Consider the rings of formal functions along X' and Z', viz. $k[x^{-1}][[t]]$ and $k[z^{-1}][[t]]$ respectively, and their spectra T_1, T_2. Consider also the spectrum T_3 of $k[[x,z,t]]/(xz-t)$, the complete local ring of V at O. Here T_1 and T_2 are disjoint, while the "overlap" of T_1 and T_3 [resp. of T_2 and T_3] is the spectrum $T_{1,3}$ of $k((x))[[t]]$ [resp. $T_{2,3}$ of $k((z))[[t]]$]. By Theorem 3.2.8, giving finite modules over T_1, T_2, T_3 that agree on the two "overlaps" is equivalent to giving a coherent module over V. The generic fibres of T_1 and T_2 are the sets $|y| \le 1$ and $|x| \le c$, and that of T_3 is $c < |x| < 1$. These three sets are disjoint, though the formal set-up provides "glue" (in the form of $T_{1,3}$ and $T_{2,3}$) connecting T_3 to T_1 and to T_2. This is a special case of Example 3.2.11. (Alternatively, one could use Theorem 3.2.12, taking V to be the projective x-line over $k[[t]]$, taking $\tilde{V}$ to be the blow-up of V at $x = t = 0$, and identifying the exceptional divisor with the projective z-line over k. See Example 3.2.13.) □

More generally, in the rigid set-up, one can consider the annulus $c^n \le |x| \le 1$ in $\mathbb{P}^1_K$ (where $K = k((t))$ and $c = |t|$ as above, and where n is a positive integer). If one writes $u = t^n/x$, then this is the intersection of the two admissible sets $|x| \le 1$ and $|u| \le 1$. This annulus is said to have *thickness* (or *épaisseur*) equal to n. The corresponding situation in the formal framework can be arrived at by taking the projective x-line V over $R = k[[t]]$; blowing this up at the point $x = t = 0$ (obtaining a parameter $z = t/x$ on the exceptional divisor E); blowing that up at the point $z = \infty$ on E (thereby obtaining a parameter $z' = t/z^{-1} = t^2/x$ on the new exceptional divisor); and repeating the process for a total of n blow-ups. The analogs of Examples 4.2.3 and 4.2.4 above can then be considered similarly.

Section 4.3. Rigid patching and constructing covers.

Rigid geometry, like formal geometry, provides a framework within which patching constructions can be carried out in order to construct covers of curves, and thereby obtain Galois groups over curves. Ideally, one would like to obtain a version of Riemann's Existence Theorem analogous to that stated for complex curves in Section 2.1. But while a kind of "Riemann's Existence Theorem" for rigid spaces was obtained by Lütkebohmert [Lü2] (see Section 4.1 above), that result does not say which Galois groups arise, due to a lack of topological information. Still, as in the formal case, one can show by a patching construction that every finite group is a Galois group of a branched cover with enough branch points, and show a "Half Riemann Existence Theorem" that is analogous to the classification of slit covers of complex curves (see Section 2.3).

Namely, Serre observed in a 1990 talk in Bordeaux that there should be a rigid proof of Theorem 3.3.1 above (on the realizability of every finite group as a Galois group over the fraction field K of a complete local domain R [Ha4]), when the base ring R is complete with respect to a non-archimedean absolute value. Given the connection between rigid and formal schemes discussed in Section 4.2 (especially in the case of complete discrete valuation rings), this would seem quite plausible. Shortly afterwards, in [Se7, §8.4.4], Serre outlined such a proof in the case that $K = \mathbb{Q}_p$. A more detailed argument was carried out by Liu for complete non-archimedean fields with an absolute value, in a manuscript that was written in 1992 and that appeared later in [Li], after circulating privately for a few years. (Concerning complete *archimedean* fields, the complex case was discussed in Section 2 above, and the real case is handled in [Se7, §8.4.3], via the complex case; cf. also [DF] for the real case.)

The rigid version of Theorem 3.3.1 is as follows:

Theorem 4.3.1. *Let K be a field that is complete with respect to a non-trivial non-archimedean absolute value. Let G be a finite group. Then there is a G-Galois irreducible branched cover $Y \to \mathbb{P}^1_K$ such that the fibre over some K-point of $\mathbb{P}^1_K$ is totally split.*

Here the totally split condition is that the fibre consists of unramified K-points. This property (which takes the place of the mock cover hypothesis of Theorem 3.3.1) forces the cover to be regular, in the sense that K is algebraically closed in the function field $K(Y)$ of Y. (Namely, if L is the algebraic closure of K in $K(Y)$, then L is contained in the integral closure in $K(Y)$ of the local ring $\mathcal{O}_\xi$ of any closed point $\xi \in \mathbb{P}^1_K$; and so it is contained in the residue field of each closed point of Y.) Thus Theorem 3.3.1 is recaptured, for such fields K.

Proof sketch of Theorem 4.3.1. The proof proceeds analogously to that of Theorem 3.3.1. Namely, first one proves the result explicitly in the special case that the group is a cyclic group. In [Se7, §8.4.4], Serre does this by using an argument involving tori [Se7, §4.2] to show that cyclic groups are Galois groups of branched covers of the line; one can then obtain a totally split fibre by twisting, e.g. as in [HV, Lemma 4.2(a)]. Or (as in [Li]) one can proceed as in the original proof for the cyclic case in the formal setting [Ha4, Lemma 2.1], which used ideas of Saltman [Slt]; cf. Proposition 3.3.3 above.

To prove the theorem in the general case, cyclic covers are patched together to produce a cover with the desired group, in a rigid analog of the proof of Theorem 3.3.1. Namely, let $g_1, \ldots, g_r$ be generators of G. For each i, let H_i be the cyclic subgroup of G generated by g_i, and let $f_i : Y_i \to \mathbb{P}^1_K$ be a H_i-Galois cover that is totally split over a point ξ_i. By the Implicit Function Theorem over complete fields, for each i there is a closed disc $\bar{D}_i$ about ξ_i such that the inverse image $f_i^{-1}(\bar{D}_i)$ is a disjoint union of copies of $\bar{D}_i$. Let D_i be the corresponding open disc about ξ_i, let $\bar{U}_i = \mathbb{P}^1_K - D_i$, and let $U_i = \mathbb{P}^1_K - \bar{D}_i$. After a change of variables, we may assume that the $\bar{U}_i$'s are pairwise disjoint affinoid sets. For each i, let $\bar{V}_i \to \bar{U}_i$ be the pullback of f_i to $\bar{U}_i$. Then $\bar{V}_i$ is an H_i-Galois cover whose restriction over $\bar{U}_i - U_i = \bar{D}_i - D_i$ is trivial. Inducing from H_i to G (by taking a disjoint union of copies, indexed by the cosets of H_i), we obtain a corresponding G-Galois disconnected cover $\bar{W}_i = \mathrm{Ind}_{H_i}^G \bar{V}_i \to \bar{U}_i$. Also, let $\bar{U}_0 = \mathbb{P}^1_K - \bigcup_{j=1}^r U_j = \bigcap_{j=1}^r \bar{D}_j$, and let $\bar{W}_0 \to \bar{U}_0$ be the trivial G-cover $\mathrm{Ind}_1^G \bar{U}_0$. We now apply rigid GAGA (see Sections 4.1

and 4.2), though for covers rather than for modules (that form following automatically, as in Theorem 3.2.4, via the General Principle 2.2.4). Namely, we patch together the covers $\bar{W}_i \to \bar{U}_i$ ($i = 0, \ldots, r$) along the overlaps $\bar{U}_i \cap \bar{U}_0 = \bar{U}_i - U_i$ ($i = 1, \ldots, r$), where they are trivial. One then checks that the resulting G-Galois cover is as desired (and in particular is irreducible, because the g_i's generate G); and this yields the theorem. □

As in Section 3.3, Theorem 4.3.1 extends to the class of large fields, such as the algebraic p-adics and the field of totally real algebraic numbers. Namely, as in the passage to Theorem 3.3.6, there is the following result of Pop:

Corollary 4.3.2. [Po4] *If k is a large field, then every finite group is the Galois group of a Galois field extension of $k(x)$. Moreover this extension may be chosen to be regular, and with a totally split fibre.*

Namely, by Theorem 4.3.1, there is a G-Galois extension of $k((t))(x)$, and this descends to a regular G-Galois extension of the fraction field of $A[x]$ with a totally split fibre over $x = 0$, for some $k(t)$-subalgebra $A \subset k((t))$ of finite type. By the Bertini-Noether Theorem [FJ, Prop. 9.29], we may assume that every specialization of A to a k-point gives a G-Galois regular field extension of $k(x)$; and such a specialization exists on $V := \operatorname{Spec} A$ since k is large and since V contains a $k((t))$-point.

The construction in the proof of Theorem 4.3.1, like the one used in proving the corresponding result using formal geometry, can be regarded as analogous to the slit cover construction of complex covers described in Section 2.3 (and see the discussion at the end of Section 3.3 for the analogy with the formal setting). In fact, rather than considering covers (and Galois groups) one at a time, a whole tower of covers (and Galois groups) can be considered, as in the "analytic half Riemann Existence Theorem" 2.3.5. In the present setting (unlike the situation over $\mathbb{C}$), the absolute Galois group G_K of the valued field K comes into play, since it acts on the geometric fundamental group (i.e. the fundamental group of the punctured line after base-change to the separable closure K^{s} of K). This construction of a tower of compatible covers has been carried out by Pop in [Po2] (where the term "half Riemann Existence Theorem" was also introduced). Also, rather than requiring K to be complete, Pop required K merely to be henselian (and cf. Example 3.3.2(d), for comments about deducing the henselian case of that result from the complete case via Artin's Approximation Theorem).

In Pop's result, as in the case of complex slit covers, one chooses as a branch locus a closed subset $S \subset \mathbb{P}^1_K$ whose base change to K^{s} consists of finitely many pairs of nearby points. That is, S is a disjoint union of two closed subsets $S = S_1 \cup S_2$ of $\mathbb{P}^1_K$ such that $S_1^{\mathrm{s}} := S_1 \times_K K^{\mathrm{s}} = \{\xi_1, \ldots, \xi_r\}$ and $S_2^{\mathrm{s}} := S_2 \times_K K^{\mathrm{s}} = \{\eta_1, \ldots, \eta_r\}$, where the ξ_i and η_j are distinct K^{s}-points, and where each ξ_i is closer to the corresponding η_i than it is to any other ξ_j. Such a set S is called *pairwise adjusted.* Note that the sets S_1^{s} and S_2^{s} are each

G_K-invariant, and that G_K acts on the sets S_1^{s} and S_2^{s} compatibly (i.e. if $\alpha \in G_K$ satisfies $\alpha(\xi_i) = \xi_j$, then $\alpha(\eta_i) = \eta_j$). Now let $U = \mathbb{P}^1_K - S$ and $U^{\mathrm{s}} = U \times_K K^{\mathrm{s}} = \mathbb{P}^1_{K^{\mathrm{s}}} - S^{\mathrm{s}}$, and recall the fundamental exact sequence

$$1 \to \pi_1(U^{\mathrm{s}}) \to \pi_1(U) \to G_K \to 1. \tag{$*$}$$

In this situation, let Π be the free profinite group $\hat{F}_r$ of rank r if the valued field K is in the equal characteristic case; this is the free product of r copies of the group $\hat{\mathbb{Z}}$, in the category of profinite groups. If K is in the unequal characteristic case with residue characteristic $p > 0$, then let Π be the free product $\hat{F}_r[p]$ of r copies of the group $\hat{\mathbb{Z}}/\mathbb{Z}_p$, in the category of profinite groups. (Note that this free product is not a pro-prime-to-p group if $r > 1$, and in particular is much larger than the free pro-prime-to-p group of rank r.) Define an action of G_K on Π by letting $\alpha \in G_K$ take the jth generator g_j of Π to $g_i^{\chi(\alpha^{-1})}$; here i is the unique index such that $\alpha(\xi_i) = \xi_j$, and $\chi : G_K \to \hat{\mathbb{Z}}^*$ is the cyclotomic character (taking $\gamma \mapsto m$ if $\gamma(\zeta) = \zeta^m$ for all roots of unity ζ). There is then the following result of Pop (and see Remark 4.3.4(c) below for an even stronger version):

Theorem 4.3.3. (Half Riemann Existence Theorem with Galois action [Po2]) *Let K be a henselian valued field of rank 1, let $S \subset \mathbb{P}^1_K$ be a pairwise adjusted subset of degree $2r$ as above, and let $U = \mathbb{P}^1_K - S$. Then the fundamental exact sequence* $(*)$ *has a quotient*

$$1 \to \Pi \to \Pi \rtimes G_K \to G_K \to 1, \tag{$**$}$$

where Π is defined as above and where the semi-direct product is taken with respect to the above action of G_K on Π.

Proof sketch of Theorem 4.3.3. In the case that the field K is complete, the proof of Theorem 4.3.3 follows a strategy that is similar to that of Theorem 4.3.1. As in Theorem 4.3.1 (and Theorem 3.3.1), the proof relies on the construction of cyclic covers that are trivial outside a small neighborhood (in an appropriate sense), and which can then be patched. The key new ingredient is that one must show that the construction is functorial, and in particular is compatible with forming towers. Concerning this last point, after passing to the maximal cyclotomic extension K^{cycl} of K, one can construct a tower of regular covers by patching together local cyclic covers that are Kummer or Artin-Schreier. These can be constructed compatibly with respect to the action of $\mathrm{Gal}(K^{\mathrm{cycl}}/K)$, since S is pairwise adjusted; and the resulting tower, viewed as a tower of covers of U, has the desired properties.

The henselian case is then deduced from the complete case. This is done by first observing that the absolute Galois groups of K and of its completion $\hat{K}$ are canonically isomorphic (because K is henselian). Then, writing $\hat{K}^{\mathrm{s}}$ for the separable closure of $\hat{K}$, it is checked that every finite branched cover of the $\hat{K}^{\mathrm{s}}$-line that results from the patching construction is defined over the separable closure K^{s} of K. (Namely, consider a finite quotient Q of Π, generated by cyclic subgroups C_i. The patching construction over $\hat{K}$ yields a Q-Galois cover $Y \to \mathbb{P}^1_{\hat{K}^{\mathrm{s}}}$ that is constructed using cyclic building blocks $Z_i \xrightarrow{C_i} \mathbb{P}^1$ which are each defined over K^{s}. Let $Z \to \mathbb{P}^1$ be the fibre product of the Z_i's; this is Galois with group $H = \prod C_i$. Pulling back the Q-cover $Y \to \mathbb{P}^1_{\hat{K}^{\mathrm{s}}}$ via $Z_{\hat{K}^{\mathrm{s}}} \xrightarrow{H} \mathbb{P}^1_{\hat{K}^{\mathrm{s}}}$ gives an unramified Q-cover Y' of the projective curve $Z_{\hat{K}^{\mathrm{s}}}$; here Y' is also a $Q \times H$-Galois branched cover of $\mathbb{P}^1_{\hat{K}^{\mathrm{s}}}$. By Grothendieck's specialization isomorphism [Gr5, XIII], Y' descends to a Q-cover of $Z_{K^{\mathrm{s}}}$ whose composition with $Z_{K^{\mathrm{s}}} \to \mathbb{P}^1_{K^{\mathrm{s}}}$ is $Q \times H$-Galois. Hence Y descends to a Q-cover of $\mathbb{P}^1_{K^{\mathrm{s}}}$.) Since the Galois actions of G_K and $G_{\hat{K}}$ are the same, the result in the general henselian case follows. $\square$

Remark 4.3.4. (a) The hypotheses of Theorem 4.3.3 are easily satisified; i.e. there are many choices of pairwise adjusted subsets. Namely, let $f \in K[x]$ be any irreducible separable monic polynomial, and let $g \in K[x]$ be chosen so that it is monic of the same degree, and so that its coefficients are sufficiently close to those of f. Then the zero locus of fg in $\mathbb{A}^1_K$ is a pairwise adjusted subset, by continuity of the roots [La, II, §2, Proposition 4]. Repeating this construction with finitely many polynomials f_i and then taking the union of the resulting sets gives a general pairwise adjusted subset. Note that in the case that K is separably closed, the construction is particularly simple: One may take an arbitrary set $S_1 = \{\xi_1, \ldots, \xi_r\}$ of K-points in $\mathbb{A}^1_K$, and any set $S_2 = \{\eta_1, \ldots, \eta_r\}$ of K-points such that each η_i is sufficiently close to ξ_i. This recovers the slit cover construction of Section 2.3 in the case $K = \mathbb{C}$.

(b) In the equal characteristic case, if K contains all of the roots of unity (of order prime to p, if $\operatorname{char} K = p \neq 0$), then Theorem 4.3.3 shows that the free profinite group $\hat{F}_r$ on r generators is a quotient of $\pi_1(U)$. (Namely, the cyclotomic character acts trivially in this case, and so the semi-direct product in $(**)$ is just a direct product.) Since arbitrarily large pairwise adjusted subsets S exist by Remark (a), this shows that $\hat{F}_r$ is a quotient of the absolute Galois group of $K(x)$ for each $r \in \mathbb{N}$. A similar result holds in the unequal characteristic case $(0,p)$ if K contains the prime-to-p roots of unity, namely that the free pro-prime-to-p group $\hat{F}'_r$ of rank $r \in \mathbb{N}$ is a quotient of $\pi_1(U)$ and of $G_{K(x)}$. But the full group $\hat{F}_r$ is *not* a quotient of $\pi_1(U)$ or $G_{K(x)}$ in the unequal characteristic case; cf. [Po2] and Remark (c) below.

(c) The result in [Po2] asserts even more. First of all, the ith generator of Π generates an inertia group over ξ_i and over η_i, for each $i = 1, \ldots, r$. This is as in the case of analytic and formal slit covers discussed at the ends of Sections 2.3 and 3.3 above. Second, in the unequal characteristic case $(0,p)$, the assertion of Theorem 4.3.3 may be improved somewhat, by replacing the group $\Pi = \hat{F}_r[p]$ by the free product of r copies of the group $\hat{\mathbb{Z}}/p^e\mathbb{Z}_p$ (in the category of profinite groups), for a certain non-negative integer e. (Specifically, $e = \max(0, e')$, where e' is the largest integer such that $|\xi_i - \eta_i| < |p|^{e'+1/(p-1)}|\xi_i - \xi_j|$

for all $i \neq j$.) This group lies in between the group $\hat{F}_r$ and its quotient $\hat{F}_r[p]$; and in the case that K contains all the prime-to-p roots of unity, this group is then a quotient of $\pi_1(U)$ and $G_{K(x)}$ (like $\hat{F}_r[p]$ but unlike $\hat{F}_r$). See [Po2] for details.

(d) The construction of cyclic extensions given in Section 3.3 can be recovered from the above result, in the case that the extension is of degree n prime to the characteristic of K. Namely, given a cyclic group $C = \langle c \rangle$ of order n, consider a primitive element for $K' := K(\zeta_n)$ as an extension of K; this corresponds to a K'-point $\xi = \xi_1$ of $\mathbb{P}^1$, and $\mathrm{Gal}(K'/K)$ acts simply transitively on the G_K-orbit $\{\xi_1, \ldots, \xi_s\}$ of ξ. Take $\eta = \eta_1$ sufficiently close to ξ to satisfy continuity of the roots [La, II, §2, Proposition 4] (and also to satisfy the inequality in Remark (c) above, in the mixed characteristic case $(0, p)$ if $p|n$); and let its orbit be $\{\eta_1, \ldots, \eta_s\}$. Let $U \subset \mathbb{P}^1_K$ be the complement of the ξ_i's and η_i's. Consider the surjection $\Pi \twoheadrightarrow C$ given by $g_j \mapsto c^{\chi(\alpha^{-1})} = c^{-\alpha(\zeta_n)}$ if $\alpha \in \mathrm{Gal}(K'/K)$ is the element taking ξ to ξ_j. Then in the quotient $C \rtimes G_K$ of $\Pi \rtimes G_K$ (and hence of $\pi_1(U)$), the action of G_K on C is trivial; i.e. the quotient is just $C \times G_K$. So it in turn has a quotient isomorphic to C; and this corresponds to the cyclic cover constructed in the proof of Proposition 3.3.3. (In the case that n is instead a power of $p = \mathrm{char}\, K$, one uses Witt vectors in the construction; and again one obtains cyclic covers of degree n, since the action of G_K via χ is automatically trivial on a p-group quotient of Π.)

(e) The main assertion in Theorem 4.3.1 above (and in Theorem 3.3.1), that every finite group G is a Galois group over $K(x)$, can be recaptured from the Half Riemann Existence Theorem. Namely, by choosing elements c_i that generate G, and applying Remark (d) separately to each c_i, one obtains a quotient of $\Pi \rtimes G_K$ of the form $G \rtimes G_K$, in which the semi-direct product is actually a direct product. So G is a quotient of $\pi_1(U)$. □

Unfortunately, the above approach (like that of Section 3.3) does not provide an explicit description, in terms of generators and relations, of the *full* fundamental group (or at least the tame fundamental group) of an arbitrary affine K-curve U. Such a full "Riemann's Existence Theorem" would generalize the explicit classical result over $\mathbb{C}$ (Corollary 2.1.2), unlike Lütkebohmert's result [Lü] discussed at the end of Section 4.1 (which is inexplicit) and the above result (which gives only a big quotient of $\pi_1(U)$).

At the moment such a full, explicit result (or even a conjecture about its exact statement) seems far out of reach, even in key special cases. For example, if K is algebraically closed of characteristic p, the profinite groups $\pi_1(\mathbb{A}^1_K)$ and $\pi_1^t(\mathbb{A}^1_K - \{0, 1, \infty\})$ are unknown. And if K is a p-adic field, the tower of all Galois branched covers of $\mathbb{P}^1_K$ remains mysterious, while little is understood about Galois branched covers of $\mathbb{P}^1_K$ with good reduction and their associated Galois groups. (Note that the covers constructed above and in Section 3.3 have models over $\mathbb{Z}_p$ in which the closed fibres are quite singular — as is clear from the mock cover construction of Section 3.3.) For $p > 3$, a wildly ramified cover of $\mathbb{P}^1_{\mathbb{Q}_p}$ cannot have good reduction over $\mathbb{Q}_p$ (or even over the maximal unramified extension of $\mathbb{Q}_p$) [Co, p.247, Remark 3]; and so $\mathbb{Z}/p$ cannot be such a Galois group. But it is unknown whether every finite group G is the Galois group of a cover of $\mathbb{P}^1_K$ with good reduction over K, for some *totally ramified* extension K of $\mathbb{Q}_p$ (depending on G); if so, this would imply that every finite group is a Galois group over the field $\mathbb{F}_p(x)$ (cf. Proposition 3.3.9).

See Section 5 for a further discussion of results in the direction of a generalized Riemann's Existence Theorem.

In the rigid patching constructions above, and in the analogous formal patching constructions in Section 3.3, the full generality of rigid analytic spaces and formal schemes is not needed in order to obtain the results in Galois theory. Namely, the rigid analytic spaces and formal schemes that arise in these proofs are induced from algebraic varieties; and so less machinery is needed in order to prove the results of these sections than might first appear. Haran and Völklein (and later Jarden) have developed an approach to patching that goes further, and which seeks to omit all unnecessary geometric objects. Namely, in [HV], the authors created a context of "algebraic patching" in which everything is phrased in terms of rings and fields (viz. the rings of functions on formal or rigid patches, and their fractions fields), and in which the geometric and analytic viewpoints are suppressed. That set-up was then used to reprove Corollary 4.3.2 above on realizing Galois groups regularly over large fields [HV, Theorem 4.4], as well as to prove additional related Galois results (in [HV], [HJ1], and [HJ2]). For covers of curves, it appears that the formal patching, rigid patching, and algebraic patching methods are essentially interchangeable, in terms of what they are capable of showing. The main differences concern the intuition and the precise machinery involved; and these are basically matters of individual mathematical taste. In other applications, it may turn out that one or another of these methods is better suited.

Section 5: Toward Riemann's Existence Theorem

Sections 3 and 4 showed how formal and rigid patching methods can be used to establish analogs of GAGA, and to realize all finite groups as Galois groups of covers, in rather general settings. This section pursues these ideas further, in the direction of a sought-after "Riemann's Existence Theorem" that would classify covers in terms of group-theoretic data, corresponding to the Galois group, the inertia groups, and how the covers fit together in a tower. Central to this section is the notion of "embedding problems", which will be used in studying this tower. In particular, Section 5.1 uses embedding problems to give the structure of the absolute Galois group of the function field of a curve over an algebraically closed field (which can be regarded as the geometric case of a conjecture of Shafarevich). Section 5.2 relates patching and embedding problems to arithmetic lifting problems, in which one considers the existence of a cover with a given Galois group and a given fibre (over a non-algebraically closed base field). In doing so, it relies on results from Section 5.1. Section 5.3 considers Abhyankar's Conjecture on fundamental groups in characteristic p, along with strengthenings and generalizations that relate to embedding problems and patching. These results move further in the direction of a full "Riemann's Existence Theorem", although the full classification of covers in terms of groups remains unknown.

Section 5.1. Embedding problems and the geometric case of Shafarevich's Conjecture.

The motivation for introducing patching methods into Galois theory was to prove results about Galois groups and fundamental groups for varieties that are not necessarily defined over $\mathbb{C}$. Complex patching methods, combined with topology, permitted a quite explicit description of the tower of covers of a given complex curve U (Riemann's Existence Theorem 2.1.1 and 2.1.2). In particular, this approach showed what the fundamental group of U is, and thus which finite groups are Galois groups of unramified covers of U. Analogous formal and rigid patching methods were applied (in Sections 3 and 4) to the study of curves over certain other coefficient fields, in particular *large* fields. Without restriction on the branch locus, it was shown that every finite group is a Galois group over the function field of the curve (Sections 3.3 and 4.3), and Pop's "Half Riemann's Existence Theorem" gave an explicit description of a big part of the tower of covers for certain special choices of branch locus (Section 4.3). Further results about Galois groups over an arbitrary affine curve have also been obtained (see Section 5.3 below), but an explicit description of the full tower of covers, and of the full fundamental group, remain out of reach for now.

Nevertheless, if one does not restrict the branch locus, then patching methods can be used to find the birational analog of the fundamental group, in the case of curves over an algebraically closed field k — i.e. to find the absolute Galois group of the function field of a k-curve X. And here, unlike the situation with the fundamental group of an affine

k-curve, the absolute Galois group turns out to be free even in characteristic $p > 0$.

In the case $k = \mathbb{C}$ and $X = \mathbb{P}^1_{\mathbb{C}}$, this result was proven in [Do] using the classical form of Riemann's Existence Theorem (see Corollary 2.1.5). For more general fields, it was proven independently by the author [Ha10] and by F. Pop [Po1], [Po3]:

Theorem 5.1.1. *Let X be an irreducible curve over an algebraically closed field k of arbitrary characteristic. Then the absolute Galois group of the function field of X is the free profinite group of rank equal to the cardinality of k.*

In particular, the absolute Galois group of $k(x)$ is free profinite of rank equal to $\operatorname{card} k$.

Theorem 5.1.1 implies the geometric case of Shafarevich's Conjecture. In the form originally posed by Shafarevich, the conjecture says that the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}^{\mathrm{ab}})$ of $\mathbb{Q}^{\mathrm{ab}}$ is free profinite of countable rank. Here $\mathbb{Q}^{\mathrm{ab}}$ denotes the maximal abelian extension of $\mathbb{Q}$, or equivalently (by the Kronecker-Weber theorem) the maximal cyclotomic extension of $\mathbb{Q}$ (i.e. $\mathbb{Q}$ with all the roots of unity adjoined). The conjecture was later generalized to say that if K is any global field and K^{cycl} is its maximal cyclotomic extension, then the absolute Galois group of K^{cycl} is free profinite of countable rank. (See Remark 3.3.8(b).) The arithmetic case of this conjecture (the case where K is a number field) is *still open*, but the geometric case (the case where K is the function field of a curve X over a finite field F) follows from Theorem 5.1.1, by considering passage to the algebraic closure $\bar{F}$ of F. Namely, in this situation, $\bar{F} = \bar{\mathbb{F}}_p$ where $p = \operatorname{char} F$, and so the function field $\bar{K}$ of $\bar{X} := X \times_F \bar{F}$ is equal to K^{cycl}; and in this case Theorem 5.1.1, applied to the $\bar{K}$-curve $\bar{X}$, asserts the conclusion of Shafarevich's Conjecture.

Theorem 5.1.1 above is proven using the notion of embedding problems. Recall that an *embedding problem* $\mathcal{E}$ for a profinite group Π is a pair of surjective group homomorphisms $(\alpha : \Pi \to G, f : \Gamma \to G)$. A *weak* [resp. *proper*] *solution* to $\mathcal{E}$ consists of a group homomorphism [resp. epimorphism] $\beta : \Pi \to \Gamma$ such that $f\beta = \alpha$:

$$\begin{array}{ccccccccc} & & & & & & \Pi & & \\ & & & & & \overset{\beta?}{\swarrow} & \downarrow{\scriptstyle\alpha} & & \\ 1 & \longrightarrow & N & \longrightarrow & \Gamma & \overset{f}{\longrightarrow} & G & \longrightarrow & 1 \end{array}$$

An embedding problem $\mathcal{E}$ is *finite* if Γ is finite; it is *split* if f has a section; it is *non-trivial* if $N = \ker f$ is non-trivial; it is a *p-embedding problem* if $\ker f$ is a p-group. A profinite group Π is *projective* if every finite embedding problem for Π has a weak solution.

In terms of Galois theory, if Π is the absolute Galois group of a field K, then giving a G-Galois field extension L of K is equivalent to giving a surjective homomorphism $\alpha : \Pi \to G$. For such an L, giving a proper solution to $\mathcal{E}$ as above is equivalent to giving a Γ-Galois field extension F of K together with an embedding of L into F as a G-Galois

K-algebra. (Here the G-action on L agrees with the one induced by restricting the action of Γ to the image of the embedding.) Giving a weak solution to $\mathcal{E}$ is the same, except that F need only be a separable K-algebra, not a field extension (and so it can be a direct product of finitely many fields). In this field-theoretic context we refer to an *embedding problem for K*.

If K is the function field of a geometrically irreducible k-scheme X, then the field extensions L and F correspond to branched covers $Y \to X$ and $Z \to X$ which are G-Galois and Γ-Galois respectively, such that Z dominates Y. Here Y is irreducible; and Z is also irreducible in the case of a proper solution. If the algebraic closure of k in the function fields of Y and Z are equal (i.e. if there is no extension of constants from Y to Z), we say that the solution is *regular*.

By considering embedding problems for a field K, or over a scheme X, one can study not only which finite groups are Galois groups over K or X, but how the extensions or covers fit together in the tower of all finite Galois groups. As a result, one can obtain information about absolute Galois groups and fundamental groups. In particular, in the key special case that X is the projective line and k is countable (e.g. if $k = \bar{\mathbb{F}}_p$), Theorem 5.1.1 follows from the following three results about embedding problems:

Theorem 5.1.2. (Iwasawa [Iw,p.567], [FJ,Cor.24.2]) *Let Π be a profinite group of countably infinite rank. Then Π is a free profinite group if and only if every finite embedding problem for Π has a proper solution.*

Theorem 5.1.3. (Serre [Se6, Prop. 1]) *If U is an affine curve over an algebraically closed field k, then the profinite group $\pi_1(U)$ is projective.*

Theorem 5.1.4. (Harbater [Ha10], Pop [Po1], [Po3]) *If k is an algebraically closed field, and K is the function field of an irreducible k-curve X, then every finite split embedding problem for K has a proper solution.*

Concerning these three results which will be used in proving Theorem 5.1.1: Theorem 5.1.2 is entirely group-theoretic (and *rank* refers to the minimal cardinality of any generating set). The proof of Theorem 5.1.3 is cohomological, and in fact the assertion in [Se6] is stated in terms of cohomological dimension (that $\text{cd}(\pi_1(X)) \leq 1$, which implies projectivity by [Se4, I, 5.9, Proposition 45]). Theorem 5.1.4 is a strengthening of Theorem 3.3.1, and like that result it is proven using patching. (Theorem 5.1.4 will be discussed in more detail below.)

Using these results, Theorem 5.1.1 can easily be shown in the case that the algebraically closed field k is countable. Namely, let Π be the absolute Galois group of $k(x)$. Then the profinite group Π has at most countable rank, since the countable field $k(x)$ has only countably many finite field extensions; and Π has infinite rank, since every finite group is a quotient of Π (as seen in Section 3.3). So Theorem 5.1.2 applies, and it suffices to show that every finite embedding problem $\mathcal{E}$ for Π is properly solvable. Say $\mathcal{E}$ is given by $(\alpha : \Pi \to G, f : \Gamma \to G)$, with f corresponding to a G-Galois branched cover $Y \to X$. This cover is étale over an affine dense open subset $U \subset X$, and α factors through $\pi_1(U)$ (since quotients of π_1 classify unramified covers). Writing this map as $\alpha_U : \pi_1(U) \to G$, consider the finite embedding problem $\mathcal{E}_U = (\alpha_U : \pi_1(U) \to G, f : \Gamma \to G)$. By Theorem 5.1.3, this has a weak solution $\beta_U : \pi_1(U) \to \Gamma$, say with image $H \subset \Gamma$ (which surjects onto G under f). Let N be the kernel of f, and Γ_1 be the semidirect product $N \rtimes H$ with respect to the conjugation action of H on N. The multiplication map $(n, h) \mapsto nh \in \Gamma$ is an epimorphism $m : \Gamma_1 \to \Gamma$, and the projection map $h : \Gamma_1 \to H$ is surjective with kernel N. The surjection $\beta_U : \pi_1(U) \to H$ corresponds to an H-Galois branched cover $Y_1 \to X$ (unramified over U). This in turn corresponds to a surjective group homomorphism $\beta : \Pi \to H$. By Theorem 5.1.4, the split embedding problem $(\beta : \Pi \to H, h : \Gamma_1 \to H)$ has a proper solution. That solution corresponds to an irreducible Γ_1-Galois cover $Z_1 \to X$ that dominates Y_1; and composing the corresponding surjection $\Pi \to \Gamma_1$ with $m : \Gamma_1 \to \Gamma$ provides a proper solution to the original embedding problem $\mathcal{E}$.

Remark 5.1.5. The above argument actually requires less than Theorem 5.1.3; viz. it suffices to use Tsen's Theorem [Ri, Proposition V.5.2] that if k is algebraically closed then the absolute Galois group of $k(x)$ has cohomological dimension 1. For then, by writing X in Theorem 5.1.1 as a branched cover of $\mathbb{P}^1_k$, it follows that the absolute Galois group of its function field is also of cohomological dimension 1 [Se4, I, 3.3, Proposition 14], and hence is projective [Se4, I, 5.9, Proposition 45]. One can then proceed as before.

But by using Theorem 5.1.3 as in the argument above, one obtains additional information about the branch locus of the solution to the embedding problem. Namely, one sees in the above argument that the H-Galois cover $Y_1 \to X$ remains étale over U. In applying Theorem 5.1.4 to pass to a Γ_1-cover (and thence to a Γ-cover), one typically obtains new branch points. But a sharp upper bound can be found on the number of additional branch points [Ha11], using Abhyankar's Conjecture (discussed in Section 5.3). □

Before turning to the general case of Theorem 5.1.1 (where k is allowed to be uncountable), we sketch the proof of Theorem 5.1.4:

Proof sketch of Theorem 5.1.4. Let Π be the absolute Galois group of K. Consider a finite split embedding problem $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ for K, with s a section of f, and with f corresponding to a G-Galois branched cover $Y \to X$. Let $N = \ker(f)$, and let $n_1, \ldots, n_r$ be generators of N. Thus Γ is generated by $s(G)$ and the n_i's. Pick r closed points $\xi_i \in X$ that are not branch points of $Y \to X$. Thus $Y \to X$ splits completely over each ξ_i, since k is algebraically closed. Let $k' = k((t))$, and let $X' = X \times_k k'$ and similarly for Y' and ξ_i'. Pick small neighborhoods X_i' around each of the points ξ_i' on X'. (Here, if one works in the rigid context, one takes t-adic closed discs. If one works in the formal context, one blows up at the points ξ_i', and proceeds as in Example 3.2.11 or 3.2.13, using

Theorem 3.2.8 or 3.2.12. See also Example 4.2.4.) Over these neighborhoods, build cyclic covers $Z_i' \to X_i'$ with group $N_i = \langle n_i \rangle$ (branched at ξ_i' and possibly other points; cf. the proof of Proposition 3.3.3, using the presence of prime-to-p roots of unity). Let $Y_0' \to X_0'$ be the restriction of $Y' \to X'$ away from the above neighborhoods (viz. over the complement of the corresponding open discs if one works rigidly, and over the general fibre of the formal completion at the complement of the ξ_i's if one works formally). Via the section s of f, the Galois group G of $Y_0' \to X_0'$ may be identified with $s(G) \subset \Gamma$. The induced Γ-Galois covers $\mathrm{Ind}_{N_i}^{\Gamma} Z_i' \to Z_i'$ and $\mathrm{Ind}_{s(G)}^{\Gamma} Y_0' \to X_0'$ agree over the (rigid or formal) overlap. Hence by (rigid or formal) GAGA, these patch together to form a Γ-Galois cover $Z' \to X'$. (In the formal case, one uses Theorem 3.2.8 rather than Theorem 3.2.1, since the agreement is not on the completion along a Zariski open set.) This cover is connected since Γ is generated by $s(G)$ and the n_i's; it dominates $Y' \to X'$ since it does on each patch; and it is branched at each ξ_i'. As in the proof of Corollary 3.3.5, one may now specialize from k' to k using that k is algebraically closed, obtaining a Γ-Galois cover $Z \to X$ that dominates $Y \to X$. This corresponds to a proper solution to $\mathcal{E}$. □

Remark 5.1.6. (a) The above proof also shows that one has some control over the position of the new branch points of $Z \to X$. Namely, the branch locus contains the points ξ_i, and these points can be taken arbitrarily among non-branch points of $Y \to X$. In particular, any given point of X can be taken to be a branch point of $Z \to X$ above (by choosing it to be one of the ξ_i's). More precise versions of this fact appear in [Ha10, Theorem 3.5] and [Po3, Theorem A], where formal and rigid methods are respectively used.

(b) As a consequence of Remark (a), it follows that the set of (isomorphism classes of) solutions to the split embedding problem has cardinality equal to that of k.

(c) The above proof of Theorem 5.1.4 also gives information about inertia of the constructed cover $Z \to X$. Namely, if $I \subset G$ is the inertia group of $Y \to X$ at a point $\eta \in Y$ over $\xi \in X$, then $s(I) \subset \Gamma$ is an inertia group of $Z \to X$ at a point $\zeta \in Z$ over η (and the other inertia groups over ξ are the conjugates of $s(I)$.

(d) Adjustments to the above construction give additional flexibility in controlling the properties of $Z \to X$. In particular, if $\mathrm{char}\, k = p > 0$ and if $I' \subset \Gamma$ is the extension of $s(I)$ by a p-group, then one may build Z so that I' is an inertia group over ξ at a point over η (with notation as in Remark (c)). In addition, rather than considering a split embedding problem, i.e. a group Γ generated by a normal subgroup N and a complement $s(G)$, one can more generally consider a group Γ generated by two subgroups H and G, where we are given a G-Galois cover $Y \to X$. The assertion then says that this cover can be modified to produce a Γ-Galois cover $Z \to X$ with control as above on the branch locus and inertia groups. In particular, one can add additional branch points to a cover, and one can modify a cover by enlarging an inertia group from a p-subgroup of the Galois group to a larger p-subgroup. (See [Ha6, Theorem 2] and [Ha13, Theorem 3.6], where formal patching is used to prove these assertions.)

(e) The ability to add branch points was used in [MR] to show that for any finite group G and any smooth connected curve X over an algebraically closed field k, there is a G-Galois branched cover $Y \to X$ such that G is the *full* group of automorphisms of Y. The idea is that if one first takes an arbitrary G-Galois cover of X (by Corollary 3.3.5); then one can adjust it by adding new branch points and thereby killing automorphisms that are not in G. □

To prove the general case of Theorem 5.1.1, one replaces Theorem 5.1.2 above by a result of Melnikov and Chatzidakis (see [Ja, Lemma 2.1]):

Theorem 5.1.7. *Let Π be a profinite group and let m be an infinite cardinal. Then Π is a free profinite group of rank m if and only if every non-trivial finite embedding problem for Π has exactly m proper solutions.*

Namely, by Remark 5.1.6(b) above, in the situation of Theorem 5.1.1 the number of proper solutions to any finite *split* embedding problem is $\operatorname{card} k$. Proceeding as in the proof of Theorem 5.1.1 in the countable case, one obtains that *every* finite embedding problem for Π has $\operatorname{card} k$ proper solutions. So Π is free profinite of that rank by Theorem 5.1.7, and this proves Theorem 5.1.1.

Remark 5.1.8. By refining the proof of Theorem 5.1.1 (in particular modifying Theorems 5.1.3 and 5.1.4 above), one can prove a tame analog of that result [Ha13, Theorem 4.9(b)]: If X is an affine curve with function field K, consider the maximal extension Ω of K that is at most tamely ramified over each point of X. Then $\operatorname{Gal}(\Omega/K)$ is a free profinite group, of rank equal to the cardinality of k. □

Theorem 5.1.4 above extends from algebraically closed fields to arbitrary large fields (cf. Section 3.3), according to the following result of Pop:

Theorem 5.1.9. (Pop [Po1, Theorem 2.7]) *If k is a large field, and K is the function field of a geometrically irreducible k-curve X, then every finite split embedding problem for K has a proper regular solution.*

Namely, the above proof of Theorem 5.1.4 showed that result for an *algebraically closed* field k by first proving it for the Laurent series field $K = k((t))$, and then specializing from K to k, using that k is algebraically closed. In order to prove Theorem 5.1.9, one does the same in this more general context, using that k is large in order to specialize from $K = k((t))$ to k (as in Sections 3.3 and 4.3). A difficulty is that since k need not be algebraically closed, one can no longer choose the extra branch points $\xi_i \in X$ arbitrarily (as one could in the above proof of Theorem 5.1.4, where ξ_i and the points of its fibre were automatically k-rational). Still, one can proceed as in the proofs of Theorems 3.3.1, 4.3.1, and 4.3.3 — viz. using cyclic covers branched at clusters of points constructed in the proof of Proposition 3.3.3.

Since an arbitrary large field k is not algebraically closed, one would also like to know that the Γ-Galois cover $Z \to X$ has the property that $Z \to Y$ is *regular* (i.e. Z and Y have the same ground field ℓ, or equivalently the algebraic closures of k in the function fields of Y and Z are equal). This can be achieved by using that in the construction using formal patching, the closed fibre of the cover $Z \to Y$ over K is a mock cover (as in the proof of Theorem 3.3.1). Alternatively, from the rigid point of view, one can observe from the patching construction (as in the proof of Theorem 4.3.1) that Z may be chosen so that $Z \to Y$ has a totally split fibre over $\eta \in Y$, if η has been chosen (in advance) to be an ℓ-point of Y that lies over a k-point ξ of X. This then implies regularity, as in Theorem 4.3.1. (If there is no such point $\eta \in Y$, then one can first base-change to a finite Galois extension $\tilde{k}$ of k where there is such a point; and then construct a regular solution $\tilde{Z} \to \tilde{Y} = Y \times_k \tilde{k}$ which is compatible with the $\mathrm{Gal}(\tilde{k}/k)$-action, and so which descends to a regular solution $Z \to Y$.)

Remarks 5.1.6(a) and (b) above no longer hold for curves over an arbitrary large field (nor does Theorem 5.1.1 — see below); but Remark 5.1.6(c) still applies in this situation. So the argument in the case of an arbitrary large field gives the following more precise form of Theorem 5.1.9 (where one looks at the actual curve X, rather than just at its function field):

Theorem 5.1.10. *Let k be a large field, let X be a geometrically irreducible smooth k-curve, let $f : \Gamma \to G$ be a surjection of finite groups with a section s, and let $Y \to X$ be a G-Galois connected branched cover of smooth curves.*

(a) Then there is a smooth connected Γ-Galois branched cover $Z \to X$ that dominates the G-Galois cover $Y \to X$, such that $Z \to Y$ is regular.

(b) Let ξ be a k-point of X which is not a branch point of $Y \to X$, and let η be a closed point of Y over ξ with decomposition group $G_1 \subset G$. Then the cover $Z \to X$ in (a) may be chosen so that it is totally split over η, and so that there is a point $\zeta \in Z$ over η whose decomposition group over ξ is $s(G_1) \subset \Gamma$.

Remark 5.1.11. (a) In [Po1], the above result was stated for a slightly smaller class of fields (those with a "universal local-global principle"); but in fact, all that was used is that the field is large. Also, the result there did not assert 5.1.10(b), though this can be deduced from the proof. The result was stated for large fields in [Po4, Main Theorem A], but only in the case that $X = \mathbb{P}^1_k$ and $Y = \mathbb{P}^1_\ell$. (Both proofs used rigid patching.) The fact that the fibre over η can be chosen to be totally split first appeared explicitly in [HJ1, Theorem 6.4], in the case that $X = \mathbb{P}^1_k$ and $Y = \mathbb{P}^1_\ell$; and in [HJ2, Proposition 4.2] if $X = \mathbb{P}^1_k$ and Y is arbitrary. The proofs there used "algebraic patching" (cf. the comments at the end of Section 4.3).

(b) A possible strengthening of Theorem 5.1.10(b) would be to allow one to specify the decomposition group of ζ as a given subgroup $G_1' \subset \Gamma$ that maps isomorphically onto $G_1 \subset G$ via f (rather than having to take $G_1' = s(G_1)$, as in the statement above). It would be interesting to know if this strengthening is true. □

As a consequence of Theorem 5.1.9, we have:

Corollary 5.1.12. *Let k be a Hilbertian large field, with absolute Galois group G_k.*

(a) Then every finite split embedding problem for G_k has a proper solution.

(b) If k is also countable, and if G_k is projective, then G_k is isomorphic to the free profinite group of countable rank.

Proof. (a) Every such embedding problem for G_k gives a split embedding problem for $G_{k(x)}$. That problem has a proper solution by Theorem 5.1.9. Since k is Hilbertian, that solution can be specialized to a proper solution of the given embedding problem.

(b) Since G_k is projective, the conclusion of part (a) implies that *every* finite embedding problem for G_k has a proper solution (as in the proof of Theorem 5.1.1 above, using semi-direct products). Also, G_k is of countably infinite rank (again as in the proof of Theorem 5.1.1). So Theorem 5.1.2 implies the conclusion. □

Remark 5.1.13. (a) Part (a) of Corollary 5.1.12 appeared in [Po3, Main Theorem B] and [HJ1, Thm.6.5(a)] . As a special case of part (b) of the corollary, one has that if k is a countable Hilbertian PAC field (see Example 3.3.7(c)), then G_k is free profinite of countable rank. This is because PAC fields are large, and because their absolute fundamental groups are projective (because they are of cohomological dimension ≤ 1 [Ax2, §14, Lemma 2]). This special case had been a conjecture of Roquette, and it was proven as above in [Po3, Thm. 1] and [HJ1, Thm. 6.6] (following a proof in [FV2] in the characteristic 0 case, using the classical complex analytic form of Riemann's Existence Theorem).

(b) As remarked in Section 3.3, it is unknown whether $\mathbb{Q}^{\mathrm{ab}}$ is large. But it is Hilbertian ([Vö, Corollary 1.28], [FJ, Theorem 15.6]) and countable (being contained in $\bar{\mathbb{Q}}$), and its absolute Galois group is projective (being of cohomological dimension 1 by [Se4, II, 3.3, Proposition 9]). So if it is indeed large, then part (b) of the corollary would imply that its absolute Galois group is free profinite of countable rank — i.e. the original (arithmetic) form of Shafarevich's Conjecture would hold. Among other things, this would imply that every finite group is a Galois group over $\mathbb{Q}^{\mathrm{ab}}$.

The solvable version of Shafarevich's Conjecture has been shown; i.e. the maximal pro-solvable quotient of $G_{\mathbb{Q}^{\mathrm{ab}}}$ is the free prosolvable group of countable rank [Iw]. More generally, if k is Hilbertian and G_k is projective, then every finite embedding problem for G_k with solvable kernel has a proper solution [Vö, Corollary 8.25]. This result does not require k to be large, and it does not use patching.

(c) It has been *conjectured* by Dèbes and Deschamps [DD] that Theorem 5.1.9 and Corollary 5.1.12 remain true even if the ground field is not large. Specifically, they conjecture that for any field k, every finite split embedding problem for $G_{k(x)}$ has a proper regular solution; and hence that if k is Hilbertian, then every finite split embedding problem for G_k has a proper solution. This is a very strong conjecture, in particular implying an affirmative answer to the Regular Inverse Galois Problem (i.e. that every finite group is a regular Galois group over $k(x)$ for every field k). But it also seems very far away from being proven. □

As mentioned above, Theorem 5.1.1 does not hold if the algebraically closed field k is replaced by an arbitrary large field. This is because if K is the function field of a k-curve X, then its absolute Galois group G_K is not even projective (much less free) if k is not separably closed. That is, not every finite embedding problem for K has a weak solution — and so certainly not a proper solution, as would be required in order to be free.

This can be seen by using the equivalence between the condition that a profinite group Π is projective and the condition that it has cohomological dimension ≤ 1 [Se4, I; 5.9, Proposition 45 and 3.4, Proposition 16]. Namely, if k is not separably closed, then its absolute Galois group G_k is non-trivial, and so G_k has cohomological dimension > 0 [Se4, I, 3.3, Corollaire 2 to Proposition 14]. Since the function field K is of finite type over k and of transcendence degree 1 over k, it follows that G_K has cohomological dimension > 1. (This is by [Se4, II, 4.2, Proposition 11] in the case that $\operatorname{cd} G_k$ is finite; and by [Ax1] and [Se4, II, 4.1, Proposition 10(ii)] if $\operatorname{cd} G_k$ is infinite.) So G_K is not projective.

But as Theorem 5.1.9 shows, every finite *split* embedding problem for G_K has a proper solution, if K is the function field of a curve over an arbitrary large field k. Thus (as in the proof of Theorem 1 above, via semi-direct products), it follows that any finite embedding problem for G_K that has a weak solution must also have a proper solution. So Theorem 5.1.9 can be regarded as saying that G_K is "as close as possible" to being free, given that it is not projective.

Section 5.2. Arithmetic lifting, embedding problems, and patching.

In realizing Galois groups over a Hilbertian field k like $\mathbb{Q}$ or $\mathbb{Q}^{\mathrm{ab}}$, the main method is to realize the group as a regular Galois group over $K = k(x)$, and then to specialize from K to k using that k is Hilbertian. That is, one constructs a Galois branched cover $Y \to \mathbb{P}^1_k$ such that k is algebraically closed in the function field of Y, and then obtains a Galois extension of k with the same group by considering an irreducible fibre of the cover over a k-point of $\mathbb{P}^1_k$ (which exists by the Hilbertian hypothesis). To date, essentially all simple groups that have been realized as Galois groups over $\mathbb{Q}$ or $\mathbb{Q}^{\mathrm{ab}}$ have been realized by this method.

The use of this method has led to the question of whether, given a finite Galois extension ℓ of a field k, there is a finite regular Galois extension L of $K = k(x)$ with the same group G, of which the given extension is a specialization. If so, then one says that the field k and group G satisfy the *arithmetic lifting property.* (Of course if one did not require regularity, then one could just take L to be $\ell(x)$.)

The question of when this property holds was first raised by S. Beckmann [Be], who showed that it does hold in the case that $k = \mathbb{Q}$ and G is either an abelian group or a symmetric group. Later, E. Black [Bl1] [Bl2] [Bl3] showed that the property holds for certain more general classes of groups over Hilbertian fields, particularly certain semi-direct products such as dihedral groups D_n with n odd. Black also conjectured that the arithmetic lifting property holds for all finite groups over all fields, and proving this has come to be known as the *Beckmann-Black problem* (or BB). It was later shown by Dèbes [Dè] that an affirmative answer to BB over *every* field would imply an affirmative answer to the Regular Inverse Galois Problem (RIGP) over every field (i.e. that for every field k and every finite group G, there is a regular Galois extension of $k(x)$ with group G). On the other hand, knowing BB for a *given* field k does not automatically give RIGP over k, since one needs to be given a Galois extension of the given field in order to apply BB.

Colliot-Thélène has considered a strong form of arithmetic lifting (or BB): Suppose we are given a field k and a finite group G, and a G-Galois k*-algebra* A (i.e. a finite direct sum of finite separable field extensions of k, on which G acts faithfully with fixed field k). In this situation, is there a regular G-Galois field extension L of $k(x)$ that specializes to A? Equivalently, suppose that H is a subgroup of G and ℓ is an H-Galois field extension of k. Then the question is whether there is a regular G-Galois field extension of $k(x)$ such that some specialization to k yields $A := \ell^{\oplus(G:H)}$ (where the copies of ℓ are indexed by the cosets of H in G). In geometric terms, the question is whether there is a regular G-Galois branched cover $Y \to \mathbb{P}^1_k$ with a given fibre $\mathrm{Ind}_H^G \operatorname{Spec} \ell$ — i.e. such that over some unramified k-point of the line, there is a point of Y with given decomposition group $H \subset G$ and given residue field ℓ (which is a given H-Galois field extension of k).

If, in the strong form of BB, one takes A to be a G-Galois field extension ℓ of k, then one recovers the original BB. At the other extreme, if one takes A to be a direct sum of copies of k (indexed by the elements of G), then one is asking the question of whether there is a G-Galois regular field extension of $k(x)$ with a totally split fibre. (Thus the strong form of BB over a *given* field k implies RIGP for that field.) In the case that k is a large field, this totally split case of strong BB does hold; indeed, this is precisely the content of Theorem 4.3.1.

Colliot-Thélène showed that the strong form of BB holds in general for large fields k:

Theorem 5.2.1. [CT] *If k is a large field, G is a finite group, and A is a G-Galois k-algebra, then there is a G-Galois regular branched cover of $X = \mathbb{P}^1_k$ whose fibre over a given k-point agrees with* Spec A *(as a G-Galois k-algebra).*

Remark 5.2.2. As noted in Remark 5.1.13(b), it is unknown whether $\mathbb{Q}^{\mathrm{ab}}$ is large. But if it is, then Theorem 5.2.1 would imply that it has the (strong) arithmetic lifting property for every finite group — and so every finite Galois group over $\mathbb{Q}^{ab}$ would be the specialization of a regular Galois branched cover of the line over $\mathbb{Q}^{ab}$. On the other hand, $\mathbb{Q}$ is not large, and so Theorem 5.2.1 does not apply to it. And although it is known that every finite solvable group is a Galois group over $\mathbb{Q}$ (Shafarevich's Theorem [NSW, Chap. IX, §5]), it is not known whether every such group is the Galois group of a regular branched cover of $\mathbb{P}^1_{\mathbb{Q}}$ — much less that the arithmetic lifting property holds for these groups over $\mathbb{Q}$. □

Colliot-Thélène's proof used a different form of patching, and relied on work of Kollár [Kol] on rationally connected varieties. The basic idea is to construct a "comb" of projective lines on a surface, i.e. a tree of $\mathbb{P}^1$'s in which one component meets all the others, none of which meet each other. A degenerate cover of the comb is then constructed by building it over the components, and the cover is then deformed to a non-degenerate cover of a nearby irreducible curve of genus 0 with the desired properties.

Colliot-Thélène's proof required that k be of characteristic 0 (because Kollár's work assumed that), but other proofs have been found that do not need this. In particular, Moret-Bailly [MB2] used a formal patching argument to prove this result. A proof using rigid patching can be obtained from Colliot-Thélène's argument by replacing the "spine" of the comb by an affinoid set U_0 as in the proof of Theorem 4.3.1, and the "teeth" of the comb by affinoids $U_1, \dots, U_r$ as in that proof (appropriately chosen). And a proof using "algebraic patching" (cf. the end of Section 4.3) has been found by Haran and Jarden [HJ2].

Yet another proof of Theorem 5.2.1 above can be obtained from Pop's result on solvability of split embedding problems over large fields (Theorem 5.1.9, in the more precise form Theorem 5.1.10 — which of course was also proven using patching). This proof, which was found by Pop and the author, requires only the special case of Theorem 5.1.10 in which the given cover of $\mathbb{P}^1_k$ is purely arithmetic (i.e. of the form $\mathbb{P}^1_\ell$; this was the case considered in [Po4] and [HJ1]). Namely, under the hypotheses of Theorem 5.2.1 above, we may write $A = \ell^{\oplus(G:H)}$, where ℓ is an H-Galois field extension of k for some subgroup $H \subset G$. Let $\Gamma = G \rtimes H$, where the semidirect product is formed with respect to the conjugation action of H on G. Thus there is a surjection $f : \Gamma \to H$ (given by second projection) with a splitting s (given by second inclusion). Consider the H-Galois cover $Y \to X$, where $X = \mathbb{P}^1_k$ and $Y = \mathbb{P}^1_\ell$. Let ξ be a k-point of X. By hypothesis, there is a closed point η on $Y = \mathbb{P}^1_\ell$ whose residue field is ℓ and whose decomposition group over ξ is H. So by Theorem 5.1.10, there is a regular connected G-Galois cover $Z \to Y$ which is totally split over η, such that the composition $Z \to X$ is Γ-Galois and such that $1 \rtimes H = s(H) \subset \Gamma$ is the decomposition group over ξ of some point $\zeta \in Z$ over η. Viewing G as a quotient of Γ via the multiplication map $m : \Gamma = G \rtimes H \to G$, we may consider the intermediate G-Galois cover $W \to X$ (i.e. $W = Z/N$, where $N = \ker m$). It is then straightforward to check that the cover $W \to X$ satisfies the conclusion of Theorem 5.2.1.

The arithmetic lifting result Theorem 5.2.1 above, and the split embedding problem result Theorem 5.2.10, both generalize Theorem 4.3.1 (that one can realize any finite group as a Galois group over a curve defined over a given large field, with a totally split fibre). In fact, those two generalizations can themselves be simultaneously generalized, by the following joint result of F. Pop and the author, concerning the solvability of a split embedding problem with a prescribed fibre. We first introduce some terminology.

As in Theorem 5.1.10, let X be a geometrically irreducible smooth curve over a field k, let $f : \Gamma \to G$ be a surjection of finite groups, and let $Y \to X$ be a G-Galois connected branched cover of smooth curves. Let ξ be an unramified k-point of X, and let η be a closed point of Y over ξ with decomposition group $G_1 \subset G$ and residue field $\ell \supset k$. Let Γ_1 be a subgroup of Γ such that $f(\Gamma_1) = G_1$, and let λ be a Γ_1-Galois field extension of k that contains ℓ. We say that this data constitutes a *fibred embedding problem* $\mathcal{E}$ for X. The problem $\mathcal{E}$ is *split* if f has a section s. A *proper solution* to a fibred embedding problem $\mathcal{E}$ as above consists of a smooth connected Γ-Galois branched cover $Z \to X$ that dominates the G-Galois cover $Y \to X$, such that there is a closed point ζ of Z over η which has residue field λ and whose decomposition group over ξ is $\Gamma_1 \subset \Gamma$. A solution to $\mathcal{E}$ is *regular* if $Z \to Y$ is regular (i.e. the algebraic closures of k in the function fields of Y and Z are equal.

Theorem 5.2.3. *Let k be a large field, let X be a geometrically irreducible smooth k-curve, and consider a fibred split embedding problem $\mathcal{E}$ as above, with data $f : \Gamma \to G$, s, $Y \to X$, $\xi \in X$, $\eta \in Y$, $G_1 \subset G$, $\lambda \supset \ell \supset k$. Assume that $\Gamma_1 = \mathrm{Gal}(\lambda/k)$ contains $s(G_1)$. Let k' be the algebraic closure of k in the function field of Y, let $X' = X \times_k k'$ and let $E = \mathrm{Gal}(Y/X') \subset G$. Assume that $s(E)$ commutes with $N_1 = \ker(f : \Gamma_1 \to G_1)$. Then $\mathcal{E}$ has a proper regular solution.*

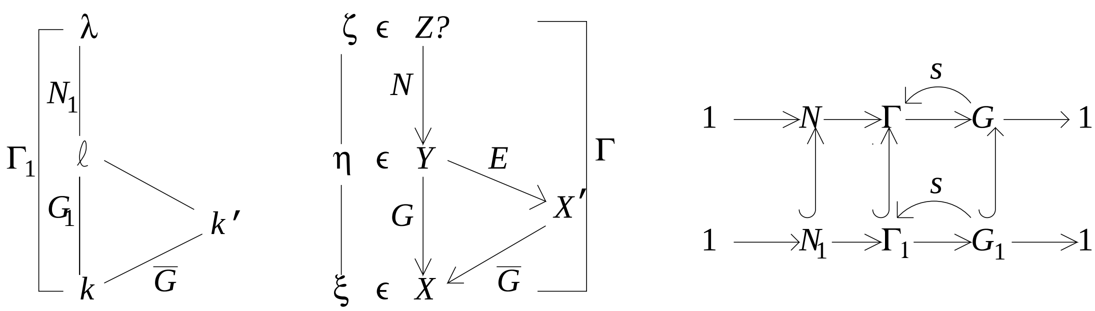

Figure 5.2.4: *The set-up in the statement of Theorem 5.2.3.*

In other words, given a split embedding problem for a curve over a large field, there is a proper regular solution with a given fibre, assuming appropriate hypothesis (on Γ_1,

E and N_1). Taking the special case $\Gamma_1 = G_1$ in Theorem 5.2.3 (i.e. taking $N_1 = 1$), one recovers Theorem 5.1.10. And taking the special case $G = 1$ in Theorem 5.2.3, one recovers Theorem 5.2.1 above. (Note that the "G" in Theorem 5.2.1 corresponds to the group Γ in Theorem 5.2.3. Also, the "A" in 5.2.1 is $\mathrm{Ind}_{\Gamma_1}^{\Gamma} \lambda = \lambda^{\oplus(\Gamma:\Gamma_1)}$, in the notation of 5.1.3.) More generally, taking $E = 1$ in Theorem 5.2.3 (but not necessarily taking G to be trivial), one obtains the result in the case that the given cover $Y \to X$ is purely arithmetic, i.e. of the form $Y = X \times_k k'$. The result in that case is a generalization of Theorem 5.2.1 — viz. instead of requiring the desired cover $Z \to X$ in Theorem 5.2.1 to be regular, it can be chosen so that the algebraic closure of k in the function field of Z is a given subfield k' of A that is Galois over k (and also X need not be $\mathbb{P}^1$). Note that in each of these special cases, the hypothesis on $s(E)$ commuting with N_1 is automatically satisfied, because either E or N_1 is trivial in each case. (On the other hand, the condition $\Gamma_1 \supset s(G_1)$ is still assumed.)

Theorem 5.2.3, like Theorem 5.2.1 above, can in fact be deduced from Theorem 5.1.10, by a strengthening of the proof of Theorem 5.2.1 given above:

Proof of Theorem 5.2.3. The G-Galois cover $Y \to X$ factors as $Y \to X' \to X$, where the E-Galois cover $Y \to X'$ is regular, and $X' \to X$ is purely arithmetic (induced by extension of constants from k to k'). Let $\bar{G} = \mathrm{Gal}(k'/k)$; we may then identify $\mathrm{Gal}(X'/X) = G/E$ with $\bar{G}$. For any field F containing k', let $X_F = X' \times_{k'} F = X \times_k F$ and let $Y_F = Y \times_{k'} F$. So we may identify $E = \mathrm{Gal}(Y_\ell/X_\ell) = \mathrm{Gal}(Y_\lambda/X_\lambda)$; and $\bar{G}$ is a quotient of $G_1 = \mathrm{Gal}(X_\ell/X)$. Since $Y_\ell = X_\ell \times_{X'} Y$, it follows that $\mathrm{Gal}(Y_\ell/X) = G_1 \times_{\bar{G}} G$ (fibre product of groups); similarly $\mathrm{Gal}(Y_\lambda/X) = \Gamma_1 \times_{\bar{G}} G$, and $Y \to X$ is the subcover of $Y_\lambda \to X$ corresponding to the second projection map $G_1 \times_{\bar{G}} G \to G$. (See Figure 5.2.5.)

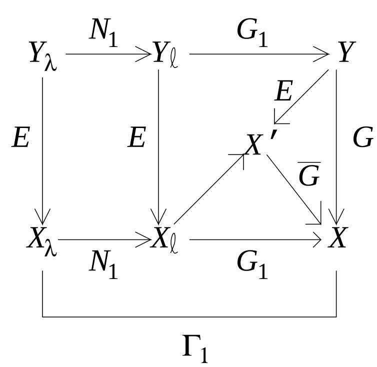

Figure 5.2.5: *The situation in the proof of Theorem 5.2.3.*

Let ξ_ℓ be the unique closed point of X_ℓ over $\xi \in X$. Then $\xi_\ell \in X_\ell$ and $\eta \in Y$ each have residue field ℓ and decomposition group G_1 over ξ. So the fibre of $Y_\ell \to Y$ over η is totally split, with each point having residue field ℓ; the points of this fibre lie over $\xi_\ell \in X_\ell$ and over $\eta \in Y$; and the local fields of X_ℓ and Y at ξ_ℓ and η (i.e. the fraction fields of the complete local rings) are isomorphic over X'. So at one of the points in this fibre (say η_ℓ), the decomposition group over $\xi \in X$ is equal to the diagonal subgroup $G_1 \times_{G_1} G_1 \subset G_1 \times_{\bar{G}} G$. (At the other points of the fibre, the decomposition group is of the form $\{(g_1, \iota(g_1)) \mid g_1 \in G_1\}$, where ι is an inner automorphism of G_1.) Similarly, there is a point $\eta_\lambda \in Y_\lambda$ over $\eta_\ell \in Y_\ell$ whose residue field is λ and whose decomposition group over $\xi \in X$ is $\Gamma_1 \times_{G_1} G_1 \subset \Gamma_1 \times_{\bar{G}} G$.

Since $E = \ker(G \to \bar{G})$, every element of $\Gamma_1 \times_{\bar{G}} G$ can uniquely be written as $(\gamma_1, f(\gamma_1)e)$, with $\gamma_1 \in \Gamma_1$ and $e \in E$. Consider the map $\sigma : \Gamma_1 \times_{\bar{G}} G \to \Gamma_1 \times_{\bar{G}} \Gamma$ given by $\sigma(\gamma_1, f(\gamma_1)e) = (\gamma_1, \gamma_1 s(e))$. Since $s(E)$ commutes with N_1, direct computation shows that σ is a homomorphism, and hence is a section of $(1, f) : \Gamma_1 \times_{\bar{G}} \Gamma \to \Gamma_1 \times_{\bar{G}} G$.

We may now apply Theorem 5.1.10 to the surjection $(1, f)$ and its section σ, to the cover $Y_\lambda \to X$, to the k-point $\xi \in X$, and to the point $\eta_\lambda \in Y_\lambda$ over ξ with decomposition group $\Gamma_1 \times_{G_1} G_1$. The conclusion of that result is that there is a smooth connected $\Gamma_1 \times_{\bar{G}} \Gamma$-Galois cover $Z_\lambda \to X$ that dominates the $\Gamma_1 \times_{\bar{G}} G$-Galois cover $Y_\lambda \to X$ with $Z_\lambda \to Y_\lambda$ regular, together with a point $\zeta_\lambda \in Z_\lambda$ whose decomposition group over ξ is $\sigma(\Gamma_1 \times_{G_1} G_1) = \Gamma_1 \times_{\Gamma_1} \Gamma_1 = \Delta_{\Gamma_1}$, the diagonal of Γ_1 in $\Gamma_1 \times_{\bar{G}} \Gamma$. Let $Z \to X$ be the intermediate Γ-Galois cover corresponding to the second projection map $\Gamma_1 \times_{\bar{G}} \Gamma \to \Gamma$, and let $\zeta \in Z$ be the image of $\zeta_\lambda \in Z_\lambda$. Then $Z \to X$ dominates the G-Galois cover $Y \to X$; the decomposition group of ζ is $\Gamma_1 \subset \Gamma$ and the residue field is λ; and ζ lies over $\eta \in Y$. So $Z \to X$ and the point ζ define a proper solution to the split embedding problem $\mathcal{E}$. The solution is regular, i.e. $Z \to Y$ is regular, since the pullback $Z_\lambda \to Y_\lambda$ is regular. □

Remark 5.2.6. (a) Theorem 5.2.3 can be regarded as a step toward an "arithmetic Riemann's Existence Theorem" for covers of curves over a large field. Namely, such a result should classify the branched covers of such a curve, in terms of how they fit together (e.g. with respect to embedding problems), and in terms of their arithmetic and their geometry, including information about decomposition groups and inertia groups (the latter of which Theorem 5.2.3 does not discuss).

(b) In Remark 5.1.11(b), it was asked if Theorem 5.1.10 can be generalized, to allow one to require the decomposition group there to be an arbitrary subgroup of Γ that maps isomorphically onto G_1 under f (rather than being required to take $s(G_1)$ for the decomposition group). If it can, then the above proof of Theorem 5.2.3 could be simplified, and the statement of Theorem 5.2.3 could be strengthened. Namely, the subgroup $\Gamma_1 \subset \Gamma$ could be allowed to be chosen more generally, viz. as any subgroup of Γ whose image under f is G_1. And the assumption that $s(E)$ commutes with N_1 could also be dropped — since one could then replace the section σ in the above proof by the section (id, s), while still requiring the decomposition group at ζ_λ to be Δ_{Γ_1}. But on the other hand there might in general be a cohomological obstruction, which would vanish if the containment and commutativity assumptions are retained. □

Section 5.3. Abhyankar's Conjecture and embedding problems.

The main theme in this manuscript has been the use of patching methods to prove results in the direction of Riemann's Existence Theorem for curves that are not necessarily defined over $\mathbb{C}$. Such a result should classify the unramified covers of such a curve U, and in particular provide an explicit description of the fundamental group of U, as a profinite group.

While the full statement of Riemann's Existence Theorem is known only for curves over an algebraically closed field of characteristic 0, partial versions have been discussed above. In particular, if one allows arbitrary branching to occur, there is the Geometric Shafarevich Conjecture (Section 5.1); and if one instead takes U to be the complement of a well chosen branch locus and if one restricts attention to a particular class of covers, then there is the Half Riemann Existence Theorem (Section 4.3).

Another way to weaken Riemann's Existence Theorem is to ask for the set $\pi_A(U)$ of finite Galois groups of unramified covers of U; i.e., for the set of finite quotients of $\pi_1(U)$, up to isomorphism. A *finitely generated* profinite group Π is in fact determined by its set of finite quotients [FJ, Proposition 15.4]; and $\pi_1(U)$ is finitely generated (as a profinite group) if the base field has characteristic 0. But in characteristic p, if U is affine, then $\pi_1(U)$ is *not* finitely generated (see below), and $\pi_A(U)$ does not determine $\pi_1(U)$. In this situation, $\pi_1(U)$ *remains unknown*; but at least $\pi_A(U)$ is known if the base field is algebraically closed. Moreover, $\pi_A(U)$ depends only on the *type* (g, r) of U (where $U = X - S$, with X a smooth connected projective curve of genus $g \geq 0$, and S is a set of $r > 0$ points of X). Namely, this 1957 conjecture of Abhyankar [Ab1] was proven by Raynaud [Ra2] and the author [Ha7] using patching and other methods:

Theorem 5.3.1. (Abhyankar's Conjecture) (Raynaud, Harbater) *Let k be an algebraically closed field of characteristic $p > 0$, and let U be a smooth connected affine curve over k of type (g, r). Then a finite group G is in $\pi_A(U)$ if and only if each prime-to-p quotient of G has a generating set of at most $2g + r - 1$ elements.*

Recall that for complex curves U of type (g, r), a finite group G is in $\pi_A(U)$ if and only if G has a generating set of at most $2g + r - 1$ elements. The same assertion is false in characteristic $p > 0$, e.g. since any affine curve has infinitely many Artin-Schreier covers (cyclic of order p), and hence has Galois groups of the form $(\mathbb{Z}/p\mathbb{Z})^s$ for arbitrarily large s. (This implies the above comment that $\pi_1(U)$ is not finitely generated.) The above theorem can be interpreted as saying that "away from p", the complex result carries over; and that every finite group consistent with this principle must occur as a Galois group over U.

In the theorem, the assertion about *every* prime-to-p quotient of G can be replaced by the same assertion about the *maximal* prime-to-p quotient of G — i.e. the group $\bar{G} := G/p(G)$, where $p(G)$ is the subgroup of G generated by the elements of p-power order (or equivalently, by the p-subgroups of G; or again equivalently, by the Sylow p-subgroups of G).

In the case that $U = \mathbb{A}^1_k$, Theorem 5.3.1 says that $\pi_A(\mathbb{A}^1_k)$ consists precisely of the *quasi-p groups*, viz. the groups G such that $G = p(G)$ (i.e. that are generated by their Sylow p-subgroups). This class of groups includes in particular all p-groups, and all finite simple groups of order divisible by p.

Remark 5.3.2. (a) Prior to Theorem 5.3.1 being proven, Serre had shown a partial result [Se6, Théorème 1]: that if Q is a quasi-p group and if $N \triangleleft Q$ is a solvable normal subgroup of Q such that the (quasi-p) group $\bar{Q} := Q/N$ is a Galois group over $\mathbb{A}^1_k$ (i.e. $\bar{Q} \in \pi_A(\mathbb{A}^1_k)$), then Q is also a Galois group over $\mathbb{A}^1_k$. Due to the solvability assumption, the proof was able to proceed cohomologically, without patching; it relied in particular on the fact that $\pi_1(U)$ is projective (Theorem 5.1.3 above, also due to Serre). Serre's result [Se6, Thm. 1] implied in particular that Theorem 5.3.1 above is true for *solvable* groups over the affine line. Serre's proof actually showed more: that if N is a p-group, then a given $\bar{Q}$-cover $Y \to \mathbb{A}^1$ can be dominated by a Q-cover (i.e. the corresponding p-embedding problem can be properly solved); but that if N has order prime-to-p, then the embedding problem need not have a proper solution (i.e. the asserted Q-Galois cover of $\mathbb{A}^1$ cannot necessarily be chosen so as to dominate the given $\bar{Q}$-Galois cover $Y \to \mathbb{A}^1$).

(b) More generally, by extending the methods of [Se6], the author showed [Ha12] that if U is any affine variety other than a point, over an arbitrary field of characteristic p, then every finite p-embedding problem for $\pi_1(U)$ has a proper solution. Moreover, this solution can be chosen so as to have prescribed local behavior. For example, if $V \subset U$ is a proper closed subset, then the proper solution over U can be chosen so that it restricts to a given weak solution over V. (Cf. Theorem 5.2.3 above, for such fibred embedding problems in a related but somewhat different context.) And if U is a curve, then the proper solution can be chosen so as to restrict to given weak solutions over the fraction fields of the complete local rings at finitely many points.

Proof sketch of Theorem 5.3.1. In the case $U = \mathbb{A}^1$, the theorem was proven by Raynaud [Ra2], using in particular rigid patching methods. The proof proceeded by induction on the order of G, and considered three cases. In Case 1, the group G is assumed to have a non-trivial normal p-subgroup N; and using Serre's result that embedding problems for $\pi_1(U)$ with p-group kernel can be properly solved (Remark (a) above), the desired conclusion for G follows from the corresponding fact for G/N. When not in Case 1, one picks a Sylow p-subgroup P, and considers all the quasi-p subgroups $Q \subset G$ such that $Q \cap P$ is a Sylow p-subgroup of Q. Case 2 is the situation in which these Q's generate G. In this case, by induction each of the Q's is a Galois group over $\mathbb{A}^1$; and using rigid patching it follows that G is also. (Or one could use formal patching for this step, viz. Theorem 3.2.8; see e.g. [HS, Theorem 6].) Case 3 is the remaining case, where Cases 1 and 2 do not apply. Then, one builds a G-Galois branched cover of the line in mixed characteristic having p-power inertia groups. The closed fibre of the semi-stable model is a reducible curve that maps down to a tree of projective lines in characteristic p. Using a careful combinatorial analysis of the situation, it turns out that over one of the terminal components of the tree (a copy of the projective line), one finds an irreducible G-Galois cover that is branched at just one point — and hence is an étale cover of the affine line, as desired. Moreover, by adjusting the cover, we may assume that the inertia groups over infinity (of the corresponding branched cover of $\mathbb{P}^1_k$) are the Sylow p-subgroups of Q. (Namely, by Abhyankar's Lemma, after pulling back by a Kummer cover $y^n - x$, we may assume that the inertia groups over infinity are p-groups. We may then enlarge this inertia to become Sylow, using Remark 5.1.6(d) above.)

The general case of the theorem was proven in [Ha7], by using the above case of the affine line, together with formal patching and embedding problems. (See also the simplified presentation in [Ha13], where more is shown.) For the proof, one first recalls that the result was shown by Grothendieck [Gr5, XIII, Cor. 2.12] in the case that the group is of order prime to p. Using this together with formal patching (Theorem 3.2.8), it is possible to reduce to the key case that $U = \mathbb{A}^1_k - \{0\}$, where $G/p(G)$ is cyclic of prime-to-p order. (For that reduction, one patches a prime-to-p cover of the original curve together with a cyclic-by-p cover of $\mathbb{A}^1_k - \{0\}$, to obtain a cover of the original curve with the desired group.) Once in this case, by group theory one can find a prime-to-p cyclic subgroup $\bar{G} \subset G$ that normalizes a Sylow p-subgroup P of G and that surjects onto $G/p(G)$. Here G is a quotient of the semi-direct product $\Gamma := p(G) \rtimes \bar{G}$ (formed with respect to the conjugation action of $\bar{G}$ on $p(G)$); so replacing G by Γ we may assume that $G = p(G) \rtimes \bar{G}$ with $\bar{G} \approx G/p(G)$. Letting $n = |\bar{G}|$, there is a $\bar{G}$-Galois étale cover $V \to U_K$ given by $y^n = x$, where $K = k((t))$ and $U_K = A^1_K - \{0\}$. Using the proper solvability of p-embedding problems with prescribed local behavior (Remark 5.3.2(b) above), one can obtain a $P \rtimes \bar{G}$-Galois étale cover $\tilde{V} \to U_K$ whose behavior over one of the (unramified) K-points ξ_K of U_K can be given in advance. Specifically, one first considers a $p(G)$-Galois étale cover $W \to \mathbb{A}^1_k$ (given by the first case of the result, with Sylow p-subgroups as inertia over ∞), and restricts to the local field at a ramification point with inertia group P (this being a P-Galois field extension of the local field $K = k((t))$ at ∞ on $\mathbb{P}^1_k$). It is this P-Galois extension of K that one uses for the prescribed local behavior over the K-point ξ_K, in applying the p-embedding result. As a consequence, the $P \rtimes \bar{G}$-Galois cover $\tilde{V} \to U_K$ (near ξ_K) has local compatibility with W (near ∞). This compatibility makes it possible for the two covers $\tilde{V}$ and W to be patched using Theorem 3.2.8 or 3.2.12 (after blowing up; see Examples 3.2.11, 3.2.13, and 4.2.4). As a result we obtain a G-Galois cover of U_K (viz. the generic fibre of a cover of $U_{k[[t]]}$). This cover is irreducible because the Galois groups of $\tilde{V}$ and W (viz. $P \rtimes \bar{G}$ and $p(G)$) together generate G. Since k is algebraically closed, one may specialize from K to k (as in

Corollary 3.3.5) to obtain the desired cover of U. □

Remark 5.3.3. (a) The proof of Theorem 5.3.1 actually shows more, concerning inertia groups: Write $U = X - S$ for a smooth connected projective k-curve X and finite set S, and let $\xi \in S$. Then in the situation of the theorem, the G-Galois étale cover of U may be chosen so that the corresponding branched cover of X is tamely ramified away from ξ. (This was referred to as the "Strong Abhyankar Conjecture" in [Ha7], where it is proven.) Note that it is necessary, in general, to allow at least one wildly ramified point. Namely, if G cannot itself be generated by $2g + r - 1$ elements or fewer, then G is not a Galois group of a tamely ramified cover of X that is étale over U, because the tame fundamental group $\pi_1^{\mathrm{t}}(U)$ is a quotient of the free profinite group on $2g + r - 1$ generators [Gr5, XIII, Cor. 2.12].

(b) It would be even more desirable, along the lines of a possible Riemann's Existence Theorem over k, to determine precisely which subgroups of G can be the inertia groups over the points of S, for a G-Galois cover of a given U (with S as in Remark (a) above). This *problem is open*, however, even in the case that $U = \mathbb{A}^1_k$. In that case, the unique branch point ∞ must be wildly ramified, since there are no non-trivial tamely ramified covers of $\mathbb{A}^1$ (by [Gr5, XIII, Cor. 2.12]). By the general theory of extensions of discrete valuation rings [Se5], any inertia group of a branched cover of a k-curve is of the form $I = P \rtimes C$, where P is a p-group (not necessarily Sylow in the Galois group) and C is cyclic of order prime to p. As noted above, it is known [Ra2] that if P is a Sylow p-subgroup of a quasi-p group Q, then there is a Q-Galois étale cover of $\mathbb{A}^1$ such that P is an inertia group over infinity (and this fact was used in the proof of the general case of Theorem 5.3.1, in order to be able to patch together the $P \rtimes \bar{G}$-cover with the $p(G)$-cover). More generally, for *any* subgroup $I \subset Q$ of the form $P \rtimes C$, a necessary condition for I to be an inertia group over ∞ for a Q-Galois étale cover $Y \to \mathbb{A}^1_k$ is that the conjugates of P generate Q. (For if not, they generate a normal subgroup $N \triangleleft Q$ such that $Y/N \to \mathbb{A}^1$ is a non-trivial tamely ramified cover; but $\mathbb{A}^1$ has no such covers, and this is a contradiction.) Abhyankar has conjectured that the converse holds (i.e. that every $I \subset Q$ satisfying the necessary condition will be an inertia group over infinity, for some Q-Galois étale cover of the line). This *remains open*, although some partial results in this direction have been found by R. Pries and I. Bouw [Pr2], [BP].

(c) The results of Sections 3.3 and 4.3 suggest that Abhyankar's Conjecture may hold for affine curves over *large* fields of characteristic p, not just over algebraically closed fields of characteristic p — since patching is possible over such fields, and various Galois realization results can be extended to these fields. But this generalization of Abhyankar's Conjecture *remains open*. The difficulty is that in the proof of Case 3 of Theorem 5.3.1 for $\mathbb{A}^1_k$, one considers a branched cover of $\mathbb{A}^1_R$, where R is a complete discrete valuation ring of mixed characteristic with residue field k. For such a cover, the semi-stable model might be defined only over a finite extension R' of R (and not over R itself); and the residue field of R' could be strictly larger than k. Thus the construction in the proof might yield only a Galois cover of the k'-line, for some finite extension k' of k.

(d) As noted above before Theorem 5.3.1, for an affine k-curve U, the fundamental group $\pi_1(U)$ is not finitely generated (as a profinite group), and is therefore not determined by $\pi_A(U)$. And indeed, the structure of $\pi_1(U)$ *is unknown*, even for $U = \mathbb{A}^1_k$ (although Theorem 5.3.4 below gives some information about how the finite quotients of π_1 "fit together"). In fact, it is easy to see that $\pi_1(\mathbb{A}^1_k)$ depends on the cardinality of the algebraically closed field k of characteristic p; viz. the p-rank of π_1 is equal to this cardinality (using Artin-Schreier extensions). Moreover, Tamagawa has shown [Tm2] that if k, k' are non-isomorphic *countable* algebraically closed fields of characteristic p with $k = \bar{\mathbb{F}}_p$, then $\pi_1(\mathbb{A}^1_k)$ and $\pi_1(\mathbb{A}^1_{k'})$ are non-isomorphic as profinite groups. (It *is unknown* whether this remains true even if k is chosen strictly larger than $\bar{\mathbb{F}}_p$.) Tamagawa also showed in [Tm2] that if $k = \bar{\mathbb{F}}_p$, then two open subsets of $\mathbb{A}^1_k$ have isomorphic π_1's if and only if they are isomorphic as schemes. More generally, given arbitrary affine curves U, U' over algebraically closed fields k, k' of non-zero characteristic, it is an *open question* whether the condition $\pi_1(U) \approx \pi_1(U')$ implies that $k \approx k'$ and $U \approx U'$. This question, which can be regarded as an algebraically closed analog of Grothendieck's anabelian conjecture for affine curves over finitely generated fields [Gr6], was essentially posed by the author in [Ha8, Question 1.9]; and the results in [Tm2] (which relied on the anabelian conjecture in the finitely generated case [Tm1], [Mo]) can be regarded as the first real progress in this direction.

(e) Theorem 5.3.1 holds only for *affine* curves, and is false for projective curves. Namely, if X is a smooth projective k-curve of genus g, then $\pi_1(X)$ is a quotient of the fundamental group of a smooth projective complex curve of genus g (which has generators $a_1, b_1, \dots, a_g, b_g$ subject to the single relation $\prod[a_i, b_i] = 1$ [Gr5, XIII, Cor. 2.12]). So if $g > 0$ and if Q is a quasi-p group whose minimal generating set has more than $2g$ generators, then Q is not in $\pi_A(X)$. Also, the p-rank of a smooth projective k-curve of genus g is at most g, and so $(\mathbb{Z}/p\mathbb{Z})^{g+1}$ is also not in $\pi_A(X)$. But both Q and $(\mathbb{Z}/p\mathbb{Z})^{g+1}$ trivially have the property that every prime-to-p quotient has at most $2g - 1$ generators (since the only prime-to-p quotient of either group is the trivial group). So both of these groups provide counterexamples to Theorem 5.3.1 over the projective curve X. (In the case of genus 0, we have $X = \mathbb{P}^1_k$, and $\pi_1(X)$ is trivial.)

(f) Another difference between the affine and projective cases concerns the relationship between π_A and π_1. As discussed in Remark (c) above, Theorem 5.3.1 gives π_A but not π_1 for an affine curve, the difficulty being that π_A does not determine π_1 because π_1 of an affine curve is not a finitely generated profinite group. On the other hand, if X is a projective curve, then $\pi_1(X)$ *is* a finitely generated profinite group, and so it is determined by $\pi_A(X)$. Unfortunately, unlike the situation for affine curves, $\pi_A(X)$ is unknown when

X is projective of genus > 1 (cf. Remark (d)), and so this does not provide a way of finding $\pi_1(X)$ in this case. A similar situation holds for the tame fundamental group $\pi_1^{\mathrm{t}}(U)$, where $U = X - S$ is an affine curve (and where the tame fundamental group classifies covers of X that are unramified over U, and at most tamely ramified over S). Namely, this group is also a finitely generated profinite group, and is a quotient of the corresponding fundamental group of a complex curve. But the structure of this group, and the set $\pi_A^{\mathrm{t}}(U)$ of its finite quotients, are *both unknown*, even for $\mathbb{P}_k^1 - \{0, 1, \infty\}$. (Note that $\pi_A^{\mathrm{t}}(\mathbb{P}_k^1 - \{0, 1, \infty\})$ is strictly smaller than the set of Galois groups of covers of $\mathbb{P}_{\mathbb{C}}^1 - \{0, 1, \infty\}$ with prime-to-p inertia, because tamely ramified covers of $\mathbb{P}_k^1$ with given degree and inertia groups will generally have lower p-rank than the corresponding covers of $\mathbb{P}_{\mathbb{C}}^1$ — and hence will have fewer unramified p-covers.) On the other hand, partial information about the structure of $\pi_A(X)$ and $\pi_A^{\mathrm{t}}(U)$ has been found by formal and rigid patching methods ([St1], [HS1], [Sa1]) and by using representation theory to solve embedding problems ([St2], [PS]). □

Following the proof of Theorem 5.3.1, Pop used similar methods to prove a stronger version of the result, in terms of embedding problems:

Theorem 5.3.4. (Pop [Po3]) *Let k be an algebraically closed field of characteristic $p > 0$, and let U be a smooth connected affine curve over k. Then every finite embedding problem for $\pi_1(U)$ that has quasi-p kernel is properly solvable.*

That is, given a finite group Γ and a quasi-p normal subgroup N of Γ, and given a Galois étale cover $V \to U$ with group $G := \Gamma/N$, there is a Galois étale cover $W \to U$ with group Γ that dominates V. Theorem 5.3.1 is contained in the assertion of Theorem 5.3.4, by taking $N = p(\Gamma)$. (On the other hand, Pop's proof of 5.3.4 relies on the fact that 5.3.1 holds in the case $U = \mathbb{A}^1$; his proof then somewhat parallels that of the general case of 5.3.1, though using rigid rather than formal methods, and performing an improved construction in order to obtain the stronger conclusion.) Note that Theorem 5.3.4 provides information about the structure of $\pi_1(U)$ (i.e. how the covers "fit together in towers"), unlike Theorem 5.3.1, which just concerned $\pi_A(U)$ (i.e. what covering groups can exist in isolation).

Actually, Theorem 5.3.4 was stated in [Po3] only for *split* embedding problems with quasi-p kernel. But one can easily deduce the general case from that one, proceeding as in the proof of Theorem 5.1.1, via Theorem 5.1.3 there. See also [CL] and [Sa2], i.e. Chapters 15 and 16 in [BLoR], for more about the proofs of Theorems 5.3.1 and 5.3.4, presented from a rigid point of view. (More about the proof of Theorem 5.3.1 can be found in [Ha9, §3].)

Remark 5.3.5. (a) Theorem 5.3.4 can be generalized from étale covers to tamely ramified covers [Ha13, Theorem 4.4]. Namely, with $G = \Gamma/N$ as above, suppose that $V \to U$ is a tamely ramified G-Galois cover of U with branch locus $B \subset U$. Then there is a Γ-Galois cover $W \to U$ that dominates V, and is tamely ramified over B and étale elsewhere over U. (Note that no assertions are made here, or in Theorem 5.3.4, about the behavior over points in the complement of U in its smooth completion.)

(b) In Theorem 5.3.4, for an embedding problem $\mathcal{E} = (\alpha : \pi_1(U) \to G, f : \Gamma \to G)$, one cannot replace the assumption that $\ker f$ is quasi-p by the assumption that Γ is quasi-p. This follows from Remark 5.3.2(a), concerning Serre's results in [Se6].

(c) Theorems 5.3.1 and 5.3.4 both deal only with *finite* Galois groups and embedding problems. It *is unknown* which *infinite* quasi-p profinite groups can arise as Galois groups, and which embedding problems with *infinite* quasi-p kernel have proper solutions. For example, let G be the free product $\mathbb{Z}_p * \mathbb{Z}_p$ (in the category of profinite groups). This is an infinite quasi-p group, and so every finite quotient of G is a Galois group over $\mathbb{A}^1$. But it *is unknown* whether G itself is a Galois group over $\mathbb{A}^1$ (or equivalently, whether G is a quotient of $\pi_1(\mathbb{A}^1)$). □

Theorem 5.3.4 raises the question of which finite embedding problems for $\pi_1(U)$ are properly solvable, where U is an affine variety (of any dimension) in characteristic p — and in particular, whether every finite embedding problem for U with a quasi-p kernel is properly solvable. For example, one can ask this for affine varieties U of finite type over an algebraically closed field k of characteristic p, i.e. whether Pop's result remains true in higher dimensions.

Abhyankar had previously posed a weaker form of this question as a conjecture, paralleling his conjecture for curves (i.e. Theorem 5.3.1). Namely, in [Ab3], he proposed that if U is the complement of a normal crossing divisor D in $\mathbb{P}^n_k$ (where k is algebraically closed of characteristic p), then $G \in \pi_A(U)$ if and only if $G/p(G) \in \pi_A(U_\mathbb{C})$, where $U_\mathbb{C}$ is an "analogous complex space". That is, if D has irreducible components $D_1, \ldots, D_r$ of degrees $d_1, \ldots, d_r$, then one takes $U_\mathbb{C}$ to be the complement in $\mathbb{P}^n_\mathbb{C}$ of a normal crossing divisor consisting of r components of degrees $d_1, \ldots, d_r$. It is known (by [Za1], [Za3], [Fu2]) that $\pi_1(U_\mathbb{C})$ is the abelian group $A(d_1, \ldots, d_r)$ on generators $g_1, \ldots, g_r$ satisfying $\sum d_i g_i = 0$ (writing additively). It is also known (by [Ab2], [Fu2]) that the prime-to-p groups in $\pi_A(U)$ are precisely the prime-to-p quotients of $A(d_1, \ldots, d_r)$. Thus Abhyankar's conjecture in [Ab3] is a special case of a more general conjecture that $G \in \pi_A(U) \Leftrightarrow G/p(G) \in \pi_A(U)$ for any affine k-variety U of finite type. This in turn would follow from an affirmative answer to the question asked in the previous paragraph.

Abhyankar also posed a local version of this conjecture in [Ab3], viz. that if $U = \operatorname{Spec} k[[x_1, \ldots, x_n]][(x_1 \cdots x_r)^{-1}]$ (where $n > 1$ and $1 \le r \le n$), then a finite group G is in $\pi_A(U)$ if and only if $G/p(G)$ is in $\pi_A(U_\mathbb{C})$; here $U_\mathbb{C} = \operatorname{Spec} \mathbb{C}[[x_1, \ldots, x_n]][(x_1 \cdots x_r)^{-1}]$. (Note that this fails if $r = 0$, since then $\pi_A(U)$ is trivial by Hensel's Lemma. It also fails if $n = 1$, since in that case the only quasi-p groups in $\pi_A(U)$ are p-groups, by the structure of Galois groups over complete discrete valuation fields [Se5].) Now $\pi_A(U_\mathbb{C})$ consists of the finite abelian groups on r generators (via Abhyankar's Lemma; cf. [HP, § 3]), and the prime-to-p groups in $\pi_A(U_{\mathbb{C}})$ are the finite abelian prime-to-p groups on r generators. So this conjecture is again equivalent to asserting that $G \in \pi_A(U) \Leftrightarrow G/p(G) \in \pi_A(U)$.

Abhyankar's higher dimensional global conjecture is easily seen to hold in some special cases, e.g. if D is a union of one or two hyperplanes (since it then reduces immediately to Theorem 5.3.1). Using patching, one can show that the higher dimensional local conjecture holds for $r = 1$ [HS2]. But perhaps surprisingly, both the global and local conjectures fail in general, because some groups that satisfy the conditions of the conjectures nevertheless fail to arise as Galois groups of covers. In particular, the global conjecture fails for $\mathbb{P}^2_k$ minus three lines crossing normally, and the local conjecture fails for $n = r = 2$ [HP]. Thus not every embedding problem with quasi-p kernel can be solved for $\pi_1(U)$, in general.

Remark 5.3.6. The main reason that the higher dimensional conjecture fails in general is that the group-theoretic reduction in the proof of the general case of Theorem 5.3.1 does not work in the more general situation. That is, it is possible that $G/p(G) \in \pi_A(U)$ but that G is not a quotient of a group $\tilde{G}$ of the form $\tilde{G} = p(G) \rtimes \bar{G}$, with $\bar{G}$ a prime-to-p group in $\pi_A(U)$. (Cf. the group-theoretic examples of Guralnick in [HP].) Moreover, even if there is such a $\tilde{G}$, it might not be possible to choose it such that $\bar{G}$ normalizes a Sylow p-subgroup of $p(G)$ (or equivalently, of G), as was done in the proof of Theorem 5.3.1. And in fact, a condition of the above type is *necessary* in order that $G \in \pi_A(U)$, if U is the complement of $x_1 \cdots x_i = 0$ (in either the local or global case; cf. [HP]).

This suggests that a group G should lie in $\pi_A(U)$ if it satisfies these additional conditions, as well as the condition that $G/p(G) \in \pi_A(U)$. One might wish to parallel the proof of the general case of Theorem 5.3.1, using higher dimensional patching (Theorem 3.2.12) together with the result on embedding problems with p-group kernel ([Ha12], which holds in arbitrary dimension). Unfortunately, there is another difficulty: The strategy for curves used that for every quasi-p group Q there is a Q-Galois étale cover of $\mathbb{A}^1_k$ such that the fibre over infinity (of the corresponding branched cover of $\mathbb{P}^1_k$) consists of a disjoint union of points whose inertia groups are Sylow p-subgroups of Q (cf. Case 1 of the proof of Theorem 5.3.1). But the higher dimensional analog of this is false; in fact, for $n > 1$, every branched cover of $\mathbb{P}^n_k$ that is étale over $\mathbb{A}^n_k$ must have the property that its fibre over the hyperplane at infinity is connected [Hrt2, III, Cor. 7.9]. This then interferes with the desired patching, on the overlap. □

One can also consider birational variants of the above questions, in studying the absolute Galois groups of $k_n := k(x_1, \ldots, x_n)$ and $k_n^* := k((x_1, \ldots, x_n))$. Here k is an algebraically closed field of characteristic $p \geq 0$; $n > 1$; and $k((x_1, \ldots, x_n))$ denotes the fraction field of $k[[x_1, \ldots, x_n]]$. Of course every finite group is a Galois group over k_n, since this is true for $k(x_1)$ (see Corollary 3.3.5) and one may base-change to k_n. Also, every finite group is a Galois group over k_n^*, by Example 3.3.2(c). But this does not determine the structure of the absolute Galois groups of k_n and k_n^*.

In the one-dimensional analog, the absolute Galois group of $k(x)$ is a free profinite group (of rank equal to the cardinality of k), by the geometric case of Shafarevich's Conjecture (Section 5.1). But for $n > 1$, the absolute Galois group of k_n has cohomological dimension > 1 [Se4, II, 4.1, Proposition 11], and so is not projective [Se4, I, 3.4, Proposition 16]. That is, not every finite split embedding problem for G_{k_n} has a weak solution; and therefore G_{k_n} is not free.

This can also be seen explicitly as in the following argument, which also applies to k_n^*:

Proposition 5.3.7. *Let k be an algebraically closed field of characteristic $p \geq 0$, let $n > 1$, and let $K = k_n$ or k_n^* as above. Then not every finite embedding problem for the absolute Galois group G_K is weakly solvable. Equivalently, there is a surjection $G \to A$ of finite groups, and an A-Galois field extension K' of K, such that K' is not contained in any H-Galois field extension L of K for any $H \subset G$.*

Proof. First suppose that char $k \neq 2$. Let G be the quaternion group of order 8, and let A be the quotient of G by its center $Z = \{\pm 1\}$. Thus $A = G/Z \approx C_2^2$, say with generators a, b which are commuting involutions. Consider the surjection $G_K \to A$ corresponding to the A-Galois field extension K' given by $u^2 = x_1, v^2 = x_2$. Suppose that this field extension is contained in an H-Galois extension L/K as in the statement of the proposition. Then A is a quotient of H. But no proper subgroup of G surjects onto A; so actually $H = G$.

Let $F = k((x_1)) \cdots ((x_n))$, and let F' [resp. E] be the compositum of F and K' [resp. F and L] in some algebraic closure of F. Thus E is a Galois field extension of F, and its Galois group G' is a subgroup of G. Moreover E contains F', which is an A-Galois field extension of F (being given by $u^2 = x_1, v^2 = x_2$). Thus A is a quotient of G', and hence $G' = G$. But the maximal prime-to-p quotient of the absolute Galois group G_F is abelian [HP, Prop. 2.4], and so G cannot be a Galois group over F (using that $p \neq 2$). This is a contradiction, proving the result in this case.

On the other hand, if char $k = 2$, then one can replace the quaternion group in the above argument by a similar group of order prime to 2. Namely, let ℓ be any odd prime. Then there is a group G of order ℓ^3 whose center Z is cyclic of order ℓ; such that $G/Z \approx C_\ell^2$; and such that no proper subgroup of G surjects onto G/Z. (See [As, 23.13]; such a group is called an *extraspecial* group of order ℓ^3.) The proof then proceeds as before. □

Remark 5.3.8. The above proof also applies to the field $K = k((x_1, \ldots, x_n))(y)$, by using the extension $u^2 = x_1, v^2 = y$. So its absolute Galois group G_K is not projective, and hence not free. (This can alternatively be seen by using [Se4, II, 4.1, Proposition 11]). Note that this field K has the property that every finite group is a Galois group over K (by Theorem 3.3.1), even though G_K is not free or even projective. In fact if $n = 1$, then every finite *split* embedding problem has a proper solution (by Theorem 5.1.9). Thus in this case, once a finite embedding problem has a weak solution, it automatically has a proper solution. In this sense, the absolute Galois group of $k((x))(y)$ is "as close as possible to being free" without being projective. □

Motivated by the above proposition and remark, it would be desirable to know whether the absolute Galois groups of $k_n := k(x_1, \ldots, x_n)$ and $k_n^* := k((x_1, \ldots, x_n))$ are "as close as possible to being free" without being projective. (Here k is still algebraically closed and $n > 1$.) In other words, does every finite split embedding problem for G_{k_n} or $G_{k_n^*}$ have a proper solution? The former case can be regarded as a birational analog of the question asked previously concerning quasi-p embedding problems in the higher dimensional Abhyankar Conjecture; it can also be considered a weak version of a higher dimensional geometric Shafarevich Conjecture. In this case, the question *remains open*, even for $\mathbb{C}(x, y)$. In the latter case, the answer is affirmative for $\mathbb{C}((x, y))$, as the following result shows. The proof follows a strategy from [HS2], viz. blowing up Spec $\mathbb{C}[[x, y]]$ at the closed point to obtain a more global object, and then patching (here using Theorem 3.2.12).

Theorem 5.3.9. *Every finite split embedding problem over $\mathbb{C}((x, y))$ has a proper solution.*

Proof. Let L be a finite Galois extension of $\mathbb{C}((x, y))$, with group G, and let Γ be a semi-direct product $N \rtimes G$ for some finite group N. Let $R = \mathbb{C}[[x, y]]$ and let S be the integral closure of R in L, and write $X^* := \operatorname{Spec} R$ and $Z^* := \operatorname{Spec} S$. We want to show that there is an irreducible normal Γ-Galois branched cover $W^* \to X^*$ that dominates the G-Galois branched cover $Z^* \to X^*$.

Case 1: S/R is ramified only over $(x = 0)$.

Let n be the ramification index of $Z^* \to X^*$ over the generic point of $(x = 0)$, and consider the normalized pullback of $Z^* \to X^*$ via Spec $R[z]/(z^n - x) \to X^*$. By Abhyankar's Lemma and Purity of Branch Locus, the resulting cover of Spec $R[z]/(z^n - x) =$ Spec $\mathbb{C}[[z, y]]$ is unramified and hence trivial. Thus $S \approx R[z]/(z^n - x)$, and G is cyclic of order n.

Now consider the projective y-line over $\mathbb{C}((x))$, and the G-Galois cover of this line $Z^\circ \to \mathbb{P}^1_{\mathbb{C}((x))}$ that is given by the constant extension $z^n = x$. Applying Pop's Theorem 5.1.10 to the split embedding problem given by this cover and the group homomorphism $\Gamma \to G$, we obtain a regular irreducible (hence geometrically irreducible) Γ-Galois cover $W^\circ \to \mathbb{P}^1_{\mathbb{C}((x))}$ that dominates $Z^\circ \to \mathbb{P}^1_{\mathbb{C}((x))}$ and is such that $W^\circ \to Z^\circ$ is totally split over $y = \infty$. Consider the normalization W of $\mathbb{P}^1_{\mathbb{C}[[x]]}$ in W°; this is a Γ-Galois cover of $\mathbb{P}^1_{\mathbb{C}[[x]]}$ that dominates Z, the normalization of $\mathbb{P}^1_{\mathbb{C}[[x]]}$ in Z°. The branch locus of $W \to \mathbb{P}^1_{\mathbb{C}[[x]]}$ consists of finitely many irreducible components. After a change of variables $y' = x^m y$ on $\mathbb{P}^1_{\mathbb{C}((x))}$, we may assume that every branch component passes through the closed point (x, y), and that no branch component other than (x) passes through any other point on the closed fibre of $\mathbb{P}^1_{\mathbb{C}[[x]]}$. Again using Abhyankar's Lemma and Purity of Branch Locus, we conclude that the restriction of W over $\mathbb{C}[y^{-1}][[x]]$ is a disjoint union of components given by $w^N = x$ for some multiple N of n, with each reduced component of the closed fibre of W being a complex line. Since $W^\circ \to Z^\circ$ is split over $y = \infty$, it follows that $N = n$. Thus the pullback of $W \to Z$ over $\mathbb{C}[y^{-1}][[x]]$ is a trivial cover.

Since the general fibre of $W \to \mathbb{P}^1_{\mathbb{C}[[x]]}$ is geometrically irreducible, the closed fibre is connected, by Zariski's Connectedness Theorem [Hrt2, III, Cor. 11.3]. So by the previous paragraph, the components of the closed fibre of W all meet at a single point over $(x = y = 0)$. So the pullback W^* of $W \to \mathbb{P}^1_{\mathbb{C}[[x]]}$ over $\operatorname{Spec}\mathbb{C}[[x, y]]$ is connected; and since W is normal, it follows that W^* is also normal and hence is irreducible. So $W^* \to \operatorname{Spec}\mathbb{C}[[x, y]]$ is an irreducible Γ-Galois cover. Moreover W^*/N is isomorphic to $\operatorname{Spec} S$ over $\operatorname{Spec} R$, since each is given by $z^n = x$. So it is a proper solution to the given embedding problem.

Note that in this case, the proof shows more: that G is the cyclic group C_n, and that over $\mathbb{C}((y))[[x]]$, the pullback of $W^* \to Z^*$ is trivial (since the same is true over $\mathbb{C}[y^{-1}][[x]]$).

Case 2: General case.

Let B be the branch locus of $Z^* \to X^*$, and let C be the tangent cone to B at the closed point (x, y). Thus C is a union of finitely many "lines" $(ax + by)$ through (x, y) in X^*. After a change of variables of the form $y' = y - cx$, we may assume that C does not contain the locus of $(y = 0)$.

Let $\tilde{X}$ be the blow-up of X^* at the closed point (x, y). Let E be the exceptional divisor; this is a copy of $\mathbb{P}^1_{\mathbb{C}}$, with parameter $t = y/x$. Let $\tau \in T$ be the closed point $(x = y = t = 0)$; this is where E meets the proper transform of $(y = 0)$. Let $\tilde{Z} \to \tilde{X}$ be the normalized pullback of $Z^* \to X^*$. By the previous paragraph, this is unramified in a neighborhood of τ except possibly along E. So over the complete local ring $\hat{\mathcal{O}}_{\tilde{X},\tau} = \mathbb{C}[[x, t]]$ of τ in $\tilde{X}$, the pullback $\tilde{Z}^* \to \tilde{X}^* := \operatorname{Spec}\hat{\mathcal{O}}_{\tilde{X},\tau}$ of $\tilde{Z} \to \tilde{X}$ is ramified only over $(x = 0)$. We will construct a Γ-Galois cover $\tilde{W} \to \tilde{X}$ dominating $\tilde{Z}$. (See Figs. 5.3.10 and 5.3.11.)

Let $\tilde{Z}^*_0$ be a connected component of $\tilde{Z}^*$. Thus $\tilde{Z}^*_0 \to \tilde{X}^*$ is Galois with group $G_0 \subset G$, and $\tilde{Z}^* = \operatorname{Ind}_{G_0}^G \tilde{Z}^*_0$. Let $\Gamma_0 \subset \Gamma$ be the subgroup generated by N and G_0 (identifying N with $N \rtimes 1 \subset \Gamma$, and G with $1 \rtimes G \subset \Gamma$). Thus $\Gamma_0 = N \rtimes G_0$. By Case 1, there is a regular irreducible normal Γ_0-Galois cover $\tilde{W}^*_0 \to \tilde{X}^*$ that dominates $\tilde{Z}^*_0$, and such that the pullback of $\tilde{W}^*_0 \to \tilde{Z}^*_0$ over $\tilde{X}' = \operatorname{Spec}\mathbb{C}((t))[[x]]$ is trivial. That is, $\tilde{W}'_0 := \tilde{W}^*_0 \times_{\tilde{X}^*} \tilde{X}'$ is the trivial N-Galois cover of $\tilde{Z}'_0 := \tilde{Z}^*_0 \times_{\tilde{X}^*} \tilde{X}'$, and the Γ_0-Galois cover $\tilde{W}'_0 \to \tilde{X}'$ is just $\operatorname{Ind}_{G_0}^{\Gamma_0} \tilde{Z}'_0$. Thus the Γ-Galois cover $\tilde{W}^* := \operatorname{Ind}_{\Gamma_0}^{\Gamma} \tilde{W}^*_0 \to \tilde{X}^*$ has the property that its pullback $\tilde{W}' := \tilde{W}^* \times_{\tilde{X}^*} \tilde{X}'$ is just $\operatorname{Ind}_{G_0}^{\Gamma} \tilde{Z}'_0 = \operatorname{Ind}_G^{\Gamma} \tilde{Z}'$, where $\tilde{Z}' = \operatorname{Ind}_{G_0}^G \tilde{Z}'_0$ is the pullback $\tilde{Z}^* \times_{\tilde{X}^*} \tilde{X}'$.

Let $U = E - \{\tau\}$, and let X' be the completion of $\tilde{X}$ along U; i.e. $X' = \operatorname{Spec}\mathbb{C}[s][[y]]$, where $s = x/y = 1/t$. Let $Z' = \tilde{Z} \times_{\tilde{X}} X'$, and let $W' = \operatorname{Ind}_G^{\Gamma} Z'$. Thus the pullback $Z' \times_{X'} \tilde{X}'$ can be identified with $\tilde{Z}' = \operatorname{Ind}_{G_0}^G \tilde{Z}'_0$ as G-Galois covers of $\tilde{X}'$; and the pullback

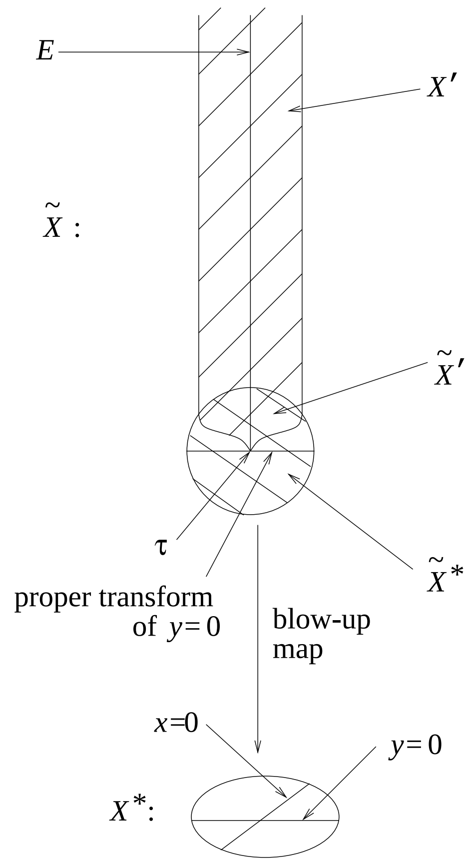

Figure 5.3.10: *Picture of the situation in Case 2 of the proof of Theorem 5.3.9. The space X^*, shown as a disc, is blown up, producing $\tilde{X}$, with an exceptional divisor E (which meets the proper transform of $y = 0$ at the point τ). The proof proceeds by building the desired cover over formal patches: X', the completion along $E - \{\tau\}$; and $\tilde{X}^*$, the completion at τ. These two patches are shaded above, with the doubly shaded region $\tilde{X}'$ being the "overlap".*

$W' \times_{X'} \tilde{X}'$ can be identified with $\tilde{W}' = \mathrm{Ind}_{G_0}^{\Gamma} \tilde{Z}'_0$, as Γ-Galois covers of $\tilde{X}'$.

Now apply the formal patching result Theorem 3.2.12, with $A = R$, $V = \tilde{V} = \tilde{X}$, $f =$ identity, and the finite set of closed points of V being just $\{\tau\}$. Using the equivalence of categories for covers, we conclude that there is a unique Γ-Galois cover $\tilde{W} \to \tilde{X}$ whose pullbacks to $\tilde{X}^*$ and to X' are given respectively by $\tilde{W}^* = \mathrm{Ind}_{\Gamma_0}^{\Gamma} \tilde{W}_0^* \to \tilde{X}^*$ and $W' \to X'$, compatibly with the above identification over $\tilde{X}'$ with $\tilde{W}' = \mathrm{Ind}_{G_0}^{\Gamma} \tilde{Z}'_0 \to \tilde{X}'$. The quotient $\tilde{W}/N$ can be identified with $\tilde{Z}$ as a G-Galois cover, since we have compatible identifications of their pullbacks over $\tilde{X}^*$, X', and their "overlap" $\tilde{X}'$, and because of the uniqueness assertion of the patching theorem. Also, $\tilde{W}$ is normal, since normality is a local property and since $\tilde{W}^*$ and W' are normal. Let $\tilde{W}_0$ be the connected component of $\tilde{W}$ whose pullback to $\tilde{X}^*$ contains $\tilde{W}_0^*$. Its Galois group Γ_1 over $\tilde{X}$ surjects onto $G = \mathrm{Gal}(\tilde{Z}/\tilde{X})$, and Γ_1 contains $\mathrm{Gal}(\tilde{W}_0^*/\tilde{X}^*) = \Gamma_0 \supset N \rtimes 1$. So Γ_1 is all of Γ, and so $\tilde{W}_0 = \tilde{W}$. That is,

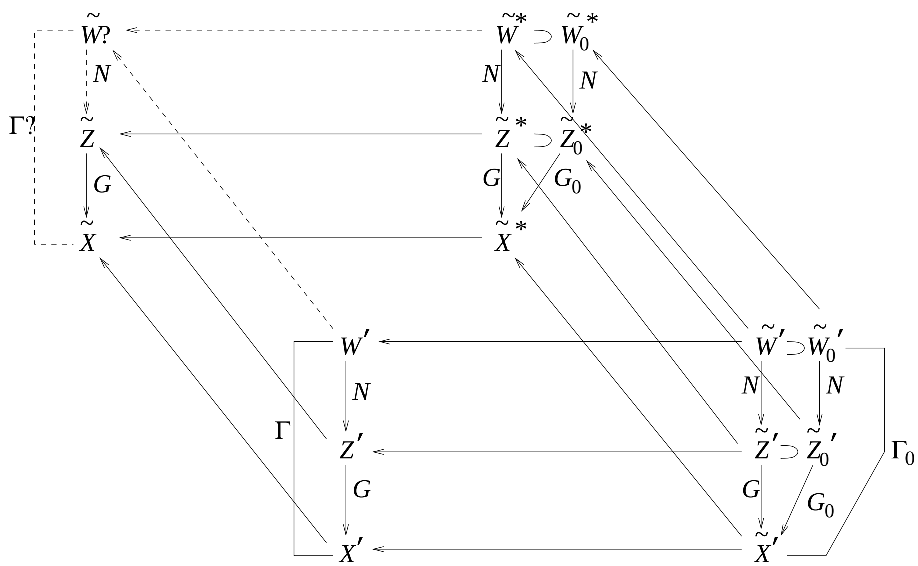

Figure 5.3.11: *Diagram illustrating the patching situation in Case 2 of the proof of Theorem 5.3.9. In order to construct a* Γ*-Galois cover* $\tilde{W} \to \tilde{X}$*, the restrictions* $W' \to X'$ *and* $\tilde{W}^* \to \tilde{X}^*$ *are first constructed, so as to induce the same "overlap"* $\tilde{W}' \to \tilde{X}'$*. Formal patching is then used to obtain* $\tilde{W}$*.*

$\tilde{W}$ is connected, and hence irreducible (being normal).

Now let $W^* \to X^*$ be the normalization of X^* in $\tilde{W}$. This is then a connected normal Γ-Galois cover that dominates Z^* (since Z^* is the normalization of X^* in $\tilde{Z}$). It is irreducible because it is connected and normal. So it provides a proper solution to the given embedding problem. □

Remark 5.3.12. (a) Observe that the above theorem would also follow from Theorem 5.1.9, if it were known that $\mathbb{C}((x,y))$ is large. (Namely, given a split embedding problem over $\mathbb{C}((x,y))$, one could apply Theorem 5.1.9 to the induced constant split embedding problem over $\mathbb{C}((x,y))(t)$; and then one could specialize the proper solution to an extension of $\mathbb{C}((x,y))$, using that that field is separably Hilbertian by Weissauer's Theorem [FJ, Theorem 14.17].) But it *is unknown* whether $\mathbb{C}((x,y))$ is large. (Cf. Example 3.3.7(d).)

(b) It would be desirable to generalize the above result, e.g. by allowing more Laurent series variables, and by replacing $\mathbb{C}$ by an algebraically closed field of arbitrary characteristic (or even by an arbitrary large field). Note that the above proof uses Kummer theory and Abhyankar's Lemma, and so one would somehow need to treat the case of wild ramification. □

The ultimate goal remains that of proving a full analog of Riemann's Existence Theorem — classifying covers via their Galois groups and inertia groups, and determining how they fit together into the tower of covers. This goal, however, has so far been achieved in full only for curves over algebraically closed fields of characteristic 0 (where it is deduced from the complex result, which relied on topological methods). As seen above, the weaker goal of finding π_1 as a profinite group, and finding absolute Galois groups of function fields, also remains open in most cases, although the absolute Galois group of the function field is known for curves over algebraically closed fields (Theorem 5.1.1), and partial results are known for other fields (e.g. Theorem 5.1.9, 5.3.4, and 5.3.9). The still weaker, but difficult, goal of finding π_A has been achieved for affine curves over algebraically closed fields of arbitrary characteristic (Theorem 5.3.1 above), and the goal of finding which groups are Galois groups over the function field is settled for curves over large fields and fraction fields of complete local rings (Theorems 3.3.1 and 3.3.6) and partially for curves over finite fields (Proposition 3.3.9). But the structure of the absolute Galois groups of most familiar fields remains undetermined (e.g. for number fields and function fields of several variables over $\mathbb{C}$), and the inverse Galois problem over $\mathbb{Q}$ remains open. The strategy used in Theorem 5.3.9 above, though, may suggest an approach to higher dimensional geometric fields; and Remark 3.3.8(a) suggests a possible strategy in the number field case. These and other patching methods described here may help further attack these open problems, on the way toward achieving a full generalization of Riemann's Existence Theorem.

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