Patching and Galois theory

David Harbater*
Dept. of Mathematics, University of Pennsylvania

Abstract: Galois theory over $\mathbb{C}(x)$ is well-understood as a consequence of Riemann's Existence Theorem, which classifies the algebraic branched covers of the complex projective line. The proof of that theorem uses analytic and topological methods, including the ability to construct covers locally and to patch them together on the overlaps. To study the Galois extensions of $k(x)$ for other fields k, one would like to have an analog of Riemann's Existence Theorem for curves over k. Such a result remains out of reach, but partial results in this direction can be proven using patching methods that are analogous to complex patching, and which apply in more general contexts. One such method is formal patching, in which formal completions of schemes play the role of small open sets. Another such method is rigid patching, in which non-archimedean discs are used. Both methods yield the realization of arbitrary finite groups as Galois groups over $k(x)$ for various classes of fields k, as well as more precise results concerning embedding problems and fundamental groups. This manuscript describes such patching methods and their relationships to the classical approach over $\mathbb{C}$, and shows how these methods provide results about Galois groups and fundamental groups.

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Section 1: Introduction

This manuscript discusses patching methods and their use in the study of Galois groups and fundamental groups. There is a particular focus on Riemann's Existence Theorem and the inverse Galois problem, and their generalizations (both known and conjectured). This first section provides an introduction, beginning with a brief overview of the topic in Section 1.1. More background about Galois groups and fundamental groups is provided in Section 1.2. Section 1.3 then discusses the overall structure of the paper, briefly indicating the content of each later section.

Section 1.1. Overview.

Galois theory is algebraic in its origins, arising from the study of polynomial equations and their solvability. But it has always had intimate connections to geometry. This is evidenced, for example, when one speaks of an "icosahedral Galois extension" — meaning a field extension whose Galois group is A_5, the symmetry group of an icosahedron.

Much progress in Galois theory relies on connections to geometry, particularly on the parallel between Galois groups and the theory of covering spaces and fundamental groups in topology. This parallel is more than an analogy, with the group-theoretic and topological approaches being brought together in the context of algebraic geometry. The realization of all finite groups as Galois groups over $\mathbb{C}(x)$ is an early example of this approach.

In recent years, this approach has drawn heavily on the notion of *patching*, i.e. on "cut-and-paste" constructions that build covers locally and then combine them to form a global cover with desired symmetries. Classically this could be performed only for spaces defined over $\mathbb{C}$, in order to study Galois groups over fields like $\mathbb{C}(x)$. But by carrying complex analytic methods over to other settings — most notably via formal and rigid geometry — results in Galois theory have now been proven for a broad array of rings and fields by means of patching methods.

This paper provides an overview of this approach to Galois theory via patching, both in classical and non-classical contexts. A key theme in both contexts is *Riemann's Existence Theorem*. In the complex case, that result provides a classification of the finite Galois extensions of the field $\mathbb{C}(x)$ and more generally of the function field K of any Riemann surface X. (In the case $K = \mathbb{C}(x)$, X is the Riemann sphere, i.e. the complex projective line $\mathbb{P}^1_{\mathbb{C}}$). This classification relies on the correspondence between these field extensions and the branched covers of X, and on the classification of the branched covers of X with given branch locus B. This correspondence between field extensions and covers in turn proceeds by proving the equivalence of covers in the algebraic, analytic, and topological senses, and then relying on topology to classify the covering spaces of the complement of a finite set $B \subset X$. In demonstrating this equivalence, one regards branched covers as being given locally over discs, where the cover breaks up into a union of cyclic components, and where agreement on the overlaps is given in order to define the cover globally. Using "complex patching" (specifically, Serre's result GAGA), such an analytic cover in fact arises from a cover of complex algebraic curves, given by polynomial equations.

This patching approach proves that every finite group is a Galois group over $\mathbb{C}(x)$ and more generally over K as above, and it provides the structure of the absolute Galois group of the field. Moreover, if one fixes a finite set of points B, the approach shows which finite groups are Galois groups of covers with that branch locus, and how those groups fit together as quotients of the fundamental group of $X - B$.

The success of this approach made it desirable to carry it over to other settings, in order to study the Galois theory of other fields K — e.g. $K = k(x)$ where k is a field other than $\mathbb{C}$, or even arithmetic fields K. In order to do this, one needs to carry over the notion of patching. This is, if K is the function field of a scheme X (e.g. $X = \text{Spec } R$, where R is an integral domain whose fraction field is K), then one would like to construct covers of X locally, with agreement on the overlaps, and then be able to assert the existence of a global cover that induces this local data. The difficulty is that one needs an appropriate topology on X. Of course, there is the Zariski topology, but that is too coarse. Namely, if U is a Zariski open subset of an irreducible scheme X, then giving a branched cover $V \to U$ is already tantamount to giving a cover over all of X, since $X - U$ is just a closed subset of lower dimension (and one can take the normalization of X in V). Instead, one needs a finer notion, which behaves more like the complex metric topology in the classical setting, and where one can speak of the ring of "holomorphic functions" on any open set in this topology.

In this manuscript, after discussing the classical form of Riemann's Existence Theorem via GAGA for complex curves, we present two refinements of the Zariski topology that allow patching constructions to take place in many (but not all) more general settings. These approaches of *formal* and *rigid patching* are roughly equivalent to each other, but they developed separately. Each relies on an analog of GAGA, whose proof parallels the proof of Serre's original GAGA. These approaches are then used to realize finite groups as Galois groups over various function fields, and to show how these groups fit together in the tower of all extensions of the field (corresponding to information about the structure of the absolute Galois group — or of a fundamental group, if the branch locus is fixed).

Underlying this entire approach is the ability to pass back and forth between algebra and geometry. This ability is based on the relationship between field extensions and covers, with Galois groups of field extensions corresponding to groups of deck transformations of covers, and with absolute Galois groups of fields playing a role analogous to fundamental groups of spaces. This relationship is reviewed in Section 1.2 below, where basic terminology is also introduced. (Readers who are familiar with this material may wish to skip §1.2.) Section 1.3 then describes the structure of this paper as a whole.

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Section 1.2. Galois groups and fundamental groups.

Traditionally, Galois theory studies field extensions by means of symmetry groups (viz. their Galois groups). Covering spaces can also be studied using symmetry groups (viz. their groups of deck transformations). In fact, the two situations are quite parallel.

In the algebraic situation, the basic objects of study are field extensions $L \supset K$. To such an extension, one associates its symmetry group, viz. the Galois group $\mathrm{Gal}(L/K)$, consisting of automorphisms of L that fix all the elements of K. If L is a finite extension of K of degree $[L : K] = n$, then the order of the Galois group is at most n; and the extension is *Galois* if the order is exactly n, i.e. if the extension is as symmetric as possible. (For finite extensions, this is equivalent to the usual definition in terms of being normal and separable.)

In the geometric situation, one considers topological covering spaces $f : Y \to X$. There is the associated symmetry group $\mathrm{Aut}(Y/X)$ of deck transformations, consisting of self-homeomorphisms ϕ of Y that preserve the map f (i.e. such that $f \circ \phi = f$). If $Y \to X$ is a finite cover of degree n, then the order of the covering group is at most n; and the extension is *"regular"* (in the terminology of topology) if the order is exactly n, i.e. if the cover is as symmetric as possible. To emphasize the parallel, we will refer to the covering group $\mathrm{Aut}(Y/X)$ as the *Galois group* of the cover, and will call the symmetric covers *Galois* rather than "regular" (and will instead reserve the latter word for another meaning that is used in connection with covers in arithmetic algebraic geometry).

The parallel extends further: If L is a finite Galois extension of K with Galois group G, then the intermediate extensions M of K are in bijection with the subgroups H of G; namely a subgroup H corresponds to its fixed field $M = L^H$, and an intermediate field M corresponds to the Galois group $H = \mathrm{Gal}(L/M) \subset G$. Moreover, M is Galois over K if and only if H is normal in G; and in that case $\mathrm{Gal}(M/K) = G/H$. Similarly, if $Y \to X$ is a finite Galois cover with group G, then the intermediate covers $Z \to X$ are in bijection with the subgroups H of G; namely a subgroup H corresponds to the quotient space $Z = Y/H$, and an intermediate cover $Z \to X$ corresponds to the Galois group $H = \mathrm{Gal}(Y/Z) \subset G$. Moreover, $Z \to X$ is Galois if and only if H is normal in G; and in that case $\mathrm{Gal}(Z/X) = G/H$. Thus in both the algebraic and geometric contexts, there is a "Fundamental Theorem of Galois Theory".

The reason behind this parallel can be illustrated by a simple example. Let X and Y be two copies of $\mathbb{C}-\{0\}$, with complex parameters x and y respectively, and let $f : Y \to X$ be given by $x = y^n$, for some integer $n > 1$. This is a degree n Galois cover whose Galois group is the cyclic group C_n of order n, where the generator of the Galois group takes

$y \mapsto \zeta_n y$ (with $\zeta_n \in \mathbb{C}$ being a primitive nth root of unity). If one views X and Y not just as topological spaces, but as copies of the affine variety $\mathbb{A}^1_{\mathbb{C}} - \{0\}$, then f is the morphism corresponding to the inclusion of function fields, $\mathbb{C}(x) \hookrightarrow \mathbb{C}(y)$, given by $x \mapsto y^n$. This inclusion is a Galois field extension of degree n whose Galois group is C_n, whose generator acts by $y \mapsto \zeta_n y$. (Strictly speaking, if the Galois group of covers acts on the left, then the Galois group of fields acts on the right.)

In this example, the Fundamental Theorem for covering spaces implies the Fundamental Theorem for the extension of function fields, since intermediate covers $Z \to X$ correspond to intermediate field extensions $M \supset \mathbb{C}(x)$ (where M is the function field of Z). More generally, one can consider Galois covers of schemes that are not necessarily defined over $\mathbb{C}$, and in that context have both algebraic and geometric forms of Galois theory.

In order to extend the idea of covering space to this setting, one needs to define a class of finite morphisms $f : Y \to X$ that generalizes the class of covering spaces (in the complex metric topology) for complex varieties. The condition of being a covering space in the Zariski topology does not do this, since an irreducible scheme X will not have any irreducible covers in this sense, other than the identity map. (Namely, if $Y \to X$ is evenly covered over a dense open set, then it is a disjoint union of copies of X, globally.) Instead one uses the notion of *finite étale covers*, i.e. finite morphisms $f : Y \to X$ such that locally at every point of Y, the scheme Y is given over X by m polynomials $f_1, \ldots, f_m$ in m variables $y_1, \ldots, y_m$, and such that the Jacobian determinant $(\partial f_i / \partial y_j)$ is locally invertible. The point is that for spaces over $\mathbb{C}$, this condition is equivalent to f having a local section near every point (by the Inverse Function Theorem, where "local section" means in the complex metric); and for a finite morphism, satisfying this latter condition is equivalent to being a finite covering space (in the complex metric sense).

For finite étale covers of an irreducible scheme X, one then has a Fundamental Theorem of Galois Theory as above. If one restricts to complex varieties, one obtains the geometric situation discussed above. And if one restricts to X of the form Spec K (for some field K), then one recovers Galois theory for field extensions (with Spec $L \to$ Spec K corresponding to a field extension $L \supset K$).

As in the classical situation, one can consider fundamental groups. Namely, let X be an irreducible normal scheme, and let K be the function field of X^{red} (or just of X, if X has no nilpotents). Also, let $\bar{K}$ be the separable closure of K (so just the algebraic closure of K, if X has characteristic 0). Then the function fields of the (reduced) finite étale Galois covers of X form a direct system of extensions of K contained in $\bar{K}$, and so the covers form an inverse system — as do their Galois groups. In the complex case, this system of groups is precisely the one obtained by taking the finite quotients of the classical topological fundamental group of X. More generally, the *algebraic* (or *étale*) *fundamental* *group* of the scheme X is defined to be the inverse limit of this inverse system of finite groups (or equivalently, the automorphism group of the inverse system of covers); this is a profinite group whose finite quotients precisely form the above inverse system. This group is denoted by $\pi_1^{\text{ét}}(X)$; it is the profinite completion of the topological fundamental group $\pi_1^{\text{top}}(X)$, in the special case of complex varieties X. (Thus, for $X = \mathbb{A}^1_{\mathbb{C}} - \{0\}$, $\pi_1^{\text{ét}}(X)$ is $\hat{\mathbb{Z}}$ rather than just $\mathbb{Z}$.) When working in the algebraic context, one generally just writes $\pi_1(X)$ for $\pi_1^{\text{ét}}(X)$.

As in the classical situation, one may also consider *branched covers.* For Riemann surfaces X, giving a branched cover of X is equivalent to giving a covering space of $X - B$, where the finite set B is the branch locus of the cover (i.e. where it is not étale). More generally, we can define a finite *branched cover* (or for short, a *cover*) of a scheme X to be a finite morphism $Y \to X$ that is generically separable. Most often, one restricts to the case that X and Y are normal integral schemes. In this case, the finite branched covers of X are in natural bijection with the finite separable field extensions of the function field of K. The notions of "Galois" and "Galois group" carry over to this situation: The *Galois group* $\mathrm{Gal}(Y/X)$ of a branched cover $f : Y \to X$ consists of the self-automorphisms of Y that preserve f. And a finite branched cover $Y \to X$ is *Galois* if X and Y are irreducible, and if the degree of the automorphism group is equal to the degree of f. Sometimes one wants to allow X or Y to be reducible, or even disconnected; and sometimes one wants to make explicit the identification of a given finite group G (e.g. the abstract group D_5) with the Galois group of a cover. In this situation, one speaks of a *G-Galois* cover $f : Y \to X$; this means a cover together with a homomorphism $\alpha : G \xrightarrow{\sim} \mathrm{Gal}(Y/X)$ such that via α, the group G acts simply transitively on a generic geometric fibre.

Thus in order to understand the Galois theory of an integral scheme X, one would like to classify the finite étale covers of X in terms of their branch loci, ramification behavior, and Galois groups; and also to describe how they fit together in the tower of covers. In the classical case of complex curves (Riemann surfaces), this is accomplished by Riemann's Existence Theorem (discussed in Section 2.1 below). A key goal is to carry this result over to more general contexts. Such a classification would in particular give an explicit description of the profinite group $\pi_1(X)$, and also of the set $\pi_A(X)$ of finite quotients of $\pi_1(X)$ (i.e. the Galois groups of finite étale covers of X). Similarly, on the generic level (where arbitrary branching is allowed), one would like to have an explicit description of the *absolute Galois group* $G_K = \mathrm{Gal}(K^{\mathrm{s}}/K)$ of the function field K of X (where K^{s} is the separable closure of K). This in turn would provide an explicit description of the finite quotients of G_K, i.e. the Galois groups of finite field extensions of K.

Beyond the above parallel between field extensions and covers, there is a second connection between those two theories, relating to *fields of definition* of covers. The issue can be illustrated by a variant on the simple example given earlier. As before, let X and

Y be two copies of $\mathbb{C} - \{0\}$, with complex parameters x and y respectively; let $n > 1$ be an integer; and this time let $f : Y \to X$ be given by $x = \pi y^n$ (where π is the usual transcendental constant). Again, the cover is Galois, with cyclic Galois group generated by $g : y \mapsto \zeta_n y$. This cover, along with its Galois action, is defined by polynomials over $\mathbb{C}$; but after a change of variables $z = \pi^{1/n} y$, the cover is given by polyomials over $\bar{\mathbb{Q}}$ (viz. $z^n = x$, and $g : z \mapsto \zeta_n z$). In fact, if X is any curve that can be defined over $\bar{\mathbb{Q}}$ (e.g. if X is the complement of finitely many $\mathbb{Q}$-points in $\mathbb{P}^1$), then any finite étale cover of the induced complex curve $X_{\mathbb{C}}$ can in fact itself be defined over $\bar{\mathbb{Q}}$ (along with its Galois action, in the G-Galois case; see Remark 2.1.6 below). And since there are only finitely many polynomials involved, it can even be defined over some number field K.

The key question here is what this number field is, in terms of the topology of the cover. By Riemann's Existence Theorem, the Galois covers of a given base X are classified; e.g. those over $\mathbb{P}^1 - \{0, 1, \infty\}$ correspond to the finite quotients of the free group on two generators. So in that case, given a finite group G together with a pair of generators, what is the number field K over which the corresponding G-Galois cover of $\mathbb{P}^1 - \{0, 1, \infty\}$ is defined? Actually, this field of definition K is not uniquely determined, although there is an "ideal candidate" for K, motivated by Galois theory. Namely, if $\omega \in G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, then ω acts on the set of (isomorphism classes of) G-Galois covers, by acting on the coefficients. If a G-Galois cover is defined over a number field K, then any $\omega \in \mathrm{Gal}(\bar{\mathbb{Q}}/K) \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ must carry this G-Galois cover to itself. So we may consider the *field of moduli* M for the G-Galois cover, defined to be the fixed field of all the ω's in $G_{\mathbb{Q}}$ that carry the G-Galois cover to itself. This is then contained in every field of definition of the G-Galois cover. Moreover it is the *intersection* of the fields of definition; and in key cases (e.g. if G is abelian or has trivial center) it is the unique minimal field of definition [CH]. One can then investigate the relationship between the (arithmetic) Galois theory of M and the (geometric) Galois theory of the given cover.

In particular, if X is a Zariski open subset of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, and if M is known to be the minimal field of definition of the given G-Galois cover, then G is a Galois group over $M(x)$ — and hence over the field M, by Hilbert's Irreducibility Theorem [FJ, Chapter 11]. The most important special case is that of finite simple groups G such that $M \subset \mathbb{Q}(\zeta_n)$ for some n, for some G-Galois cover of $X = \mathbb{P}^1 - \{0, 1, \infty\}$. Many examples of such simple groups and covers have been found, using the technique of *rigidity*, developed in work of Matzat, Thompson, Belyi, Fried, Feit, Shih, and others. In each such case, the group has thus been realized as a Galois group over some explicit $\mathbb{Q}(\zeta_n)$, the most dramatic example being Thompson's realization [Th], over $\mathbb{Q}$ itself, of the *monster group*, the largest of the 26 sporadic finite simple groups (having order $\approx 8 \cdot 10^{53}$). See the books [Se7], [Vö], and [MM] for much more about rigidity. (Note that the name *rigidity* is related to the same term in the theory of local systems and differential equations [Ka], but is unrelated to the notion of *rigid spaces* discussed elsewhere in this paper.)

Thus Galois theory appears in the theory of covering spaces in two ways — in a geometric form, coming from the parallel between covering groups and Galois groups of field extensions, and in an arithmetic form, coming from fields of definition. This situation can be expressed in another way, via the *fundamental exact sequence*

$$1 \to \pi_1(\bar{X}) \to \pi_1(X) \to G_k \to 1.$$

Here X is a geometrically connected variety over a field k; G_k is the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$; and $\bar{X} = X \times_k \bar{k}$, the space obtained by regarding X over $\bar{k}$. The kernel $\pi_1(\bar{X})$ is the geometric part of the fundamental group, while the cokernel $G_k = \pi_1(\mathrm{Spec}\, k)$ is the arithmetic part. There is a natural outer action of G_k on $\pi_1(\bar{X})$, obtained by lifting an element of G_k to $\pi_1(X)$ and conjugating; and this action corresponds to the action of the absolute Galois group on covers, discussed just above in connection with fields of moduli. Moreover, this exact sequence splits if X has a point ξ defined over k (by applying π_1 to $\mathrm{Spec}\, k = \{\xi\} \hookrightarrow X$), and in that case there is a true action of G_k on $\pi_1(\bar{X})$.

For a given cover $Y \to X$ of irreducible varieties over a base field k, one can separate out the arithmetic and geometric parts by letting ℓ be the algebraic closure of the function field of X in the function field of Y. The given cover then factors as $Y \to X_\ell \to X$, where $X_\ell = X \times_k \ell$. Here $X_\ell \to X$ is a purely arithmetic extension, coming just from an extension of constants; one sometimes refers to it as a *constant* extension. The cover $Y \to X_\ell$ is purely geometric, in the sense that the base field ℓ of X_ℓ is algebraically closed in the function field of Y; and so Y is in fact geometrically irreducible (i.e. even $Y \times_\ell \bar{k}$ is irreducible). One says that $Y \to X_\ell$ is a *regular* cover of ℓ-curves (or that Y is "regular over ℓ"; note that this use of the word "regular" is unrelated to the topological notion of "regularity" mentioned at the beginning of this Section 1.2.) Algebraically, we say that the function field of Y is a *regular* extension of that of X_ℓ, as an ℓ-algebra (or that the function field of Y is "regular over ℓ"). Of course if a field k is algebraically closed, then every cover of a k-variety (and every Galois extension of $k(x)$) is automatically regular over k.

This paper will discuss methods of patching to construct complex covers, and how those methods can be carried over to covers defined over various other classes of fields. One of those classes will be that of algebraically closed fields. But there will be other classes as well (particularly complete fields and "large" fields), for which the fields need not be algebraically closed. In those settings, we will be particularly interested in regular covers. In particular, in carrying over to those situations the fact that every finite group is a Galois group over $\mathbb{C}(x)$, it will be of interest to show that a field k has the property that every finite group is the Galois group of a *regular* extension of $k(x)$; this is the "regular inverse Galois problem" over k.

Section 1.3. Structure of this manuscript.

This paper is intended as an introduction to patching methods and their use in Galois theory. The main applications are to Riemann's Existence Theorem and related problems, particularly finding fundamental groups and solving the inverse Galois problem over curves. This is first done in Section 2 in the classical situation of complex curves (Riemann surfaces), using patching in the complex topology. Guided by the presentation in Section 2, later sections describe non-classical patching methods that apply to curves over other fields, and use these methods to obtain analogs of results and proofs that were presented in Section 2. In particular, Sections 3 and 4 each parallel Section 2, with Section 3 discussing patching methods using formal schemes, and Section 4 discussing patching using rigid analytic spaces. In each of these cases, the context provides enough structure to carry over results and proofs from the classical situation of Section 2 to the new situation. Although a full analog of Riemann's Existence Theorem remains unknown in the non-classical settings, the partial analogs that are obtained are sufficient to solve the geometric inverse Galois problem in these settings. Further results about Galois groups and fundamental groups are presented in Section 5, using both formal and rigid methods.

Section 2 begins in §2.1 with a presentation of Riemann's Existence Theorem for complex curves (Theorem 2.1.1). There, an equivalence between algebraic, analytic, and topological notions of covers provides an explicit classification of the covers of a given base (Corollary 2.1.2). As a consequence, one solves the inverse Galois problem over $\mathbb{C}(x)$ (Corollary 2.1.4). Section 2.2 shows how Riemann's Existence Theorem follows from Serre's result GAGA (Theorem 2.1.1), which gives an equivalence between coherent sheaves in the algebraic and analytic settings (basically, between the set-up in Hartshorne [Hrt2] and the one in Griffiths-Harris [GH]). The bulk of §2.2 is devoted to proving GAGA, by showing that the two theories behave in analogous ways (e.g. that their cohomology theories agree, and that sufficient twisting provides a sheaf that is generated by its global sections). This proof follows that of Serre [Se3]. Specific examples of complex covers are considered in §2.3, including ones obtained by taking copies of the base and pasting along slits; these are designed to emphasize the "patching" nature of GAGA and Riemann's Existence Theorem, and to motivate what comes after.

Section 3 treats *formal patching*, a method to extend complex patching to more general situations. The origins of this approach, going back to Zariski, are presented in §3.1, along with the original motivation of "analytic continuation" along subvarieties. Related results of Ferrand-Raynaud and Artin, which permit patching constructions consistent with Zariski's original point of view, are also presented here. Grothendieck's extension of Zariski's viewpoint is presented in §3.2, where formal schemes are discussed. The key result presented here is Grothendieck's Existence Theorem, or GFGA (Theorem 3.2.1), which is a formal analog of GAGA. We present a proof here which closely parallels Serre's proof of complex GAGA that appeared in Section 2. Afterwards, a strengthening of this result, due to the author, is shown, first for curves (Theorem 3.2.8) and then in higher dimensions (Theorem 3.2.12). Applications to covers and Galois theory are then given in §3.3. These include the author's result that every finite group can be regularly realized over the fraction field of a complete local ring other than a field (Theorem 3.3.1); the corollary that the same is true for algebraically closed fields of arbitrary characteristic (Corollary 3.3.5); and Pop's extension of this corollary to "large fields" (Theorem 3.3.6). There is also an example that illustrates the connection to complex "slit" covers that were considered in §2.3.

Section 4 considers a parallel approach, viz. *rigid patching.* Tate's original view of this approach is presented in §4.1. Unlike formal patching, which is motivated by considerations of abstract varieties and schemes, this viewpoint uses an intuition that remains closer to the original analytic approach. On the other hand, there are technical difficulties to be overcome, relating to the non-uniqueness of analytic continuation with respect to a non-archimedean metric. Tate's original method of dealing with this (via the introduction of "rigidifying data") is given in §4.1, and the status of rigid GAGA from this point of view is discussed. Then §4.2 presents a reinterpretation of rigid geometry from the point of view of formal geometry, along the lines introduced by Raynaud and worked out later by him and by Bosch and Lütkebohmert. This point of view allows rigid GAGA to "come for free" as a consequence of the formal version. It also establishes a partial dictionary between the formal and rigid approaches, allowing one to use the formal machinery together with the rigid intuition. Applications to covers and Galois theory are then given in §4.3 — in particular the regular realization of groups over complete fields (Theorem 4.3.1, paralleling Theorem 3.3.1); and Pop's "Half Riemann Existence Theorem" for henselian fields (Theorem 4.3.3), classifying "slit covers" in an arithmetic context.

Section 5 uses both formal and rigid methods to consider results that go further in the direction of a full Riemann's Existence Theorem in general contexts. In order to go beyond the realization of individual Galois groups, §5.1 discusses *embedding problems* for the purpose of seeing how Galois groups "fit together" in the tower of all covers. In particular, a result of the author and Pop is presented, giving the structure of the absolute Galois group of the function field of a curve over an algebraically closed field (Theorem 5.1.1). This result relies on showing that finite embedding problems over such curves have proper solutions. That fact about embedding problems does not extend to more general fields, but we present Pop's result that it holds for *split* embedding problems over large fields (Theorem 5.1.9). Section 5.2 presents Colliot-Thélène's result on the existence of covers of the line with given Galois group and a given fibre, in the case of a large base field (Theorem 5.2.1). Both this result and Pop's Theorem 5.1.9 can be subsumed by a single result, due to the author and Pop; this appears as Theorem 5.2.3. The classification of covers with given branch locus is taken up in §5.3, where Abhyankar's Conjecture (Theorem 5.3.1, proven by Raynaud and the author) is discussed, along with Pop's strengthening in terms of embedding problems (Theorem 5.3.4). Possible generalizations to higher dimensional spaces are discussed, along with connections to embedding problems for such spaces and their function fields. As a higher dimensional local application, it is shown that every finite split embedding problem over $\mathbb{C}((x,y))$ has a proper solution. But as discussed there, most related problems in higher dimension, including the situation for the rational function field $\mathbb{C}(x,y)$, remain open.

This manuscript is adapted, in part, from lectures given by the author at workshops at MSRI during the fall 1999 semester program on Galois groups and fundamental groups. Like those lectures, this paper seeks to give an overall view of its subject to beginners and outsiders, as well as to researchers in Galois theory who would benefit from a general presentation, including new and recent results. It follows an approach that emphasizes the historical background and motivations, the geometric intuition, and the connections between various approaches to patching — in particular stressing the parallels between the proofs in the complex analytic and formal contexts, and between the frameworks in the formal and rigid situations. The manuscript ties together results that have appeared in disparate locations in the literature, and highlights key themes that have recurred in a variety of contexts. In doing so, the emphasis is on presenting the main themes first, and afterwards discussing the ingredients in the proofs (thus following, to some extent, the organization of a lecture series).

Certain results that have been stated in the literature in a number of special cases are given here in a more natural, or more general, setting (e.g. see Theorems 3.2.8, 3.2.12, 5.1.9, 5.1.10, and 5.2.3). Quite a number of remarks are given, describing open problems, difficulties, new directions, and alterative versions of results or proofs. In particular, there is a discussion in Section 5.3 of the higher dimensional situation, which is just beginning to be understood. A new result in the local case is shown there (Theorem 5.3.9), and the global analog is posed.

The only prerequisite for this paper is a general familiarity with concepts in algebraic geometry along the lines of Hartshorne [Hrt2], although some exposure to arithmetic notions would also be helpful. Extensive references are provided for further exploration, in particular the books on inverse Galois theory by Serre [Se7], Völklein [Vö], and Malle-Matzat [MM], and the book on fundamental groups in algebraic geometry edited by Bost, Loeser, and Raynaud [BLoR].

Section 2: Complex patching

This section presents the classical use of complex patching methods in studying Galois branched covers of Riemann surfaces, and it motivates the non-classical patching methods discussed in the later sections of this manuscript. Section 2.1 begins with the central result, Riemann's Existence Theorem, which classifies covers. In its initial version, it shows the equivalence between algebraic covers and topological covers; but since topological covers can be classified group-theoretically, so can algebraic covers. It is the desire to classify algebraic covers (and correspondingly, field extensions) so concretely that provides much of the motivation in this manuscript.

The key ingredient in the proof of Riemann's Existence Theorem is Serre's result GAGA. This is proven in Section 2.2, using an argument that will itself motivate the proof of a key result in Section 3 (formal GAGA). Some readers may wish to skip Section 2.2 on first reading, and go directly to Section 2.3, where examples of Riemann's Existence Theorem are given. These examples show how topological covers can be constructed by building them locally and then patching; and the "slit cover" example there will motivate constructions that will appear in analogous contexts later, in Sections 3 and 4.

Section 2.1. Riemann's Existence Theorem

Algebraic varieties over the complex numbers can be studied topologically and analytically, as well as algebraically. This permits the use of tools that are not available for varieties over more general fields and rings. But in order to use these tools, one needs a link between the objects that exist in the algebraic, analytic, and topological categories. In the case of fundamental groups, this link is the correspondence between covers in the three settings. Specifically, in the case of complex algebraic curves, the key result is

Theorem 2.1.1. (Riemann's Existence Theorem) *Let X be a smooth connected complex algebraic curve, which we also regard as a complex analytic space and as a topological space with respect to the classical topology. Then the following categories are equivalent:*
(i) Finite étale covers of the variety X;
(ii) Finite analytic covering maps of X;
(iii) Finite covering spaces of the topological space X.

(Strictly speaking, one should write X^{an} in (ii) and X^{top} in (iii), for the associated analytic and topological spaces.)

Using this theorem, results about topological fundamental groups, which can be obtained via loops or covering spaces, can be translated into results about étale covers and étale fundamental groups. In particular, there is the following corollary concerning Zariski open subsets of the complex projective line (corresponding analytically to complements of finite sets in the Riemann sphere):

Corollary 2.1.2. (Explicit form of Riemann's Existence Theorem) *Let $r \geq 0$, let $\xi_1, \ldots, \xi_r \in \mathbb{P}^1_{\mathbb{C}}$, and let $X = \mathbb{P}^1_{\mathbb{C}} - \{\xi_1, \ldots, \xi_r\}$. Let G be a finite group, and let $\mathcal{C}$ be the set of equivalence classes of r-tuples $\underline{g} = (g_1, \ldots, g_r) \in G^r$ such that $g_1, \ldots, g_r$ generate G and satisfy $g_1 \cdots g_r = 1$. Here we declare two such r-tuples $\underline{g}, \underline{g}'$ to be equivalent if they are uniformly conjugate (i.e. if there is an $h \in G$ such that for $1 \leq i \leq r$ we have $g_i' = hg_ih^{-1}$). Then there is a bijection between the G-Galois connected finite étale covers of X and the elements of $\mathcal{C}$. Moreover this correspondence is functorial under the operation of taking quotients of G, and also under the operation of deleting more points from $\mathbb{P}^1_{\mathbb{C}}$.*

Namely, the topological fundamental group of X is given by

$$\pi_1^{\text{top}}(X) = \langle x_1, \ldots, x_r \mid x_1 \cdots x_r = 1 \rangle,$$

where the x_i's correspond to loops around the ξ_i's, from some base point $\xi_0 \in X$. The fundamental group can be identified with the Galois group (of deck transformations) of the universal cover, and the finite quotients of π_1 can be identified with pointed finite Galois covers of X. To give a quotient map $\pi_1 \twoheadrightarrow G$ is equivalent to giving the images of the x_i's, i.e. giving g_i's as above. Making a different choice of base point on the cover (still lying over ξ_0) uniformly conjugates the g_i's. So the elements of $\mathcal{C}$ are in natural bijection with G-Galois connected covering spaces of X; and by Riemann's Existence Theorem these are in natural bijection with G-Galois connected finite étale covers of X.

In the situation of Corollary 2.1.2, the uniform conjugacy class of $(g_1, \ldots, g_r)$ is called the *branch cycle description* of the corresponding cover of X [Fr1]. It has the property that the corresponding *branched* cover of $\mathbb{P}^1_{\mathbb{C}}$ contains points $\eta_1, \ldots, \eta_r$ over $\xi_1, \ldots, \xi_r$ respectively, such that g_i generates the inertia group of η_i over ξ_i. (See [Fr1] and Section 2.3 below for a further discussion of this.)

The corollary can also be stated for more general complex algebraic curves. Namely if X is obtained by deleting r points from a smooth connected complex projective curve of genus γ, then the topological fundamental group of X is generated by elements $x_1, \ldots, x_r$, $y_1, \ldots, y_\gamma$, $z_1, \ldots, z_\gamma$, subject to the relation $x_1 \cdots x_r [y_1, z_1] \cdots [y_\gamma, z_\gamma] = 1$, where the y's and z's correspond to loops around the "handles" of the topological surface X. The generalization of the corollary then replaces $\mathcal{C}$ by the set of equivalence classes of $(r + 2\gamma)$-tuples of generators that satisfy this longer relation.

Note that the above results are stated only for finite covers, whereas the topological results are a consequence of the fact that the fundamental group is isomorphic to the Galois group of the universal cover (which is of infinite degree, unless $X = \mathbb{P}^1_{\mathbb{C}}$ or $\mathbb{A}^1_{\mathbb{C}}$). Unfortunately, the universal cover is *not* algebraic — e.g. if E is a complex elliptic curve, then the universal covering map $\mathbb{C} \to E$ is not a morphism of varieties (only of topological spaces and of complex analytic spaces). As a result, in algebraic geometry there is no "universal étale cover"; only a "pro-universal cover", consisting of the inverse system of finite covers. The *étale fundamental group* is thus defined to be the automorphism group of this inverse system; and for complex varieties, this is then the profinite completion of the topological fundamental group. By Corollary 2.1.2 and this definition, we have the following result, which some authors also refer to as "Riemann's Existence Theorem":

Corollary 2.1.3. *Let $r \geq 1$, and let $S = \{\xi_1, \ldots, \xi_r\}$ be a set of r distinct points in $\mathbb{P}^1_{\mathbb{C}}$. Then the étale fundamental group of $X = \mathbb{P}^1_{\mathbb{C}} - S$ is the profinite group Π_r on generators $x_1, \ldots, x_r$ subject to the single relation $x_1 \cdots x_r = 1$. This is isomorphic to the free profinite group on $r - 1$ generators.*

Also note that there is a bijection between finite field extensions of $\mathbb{C}(x)$ and connected finite étale covers of (variable) Zariski open subsets of $\mathbb{P}^1_{\mathbb{C}}$. The reverse direction is obtained by taking function fields; and the forward direction is obtained by considering the integral closure of $\mathbb{C}[x]$ in the extension field, taking its spectrum and the corresponding morphism to the complex affine line, and then deleting points where the morphism is not étale. Under this bijection, Galois field extensions correspond to Galois finite étale covers. The corresponding statements remain true for general complex connected projective curves and their function fields.

Reinterpreting Corollary 2.1.2 via this bijection, we obtain a correspondence between field extensions and tuples of group elements (which is referred to as the "algebraic version of Riemann's Existence Theorem" in [Vö, Thm.2.13]). From this point of view, we obtain as an easy consequence the following result in the Galois theory of field extensions:

Corollary 2.1.4. *The inverse Galois problem holds over $\mathbb{C}(x)$. That is, for every finite group G, there is a finite Galois field extension K of $\mathbb{C}(x)$ such that the Galois group $\mathrm{Gal}(K/\mathbb{C}(x))$ is isomorphic to G.*

In this context, we say for short that "every finite group is a Galois group over $\mathbb{C}(x)$".

Corollary 2.1.4 is immediate from Corollary 2.1.2, since for every finite group G we may pick a set of generators $g_1, \ldots, g_r \in G$ whose product is 1, and a set of distinct points $\xi_1, \ldots, \xi_r \in \mathbb{P}^1_{\mathbb{C}}$; and then obtain a connected G-Galois étale cover of $X = \mathbb{P}^1_{\mathbb{C}} - \{\xi_1, \ldots, \xi_r\}$. The corresponding extension of function fields is then the desired extension K of $\mathbb{C}(x)$. (Similarly, if K_0 is any field of transcendence degree 1 over $\mathbb{C}$, we may prove the inverse Galois problem over K_0 by applying the generalization of Corollary 2.1.2 to the complex projective curve with function field K_0, minus r points.)

Even more is true:

Corollary 2.1.5. *The absolute Galois group of $\mathbb{C}(x)$ is a free profinite group, of rank equal to the cardinality of $\mathbb{C}$.*

This follows from the fact that the correspondences in Corollary 2.1.2 are compatible with quotient maps and with deleting more points. For then, one can deduce that the absolute Galois group is the inverse limit of the étale fundamental groups of $\mathbb{P}^1_{\mathbb{C}} - S$, where S ranges over finite sets of points. The result then follows from Corollary 2.1.3, since $\pi_1^{\mathrm{et}}(\mathbb{P}^1_{\mathbb{C}} - S)$ is free profinite on $\operatorname{card} S - 1$ generators; see [Do] for details.

Remark 2.1.6. Corollaries 2.1.4 and 2.1.5 also hold for $\bar{\mathbb{Q}}(x)$, and this fact can be deduced from a refinement of Riemann's Existence Theorem. Namely, consider a smooth curve V defined over $\bar{\mathbb{Q}}$, and let $X = V_{\mathbb{C}}$ be the base change of V to $\mathbb{C}$ (i.e. the "same" curve, viewed over the complex numbers). Then every finite étale cover of X is induced from a finite étale cover of V (along with its automorphism group). In particular, take V to be an open subset of $\mathbb{P}^1$. Then there is a bijection between topological covering spaces of $S^2 - (r \text{ points})$ and finite étale covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - (r \text{ points})$, where S^2 is the sphere. This yields the analogs of Corollaries 2.1.4 and 2.1.5 for $\bar{\mathbb{Q}}$.

This refinement of Riemann's Existence Theorem can be proven by first observing that a finite étale Galois cover $f : Y \to X$ is defined over some subalgebra $A \subset \mathbb{C}$ that is of finite type over $\bar{\mathbb{Q}}$. That is, there are finitely many equations that define the cover (along with its automorphism group, and the property of being étale); and their coefficients all lie in such an A, thereby defining a finite étale Galois cover $f_A : \mathcal{Y} \to \mathcal{X} := X \times_{\bar{\mathbb{Q}}} A$. This cover can be regarded as a *family* of covers of X, parametrized by $T := \operatorname{Spec} A$. The inclusion $i : A \hookrightarrow \mathbb{C}$ defines a $\mathbb{C}$-point ξ of T, and the fibre over this point is (tautologically) the given cover $f : Y \to X$. Meanwhile, let κ be a $\bar{\mathbb{Q}}$-point of T, and consider the corresponding fibre $g : W \to V$. Both ξ and κ define $\mathbb{C}$-points on $T_{\mathbb{C}} = T \times_{\bar{\mathbb{Q}}} \mathbb{C}$, corresponding to two fibres of a connected family of finite étale covers of X. But in any continuous connected family of covering spaces of a constant base, the fibres are the same (because $\pi_1(X_1 \times X_2) = \pi_1(X_1) \times \pi_1(X_2)$ in topology). Thus the complex cover induced by $g : W \to V$ agrees with $f : Y \to X$, as desired.

Using ideas of this type, Grothendieck proved a stronger result [Gr5, XIII, Cor. 2.12], showing that Riemann's Existence Theorem carries over from $\mathbb{C}$ to any algebraically closed field of characteristic 0. But in fact, Corollaries 2.1.4 and 2.1.5 even carry over to characteristic $p > 0$; see Sections 3.3 and 5.1 below. □

The assertions in Corollaries 2.1.2-2.1.5 above (and the analogous results for $\bar{\mathbb{Q}}(x)$ mentioned in the above remark) are purely algebraic in nature. It would therefore be desirable to have purely algebraic proofs of these assertions — and this would also have the consequence of permitting generalizations of these results to a variety of other contexts beyond those considered in [Gr5, XIII]. Unfortunately, no purely algebraic proofs of these results are known. Instead, the only known proofs rely on Riemann's Existence Theorem 2.1.1, which (because it states an equivalence involving algebraic, analytic, and topological objects) is inherently non-algebraic in nature.

Concerning the proof of Riemann's Existence Theorem 2.1.1, the easy part is the equivalence of (ii) and (iii). Namely, there is a forgetful functor from the category in (ii) to the one in (iii). We wish to show that the functor induces a surjection on isomorphism classes of objects, and bijections on morphisms between corresponding pairs of objects. (Together these automatically guarantee injectivity on isomorphism classes of objects.) In the case of objects, consider a topological covering space $f : Y \to X$. The space X is evenly covered by Y; i.e. X is a union of open discs D_i such that $f^{-1}(D_i)$ is a disjoint union of finitely many connected open subsets D_{ij} of Y, each mapping homeomorphically onto D_i. By giving each D_{ij} the same analytic structure as D_i, and using the same identifications on the overlapping D_{ij}'s as on the overlapping D_i's, we give Y the structure of a Riemann surface, i.e. a complex manifold of dimension 1; and $f : Y \to X$ is an object in (ii) whose underlying topological cover is the one we were given. This shows surjectivity on isomorphism classes. Injectivity on morphisms is trivial, and surjectivity on morphisms follows from surjectivity on objects, since if $f : Y \to X$ and $g : Z \to X$ are analytic covering spaces and if $\phi : Y \to Z$ is an morphism of topological covers (i.e. $g\phi = f$), then ϕ is itself a topological cover (of Z), hence a morphism of analytic curves and thus of analytic covers.

With regard to the equivalence of (i) and (ii), observe first of all that while the objects in (ii) and (iii) are covering spaces with respect to the metric topology, those in item (i) are finite étale covers rather than covering spaces with respect to the Zariski topology (since those are all trivial, because non-empty Zariski open subsets are dense). And indeed, if one forgets the algebraic structure and retains just the analytic (or topological) structure, then a finite étale cover of complex curves is a covering space in the metric topology, because of the Inverse Function Theorem. Thus every object in (i) yields an object in (ii). (Note also that finite étale covers can be regarded as "covering spaces in the étale topology", making the parallels between (i), (ii), (iii) look a bit closer.)

The deeper and more difficult part of the proof of Riemann's Existence Theorem is going from (ii) to (i) — and in particular, showing that every finite analytic cover of an algebraic curve is itself algebraic. One approach to this is to show that every *compact* Riemann surface (i.e. compact one-dimensional complex manifold) is in fact a complex algebraic curve, with enough meromorphic functions to separate points. See [Vö, Chaps. 5,6] for a detailed treatment of this approach. Another approach is to use Serre's result GAGA ("géométrie algébrique et géométrie analytique"), from the paper [Se3] of the same name. That result permits the use of "analytic patching" in complex algebraic geometry; i.e. constructing analytic objects locally so as to agree on overlaps, and then concluding that there is a global algebraic object that induces the local structures compatibly. It is this approach that we describe next, and it is this approach that motivates much of the discussion in the later parts of this paper.

Section 2.2. GAGA

Serre's result GAGA [Se3] permits the construction of sheaves of modules over a complex projective algebraic curve, in the Zariski topology, by constructing the sheaf analytically, in the classical complex metric topology. From this assertion about sheaves of modules, the corresponding result follows for sheaves of algebras, and therefore for covers. This is turn leads to a proof of Riemann's Existence Theorem, as discussed below.

GAGA allows one to pass from an object whose definition is inherently infinite in nature (viz. an analytic space, where functions are defined in terms of limits) to one whose definition is finite in nature (viz. an algebraic variety, based on polynomials). Intuitively, the idea is that the result is stated only for spaces that are projective, and hence compact (in the metric topology); and this compactness provides the finiteness condition that permits us to pass from the analytic to the algebraic.

To make this more precise, let X be a complex algebraic variety, with the Zariski topology, and let $\mathcal{O} = \mathcal{O}_X$ be its structure sheaf — so that $(X, \mathcal{O})$ is a locally ringed space. Meanwhile, let X^{h} be the space X with the complex metric topology, and let $\mathcal{H} = \mathcal{H}_X$ be the corresponding structure sheaf, which assigns to any metric open set $U \subset X$ the ring $\mathcal{H}(U)$ of holomorphic functions on U. So $(X^{\mathrm{h}}, \mathcal{H})$ is also a locally ringed space, called the complex analytic space associated to $(X, \mathcal{O})$; this is a Riemann surface if X is a smooth complex algebraic curve.

The sheaves considered in GAGA satisfy a finiteness condition, in both the algebraic and the analytic situations. Recall that for a scheme X with structure sheaf $\mathcal{O}$, a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is *locally of finite type* if it is locally generated by finitely many sections. It is *locally of finite presentation* if it is locally of finite type and moreover in a neighborhood of each point there is *some* finite generating set of sections whose module of relations is finitely generated. This condition is the same as saying that $\mathcal{F}$ is locally (in the Zariski topology) of the form $\mathcal{O}^m \to \mathcal{O}^n \to \mathcal{F} \to 0$. The sheaf $\mathcal{F}$ is *coherent* if the above condition holds for *every* finite generating set of sections in some neighborhood of any given point. If X is locally Noetherian (e.g. if it is a complex algebraic variety), then locally any submodule of a finitely generated module is finitely generated; and so in this situation, coherence is equivalent to local finite presentation.

There are similar definitions for complex analytic spaces. Specifically, let X be a complex algebraic variety with associated analytic space X^{h}, and let $\mathcal{F}$ be a sheaf of $\mathcal{H}$-modules. Then the conditions of $\mathcal{F}$ being *locally of finite type*, *locally of finite presentation*, and *coherent* are defined exactly as above, but with respect to $\mathcal{H}$ and the complex metric topology rather than with respect to $\mathcal{O}$ and the Zariski topology. As before, saying that a sheaf of $\mathcal{H}$-modules $\mathcal{F}$ is locally of finite presentation is the same as saying that it locally has the form $\mathcal{H}^m \to \mathcal{H}^n \to \mathcal{F} \to 0$ (in the metric topology). And it is again the case that for such a space, being coherent is equivalent to being locally of finite presentation; but the reason for this is subtler than before because X^{h} is not locally Noetherian (i.e. the ring of holomorphic functions on a small open set is not Noetherian). In this situation, the equivalence between the two conditions follows from a result of Oka [GH, pp.695-696]: If $r_1, \ldots, r_e$ generate the module of relations among a collection of sections of $\mathcal{H}^n$ in the stalk over a point, then they generate the module of relations among those sections in some metric open neighborhood of the point. Oka's result implies the equivalence between coherence and local finite presentation, because the stalks of $\mathcal{H}$ are Noetherian [GH, p.679]. (For a proof of Oka's result, see [Ca2, XV, §§4-5]; note that the terminology there is somewhat different.)

The main point of GAGA is that every coherent sheaf of $\mathcal{H}$-modules on X^{h} comes from a (unique) coherent sheaf of $\mathcal{O}$-modules on X, via a natural passage from $\mathcal{O}$-modules to $\mathcal{H}$-modules. More precisely, we may associate, to any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X, a sheaf $\mathcal{F}^{\mathrm{h}}$ of $\mathcal{H}$-modules on X^{h}. Following [Se3], this is done as follows: First, let $\mathcal{O}'$ be the sheaf of rings on X^{h} given by $\mathcal{O}'(U) = \varinjlim_{V} \mathcal{O}(V)$, where V ranges over the set Z_U of Zariski open subsets $V \subset X$ such that $V \supset U$. (For example, if U is an open disc in $\mathbb{P}^1_{\mathbb{C}}$, then $\mathcal{O}'(U)$ is the ring of rational functions with no poles in U.) Similarly, for every sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X, we can define a sheaf $\mathcal{F}'$ of $\mathcal{O}'$-modules on X^{h} by $\mathcal{F}'(U) = \varinjlim_{V} \mathcal{F}(V)$, where again V ranges over Z_U. Then define $\mathcal{F}^{\mathrm{h}}$, a sheaf of $\mathcal{H}$-modules on X^{h}, by $\mathcal{F}^{\mathrm{h}}(U) = \mathcal{F}'(U) \otimes_{\mathcal{O}'} \mathcal{H}$. For example, $\mathcal{O}^{\mathrm{h}} = \mathcal{H}$. The assignment $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{h}}$ is an exact functor; so if $\mathcal{O}^m \to \mathcal{O}^n \to \mathcal{F} \to 0$ on a Zariski open subset U, then we also have $\mathcal{H}^m \to \mathcal{H}^n \to \mathcal{F}^{\mathrm{h}} \to 0$. Thus if $\mathcal{F}$ is coherent, then so is $\mathcal{F}^{\mathrm{h}}$.

Theorem 2.2.1. **(GAGA)** [Se3] *Let X be a complex projective variety. Then the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{h}}$, from the category of coherent $\mathcal{O}_X$-modules to the category of coherent $\mathcal{H}_X$-modules, is an equivalence of categories.*

There are two main ingredients in proving GAGA. The first of these (Theorem 2.2.2 below) is that that functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{h}}$ preserves cohomology. This result, due to Serre [Se3, §12, Théorème 1], allows one to pass back and forth more freely between the algebraic and analytic settings. Namely, on $(X, \mathcal{O})$ and $(X^{\mathrm{h}}, \mathcal{H})$, as on any locally ringed space, we can consider Čech cohomology of sheaves. In fact, given any topological space X, a sheaf of abelian groups $\mathcal{F}$ on X, and an open covering $\mathcal{U} = \{U_\alpha\}$ of X, we can define the ith Čech cohomology group $\check{H}^i(\mathcal{U}, \mathcal{F})$ as in [Hrt2, Chap III, §4]. We then define $H^i(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ ranges over all open coverings of X in the given topology. For schemes X and coherent (or quasi-coherent) sheaves $\mathcal{F}$, this Čech cohomology agrees with the (derived functor) cohomology considered in Hartshorne [Hrt2, Chap. III, §2], because of [Hrt2, Chap. III, Theorem 4.5]. Meanwhile, for analytic spaces, Čech cohomology is the cohomology considered in Griffiths-Harris [GH, p.39], and also agrees with analytic derived functor cohomology (cf. [Hrt2, p.211]).

Theorem 2.2.2. [Se3] *Let X be a complex projective variety, and $\mathcal{F}$ a coherent sheaf on*

X. Then the natural map $\varepsilon : H^q(X, \mathcal{F}) \to H^q(X^{\mathrm{h}}, \mathcal{F}^{\mathrm{h}})$ is an isomorphism for every $q \geq 0$.

The second main ingredient in the proof of GAGA is the following result of Serre and Cartan:

Theorem 2.2.3. *Let $X = \mathbb{P}^r_{\mathbb{C}}$ or $(\mathbb{P}^r_{\mathbb{C}})^{\mathrm{h}}$, and let $\mathcal{M}$ be a coherent sheaf on X. Then for $n \gg 0$, the twisted sheaf $\mathcal{M}(n)$ is generated by finitely many global sections.*

In the algebraic case (i.e. for $X = \mathbb{P}^r_{\mathbb{C}}$), this is due to Serre, and is from his paper "FAC" [Se2]; cf. [Hrt2, Chap. II, Theorem 5.17]. In the analytic case (i.e. for $X = (\mathbb{P}^r_{\mathbb{C}})^{\mathrm{h}}$), this is Cartan's "Theorem A" [Ca, exp. XVIII]; cf. [GH, p.700]. Recall that the condition that a sheaf $\mathcal{F}$ is generated by finitely many global sections means that it is a quotient of a free module of finite rank; i.e. that there is a surjection $\mathcal{O}^N \twoheadrightarrow \mathcal{F}$ in the algebraic case, and $\mathcal{H}^N \twoheadrightarrow \mathcal{F}$ in the analytic case, for some integer $N > 0$ (where the exponent indicates a direct sum of N copies).

Proof of Theorem 2.2.1 (GAGA). The proof will rely on Theorems 2.2.2 and 2.2.3 above, the proofs of which will be discussed afterwards.

Step 1: We show that the functor $\mathcal{F} \to \mathcal{F}^{\mathrm{h}}$ induces bijections on morphisms. That is, if $\mathcal{F}, \mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then the natural map $\phi : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{H}_X}(\mathcal{F}^{\mathrm{h}}, \mathcal{G}^{\mathrm{h}})$ is an isomorphism of groups.

To accomplish this, let $\mathcal{S} = \underline{\mathit{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$; i.e. $\mathcal{S}$ is the sheaf of $\mathcal{O}_X$-modules associated to the presheaf $U \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}(U), \mathcal{G}(U))$. Similarly, let $\mathcal{T} = \underline{\mathit{Hom}}_{\mathcal{H}_X}(\mathcal{F}^{\mathrm{h}}, \mathcal{G}^{\mathrm{h}})$. There is then a natural morphism $\iota : \mathcal{S}^{\mathrm{h}} \to \mathcal{T}$ of (sheaves of) $\mathcal{H}$-modules, inducing $\iota_* : H^0(X^{\mathrm{h}}, \mathcal{S}^{\mathrm{h}}) \to H^0(X^{\mathrm{h}}, \mathcal{T})$. Let $\varepsilon : H^0(X, \mathcal{S}) \to H^0(X^{\mathrm{h}}, \mathcal{S}^{\mathrm{h}})$ be as in Theorem 2.2.2 above. With respect to the identifications $H^0(X, \mathcal{S}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $H^0(X^{\mathrm{h}}, \mathcal{T}) = \mathrm{Hom}_{\mathcal{H}_X}(\mathcal{F}^{\mathrm{h}}, \mathcal{G}^{\mathrm{h}})$, the composition $\iota_*\varepsilon : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{H}_X}(\mathcal{F}^{\mathrm{h}}, \mathcal{G}^{\mathrm{h}})$ is the natural map ϕ taking $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ to $f^{\mathrm{h}} \in \mathrm{Hom}_{\mathcal{H}_X}(\mathcal{F}^{\mathrm{h}}, \mathcal{G}^{\mathrm{h}})$. We want to show that ϕ is an isomorphism. Since ε is an isomorphism (by Theorem 2.2.2), it suffices to show that ι_* is also — which will follow from showing that $\iota : \mathcal{S}^{\mathrm{h}} \to \mathcal{T}$ is an isomorphism. That in turn can be checked on stalks. Here, the stalks of $\mathcal{S}$, $\mathcal{S}^{\mathrm{h}}$, and $\mathcal{T}$ at a point ξ are given by

$$\mathcal{S}_\xi = \mathrm{Hom}_{\mathcal{O}_{X,\xi}}(\mathcal{F}_\xi, \mathcal{G}_\xi), \quad \mathcal{S}^{\mathrm{h}}_\xi = \mathrm{Hom}_{\mathcal{O}_{X,\xi}}(\mathcal{F}_\xi, \mathcal{G}_\xi) \otimes_{\mathcal{O}_{X,\xi}} \mathcal{H}_{X,\xi},$$

$$\mathcal{T}_\xi = \mathrm{Hom}_{\mathcal{H}_{X,\xi}}(\mathcal{F}_\xi \otimes_{\mathcal{O}_{X,\xi}} \mathcal{H}_\xi, \mathcal{G}_\xi \otimes_{\mathcal{O}_{X,\xi}} \mathcal{H}_\xi).$$

Now the local ring $\mathcal{H}_{X,\xi}$ is flat over $\mathcal{O}_{X,\xi}$, since the inclusion $\mathcal{O}_{X,\xi} \hookrightarrow \mathcal{H}_{X,\xi}$ becomes an isomorphism upon completion (with both rings having completion $\mathbb{C}[[\underline{x}]]$, where $\underline{x} = (x_1, \ldots, x_n)$ is a system of local parameters at ξ). So by [Bo, I, §2.10, Prop. 11], we may "pull the tensor across the Hom" here; i.e. $\mathcal{S}^{\mathrm{h}}_\xi \to \mathcal{T}_\xi$ is an isomorphism.

Step 2: We show that the functor $\mathcal{F} \to \mathcal{F}^{\mathrm{h}}$ is essentially surjective, i.e. is surjective on isomorphism classes (and together with Step 1, this implies that it is bijective on isomorphism classes).

First we reduce to the case $X = \mathbb{P}^r_{\mathbb{C}}$, by taking an embedding $j : X \hookrightarrow \mathbb{P}^r_{\mathbb{C}}$, considering the direct image sheaf $j_*\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$, and using that $(j_*\mathcal{F})^{\mathrm{h}}$ is canonically isomorphic to $j_*(\mathcal{F}^{\mathrm{h}})$. Next, say that $\mathcal{M}$ is a coherent sheaf on X^{h}, i.e. a coherent $\mathcal{H}$-module. By Theorem 2.2.3, there is a surjection $\mathcal{H}^M \to \mathcal{M}(m) \to 0$ for some integers m, M; and so $\mathcal{H}(-m)^M \to \mathcal{M} \to 0$. Let the sheaf $\mathcal{N}$ be the kernel of this latter surjection. Then $\mathcal{N}$ is a coherent $\mathcal{H}$-module, and so there is similarly a surjection $\mathcal{H}(-n)^N \to \mathcal{N} \to 0$ for some n, N. Combining, we have an exact sequence $\mathcal{H}(-n)^N \xrightarrow{g} \mathcal{H}(-m)^M \to \mathcal{M} \to 0$. Now $\mathcal{H}(-n)^N = (\mathcal{O}(-n)^N)^{\mathrm{h}}$ and $\mathcal{H}(-m)^M = (\mathcal{O}(-m)^M)^{\mathrm{h}}$. So by Step 1, $g = f^{\mathrm{h}}$ for some $f \in \mathrm{Hom}(\mathcal{O}(-n)^N), \mathcal{O}(-m)^M)$. Let $\mathcal{F} = \mathrm{cok} f$. So $\mathcal{O}(-n)^N \xrightarrow{f} \mathcal{O}(-m)^M \to \mathcal{F} \to 0$ is exact, and hence so is $\mathcal{H}(-n)^N \xrightarrow{g} \mathcal{H}(-m)^M \to \mathcal{F}^{\mathrm{h}} \to 0$, using $g = f^{\mathrm{h}}$. Thus $\mathcal{M} \approx \mathcal{F}^{\mathrm{h}}$. □

Having proven GAGA, we now use it to finish the proof of Riemann's Existence Theorem for complex algebraic curves X. Two steps are needed. The first is to pass from an assertion about *modules* over a projective curve (or a projective variety) X to an assertion about *branched covers* of X. The second step is to pass from branched covers of a projective curve X to (unramified) covering spaces over a Zariski open subset of X.

For the first of these steps, observe that the equivalence between algebraic and analytic coherent modules, stated in GAGA, automatically implies the corresponding equivalence between algebraic and analytic coherent *algebras* (i.e. sheaves of algebras that are coherent as sheaves of modules). The reason is that an R-algebra A is an R-module together with some additional structure, given by module homomorphisms (viz. a product map $\mu : A \otimes_R A \to A$ and an identity $1 : R \to A$) and relations which can be given by commutative diagrams (corresponding to the associative, commutative, distributive, and identity properties); and the same holds locally for sheaves of algebras. The equivalence of categories $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{h}}$ in GAGA is compatible with tensor product (i.e. it is an equivalence of tensor categories); so the additional algebra structure carries over under the equivalence — and thus the analog of GAGA for coherent algebras holds. Under this equivalence, generically separable $\mathcal{O}_X$-algebras correspond to generically separable $\mathcal{H}_X$-algebras (using that $\mathcal{H}(U)$ is faithfully flat over $\mathcal{O}(U)$ for a Zariski open subset $U \subset X$, because the inclusion of stalks becomes an isomorphism upon completion). So taking spectra, we also obtain an equivalence between *algebraic branched covers* and *analytic branched covers.*

This formal argument can be summarized informally in the following

General Principle 2.2.4. *An equivalence of tensor categories of modules induces a corresponding equivalence of categories of algebras, of branched covers, and of Galois branched covers for any given finite Galois group.*

The last point (about Galois covers) holds because an equivalence of categories between covers automatically preserves the Galois group.

In order to obtain Riemann's Existence Theorem, one more step is needed, viz. passage from branched covers of a curve X to étale (or unramified) covers of an open subset of X. For this, recall that an algebraic branched cover is locally a covering space (in the metric topology) precisely where it is étale, by the Inverse Function Theorem. Conversely, an étale cover of a Zariski open subset of X extends to an algebraic branched cover of X (by taking the normalization in the function field of the cover). Such an extension also exists for analytic covers of curves, since it exists *locally* over curves. (Namely, a finite covering space of the punctured disc $0 < |z| < 1$ extends to an analytic branched cover of the disc $|z| < 1$, since the covering map — being bounded and holomorphic — has a removable singularity [Ru, Theorem 10.20].) Thus the above equivalence for branched covers induces an equivalence of categories between finite étale covers of a smooth complex algebraic curve X, and finite analytic covering maps to X^{h}. That is, the categories (i) and (ii) in Riemann's Existence Theorem are equivalent; and this completes the proof of that theorem.

Apart from Riemann's Existence Theorem, GAGA has a number of other applications, including several proven in [Se3]. Serre showed there that if V is a smooth projective variety over a number field K, and if X is the complex variety obtained from V via an embedding $j : K \hookrightarrow \mathbb{C}$, then the Betti numbers of X are independent of the choice of j [Se3, Cor. to Prop. 12]. Serre also used GAGA to obtain a proof of Chow's Theorem [Ch] that every closed analytic subset of $\mathbb{P}^n_{\mathbb{C}}$ is algebraic [Se3, Prop. 13], as well as several corollaries of that result. In addition, he showed that if X is a projective algebraic variety, then the natural map $H^1(X, \mathrm{GL}_n(\mathbb{C})) \to H^1(X^{\mathrm{h}}, \mathrm{GL}_n(\mathbb{C})^{\mathrm{h}})$ is bijective [Se3, Prop. 18]. As a consequence, the set of isomorphism classes of rank n algebraic vector bundles over X (in the Zariski topology) is in natural bijection with the set of isomorphism classes of rank n analytic vector bundles over X^{h} (in the metric topology). In a way, this is surprising, since the corresponding assertion for *covers* is false (because all covering spaces in the Zariski topology are trivial, over an irreducible complex variety).

Having completed the proofs of GAGA and Riemann's Existence Theorem, we return to the proofs of Theorems 2.2.2 and 2.2.3.

Proof of Theorem 2.2.2. First we reduce to the case $X = \mathbb{P}^r_{\mathbb{C}}$ as in Step 2 of the proof of Theorem 2.2.1, using that $H^q(X, \mathcal{F}) = H^q(\mathbb{P}^r_{\mathbb{C}}, j_*\mathcal{F})$ if $j : X \hookrightarrow \mathbb{P}^r_{\mathbb{C}}$, and similarly for X^{h}.

Second, we verify the result directly for the case $\mathcal{F} = \mathcal{O}$ and $\mathcal{F}^{\mathrm{h}} = \mathcal{H}$, for all $q \geq 0$. The case $q = 0$ is clear, since then both sides are just $\mathbb{C}$, because X is projective (and hence compact). On the other hand if $q > 0$, then $H^q(X, \mathcal{O}) = 0$ by the (algebraic) cohomology of projective space [Hrt2, Chap. III, Theorem 5.1], and $H^q(X^{\mathrm{h}}, \mathcal{H}) = 0$ via Dolbeault's Theorem [GH, p.45].

Third, we verify the result for the sheaf $\mathcal{O}(n)$ on $X = \mathbb{P}^r_{\mathbb{C}}$. This step uses induction on the dimension r, where the case $r = 0$ is trivial. Assuming the result for $r - 1$, we need to show it for r. This is done by induction on $|n|$; for ease of presentation, assume $n > 0$ (the other case being similar). Let E be a hyperplane in $\mathbb{P}^r_{\mathbb{C}}$; thus $E \approx \mathbb{P}^{r-1}_{\mathbb{C}}$. Tensoring the exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_E \to 0$ with $\mathcal{O}(n)$, we obtain an associated long exact sequence $(*)$ which includes, in part:

$$H^{q-1}(E, \mathcal{O}_E(n)) \to H^q(X, \mathcal{O}(n-1)) \to H^q(X, \mathcal{O}(n)) \to H^q(E, \mathcal{O}_E(n)) \to H^q(X, \mathcal{O}(n-1))$$

Similarly, replacing $\mathcal{O}$ by $\mathcal{H}$, we obtain an analogous long exact sequence $(*)^{\mathrm{h}}$; and there are (commuting) maps ε from each term in $(*)$ to the corresponding term in $(*)^{\mathrm{h}}$. By the inductive hypotheses, the map ε is an isomorphism on each of the outer four terms above. So by the Five Lemma, ε is an isomorphism on $H^q(X, \mathcal{O}(n))$.

Fourth, we handle the general case. By a vanishing theorem of Grothendieck ([Gr1]; see also [Hrt2, Chap. III, Theorem 2.7]), the qth cohomology vanishes for a Noetherian topological space of dimension n if $q > n$. (Cf. [Hrt2, p.5] for the definition of *dimension*.) So we can proceed by descending induction on q. Since $\mathcal{F}$ is coherent, it is a quotient of a sheaf $\mathcal{E} = \bigoplus_i \mathcal{O}(n_i)$ [Hrt2, Chap. II, Cor. 5.18], say with kernel $\mathcal{N}$. The associated long exact sequence includes, in part:

$$H^q(X, \mathcal{N}) \to H^q(X, \mathcal{E}) \to H^q(X, \mathcal{F}) \to H^{q+1}(X, \mathcal{N}) \to H^{q+1}(X, \mathcal{E})$$

The (commuting) homomorphisms ε map from these terms to the corresponding terms of the analogous long exact sequence of coherent $\mathcal{H}$-modules on X^{h}. On the five terms above, the second map ε is an isomorphism by the previous step; and the fourth and fifth maps ε are isomorphisms by the descending inductive hypothesis. So by the Five Lemma, the middle ε map is surjective. This gives the surjectivity part of the result, for an arbitrary coherent sheaf $\mathcal{F}$. In particular, surjectivity holds with $\mathcal{F}$ replaced by $\mathcal{N}$. That is, on the first of the five terms in the exact sequence above, the map ε is surjective. So by the Five Lemma, the middle ε is injective; so it is an isomorphism. □

Concerning Theorem 2.2.3, that result is equivalent to the following assertion:

Theorem 2.2.5. *Let $X = \mathbb{P}^r_{\mathbb{C}}$ or $(\mathbb{P}^r_{\mathbb{C}})^{\mathrm{h}}$, and let $\mathcal{M}$ be a coherent sheaf on X. Then there is an n_0 such that for all $n \geq n_0$ and all $q > 0$, we have $H^q(X, \mathcal{M}(n)) = 0$.*

In the algebraic setting, Theorem 2.2.5 is due to Serre; cf. [Hrt2, Chap. III, Theorem 5.2]. In the analytic setting, this is Cartan's "Theorem B" ([Se1], exp. XVIII of [Ca2]); cf. also [GH, p.700].

The proof of Theorem 2.2.3 is easier in the algebraic situation than in the analytic one. In the former case, the proof proceeds by choosing generators of stalks $\mathcal{M}_\xi$; multiplying each by an appropriate monomial to get a global section of some $\mathcal{M}(n)$; and using quasi-compactness to require only finitely many sections overall (also cf. [Hrt2, Chapter II, proof of Theorem 5.17]). But this strategy fails in the analytic case because the local sections are not rational, or even meromorphic; and so one cannot simply clear denominators to get a global section of a twisting of $\mathcal{M}$.

The proof in the analytic case proves Cartan's Theorems A and B (i.e. 2.2.3 and 2.2.5) together, by induction on r. Denoting these assertions in dimension r by (A_r) and (B_r), the proof in ([Se1], exp. XIX of [Ca3]) proceeds by showing that $(\mathrm{A}_{r-1}) + (\mathrm{B}_{r-1}) \Rightarrow (\mathrm{A}_r)$ and that $(\mathrm{A}_r) \Rightarrow (\mathrm{B}_r)$. Since the results are trivial for $r = 0$, the two theorems then follow; and as a result, GAGA follows as well. Serre's later argument in [Se3] is a variant on this inductive proof that simultaneously proves GAGA and Theorems A and B (i.e. Theorems 2.2.1, 2.2.3, and 2.2.5 above).

Theorems A and B were preceded by a non-projective version of those results, viz. for polydiscs in $\mathbb{C}^r$, and more generally for Stein spaces (exp. XVIII and XIX of [Ca2]; cf. also [GuR, pp. 207, 243]). There too, the two theorems are essentially equivalent. Also, no twisting is needed for Theorem B in the earlier version because the spaces were not projective there.

The proof of Theorem A in this earlier setting uses an "analytic patching" argument, applied to overlapping compact sets K', K'' on a Stein space X. In that situation, one considers metric neighborhoods U', U'' of K', K'' respectively, and one chooses generating sections $f'_1, \dots, f'_k \in \mathcal{M}(U')$ and $f''_1, \dots, f''_k \in \mathcal{M}(U'')$ for the given sheaf $\mathcal{M}$ on U', U'' respectively. From this data, one produces generating sections $g_1, \dots, g_k \in \mathcal{M}(U)$, where U is an open neighborhood of $K = K' \cup K''$. This is done via Cartan's Lemma on matrix factorization, which says (for appropriate choice of K', K'') that every element $A \in GL_n(K' \cap K'')$ can be factored as a product of an element $B \in GL_n(K')$ and an element $C \in GL_n(K'')$. That lemma, which can be viewed as a multiplicative matrix analog of Cousin's Theorem [GuRo, p.32], had been proven earlier in [Ca1], with this application in mind; and a special case had been shown even earlier in [Bi]. See also [GuRo, Chap. VI, §E]. (Cartan's Lemma is also sometimes called Cartan's "attaching theorem", where *attaching* is used in essentially the same sense as *patching* here.)

Cartan's Lemma can be used to prove these earlier versions of Theorems A and B by taking bases f'_i and f''_i over U' and U'', and letting A be the transition matrix between them (i.e. $\vec{f'} = A\vec{f''}$, where $\vec{f'}$ and $\vec{f''}$ are the column vectors with entries f'_i and f''_i respectively). The generators g_i as above can then be defined as the sections that differ from the f''s by B^{-1} and from the f'''s by C (i.e. $\vec{g} = B^{-1}\vec{f} = C\vec{f''}$). The g_i's are then well-defined wherever either the f''s or f'''s are — and hence in a neighborhood of $K = K' \cup K''$. This matrix factorization strategy also appears elsewhere, e.g. classically, concerning the Riemann-Hilbert problem, in which one attempts to find a system of linear differential equations whose monodromy representation of the fundamental group is a given representation (this being a differential analog of the inverse Galois problem for covers).

This use of Cartan's Lemma also suggests another way of restating GAGA, in the case where a projective variety X is covered by two open subsets X_1, X_2 that are strictly contained in X. The point is that if one gives coherent analytic (sheaves of) modules over X_1 and over X_2 together with an isomorphism on the overlap, then there is a unique coherent *algebraic* module over X that induces the given data compatibly. Of course by definition of coherent sheaves, there is such an *analytic* module over X (and similarly, we can always reduce to the case of two metric open subsets X_1, X_2); but the assertion is that it is algebraic.

To state this compactly, we introduce some categorical terminology. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are categories, with functors $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$, then the 2-*fibre product* of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{C}$ (with respect to f, g) is the category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ in which an *object* is a pair $(A, B) \in \mathcal{A} \times \mathcal{B}$ together with an isomorphism $\iota : f(A) \xrightarrow{\sim} g(B)$ in $\mathcal{C}$; and in which a *morphism* $(A, B; \iota) \to (A', B'; \iota')$ is a pair of morphisms $A \to A'$ and $B \to B'$ that are compatible with the ι's. For any variety [resp. analytic space] X, let $\mathfrak{M}(X)$ denote the category of algebraic [resp. analytic] coherent modules on X. (Similarly, for any ring R, we write $\mathfrak{M}(R)$ for the category of finitely presented R-modules. This is the same as $\mathfrak{M}(\mathrm{Spec}\, R)$.) In this language, GAGA and its generalizations to algebras and covers can be restated as:

Theorem 2.2.6. *Let X be a complex projective algebraic variety, with metric open subsets X_1, X_2 such that $X = X_1 \cup X_2$; let X_0 be their intersection. Then the natural base change functor*

$$\mathfrak{M}(X) \to \mathfrak{M}(X_1) \times_{\mathfrak{M}(X_0)} \mathfrak{M}(X_2)$$

is an equivalence of categories. Moreover the same holds if $\mathfrak{M}$ is replaced by the category of finite algebras, or of finite branched covers, or of Galois covers with a given Galois group.

Here X is regarded as an algebraic variety, and the X_i's as analytic spaces (so that the left hand side of the equivalence consists of algebraic modules, and the objects on the right hand side consist of analytic modules). In the case of curves, each X_i is contained in an affine open subset U_i, so coherent sheaves of modules on X_i can be identified with coherent modules over the ring $\mathcal{H}(X_i)$; thus we may identify the categories $\mathfrak{M}(X_i)$ and $\mathfrak{M}(\mathcal{H}(X_i))$.

The approach in Theorem 2.2.6 will be useful in considering analogs of GAGA in Sections 3 and 4 below.

Section 2.3. Complex patching and constructing covers.

Consider a Zariski open subset of the Riemann sphere, say $U = \mathbb{P}^1_{\mathbb{C}} - \{\xi_1, \ldots, \xi_r\}$. By Riemann's Existence Theorem, every finite covering space of U is given by an étale morphism of complex algebraic curves. Equivalently, every finite branched cover of $\mathbb{P}^1_{\mathbb{C}}$, branched only at $S = \{\xi_1, \ldots, \xi_r\}$, is given by a finite dominating morphism from a smooth complex projective curve Y to $\mathbb{P}^1_{\mathbb{C}}$. As discussed in Section 2.1, passage from topological to analytic covers is the easier step, but it requires knowledge of what topological covering spaces exist (essentially via knowledge of the fundamental group, which is understood via loops). Passage from analytic covers to algebraic covers is deeper, and can be achieved using GAGA, as discussed in Section 2.2.

Here we consider how covers can be constructed from this point of view using complex analytic patching, keeping an eye on possible generalizations. In particular, we raise the question of how to use these ideas to understand covers of curves that are not defined over the complex numbers.

We begin by elaborating on the bijection described in Corollary 2.1.2.

Taking $U = \mathbb{P}^1_{\mathbb{C}} - \{\xi_1, \ldots, \xi_r\}$ as above, choose a base point $\xi_0 \in U$. The topological fundamental group $\pi_1(U, \xi_0)$ is then the discrete group $\langle x_1, \ldots, x_r \,|\, x_1 \cdots x_r = 1 \rangle$, as discussed in Section 2.1. Up to isomorphism, the fundamental group is independent of the choice of ξ_0, and so the mention of the base point is often suppressed; but fixing a base point allows us to analyze the fundamental group more carefully. Namely, we may choose a "bouquet of loops" at ξ_0 (in M. Fried's terminology [Fr1]), consisting of a set of counterclockwise loops $\sigma_1, \ldots, \sigma_r$ at ξ_0, where σ_j winds once around ξ_j and winds around no other ξ_k; where the support of the σ_j's are disjoint except at ξ_0; where $\sigma_1 \cdots \sigma_r$ is homotopic to the identity; and where the homotopy classes of the σ_j's (viz. the x_j's) generate $\pi_1(U, \xi_0)$. In particular, we can choose σ_j to consist of a path ϕ_j from ξ_0 to a point ξ_j' near ξ_j, followed by a counterclockwise loop λ_j around ξ_j, followed by ϕ_j^{-1}. The term "bouquet" is natural with this choice of loops (e.g. in the case that $\xi_0 = 0$ and $\xi_j = e^{j\pi i/r}$, with $j = 1, \ldots, r$, and where each ϕ_j is a line segment from ξ_0 to $(1-\varepsilon)\xi_j$ for some small positive value of ε).

Let $f : V \to U$ be a finite Galois covering space, say with Galois group G. Then $\pi_1(V)$ is a subgroup N of finite index in $\pi_1(U)$, and $G = \pi_1(U)/N$. Let $g_1, \ldots, g_r \in G$ be the images of $x_1, \ldots, x_r \in \pi_1(U)$, and let m_j be the order of g_j. Thus $(g_1, \ldots, g_r)$ is the branch cycle description of $V \to U$; i.e. the G-Galois cover $V \to U$ corresponds to the uniform conjugacy class of $(g_1, \ldots, g_r)$ in Corollary 2.1.2. By Riemann's Existence Theorem 2.1.1, the cover $V \to U$ can be given by polynomial equations and regarded as a finite étale cover. Taking the normalization of $\mathbb{P}^1_{\mathbb{C}}$ in V, we obtain a smooth projective curve Y containing V as a Zariski open subset, and a G-Galois connected branched covering map $f : Y \to \mathbb{P}^1_{\mathbb{C}}$ which is branched only over $S = \{\xi_1, \ldots, \xi_r\}$.

In the above notation, with $\sigma_j = \phi_j \lambda_j \phi_j^{-1}$ (and multiplying paths from left to right), we can extend ϕ_j to a path ψ_j from ξ_0 to ξ_j in $\mathbb{P}^1_{\mathbb{C}}$. The path ψ_j can be lifted to a path $\tilde{\psi}_j$ in Y from a base point $\eta_0 \in Y$ over ξ_0, to a point $\eta_j \in Y$ over ξ_j. The element g_j generates the inertia group A_j of η_j (i.e. the stablizer of η_j in the group G). If X_j is a simply connected open neighborhood of ξ_j that contains no other ξ_k, then the topological fundamental group of $X_j - \{\xi_j\}$ is isomorphic to $\mathbb{Z}$. So $f^{-1}(X_j)$ is a union of homeomorphic connected components, each of which is Galois and cyclic of order m_j over X_j, branched only at ξ_j. The component Y_j of $f^{-1}(X_j)$ that contains η_j has stablizer $A_j = \langle g_j \rangle \subset G$, and by Kummer theory it is given by an equation of the form $s_j^{m_j} = t_j$, if t_j is a uniformizer on X_j at ξ_j. Moreover g_j acts by $g_j(s_j) = e^{2\pi i/m_j} s_j$. So $f^{-1}(X_j)$ is a (typically disconnected) G-Galois cover of X_j, consisting of a disjoint union of copies of the m_j-cyclic cover $Y_j \to X_j$, indexed by the left cosets of A_j in G. We say that $f^{-1}(X_j)$ is the G-Galois branched cover of X_j that is *induced* by the A_j-Galois cover $Y_j \to X_j$; and we write $f^{-1}(X_j) = \mathrm{Ind}_{A_j}^G Y_j$. Similarly, if U' is a simply connected open subset of U (and so U' does not contain any branch points ξ_j), then $f^{-1}(U')$ is the trivial G-Galois cover of U', consisting of $|G|$ copies of U' permuted simply transitively by the elements of G; this cover is $\mathrm{Ind}_1^G U'$.

Since the complex affine line is simply connected, the smallest example of the above situation is the case $r = 2$. By a projective linear change of variables, we may assume that the branch points are at $0, \infty$. The fundamental group of $U = \mathbb{P}^1_{\mathbb{C}} - \{0, \infty\}$ is infinite cyclic, so a finite étale cover is cyclic, say with Galois group C_m; and the cover has branch cycle description $(g, g^{-1}) = (g, g^{m-1})$ for some generator g of the cyclic group C_m. This cover is given over U by the single equation $y^m = x$. So no patching is needed in this case. (If we instead take two branch points $x = c_0, x = c_1$, with $c_0, c_1 \in \mathbb{C}$, then the equation is $y^m = (x - c_0)(x - c_1)^{m-1}$ over $\mathbb{P}^1_{\mathbb{C}}$ minus the two branch points.)

The next simplest case is that of $r = 3$. This is the first really interesting case, and in fact it is key to understanding cases with $r > 3$. By a projective linear transformation we may assume that the branch locus is $\{0, 1, \infty\}$. We consider this case next in more detail:

Example 2.3.1. We give a "recipe" for constructing Galois covers of $U = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ via patching, in terms of the branch cycle description of the given cover.

The topological fundamental group of U is $\langle \alpha, \beta, \gamma \,|\, \alpha\beta\gamma = 1 \rangle$, and this is isomorphic to the free group on two generators, viz. α, β. If we take $z = 1/2$ as the base point for the fundamental group, then these generators can be taken to be counterclockwise loops at $1/2$ around $0, 1$, respectively. The paths ψ_0, ψ_1 as above can be taken to be the real line segments connecting the base point to $0, 1$ respectively, and ψ_∞ can be taken to be the vertical path from $1/2$ to "$1/2 + i\infty$".

Let G be a finite group generated by two elements a, b. Let $c = (ab)^{-1}$, so that $abc = 1$. Consider the connected G-Galois covering space $f : V \to U$ with branch cycle description (a, b, c), and the corresponding branched cover $Y \to P^1_{\mathbb{C}}$ branched at S. As above, after choosing a base point $\eta \in Y$ over $1/2 \in \mathbb{P}^1_{\mathbb{C}}$ and lifting the paths ψ_j, we obtain points η_0, η_1, η_∞ over $0, 1, \infty$, with cyclic stabilizers $A_0 = \langle a \rangle$, $A_1 = \langle b \rangle$, $A_\infty = \langle c \rangle$ respectively. Let ι be the path in $\mathbb{P}^1_{\mathbb{C}}$ from 0 to 1 corresponding to the real interval $[0, 1]$, and let $\tilde{\iota}$ be the unique path in Y that lifts ι and passes through η. Observe that the initial point of $\tilde{\iota}$ is η_0, and the final point is η_1.

Consider the simply connected neighborhoods $X_0 = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z < 2/3\}$ of 0, and

$X_1 = \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 1/3\}$ of 1. We have that $X_0 \cup X_1 = \mathbb{C}$, and $U' := X_0 \cap X_1$ is contained in U. Also, $U = U_0 \cup U_1$, where $U_j = X_j - \{j\}$ for $j = 0, 1$. By the above discussion, $f^{-1}(X_0) = \mathrm{Ind}_{A_0}^G Y_0$, where $Y_0 \to X_0$ is a cyclic cover branched only at 0, and given by the equation $y_0^m = x$ (where m is the order of a). Similarly $f^{-1}(X_1) = \mathrm{Ind}_{A_1}^G Y_1$, where the branched cover $Y_1 \to X_1$ is given by $y_1^n = x - 1$ (where n is the order of b). Since the overlap $U' = X_0 \cap X_1$ does not meet the branch locus S, we have that $f^{-1}(U')$ is the trivial G-Galois cover $\mathrm{Ind}_1^G U'$. These induced covers have connected components that are respectively indexed by the left cosets of $A_0, A_1, 1$; and the identity coset corresponds to the component respectively containing η_0, η_1, η. Observe that the identity component of $\mathrm{Ind}_1^G U'$ is contained in the identity components of the other two induced covers, because $\tilde{\iota}$ passes through η_0, η_1, η.

Turning this around, we obtain the desired "patching recipe" for constructing the G-Galois cover of U with given branch cycle description (a, b, c): Over the above open sets U_0 and U_1, take the induced covers $\mathrm{Ind}_{A_0}^G V_0$ and $\mathrm{Ind}_{A_1}^G V_1$, where $V_0 \to U_0$ and $V_1 \to U_1$ are respectively given by $y_0^m = x$ and $y_1^n = x - 1$, and where $A_0 = \langle a \rangle$, $A_1 = \langle b \rangle$. Pick a point η over $1/2$ on the identity components of each of these two induced covers; thus $g(\eta)$ is a well-defined point on each of these induced covers, for any $g \in G$. The induced covers each restrict to the trivial G-Galois cover on the overlap $U' = U_0 \cap U_1$; now paste together the components of these trivial covers by identifying, for each $g \in G$, the component of $\mathrm{Ind}_{A_0}^G V_0$ containing $g(\eta)$ with the component of $\mathrm{Ind}_{A_1}^G V_1$ containing that point. The result is the desired cover $V \to U$. □

The above example begins with a group G and a branch cycle description (a, b, c), and constructs the cover $V \to U = \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ with that branch cycle description. In doing so, it gives the cover locally in terms of equations over two topological open discs U_0 and U_1, and instructions for patching on the overlap. Thus it gives the cover analytically (not algebraically, since the U_i's are not Zariski open subsets).

The simplest specific instance of the above example uses the cyclic group $C_3 = \langle g \rangle$ of order 3, and branch cycle description (g, g, g). Over U_0 the cover is given by (one copy of) $y_0^3 = x$; and over U_1 it is given by $y_1^3 = x - 1$. Here, over U_i, the generator g acts by $g(y_i) = \zeta_3 y_i$, where $\zeta_3 = e^{2\pi i/3}$. By GAGA, the cover can be described algebraically, i.e. by polynomials over Zariski open sets. And in this particular example, this can even be done globally over U, by the single equation $z^3 = x(x-1)$ (where $g(z) = \zeta_3 z$). Here $z = y_0 f_0(x)$ on U_0, where $f_0(x)$ is the holomorphic function on U_0 such that $f_0(0) = -1$ and $f_0^3 = x - 1$; explicitly, $f_0(x) = -1 + \frac{1}{3}x + \frac{1}{9}x^2 + \cdots$ in a neighborhood of $x = 0$. Similarly, $z = y_1 f_1(x)$ on U_1, where $f_1(x)$ is the holomorphic function on U_1 such that $f_1(1) = 1$ and $f_1^3 = x$; here $f_1(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \cdots$ in a neighborhood of $x = 1$. (Note that for this very simple cover, the global equation can be written down by inspection. But in general, for non-abelian groups, the global polynomial equations are not at all obvious from the local ones, though by GAGA they must exist.)

Example 2.3.1 requires GAGA in order to pass from the analytic equations (locally, on metric open subsets) to algebraic equations that are valid on a Zariski open dense subset. In addition, it uses ideas of topology — in particular, knowledge of the fundamental group, and the existence of open sets that overlap and together cover the space U. In later sections of this paper, we will discuss the problem of performing analogous constructions over fields other than $\mathbb{C}$, in order to understand covers of algebraic curves over those fields. For that, we will see that often an analog of GAGA exists — and that analog will permit passage from "analytic" covers to algebraic ones. A difficulty that has not yet been overcome, however, is how to find analogs of the notions from topology — both regarding explicit descriptions of fundamental groups and regarding the need for having overlapping open sets (which in non-archimedean contexts do not exist in a non-trivial way). One way around this problem is to consider only certain types of covers, for which GAGA alone suffices (i.e. where the information from topology is not required). The next example illustrates this.

Example 2.3.2. Let G be a finite group, with generators $g_1, \ldots, g_r$ (whose product need not be 1). Let $S = \{\xi_1, \ldots, \xi_{2r}\}$ be a set of $2r$ distinct points in $\mathbb{P}^1_{\mathbb{C}}$, and consider the G-Galois covering space $V \to U = \mathbb{P}^1_{\mathbb{C}} - S$ with branch cycle description

$$(g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_r, g_r^{-1}), \qquad (*)$$

with respect to a bouquet of loops $\sigma_1, \ldots, \sigma_{2r}$ at a base point $\xi_0 \in U$. Let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be the corresponding branched cover. This cover is well defined since the product of the entries of $(*)$ is 1, and it is connected since the entries of $(*)$ generate G. The cover can be obtained by a "cut-and-paste" construction as follows: Choose disjoint simple (i.e. non-self-intersecting) paths $s_1, \ldots, s_r$ in $\mathbb{P}^1_{\mathbb{C}}$, where s_j begins at ξ_{2j-1} and ends at ξ_{2j}. Take $|G|$ distinct copies of $\mathbb{P}^1_{\mathbb{C}}$, indexed by the elements of G. Redefine the topology on the disjoint union of these copies by identifying the right hand edge of a "slit" along s_j on the gth copy of $\mathbb{P}^1_{\mathbb{C}}$ to the left hand edge of the "slit" along s_j on the gg_jth copy of $\mathbb{P}^1_{\mathbb{C}}$ (with the orientation as one proceeds along the slits). The resulting space maps to $\mathbb{P}^1_{\mathbb{C}}$ in the obvious way, and away from S it is the G-Galois covering space of $\mathbb{P}^1_{\mathbb{C}} - S$ with branch cycle description $(*)$. Because of this construction, we will call covers of this type *slit covers* [Ha1, 2.4]. (The corresponding branch cycle descriptions $(*)$ have been referred to as "Harbater-Mumford representatives" [Fr3].)

Now choose disjoint simply connected open subsets $X_j \subset \mathbb{P}^1_{\mathbb{C}}$ for $j = 1, \ldots, r$, such that $\xi_{2j-1}, \xi_{2j} \in X_j$. (If ξ_{2j-1} and ξ_{2j} are sufficiently close for all j, relative to their distances to the other ξ_k's, then the X_j's can be taken to be discs.) In the above cut-and-paste construction, the paths $s_1, \ldots, s_r$ can be chosen so that the support of s_j is contained in X_j, for each j. Each X_j contains a strictly smaller simply connected open set X_j^* (e.g. a smaller disc) which also contains the support of s_j, and whose closure $\bar{X}_j^*$ is contained in

X_j. Let $U' = \mathbb{P}^1_{\mathbb{C}} - \bigcup \bar{X}^*_j$. In the cut-and-paste construction of $V \to U$, we have that the topology of the disjoint union of the $|G|$ copies of $\mathbb{P}^1_{\mathbb{C}}$ is unaffected outside of the union of the X^*_j's; and so the restriction of $V \to U$ to U' is a trivial cover, viz. $\mathrm{Ind}_1^G U'$. Suppose that ξ_j is not the point $x = \infty$ on $\mathbb{P}^1_{\mathbb{C}}$; thus ξ_j corresponds to a point $x = c_j$, with $c_j \in \mathbb{C}$. Let m_j be the order of g_j, and let A_j be the subgroup of G generated by g_j. Then the restriction of $V \to U$ to $U_j = X_j \cap U$ is given by $\mathrm{Ind}_{A_j}^G V_j$, where $V_j \to U_j$ is the A_j-Galois étale cover given by $y_j^{m_j} = (x - c_{2j-1})(x - c_{2j})^{m_j - 1}$ (as in the two branch point case, discussed just before Example 2.3.1).

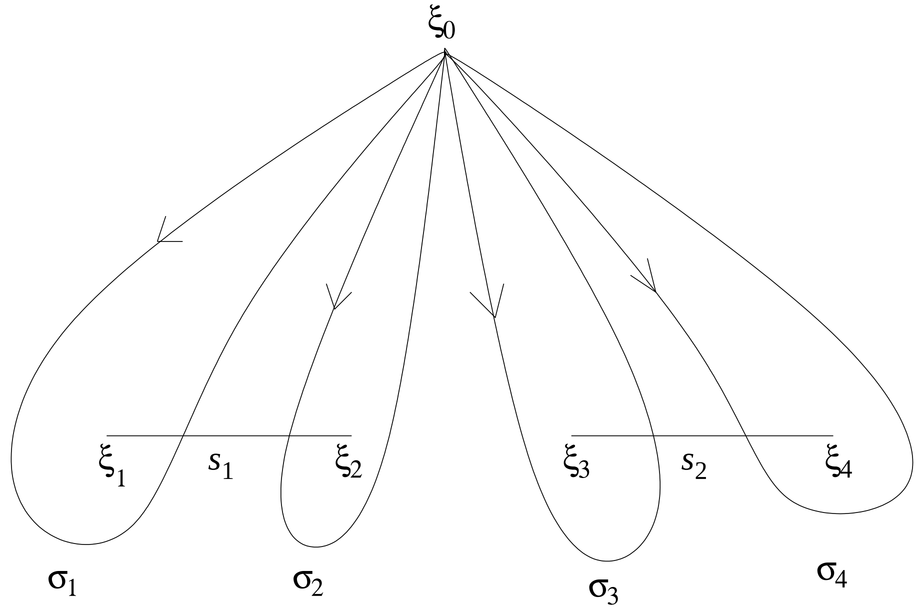

Figure 2.3.3: *Base of a slit cover of $\mathbb{P}^1_{\mathbb{C}}$ with slits s_1 from ξ_1 to ξ_2, and s_2 from ξ_3 to ξ_4; and with generators g_1, g_2, corresponding to the loops σ_1, σ_3, respectively. (Here the inverses g_1^{-1}, g_2^{-1} correspond to the loops σ_2, σ_4.)*

As a result, we obtain the following recipe for obtaining slit covers by analytic patching: Given G and generators $g_1, \dots, g_r$ (whose product need not be 1), let $A_j = \langle g_j \rangle$, and let m_j be the order of g_j. Take r disjoint open discs X_j, choose smaller open discs $X^*_j \subset X_j$, and for each j pick two points $\xi_{2j-1}, \xi_{2j} \in X^*_j$. Over $U_j = X_j - \{\xi_{2j-1}, \xi_{2j}\}$, let V_j be the A_j-Galois cover given by $y_j^{m_j} = (x - c_{2j-1})(x - c_{2j})^{m_j - 1}$. The restriction of V_j to $O_j := X_j - \bar{X}^*_j$ is trivial, and we identify it with the A_j-Galois cover $\mathrm{Ind}_1^{A_j} O_j$. This identifies the restriction of $\mathrm{Ind}_{A_j}^G V_j$ over O_j with $\mathrm{Ind}_1^G O_j$ — which is also the restriction of the trivial cover $\mathrm{Ind}_1^G U'$ of $U' = \mathbb{P}^1_{\mathbb{C}} - \bigcup \bar{X}^*_j$ to O_j. Taking the union of the trivial G-Galois cover $\mathrm{Ind}_1^G U'$ of U' with the induced covers $\mathrm{Ind}_{A_j}^G V_j$, with respect to these identifications, we obtain the slit G-Galois étale cover of $U = \mathbb{P}^1_{\mathbb{C}} - \{\xi_1, \dots, \xi_{2r}\}$ with description $(g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_r, g_r^{-1})$. □

The slit covers that occur in Example 2.3.2 can also be understood in terms of degeneration of covers — and this point of view will be useful later on, in more general settings. Consider the G-Galois slit cover $V \to U = \mathbb{P}^1_{\mathbb{C}} - S$ with branch cycle description $(*)$ as in Example 2.3.2; here $S = \{\xi_1, \ldots, \xi_{2r}\}$ and s_j is a simple path connecting ξ_{2j-1} to ξ_{2j}, with the various s_j's having disjoint support. This cover may be completed to a G-Galois branched cover $Y \to \mathbb{P}^1_{\mathbb{C}}$, with branch locus S, by taking the normalization of $\mathbb{P}^1_{\mathbb{C}}$ in (the function field of) V. Now deform this branched cover by allowing each point ξ_{2j} to move along the path s_j backwards toward ξ_{2j-1}. This yields a one (real) parameter family of irreducible G-Galois slit covers $Y_t \to \mathbb{P}^1_{\mathbb{C}}$, each of which is trivial outside of a union of (shrinking) simply connected open sets containing ξ_{2j-1} and (the moving) ξ_{2j}. In the limit, when ξ_{2j} collides with ξ_{2j-1}, we obtain a finite map $Y_0 \to \mathbb{P}^1_{\mathbb{C}}$ which is unramified away from $S' := \{\xi_1, \xi_3, \ldots, \xi_{2r-1}\}$, such that Y_0 is connected; G acts on Y_0 over $\mathbb{P}^1_{\mathbb{C}}$, and acts simply transitively away from S'; and the map is a trivial cover away from S'. In fact, Y_0 is a union of $|G|$ copies of $\mathbb{P}^1_{\mathbb{C}}$, indexed by the elements of G, such that the gth copy meets the gg_jth copy over ξ_{2j-1}. The map $Y_0 \to \mathbb{P}^1_{\mathbb{C}}$ is a *mock cover* [Ha1, §3], i.e. is finite and generically unramified, and such that each irreducible component of Y_0 maps isomorphically onto the base (here, $\mathbb{P}^1_{\mathbb{C}}$). This degeneration procedure can be reversed: starting with a connected G-Galois mock cover which is built in an essentially combinatorial manner in terms of the data $g_1, \ldots, g_r$, one can then deform it near each branch point to obtain an irreducible G-Galois branched cover branched at $2r$ points with branch cycle description $(g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_r, g_r^{-1})$. This is one perspective on the key construction in the next section, on formal patching.

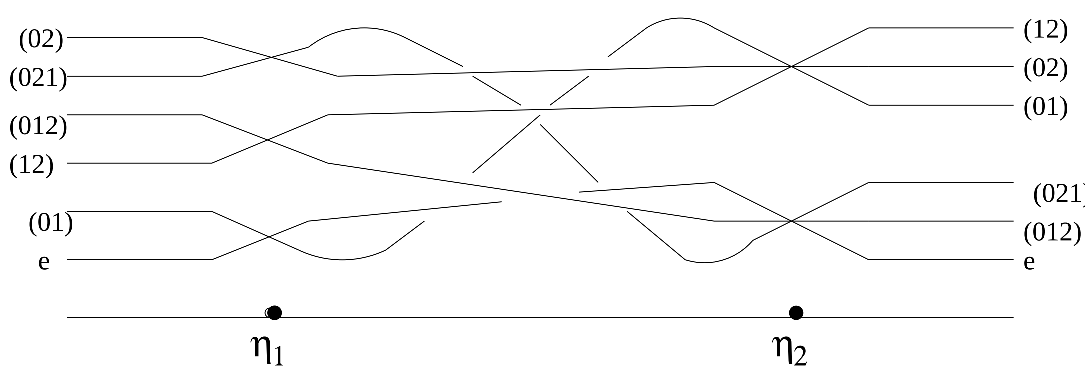

Figure 2.3.4: *A mock cover of the line, with Galois group S_3, branched at two points η_1, η_2. The sheets are labeled by the elements of S_3. The cyclic subgroups $\langle(01)\rangle, \langle(012)\rangle$ are the stablizers on the identity sheet over η_1, η_2, respectively.*

As discussed before Example 2.3.2, slit covers do not require topological input — i.e. knowledge of the explicit structure of topological fundamental groups, or the existence of overlapping open discs containing different branch points — unlike the general three-point cover in Example 2.3.1. Without this topological input, for general covers one obtains only the equivalence of algebraic and analytic covers in Riemann's Existence Theorem — and in particular, we do not obtain the corollaries to Riemann's Existence Theorem in Section 2.1. But one can obtain those corollaries as they relate to slit covers, without topological input. Since only half of the entries of the branch cycle description can be specified for a slit cover, such results can be regarded as a "Half Riemann Existence Theorem"; and can be used to motivate analogous results about fundamental groups for curves that are not defined over $\mathbb{C}$, where there are no "loops" or overlapping open discs. (Indeed, the term "half Riemann Existence Theorem" was first coined by F. Pop to refer to such an analogous result [Po2, Main Theorem]; cf. §4.3 below). In particular, we have the following variant on Corollary 2.1.3:

Theorem 2.3.5. (Analytic half Riemann Existence Theorem) *Let $r \geq 1$, let $S = \{\xi_1, \ldots, \xi_{2r}\}$ be a set of $2r$ distinct points in $\mathbb{P}^1_{\mathbb{C}}$, and let $U = \mathbb{P}^1_{\mathbb{C}} - S$. Let $\hat{F}_r$ be the free profinite group on generators $x_1, \ldots, x_r$. Then $\hat{F}_r$ is a quotient of the étale fundamental group of U.*

Namely, let G be any finite quotient of $\hat{F}_r$. That is, G is a finite group together with generators $g_1, \ldots, g_r$. Consider the G-Galois slit cover with branch cycle description $(g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_r, g_r^{-1})$. As G and its generators vary, these covers form an inverse subsystem of the full inverse system of covers of U; and the inverse limit of their Galois groups is $\hat{F}_r$.

Here, in order for this inverse system to make sense, one can first fix a bouquet of loops around the points of S; or one can fix a set of disjoint simple paths s_j from ξ_{2j-1} to ξ_{2j} and consider the corresponding set of slit covers. But to give a non-topological proof of this result (which of course is a special case of Corollary 2.1.3), one can instead give compatible local Kummer equations for the slit covers and then use GAGA; or one can use the deformation construction starting from mock covers, as sketched above. These approaches are in fact equivalent, and will be discussed in the next section in a more general setting.

Observe that the above "half Riemann Existence Theorem" is sufficient to prove the inverse Galois problem over $\mathbb{C}(x)$, which appeared above, as Corollary 2.1.4 of (the full) Riemann's Existence Theorem. Namely, for any finite group G, pick a set of r generators of G (for some r), and pick a set S of $2r$ points in $\mathbb{P}^1_{\mathbb{C}}$. Then G is the Galois group of an unramified Galois cover of $\mathbb{P}^1_{\mathbb{C}} - S$; and taking function fields yields a G-Galois field extension of $\mathbb{C}(x)$.

The above discussion relating to Example 2.3.2 brings up the question of constructing covers of algebraic curves defined over fields other than $\mathbb{C}$, and of proving at least part of Riemann's Existence Theorem for curves over more general fields. Even if the topological input can be eliminated (as discussed above), it is still necessary to have a form of GAGA to pass from "analytic" objects to algebraic ones. The "analytic" objects will be defined over a topology that is finer than the Zariski topology, and with respect to which modules and covers can be constructed locally and patched. It will also be necessary to have a structure sheaf of "analytic" functions on the space under this topology.

One initially tempting approach to this might be to use the étale topology; but unfortunately, this does not really help. One difficulty with this is that a direct analog of GAGA does not hold in the étale topology. Namely, in order to descend a module from the étale topology to the Zariski topology, one needs to satisfy a descent criterion [Gr5, Chap. VIII, §1]. In the language of Theorem 2.2.6, this says that one needs not just agreement on the overlap $X_1 \times_X X_2$ between the given étale open sets, but also on the "self-overlaps" $X_1 \times_X X_1$ and $X_2 \times_X X_2$, which together satisfy a compatibility condition. (See also [Gr3], in which descent is viewed as a special case of patching, or "recollement".) A second difficulty is that in order to give étale open sets $X_i \to X$, one needs to understand covers of X; and so this introduces an issue of circularity into the strategy for studying and constructing covers.

Two other approaches have proven quite useful, though, for large classes of base fields (though not for all fields). These are the Zariski-Grothendieck notion of formal geometry, and Tate's notion of rigid geometry. Those approaches will be discused in the following sections.

Section 3: Formal patching

This section and the next describe approaches to carrying over the ideas of Section 2 to algebraic curves that are defined over fields other than $\mathbb{C}$. The present section uses formal schemes rather than complex curves, in order to obtain analogs of complex analytic notions that can be used to obtain results in Galois theory. The idea goes back to Zariski; and his notion of a "formal holomorphic function", which uses formal power series rather than convergent power series, is presented in Section 3.1. Grothendieck's strengthening of this notion is presented in Section 3.2, including his formal analog of Serre's result GAGA (and the proof presented here parallels that of GAGA, presented in Section 2.2). These ideas are used in Section 3.3 to solve the geometric inverse Galois problem over various fields, using ideas motivated by the slit cover construction of Section 2.3. Further applications of these ideas are presented later, in Section 5.

Section 3.1. Zariski's formal holomorphic functions.

In order to generalize analytic notions to varieties over fields other than $\mathbb{C}$, one needs to have "small open neighborhoods", and not just Zariski open sets. One also needs to have a notion of ("analytic") functions on those neighborhoods.

Unfortunately, if there is no metric on the ground field, then one cannot consider discs around the origin in $\mathbb{A}^1_k$, for example, or the rings of power series that converge on those discs. But one can consider the ring of *all* formal power series, regarded as analytic (or holomorphic) functions on the spectrum of the complete local ring at the origin (which we regard as a "very small neighborhood" of that point). And in general, given a variety V and a point $\nu \in V$, we can consider the elements of the complete local ring $\hat{\mathcal{O}}_{V,\nu}$ as holomorphic functions on Spec $\hat{\mathcal{O}}_{V,\nu}$.

While this point of view can be used to study local behaviors of varieties near a point, it does not suffice in order to study more global behaviors locally and then to "patch" (as one would want to do in analogs of GAGA and Riemann's Existence Theorem), because these "neighborhoods" each contain only one closed point. The issue is that a notion of "analytic continuation" of holomorphic functions is necessary for that, so that holomorphic functions near one point can also be regarded as holomorphic functions near neighboring points.

This issue was Zariski's main focus during the period of 1945-1950, and it grew out of ideas that arose from his previous work on resolution of singularities. The question was how to extend a holomorphic function from the complete local ring at a point $\nu \in V$ to points in a neighborhood. As he said later in a preface to his collected works [Za5, pp. xii-xiii], "I sensed the probable existence of such an extension provided the analytic continuation were carried out along an algebraic subvariety W of V." That is, if W is a Zariski closed subset of V, then it should make sense to speak of "holomorphic functions" in a "formal neighborhood" of W in V.

These formal holomorphic functions were defined as follows ([Za4, Part I]; see also [Ar5, p.3]): Let W be a Zariski closed subset of a variety V. First, suppose that V is affine, say with ring of functions R, so that $W \subset V$ is defined by an ideal $I \subset R$. Consider the ring of rational functions g on V that are regular along W; this is a metric space with respect to the I-adic metric. The space of *strongly holomorphic functions f along W* (in V) is defined to be the metric completion of this space (viz. it is the space of equivalence classes of Cauchy sequences of such functions g). This space is also a ring, and can be identified with the inverse limit $\varprojlim R/I^n$.

More generally, whether or not V is affine, one can define a *(formal) holomorphic function along W* to be a function given locally in this manner. That is, it is defined to be an element $\{f_\omega\} \in \prod_{\omega \in W} \hat{\mathcal{O}}_{V,\omega}$ such that there is a Zariski affine open covering $\{V_i\}_{i \in I}$ of V together with a choice of a strongly holomorphic function $\{f_i\}_{i \in I}$ along $W_i := W \cap V_i$ in V_i (for each $i \in I$), such that f_ω is the image of f_i in $\hat{\mathcal{O}}_{V,\omega}$ whenever $\omega \in W_i$. These functions also form a ring, denoted $\hat{\mathcal{O}}_{V,W}$. Note that $\hat{\mathcal{O}}_{V,W}$ is the complete local ring $\hat{\mathcal{O}}_{V,\omega}$ if $W = \{\omega\}$. Also, if U is an affine open subset of W, and $U = \tilde{U} \cap W$ for some open subset $\tilde{U} \subset V$, then the ring $\hat{\mathcal{O}}_{\tilde{U},U}$ depends only on U, and not on the choice of $\tilde{U}$; so we also denote this ring by $\hat{\mathcal{O}}_{V,U}$, and call it the ring of holomorphic functions along U in V.

Remark 3.1.1. Nowadays, if I is an ideal in a ring R, then the *I-adic completion* of R is defined to be the inverse limit $\varprojlim R/I^n$. This modern notion of formal completion is equivalent to Zariski's above notion of metric completion via Cauchy sequences, which he first gave in [Za2, §5]. But Zariski's approach more closely paralleled completions in analysis, and fit in with his view of formal holomorphic functions as being analogs of complex analytic functions. (Prior to his giving this definition, completions of rings were defined only with respect to *maximal* ideals.) In connection with his introduction of this definition, Zariski also introduced the class of rings we now know as *Zariski rings* (and which Zariski had called "semi-local rings"): viz. rings R together with a non-zero ideal I such that every element of $1+I$ is a unit in R [Za2, Def. 1] . Equivalently [Za2, Theorem 5], these are the I-adic rings such that I is contained in (what we now call) the Jacobson radical of R. Moreover, every I-adically complete ring is a Zariski ring [Za2, Cor. to Thm. 4]; so the ring of strictly holomorphic functions on a closed subset of an affine variety is a Zariski ring. □

A deep fact proven by Zariski [Za4, §9, Thm. 10] is that every holomorphic function along a closed subvariety of an affine scheme is strongly holomorphic. So those two rings of functions agree, in the affine case; and the ring of holomorphic functions along $W =$ Spec R/I in $V =$ Spec R can be identified with the formal completion $\varprojlim R/I^n$ of R with respect to I.

Example 3.1.2. Consider the x-axis $W \approx \mathbb{A}^1_k$ in the x,t-plane $V = \mathbb{A}^2$. Then W is defined by the ideal $I = (t)$, and the ring of holomorphic functions along W in V is $A_1 := k[x][[t]]$. Note that every element of A_1 can be regarded as an element in $\hat{\mathcal{O}}_{W,\nu}$ for every point $\nu \in W$; and in this way can be regarded as an analytic continuation of (local) functions along the x-axis. Intuitively, the spectrum S_1 of A_1 can be viewed as a thin tubular neighborhood of W in V, which "pinches down" as $x \to \infty$. For example, observe that the elements x and $x-t$ are non-units in A_1, and so each defines a proper ideal of A_1; and correspondingly, their loci in $S_1 =$ Spec A_1 are non-empty (and meet the x-axis at the origin). On the other hand, $1-xt$ is a unit in A_1, with inverse $1+xt+x^2t^2\cdots$, so its locus in S_1 is empty; and geometrically, its locus in V (which is a hyperbola) approaches the x-axis only as $x \to \infty$, and so misses the ("pinched down") spectrum of A_1. One can similarly consider the ring $A_2 = k[x^{-1}][[t]]$; its spectrum S_2 is a thin neighborhood of $\mathbb{P}^1_k - (x=0)$ which "pinches down" near $x=t=0$. (See Figure 3.1.4.) □

Example 3.1.3. Let V' be the complement of the t-axis ($x=0$) in the x,t-plane $\mathbb{A}^2$, and let $W' \subset V'$ be the locus of $t=0$. Then the ring of holomorphic functions along W' in V' is $A_0 := k[x,x^{-1}][[t]]$. Geometrically, this is a thin tubular neighborhood of W' in V', which "pinches down" in two places, viz. as x approaches either 0 or ∞. (Again, see Figure 3.1.4.) Observe that Spec A_0 is *not* a Zariski open subset of Spec A_1, where A_1 is as in Example 3.1.2. In particular, A_0 is much larger than the ring $A_1[x^{-1}]$; e.g. $\sum_{n=1}^{\infty} x^{-n}t^n$ is an element of A_0 but not of $A_1[x^{-1}]$. Intuitively, $S_0 :=$ Spec A_0 can be viewed as an "analytic open subset" of $S_1 =$ Spec A_1 but not a Zariski open subset — and similarly for S_0 and $S_2 =$ Spec A_2 in Example 3.1.2. Moreover S_0 can be regarded as the "overlap" of S_1 and S_2 in $\mathbb{P}^1_{k[[t]]}$. This will be made more precise below. □

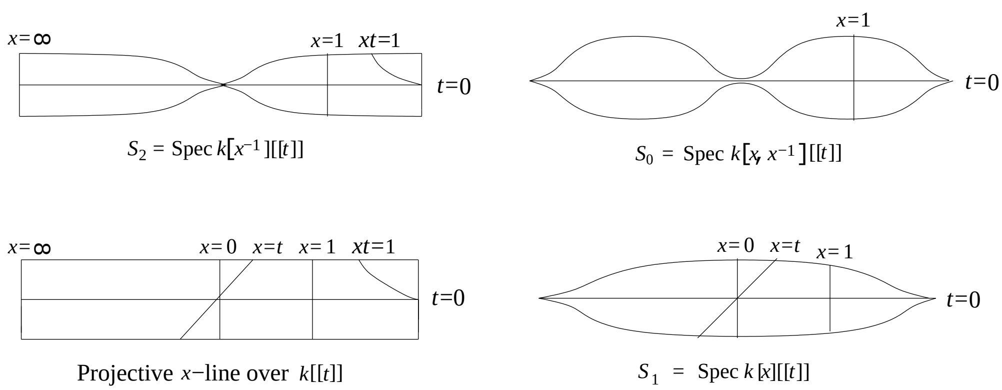

Figure 3.1.4: *A covering of* $\mathbb{P}^1_{k[[t]]}$ *(lower left) by two formal patches,* $S_1 = \operatorname{Spec} k[x][[t]]$ *and* $S_2 = \operatorname{Spec} k[1/x][[t]]$. *The "overlap"* S_0 *is* $\operatorname{Spec} k[x, 1/x][[t]]$. *See Examples 3.1.2 and 3.1.3.*

Remark 3.1.5. Just as the ring $A_0 = k[x, x^{-1}][[t]]$ in Example 3.1.3 is much larger than $A_1[x^{-1}]$, where $A_1 = k[x][[t]]$ as in Example 3.1.2, it is similarly the case that the ring A_1 is much larger than the ring $T := k[[t]][x]$ (e.g. $\sum_{n=1}^{\infty} x^n t^n$ is in A_1 but not in T). The scheme Spec T can be identified with the affine line over the complete local ring $k[[t]]$, and is a Zariski open subset of $\mathbb{P}^1_{k[[t]]}$ (given by $x \neq \infty$). This projective line over $k[[t]]$ can be viewed as a thin but uniformly wide tubular neighborhood of the projective x-line $\mathbb{P}^1_k$, and its affine open subset Spec T can correspondingly be viewed as a uniformly wide thin tubular neighborhood of the x-axis $\mathbb{A}^1_k$ (with no "pinching down" near infinity). As in Example 2, we have here that Spec A_1 is not a Zariski open subset of Spec T, and instead it can be viewed as an "analytic open subset" of Spec T. □

Using these ideas, Zariski proved his Fundamental Theorem on formal holomorphic functions [Za4, §11, p.50]: If $f : V' \to V$ is a projective morphism of varieties, with V normal and with the function field of V algebraically closed in that of V', and if $W' = f^{-1}(W)$ for some closed subset $W \subset V$, then the natural map $\hat{\mathcal{O}}_{V,W} \to \hat{\mathcal{O}}_{V',W'}$ is an isomorphism. (See [Ar5, pp.5-6] for a sketch of the proof.) This result in turn yielded Zariski's Connectedness Theorem [Za4, §20, Thm. 14] (cf. also [Hrt2, III, Cor. 11.3]), and implied Zariski's Main Theorem (cf. [Hrt2, III, Cor. 11.4]).

The above general discussion suggests that it should be possible to prove an analog of GAGA that would permit patching of modules using formal completions. And indeed, there is the following assertion, which is essentially a result of Ferrand and Raynaud (cf. [FR, Prop. 4.2]). Here the notation is as at the end of Section 2.2 above, and this result can be viewed as analogous to the version of GAGA given by Theorem 2.2.6.

Proposition 3.1.6. (Ferrand-Raynaud) *Let R be a Noetherian ring, let V be the affine scheme* $\operatorname{Spec} R$*, let W be a closed subset of V, and let $V^\circ = V - W$. Let R^* be the ring of holomorphic functions along W in V, and let $W^* =$* $\text{Spec } R^*$*. Also let $W^\circ = W^* \times_V V^\circ$. Then the base change functor*

$$\mathfrak{M}(V) \to \mathfrak{M}(W^*) \times_{\mathfrak{M}(W^\circ)} \mathfrak{M}(V^\circ)$$

is an equivalence of categories.

Here R^* is the I-adic completion of R, where I is the ideal of W in V. Intuitively, we regard $W^* = \text{Spec } R^*$ as a "formal neighborhood" of W in V, and we regard W° as the "intersection" of W^* with V° (i.e. the "complement" of W in W^*).

Remark 3.1.7. The above result is essentially a special case of the assertion in [FR, Prop. 4.2]. That result was stated in terms of cartesian diagram of categories, which is equivalent to an assertion concerning 2-fibre products (i.e. the way Proposition 3.1.6 above is stated). The main difference between the above result and [FR, Prop. 4.2] is that the latter result allows W^* more generally to be any scheme for which there is a flat morphism $f : W^* \to V$ such that the pullback $f_W : W^* \times_V W \to W$ is an isomorphism — which is the case in the situation of Proposition 3.1.6 above. Actually, though, [FR, Prop. 4.2] assumes that $f : W^* \to V$ is *faithfully* flat (unlike the situation in Proposition 3.1.6). But this extra faithfulness hypothesis is unnecessary for their proof; and in any event, given a flat morphism $f : W^* \to V$ such that f_W is an isomorphism, one can replace W^* by the disjoint union of W^* and V°, which is then faithfully flat — and applying [FR, Prop. 4.2] to that new W^* gives the desired conclusion for the original W^*. □

The following result of Artin [Ar4, Theorem 2.6] generalizes Proposition 3.1.6:

Proposition 3.1.8. *In the situation of Proposition 3.1.6, let $\tilde{V}$ be a scheme and let $f : \tilde{V} \to V$ be a morphism of finite type. Let $\tilde{W}^*, \tilde{V}^\circ, \tilde{W}^\circ$ be the pullbacks of W^*, V°, W° with respect to f. Then the base change functor*

$$\mathfrak{M}(\tilde{V}) \to \mathfrak{M}(\tilde{W}^*) \times_{\mathfrak{M}(\tilde{W}^\circ)} \mathfrak{M}(\tilde{V}^\circ)$$

is an equivalence of categories.

Note that $\tilde{V}^\circ = \tilde{V} - \tilde{W}$ in Proposition 3.1.8, where $\tilde{W} = f^{-1}(W)$.

As an example of this result, let V be a smooth n-dimensional affine scheme over a field k, let W be a closed point ω of V, and let $\tilde{V}$ be the blow-up of V at W. So $\tilde{W}$ is a copy of $\mathbb{P}_k^{n-1}$; $W^* = \operatorname{Spec} \hat{\mathcal{O}}_{V,\omega}$; and $\tilde{W}^*$ is the spectrum of a "uniformly wide tubular neighborhood" of $\tilde{W}$ in $\tilde{V}$. Here $\tilde{W}^*$, which is irreducible, can be viewed as a "twisted version" of $\mathbb{P}^{n-1}_{k[[s]]}$; cf. [Hrt2, p.29, Figure 3] for the case $n = 2$. According to Proposition 3.1.8, giving a coherent module on $\tilde{V}$ is equivalent to giving such modules on $\tilde{W}^*$ and on the complement of $\tilde{W}$, with agreement on the "overlap" $\tilde{W}^\circ$.

While the above two propositions required V to be affine, this hypothesis can be dropped if W is finite:

Corollary 3.1.9. *Let V be a Noetherian scheme, and let W be a finite set of closed points in V. Let R^* be the ring of holomorphic functions along W in V, let $W^* = \operatorname{Spec} R^*$, let $V^\circ = V - W$, and let $W^\circ = W^* \times_V V^\circ$.*

a) Then the base change functor

$$\mathfrak{M}(V) \to \mathfrak{M}(W^*) \times_{\mathfrak{M}(W^\circ)} \mathfrak{M}(V^\circ)$$

is an equivalence of categories.

b) Let $\tilde{V}$ be a scheme and let $f : \tilde{V} \to V$ be a morphism of finite type. Let $\tilde{W}^*, \tilde{V}^\circ, \tilde{W}^\circ$ be the pullbacks of W^*, V°, W° with respect to f. Then the base change functor

$$\mathfrak{M}(\tilde{V}) \to \mathfrak{M}(\tilde{W}^*) \times_{\mathfrak{M}(\tilde{W}^\circ)} \mathfrak{M}(\tilde{V}^\circ)$$

is an equivalence of categories.

Proof sketch. For part (a), we may cover V by finitely many affine open subsets $V_i = \operatorname{Spec} R_i$, with R_i Noetherian. Applying Proposition 3.1.6 to each V_i and $W_i := V_i \cap W$, we obtain equivalences over each V_i. These equivalences agree on the overlaps $V_i \cap V_j$ (since each is given by base change), and so together they yield the desired equivalence over V, in part (a). Part (b) is similar, using Proposition 3.1.8. □

Unfortunately, while the above results are a kind of GAGA, permitting the patching of modules, they do not directly help to construct covers (via the General Principle 2.2.4); and so they do not directly help prove an analog of Riemann's Existence Theorem. The reason is that these results require that a module be given over a Zariski open subset V° (or $\tilde{V}^\circ$), viz. the complement of the given closed subset W (or $\tilde{W}$). And a normal cover $Z \to V$ is determined by its restriction to a dense open subset V° (viz. it is the normalization of V in the function field of the cover — which is the same as the function field of the restriction). So these results provide a cover $Z \to V$ only in circumstances in which one already has the cover in hand.

Instead, in order to use Zariski's approach to obtain results about covers, we will focus on spaces such as $\mathbb{P}^1_{k[[t]]}$ (and see the discussion in the Remark 3.1.5 above). In that situation, Grothendieck has proven a "formal GAGA", which we discuss next. That result yields a version of Riemann's Existence Theorem for many fields other than $\mathbb{C}$. Combining that approach with the above results of Ferrand-Raynaud and Artin yields even stronger versions of "formal GAGA"; and those formal patching results have been used to prove a number of results concerning covers and fundamental groups over various fields (as will be discussed later).

Section 3.2. Grothendieck's formal schemes.

Drawing on Zariski's notion of formal holomorphic functions, Grothendieck introduced the notion of *formal scheme*, and provided a framework for proving a "formal GAGA" that is sufficient for establishing formal analogs of (at least parts of) Riemann's Existence Theorem. In his paper of the same name [Gr2], Grothendieck announced his result GFGA ("géometrie formelle et géométrie algébrique"), and sketched how it leads to results about covers and fundamental groups of curves. The details of this GFGA result appeared later in EGA [Gr4, III, Cor. 5.1.6], and the result in that form has become known as Grothendieck's Existence Theorem. In SGA 1 [Gr5], the details about the results on covers and fundamental groups appeared.

To begin with, fix a Zariski closed subset W of a scheme V. Let $\mathcal{O}_{\mathfrak{V}} = \mathcal{O}_{\mathfrak{V},W}$ be the sheaf of holomorphic functions along W in V. That is, for every Zariski open subset $U \subset W$, let $\mathcal{O}_{\mathfrak{V}}(U)$ be the ring $\hat{\mathcal{O}}_{V,U}$ of holomorphic functions along U in V. Thus $\mathcal{O}_{\mathfrak{V}} = \varprojlim_n \mathcal{O}_V/\mathcal{I}^{n+1}$, where $\mathcal{I}$ is the sheaf of ideals of $\mathcal{O}_V$ defining W in V. The ringed space $\mathfrak{V} := (W, \mathcal{O}_{\mathfrak{V}})$ is defined to be the *formal completion of* V *along* W.

The simplest example of this takes V to be the affine t-line over a field k, and W to be the point $t = 0$. Here we may identify $\mathcal{O}_{\mathfrak{V}}$ with the ring $k[[t]] = \varprojlim_n O_n$, where $O_n = k[t]/(t^{n+1})$. Here $n = 0$ corresponds to W, and $n > 0$ to infinitesimal thickenings of W. The kernel I_m of $O_m \to O_0$ is the ideal tO_m, and the kernel of $O_m \to O_n$ is $t^{n+1}O_m = I_m^{n+1}$.

As a somewhat more general example, let A be a ring that is complete with respect to an ideal I. Then $W = \operatorname{Spec} A/I$ is a closed subscheme of $V = \operatorname{Spec} A$, consisting of the prime ideals of A that are open in the I-adic topology. The formal completion $\mathfrak{V} = (W, \mathcal{O}_{\mathfrak{V}})$ of V along W consists of the underlying topological space W together with a structure sheaf whose ring of global sections is A. This formal completion is also called the *formal spectrum* of A, denoted $\operatorname{Spf} A$. (For example, if $A = k[x][[t]]$ and $I = (t)$, then the underlying space of $\operatorname{Spf} A$ is the affine x-line over k, and its global sections are $k[x][[t]]$.)

Note that the above definition of formal completion relies on the idea that the geometry of a space is captured by the structure sheaf on it, rather than on the underlying topological space. Indeed, the underlying topological space of $\mathfrak{V}$ is the same as that of W; but the structure sheaf $\mathcal{O}_{\mathfrak{V}}$ incorporates all of the information in the spectra of $\hat{\mathcal{O}}_{V,U}$ — and thus it reflects the local geometry of V near W.

More generally, suppose we are given a topological space X and a sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$ on X. Suppose also that $\mathcal{O}_{\mathfrak{X}} = \varprojlim_n \mathcal{O}_n$, where $\{\mathcal{O}_n\}_n$ is an inverse system of sheaves of rings on X such that $(X, \mathcal{O}_n)$ is a scheme X_n for each n; and that for $m \geq n$, the homomorphism $\mathcal{O}_m \to \mathcal{O}_n$ is surjective with kernel $\mathcal{I}_m^{n+1}$, where $\mathcal{I}_m = \ker(\mathcal{O}_m \to \mathcal{O}_0)$. Then the ringed space $\mathfrak{X} := (X, \mathcal{O}_{\mathfrak{X}})$ is a *formal scheme*. In particular, in the situation above for $W \subset V$ (taking $\mathcal{O}_n = \mathcal{O}_V/\mathcal{I}^{n+1}$), the formal completion $\mathfrak{V} = (W, \mathcal{O}_{\mathfrak{V}})$ of V along W is a formal scheme.

If W is a closed subset of a scheme V, with formal completion $\mathfrak{V}$, then to every sheaf $\mathcal{F}$ of $\mathcal{O}_V$-modules on V we may canonically associate a sheaf $\hat{\mathcal{F}}$ of $\mathcal{O}_{\mathfrak{V}}$ modules on $\mathfrak{V}$. Namely, for every n let $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{O}_V/\mathcal{I}^{n+1}$, where $\mathcal{I}$ is the sheaf of ideals defining W. Then let $\hat{\mathcal{F}} = \varprojlim_n \mathcal{F}_n$. Note that $\mathcal{O}_{\mathfrak{V}} = \hat{\mathcal{O}}_V$. Also observe that if $\mathcal{F}$ is a coherent $\mathcal{O}_V$-module, then $\hat{\mathcal{F}}$ is a coherent $\mathcal{O}_{\mathfrak{V}}$-module (i.e. it is locally of the form $\mathcal{O}_{\mathfrak{V}}^m \to \mathcal{O}_{\mathfrak{V}}^n \to \hat{\mathcal{F}} \to 0$).

Theorem 3.2.1. (GFGA, Grothendieck Existence Theorem) *Let A be a Noetherian ring that is complete with respect to a proper ideal I, let V be a proper A-scheme, and let $W \subset V$ be the inverse image of the locus of I. Let $\mathfrak{V} = (W, \mathcal{O}_{\mathfrak{V}})$ be the formal completion of V along W. Then the functor $\mathcal{F} \mapsto \hat{\mathcal{F}}$, from the category of coherent $\mathcal{O}_V$-modules to the category of coherent $\mathcal{O}_{\mathfrak{V}}$-modules, is an equivalence of categories.*

Before turning to the proof of Theorem 3.2.1, we discuss its content and give some examples, beginning with

Corollary 3.2.2. [Gr2, Cor. 1 to Thm. 3] *In the situation of Theorem 3.2.1, the natural map from closed subschemes of V to closed formal subschemes of $\mathfrak{V}$ is a bijection.*

Namely, such subschemes [resp. formal subschemes] correspond bijectively to coherent subsheaves of $\mathcal{O}_V$ [resp. of $\mathcal{O}_{\mathfrak{V}}$]. So this is an immediate consequence of the theorem.

This corollary may seem odd, for example in the case where V is a curve over a complete local ring A, and W is thus a curve over the residue field of A — since then, the only reduced closed subsets of W (other than W itself) are finite sets of points. But while distinct closed subschemes of V can have the same intersection with the topological space W, the structure sheaves of their restrictions will be different, and so the induced formal schemes will be different.

Theorem 3.2.1 can be viewed in two ways: as a thickening result (emphasizing the inverse limit point of view), and as a patching result (emphasizing the analogy with the classical GAGA of Section 2.2).

From the point of view of thickening, given $W \subset V$ defined by a sheaf of ideals $\mathcal{I}$, we have a sequence of subschemes $V_n = \underline{\text{Spec}}\, \mathcal{O}_V/\mathcal{I}^{n+1}$. Each V_n has the same underlying topological space (viz. that of $W = V_0$), but has a different structure sheaf. The formal completion $\mathfrak{V}$ of V along W can be regarded as the direct limit of the schemes V_n. What Theorem 3.2.1 says is that under the hypotheses of that result, to give a coherent sheaf $\mathcal{F}$ on V is equivalent to giving a compatible set of coherent sheaves $\mathcal{F}_n$ on the V_n's (i.e. the restrictions of $\mathcal{F}$ to the V_n's). The hard part (cf. the proof below) is to show the existence of a coherent sheaf $\mathcal{F}$ that restricts to a given compatible set of coherent sheaves $\mathcal{F}_n$. And later, the result will tell us that to give a branched cover of V is equivalent to giving a compatible system of covers of the V_n's.

On the other hand, the point of view of patching is closer to that of Zariski's work on formal holomorphic functions. Given $W \subset V$, we can cover W by affine open subsets U_i. By definition, giving a coherent formal sheaf on W amounts to giving finitely presented modules over the rings $\hat{\mathcal{O}}_{V,U_i}$ that are compatible on the overlaps (i.e. over the rings $\hat{\mathcal{O}}_{V,U_{ij}}$, where $U_{ij} = U_i \cap U_j$). So Theorem 3.2.1 says that to give a coherent sheaf $\mathcal{F}$ on V is equivalent to giving such modules locally (i.e. the pullbacks of $\mathcal{F}$ to the "formal neighborhoods" Spec $\hat{\mathcal{O}}_{V,U_i}$ with agreements on the "formal overlaps" Spec $\hat{\mathcal{O}}_{V,U_{ij}}$). The same principle will be applied later to covers.

Example 3.2.3. Let k be a field, let $A = k[[t]]$, and let $V = \mathbb{P}^1_A$, the projective x-line over A. So W is the projective x-line over k. Let $\mathfrak{V}$ be the formal completion of V at W. Theorem 3.2.1 says that giving a coherent $\mathcal{O}_V$-module is equivalent to giving a coherent $\mathcal{O}_{\mathfrak{V}}$-module.

From the perspective of thickening, to give a coherent $\mathcal{O}_{\mathfrak{V}}$-module $\mathcal{F}$ amounts to giving an inverse system of coherent modules $\mathcal{F}_n$ over the V_n's, where V_n is the projective x-line over $k[t]/(t^{n+1})$. Each finite-level thickening $\mathcal{F}_n$ gives more and more information about the given module, and in the limit, the theorem says that the full $\mathcal{O}_V$-module $\mathcal{F}$ is determined.

For the patching perspective, cover W by two open sets U_1 (where $x \neq \infty$) and U_2 (where $x \neq 0$), each isomorphic to the affine k-line. The corresponding rings of holomorphic functions are $k[x][[t]]$ and $k[x^{-1}][[t]]$, while the ring of holomorphic functions along the overlap $U_0 : (x \neq 0, \infty)$ is $k[x, x^{-1}][[t]]$. As in Examples 3.1.2 and 3.1.3, the spectra S_1, S_2 of the first two of these rings can be viewed as tubular neighborhoods of the two affine lines, pinching down near $x = \infty$ and near $x = 0$ respectively. The spectrum S_0 of the third ring (the "formal overlap") can be viewed as a tubular neighborhood that pinches down near both 0 and ∞. (See Figure 3.1.4.) These spectra can be viewed as "analytic open subsets" of V, which cover V (in the sense that the disjoint union $S_1 \cup S_2$ is faithfully flat over V) — and the theorem says that giving coherent modules over S_1 and S_2, which agree over S_0, is equivalent to giving a coherent module over V. □

From the above patching perspective, Theorem 3.2.1 can be rephrased as follows, in a form that is useful in the case of relative dimension 1. In order to be able to apply it to Galois theory (in Section 3.3 below), we state it as well for algebras and covers.

Theorem 3.2.4. *In the situation of Theorem 3.2.1, suppose that U_1, U_2 are affine open subsets of W such that $U_1 \cup U_2 = W$, with intersection U_0. For $i = 0, 1, 2$, let S_i be the spectrum of the ring of holomorphic functions along U_i in V. Then the base change functor*

$$\mathfrak{M}(V) \to \mathfrak{M}(S_1) \times_{\mathfrak{M}(S_0)} \mathfrak{M}(S_2)$$

is an equivalence of categories. Moreover the same holds if $\mathfrak{M}$ is replaced by the category of finite algebras, or of finite branched covers, or of Galois covers with a given Galois group.

Compare this with the restatement of the classical GAGA at Theorem 2.2.6, and with the results of Ferrand-Raynaud and Artin (Propositions 3.1.6 and 3.1.8). See also Figure 3.1.4 for an illustration of this result in the situation of the above example. As in Theorem 2.2.6, the above assertions for algebras and covers follow formally from the result for modules, via the General Principle 2.2.4. (Cf. also [Ha2, Proposition 2.8].)

Remarks 3.2.5. (a) Theorem 3.2.1 does not hold if the properness hypothesis on V is dropped. For example, Corollary 3.2.2 is false in the case that $A = k[[t]]$ and $V = \mathbb{A}^1_A$ (since the subscheme $(1 - xt)$ in V induces the same formal subscheme of $\mathcal{V}$ as the empty set). Similarly, Theorem 3.2.4 does not hold as stated if V is not proper over A (and note that $S_1 \cup S_2$ is not faithfully flat over V in this situation). But a variant of Theorem 3.2.4 does hold if V is affine: namely there is still an equivalence if $\mathfrak{M}(V)$ is replaced by $\mathfrak{M}(S)$, where S is the ring of holomorphic functions along W in V. This is essentially a restatement of Zariski's result that holomorphic functions on an affine open subset of W are strongly holomorphic. It is also analogous to the version of Cartan's Theorem A for Stein spaces [Ca2] (cf. the discussion near the end of Section 2.2 above).

(b) The main content of Theorem 3.2.1 (or Theorem 3.2.4) can also be phrased in affine terms in the case of relative dimension 1. For instance, in the situation of the above example with $A = k[[t]]$ and $V = \mathbb{P}^1_A$, a coherent module $\mathcal{M}$ over V is determined up to twisting by its restriction to $\mathbb{A}^1_A = \operatorname{Spec} A[x]$. Letting S_0, S_1, S_2 be as in the example, and restricting to the Zariski open subset $\mathbb{A}^1_A$, we obtain an equivalence of categories

$$\mathfrak{M}(R) \to \mathfrak{M}(R_1) \times_{\mathfrak{M}(R_0)} \mathfrak{M}(R_2) \qquad (*)$$

where $R = k[[t]][x]$; $R_1 = k[x][[t]]$; $R_2 = k[x^{-1}][[t]][x]$; and $R_0 = k[x, x^{-1}][[t]]$). (Here we adjoin x in the definition of R_2 because of the restriction to $\mathbb{A}^1_A$.) In this situation, one can directly prove a formal version of Cartan's Lemma, viz. that every element of $\mathrm{GL}_n(R_0)$ can be written as the product of an element of $\mathrm{GL}_n(R_1)$ and an element of $\mathrm{GL}_n(R_2)$. This immediately gives the analog of ($*$) for the corresponding categories of finitely generated *free* modules, by applying this formal Cartan's Lemma to the transition matrix between the bases over R_1 and R_2. (Cf. the discussion in Section 2.2 above, and also [Ha2, Prop. 2.1] for a general result of this form.) Moreover, combining this formal Cartan's Lemma with the fact that every element of R_0 is the sum of an element of R_1 and an element of R_2, one can deduce all of ($*$), and thus essentially all of Theorem 3.2.1 in this situation. (See [Ha2, Proposition 2.6] for the general result, and see also Remark 1 after the proof of Corollary 2.7 there.)

(c) Using the approach of Remark (b), one can also prove analogous results where Theorem 3.2.1 does not apply. For example, let A and B be subrings of $\mathbb{Q}$, let $D = A \cap B$, and let C be the subring of $\mathbb{Q}$ generated by A and B. (For instance, take $A = \mathbb{Z}[1/2]$ and $B = \mathbb{Z}[1/3]$, so $C = \mathbb{Z}[1/6]$ and $D = \mathbb{Z}$.) Then "Cartan's Lemma" applies to the four rings $A[[t]]$, $B[[t]]$, $C[[t]]$, $D[[t]]$ (as can be proven by constructing the coefficients of the entries of the factorization, inductively). So by [Ha2, Proposition 2.6]), giving a finitely generated module over $D[[t]]$ is equivalent to giving such modules over $A[[t]]$ and $B[[t]]$ together with an isomorphism between the modules they induce over $C[[t]]$.

Another example involves the ring of convergent arithmetic power series $\mathbb{Z}\{t\}$, which consists of the formal power series $f(t) \in \mathbb{Z}[[t]]$ such that f converges on the complex disc $|t| < 1$. (Under the analogy between $\mathbb{Z}$ and $k[x]$, the ring $\mathbb{Z}[[t]]$ is analogous to $k[x][[t]]$, and the ring $\mathbb{Z}\{t\}$ is analogous to $k[[t]][x]$.) Then with A, B, C, D as in the previous paragraph, "Cartan's Lemma" applies to $A[[t]]$, $B\{t\}$, $C[[t]]$, $D\{t\}$ [Ha2, Prop. 2.3]. As a consequence, the analog of Theorem 3.2.4 holds for these rings: viz. giving a fintely presented module over $D\{t\}$ is equivalent to giving such modules over $A[[t]]$ and $B\{t\}$ together with an isomorphism between the modules they induce over $C[[t]]$ [Ha5, Theorem 3.6]. □

The formal GAGA (Theorem 3.2.1) above can be proved in a way that is analogous to the proof of the classical GAGA (as presented in Section 2.2). In particular, there are two main ingredients in the proof. The first is:

Theorem 3.2.6. (Grothendieck) *In the situation of Theorem 3.2.1, if $\mathcal{F}$ is a coherent sheaf on V, then the natural map $\varepsilon : H^q(V, \mathcal{F}) \to H^q(\mathfrak{V}, \hat{\mathcal{F}})$ is an isomorphism for every $q \geq 0$.*

This result was announced in [Gr2, Cor. 1 to Thm. 2] and proven in [Gr4, III, Prop. 5.1.2]. Here the formal H^q's can (equivalently) be defined either via Čech cohomology or by derived functor cohomology. The above theorem is analogous to Theorem 2.2.2, concerning the classical case; and like that result, it is proven by descending induction on q (using that $H^q = 0$ for q sufficiently large). As in Section 2.2, it is the key case $q = 0$ that is used in proving GAGA. That case is known as Zariski's Theorem on Formal Functions [Hrt2, III, Thm. 11.1]; it generalizes the original version of Zariski's Fundamental Theorem on formal holomorphic functions [Za4, §11, p.50], which is the case $q = 0$ and $\mathcal{F} = \mathcal{O}_V$, and which was discussed in Section 3.1 above.

The second key ingredient in the proof of Theorem 3.2.1 is analogous to Theorem 2.2.3:

Theorem 3.2.7. *In the situation of Theorem 3.2.1 (with V assumed projective over A), let $\mathcal{M}$ be a coherent $\mathcal{O}_V$-module or a coherent $\mathcal{O}_{\mathfrak{V}}$-module. Then for $n \gg 0$ the twisted sheaf $\mathcal{M}(n)$ is generated by finitely many global sections.*

Once one has Theorems 3.2.6 and 3.2.7 above, the projective case of Theorem 3.2.1 follows from them in exactly the same manner that Theorem 2.2.1 (classical GAGA) followed from Theorems 2.2.2 and 2.2.3 there. The proper case can then be deduced from the projective case using Chow's Lemma [Gr4, II, Thm. 5.6.1]; cf. [Gr4, III, 5.3.5] for details.

Concerning why Theorem 3.2.7 holds:

Proof sketch of 3.2.7. In the algebraic case (i.e. for $\mathcal{O}_V$-modules), the assertion is again Serre's result [Hrt2, Chap. II, Theorem 5.17]; cf. Theorem 2.2.3 above in the algebraic case. In the formal case (i.e. for $\mathcal{O}_{\mathfrak{V}}$-modules), the assertion is a formal analog of Cartan's Theorem A (cf. the analytic case of Theorem 2.2.3). The key point in proving this formal analog (as in the analytic version) is to obtain a twist that will work for a given sheaf, even though the sheaf is not algebraic and we cannot simply clear denominators (as in the algebraic proof).

To do this, first recall that a formal sheaf $\mathcal{M}$ corresponds to an inverse system $\{\mathcal{M}_i\}$ of sheaves on the finite thickenings V_i. By the result in the algebraic case (applied to V_i), we have that for each i there is an n such that $\mathcal{M}_i(n)$ is generated by finitely many global sections. But we need to know that there is a single n that works for *all* i, and with compatible finite sets of global sections. The strategy is to pick a finite set of generating sections for $\mathcal{M}_0(n)$ for some n (and these will exist if n is chosen sufficiently large); and then inductively to lift them to sections of the $\mathcal{M}_i(n)$'s, in turn. If this is done, Theorem 3.2.7 follows, since the lifted sections automatically generate, by Nakayama's Lemma.

In order to carry out this inductive lifting, first reduce to the case $V = \mathbb{P}^m_A$ for some m, as in Section 2.2 (viz. embedding the given V in some $\mathbb{P}^m_A$ and extending the module by 0). Now let $\operatorname{gr} A$ be the associated graded ring to A and let $\operatorname{gr} \mathcal{O} = (R/I)\mathcal{O} \oplus (I/I^2)\mathcal{O} \oplus \cdots$ (where $\mathcal{O} = \mathcal{O}_V$). Also write $\operatorname{gr} \mathcal{M} = \mathcal{M}_0 \oplus (I/I^2)\mathcal{M}_1 \oplus \cdots$. Since $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{V}}$-module, it follows that $\operatorname{gr} \mathcal{M}$ is a coherent $\operatorname{gr} \mathcal{O}$-module on $\mathbb{P}^m_{\operatorname{gr} \mathcal{O}}$. So by the algebraic analog of Cartan's Theorem B (i.e. by Serre's result [Hrt2, III, Theorem 5.2]), there is an integer n_0 such that for all $n \geq n_0$, $H^1(\mathbb{P}^m_{\operatorname{gr} \mathcal{O}}, \operatorname{gr} \mathcal{M}(n)) = 0$. But $\operatorname{gr} \mathcal{M}(n) = \bigoplus_i (I^i/I^{i+1})\mathcal{M}_i(n)$, and so each $H^1(\mathbb{P}^m_{A/I^{i+1}}, (I^i/I^{i+1})\mathcal{M}_i(n)) = 0$. By the long exact sequence associated to the short exact sequence $0 \to (I^i/I^{i+1})\mathcal{M}_i(n) \to \mathcal{M}_i(n) \to \mathcal{M}_{i-1}(n) \to 0$, this H^1 is the obstruction to lifting sections of $\mathcal{M}_{i-1}(n)$ to sections of $\mathcal{M}_i(n)$. So choosing such an

n which is also large enough so that $\mathcal{M}_0(n)$ is generated by its global sections, we can carry out the liftings inductively and thereby obtain the formal case of Theorem 3.2.7. (Alternatively, one can proceed as in Grothendieck [Gr4, III, Cor. 5.2.4], to prove this formal analog of Cartan's Theorem A via a formal analog of Cartan's Theorem B [Gr4, III, Prop. 5.2.3].) □

As indicated above, Grothendieck's Existence Theorem is a strong enough form of "formal GAGA" to be useful in proving formal analogs of (at least parts of) the classical Riemann Existence Theorem. (This will be discussed further in Section 3.3.) But for certain purposes, it is useful to have a variant of Theorem 3.2.4 that allows U_1 and S_1 to be more local. Namely, rather than taking U_1 to be an affine open subset of the closed fibre, and S_1 its formal thickening, we would instead like to take U_1 to be the spectrum of the complete local ring in the closed fibre at some point ω, and S_1 its formal thickening (viz. the spectrum of the complete local ring at ω in V). In the relative dimension 1 case, the "overlap" U_0 of U_1 and U_2 is then the spectrum of the fraction field of the complete local ring at ω in the closed fibre, and S_0 is its formal thickening.

More precisely, in the case that V is of relative dimension 1 over A, there is the following formal patching result. First we introduce some notation and terminology. If ω is a closed point of a variety V_0, then $\mathcal{K}_{V_0,\omega}$ denotes the total ring of fractions of the complete local ring $\hat{\mathcal{O}}_{V_0,\omega}$ (and thus the fraction field of $\hat{\mathcal{O}}_{V_0,\omega}$, if the latter is a domain). Let A be a complete local ring with maximal ideal $\mathfrak{m}$, let V be an A-scheme, and let V_n be the fibre of V over $\mathfrak{m}^{n+1}$ (regarding $V_n \subset V_{n+1}$). Let $\omega \in V_0$, and let ω' denote $\operatorname{Spec} \mathcal{K}_{V_0,\omega}$. Then *the ring of holomorphic functions in V at ω'* is defined to be $\hat{\mathcal{O}}_{V,\omega'} := \varprojlim \mathcal{K}_{V_n,\omega}$. (For example, if $A = k[[t]]$ and V is the affine x-line over A, and if ω is the point $x = t = 0$, then $\omega' = \operatorname{Spec} k((x))$, $\mathcal{K}_{V_n,\omega} = k((x))[t]/(t^{n+1})$, and the ring of holomorphic functions at ω' is $\hat{\mathcal{O}}_{V,\omega'} = k((x))[[t]]$.)

Theorem 3.2.8. *Let V be a proper curve over a complete local ring A, let V_0 be the fibre over the closed point of $\operatorname{Spec} A$, let W be a non-empty finite set of closed points of V_0, and let $U = V_0 - W$. Let W^* be the union of the spectra of the complete local rings $\hat{\mathcal{O}}_{V,\omega}$ for $\omega \in W$. Let $U^* = \operatorname{Spec} \hat{\mathcal{O}}_{V,U}$, and let $W'^* = \bigcup_{\omega \in W} \operatorname{Spec} \hat{\mathcal{O}}_{V,\omega'}$, where $\omega' = \operatorname{Spec} \mathcal{K}_{V_0,\omega}$ as above. Then the base change functor*

$$\mathfrak{M}(V) \to \mathfrak{M}(W^*) \times_{\mathfrak{M}(W'^*)} \mathfrak{M}(U^*)$$

is an equivalence of categories. The same holds for finite algebras and for (Galois) covers.

This result appeared as [Ha6, Theorem 1], in the special case that V is regular, $A = k[[t_1, \dots, t_n]]$ for some field k and some $n \geq 0$, and where attention is restricted to projective modules. The proof involved showing that the appropriate form of Cartan's Lemma is satisfied. In the form above, the result appeared at [Pr1, Theorem 3.4]. There, it was assumed that the complete local ring A is a discrete valuation ring, but that hypothesis was not necessary for the proof there. Namely, the proof there first showed the result for $A/\mathfrak{m}^n$, where $\mathfrak{m}$ is the maximal ideal of A, using Corollary 3.1.9(a) (to the result of Ferrand and Raynaud [FR]); and afterwards used Grothendieck's Existence Theorem (Theorem 3.2.1 above) to pass to A. (This use of [FR] was suggested by L. Moret-Bailly.)

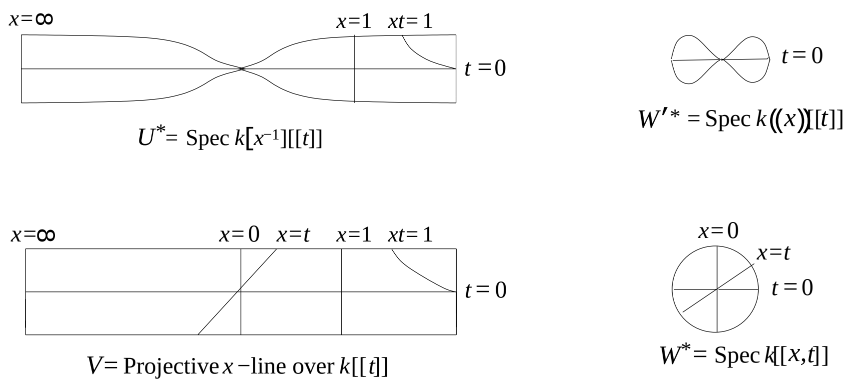

Figure 3.2.9: *Example 3.2.10 of Theorem 3.2.8, with* $V = \mathbb{P}^1_{k[[t]]}$, $W =$ *one point. Here* $\mathbb{P}^1_{k[[t]]}$ *is covered by the small patch* $W^* = \operatorname{Spec} k[[x,t]]$ *and the larger patch* $U^* = \operatorname{Spec} k[1/x][[t]]$*; their overlap is* $W'^* = \operatorname{Spec} k((x))[[t]]$ *(upper right). Compare Fig. 3.1.4.*

Example 3.2.10. Let k be a field, let $A = k[[t]]$, and let $V = \mathbb{P}^1_A$ (the projective x-line over $k[[t]]$), with closed fibre $V_0 = \mathbb{P}^1_k$ over $(t = 0)$. Let W consist of the single point ω where $x = t = 0$. In the notation of Theorem 3.2.8, $W^* = \operatorname{Spec} k[[x,t]]$, which can be viewed as a "small neighborhood" of ω. The formal completion of V along $U := V_0 - W$ is $U^* = \operatorname{Spec} k[1/x][[t]]$, whose "overlap" with W^* is $W'^* = \operatorname{Spec} k((x))[[t]]$. (See Figure 3.2.9.) According to Theorem 3.2.8, giving a coherent module on V is equivalent to giving finite modules over W^* and over U^* together with an isomorphism on their pullbacks ("restrictions") to W'^*. The same holds for covers; and this permits modifying a branched cover of V near ω, e.g. by adding more inertia there (see Remarks 5.1.6(d,e)). $\square$

Example 3.2.11. Let k, A be as in Example 3.2.10, and let V be an irreducible normal curve over A, with closed fibre V_0. Then V_0 is a k-curve which is connected (by Zariski's Connectedness Theorem [Za4, §20, Thm. 14], [Hrt2, III, Cor. 11.3]) but not necessarily irreducible; let $V_1, \dots, V_r$ be its irreducible components. The singular locus of V is a finite subset of V_0, and it includes all the points where irreducible components V_i of V_0 intersect. Let W be a finite subset of V_0 that contains this singular locus, and contains at least one smooth point on each irreducible component V_i of V_0. For $i = 1, \dots, r$ let $W_i = V_i \cap W$, let $U_i = V_i - W_i$, and consider the ring $\hat{\mathcal{O}}_{V,U_i}$ of holomorphic functions along U_i. Also, for each point ω in W, we may consider its complete local ring $\hat{\mathcal{O}}_{V,\omega}$ in V. According to Theorem 3.2.8, giving a coherent module on V is equivalent to giving finite modules over each $\hat{\mathcal{O}}_{V,U_i}$ and over each $\hat{\mathcal{O}}_{V,\omega}$ together with isomorphisms on the "overlaps". See [HS] for a formalization of this set-up. □

Theorem 3.2.8 above can be generalized to allow V to be higher dimensional over the base ring A. In addition, by replacing the result of Ferrand-Raynaud (Proposition 3.1.6) by the related result of Artin (Proposition 3.1.8), one can take a proper morphism $\tilde{V} \to V$ and work over $\tilde{V}$ rather than over V itself. Both of these generalizations are accomplished in the following result:

Theorem 3.2.12. *Let $(A, \mathfrak{m})$ be a complete local ring, let V be a proper A-scheme, and let $f : \tilde{V} \to V$ be a proper morphism. Let W be a finite set of closed points of V; let $\tilde{W} = f^{-1}(W) \subset \tilde{V}$; let $W^* = \bigcup_{\omega \in W} \operatorname{Spec} \hat{\mathcal{O}}_{V,\omega}$; and let $\tilde{W}^* = \tilde{V} \times_V W^*$. Let $\tilde{\mathcal{U}}$ [resp. $\tilde{\mathcal{U}}^*$] be the formal completion of $\tilde{V} - \tilde{W}$ [resp. of $\tilde{W}^* - \tilde{W}$] along its fibre over $\mathfrak{m}$. Then the base-change functor*

$$\mathfrak{M}(\tilde{V}) \to \mathfrak{M}(\tilde{W}^*) \times_{\mathfrak{M}(\tilde{\mathcal{U}}^*)} \mathfrak{M}(\tilde{\mathcal{U}})$$

is an equivalence of categories. The same holds for finite algebras and for (Galois) covers.

Note that the scheme $U^* = \operatorname{Spec} \hat{\mathcal{O}}_{V,U}$ in the statement of Theorem 3.2.8 is replaced in Theorem 3.2.12 by a formal scheme, because the complement of W in the closed fibre of V will no longer be affine, if V is not a curve over its base ring (and so the ring $\hat{\mathcal{O}}_{V,U}$ of Theorem 4 would not be defined here). Similarly, the scheme W'^* in Theorem 3.2.8 is also replaced by a formal scheme in Theorem 3.2.12.

Proof. For $n \geq 0$ let $\tilde{V}_n$ and $\tilde{W}_n^*$ be the pullbacks of $\tilde{V}$ and $\tilde{W}^*$, respectively, over $A_n := A/\mathfrak{m}^{n+1}$. Also, let $\tilde{U}_n = \tilde{V}_n - \tilde{W}$ and $\tilde{U}_n^* = \tilde{W}_n^* - \tilde{W}$; thus the formal schemes $\tilde{\mathcal{U}}, \tilde{\mathcal{U}}^*$ respectively correspond to the inverse systems $\{\tilde{U}_n\}_n, \{\tilde{U}_n^*\}_n$.

For every n, we have by Corollary 3.1.9(b) (to Artin's result, Proposition 3.1.8) that the base change functor

$$\mathfrak{M}(\tilde{V}_n) \to \mathfrak{M}(\tilde{W}_n^*) \times_{\mathfrak{M}(\tilde{U}_n^*)} \mathfrak{M}(\tilde{U}_n)$$

is an equivalence of categories. By definition of coherent modules over a formal scheme, we have that $\mathfrak{M}(\tilde{\mathcal{U}}) = \varprojlim \mathfrak{M}(\tilde{U}_n)$ and $\mathfrak{M}(\tilde{\mathcal{U}}^*) = \varprojlim \mathfrak{M}(\tilde{U}_n^*)$. Moreover, $\tilde{V}$ is proper over A; so Grothendieck's Existence Theorem (Theorem 3.2.1 above) implies that the functor $\mathfrak{M}(\tilde{V}) \to \varprojlim \mathfrak{M}(\tilde{V}_n)$ is an equivalence of categories. So it remains to show that the corresponding assertion holds for $\mathfrak{M}(\tilde{W}^*)$; i.e. that $\mathfrak{M}(\tilde{W}^*) \to \varprojlim \mathfrak{M}(\tilde{W}_n^*)$ is an equivalence of categories.

It suffices to prove this equivalence in the case that W consists of just one point ω; and we now assume that. Let $T = \hat{\mathcal{O}}_{V,\omega}$, and let $\mathfrak{m}_\omega$ be the maximal ideal of T, corresponding to the closed point ω. Also, let $\mathfrak{n} = \mathfrak{m}T \subset T$ (where $\mathfrak{m}$ still denotes the maximal ideal of A). Thus $\mathfrak{n} \subset \mathfrak{m}_\omega$, and so T is complete with respect to $\mathfrak{n}$. Also, $\tilde{W}^*$ is proper over the Noetherian $\mathfrak{n}$-adically complete ring T, and $\tilde{W}_n^*$ is the pullback of $\tilde{W}^* \to W^* = \operatorname{Spec} T$ over $T/\mathfrak{n}^{n+1}$. So it follows from Grothendieck's Existence Theorem 3.2.1 that the desired equivalence $\mathfrak{M}(\tilde{W}^*) \to \varprojlim \mathfrak{M}(\tilde{W}_n^*)$ holds. This proves the result in the case of modules.

The analogs for algebras, covers, and Galois covers follow as before using the General Principle 2.2.4. □

Example 3.2.13. Let k, A be as in Examples 3.2.10 and 3.2.11, and let $V = \mathbb{P}_k^n$ for some $n \geq 1$, with homogeneous coordinates $x_0, \ldots, x_n$. Let W consist of the closed point ω of V where $x_1 = \cdots = x_n = t = 0$, and let $f : \tilde{V} \to V$ be the blow-up of V at ω. Let $V_0 = \mathbb{P}_k^n$ be the closed fibre of V over $(t = 0)$. For $i = 1, \ldots, n$, let U_i be the affine open subset of V_0 given by $x_i \neq 0$, and consider the ring $\hat{\mathcal{O}}_{V,U_i}$ of holomorphic functions along U_i in V. Also consider the complete local ring $\hat{\mathcal{O}}_{V,\omega} = k[[x_1, \ldots, x_n, t]]$ at ω in V, and consider the pullback $\tilde{W}^*$ of $\tilde{V}$ over $\hat{\mathcal{O}}_{V,\omega}$ (whose fibre over the closed point ω is a copy of $\mathbb{P}_k^n$). According to Theorem 3.2.12, giving a coherent module over V is equivalent to giving finite modules over the rings $\hat{\mathcal{O}}_{V,U_i}$, and a coherent module over $\tilde{W}^*$, together with compatible isomorphisms on the overlaps. (This uses that giving a coherent module on the formal completion of $V - W$ along its closed fibre is equivalent to giving compatible modules over the completions at the U_i's; here we also identify $\tilde{V} - f^{-1}(W)$ with $V - W$.)

In particular, if $n = 1$, then $\tilde{V}$ is an irreducible A-curve whose closed fibre consists of two projective lines meeting at one point (one being the proper transform of the given line V_0, and the other being the exceptional divisor). This one-dimensional case is also within the context of Example 3.2.11, and so Theorem 3.2.8 could instead be used. (See also the end of Example 4.2.4 below.) □

Remark 3.2.14. The above formal patching results (Theorems 3.2.4, 3.2.8, 3.2.12) look similar, though differing in terms of what types of "patches" are allowed. In each case, we are given a proper scheme V over a complete local ring A, and the assertion says that if a module is given over each of two patches (of a given form), with agreement on the "overlap", then there is a unique coherent module over V that induces them compatibly. Theorem 3.2.4 (a reformulation of Grothendieck's Existence Theorem) is the basic version of formal patching, modeled after the classical result GAGA in complex patching (see Theorem 2.2.6, where two metric open sets are used as patches). In Theorem 3.2.4, the patches correspond to thickenings along Zariski open subsets of the closed fibre of V; see Example 3.2.3 above and see Figure 3.1.4 for an illustration. This basic type of formal patching will be sufficient for the results of Section 3.3 below, on the realization of Galois groups, via "slit covers".

More difficult results about fundamental groups, discussed in Section 5 below, require Theorems 3.2.8 or 3.2.12 instead of Theorem 3.2.4 (e.g. Theorem 5.1.4 and Theorem 5.3.1 use Theorem 3.2.8, while Theorem 5.3.9 uses Theorem 3.2.12). In Theorem 3.2.8 above, one of the patches is allowed to be much smaller than in Theorem 3.2.4, viz. the spectrum of the complete local ring at a point, if the closed fibre is a curve; see Examples 3.2.10 and 3.2.11 above, and see Figure 3.2.9 above for an illustration. Theorem 3.2.12 is still more general, allowing the closed fibre to have higher dimension, and also allowing a more general choice of "small patch" because of the choice of a proper morphism $\tilde{V} \to V$; see Example 3.2.13 above. The advantage of these stronger results is that the overlap of the patches is "smaller" than in the situation of Theorem 3.2.4, and therefore less agreement is required between the given modules. This gives greater applicability to the patching method, in constructing modules or covers with given properties. (Recall that the similar-looking patching results at the end of Section 3.1, which allow the construction of modules by prescribing them along and away from a given closed set, do not directly give results for covers; but they were used, together with Grothendieck's Existence Theorem, in proving Theorems 3.2.8 and 3.2.12 above.) □

Section 3.3. Formal patching and constructing covers.

The methods of Section 3.2 allow one to construct covers of algebraic curves over various fields other than the complex numbers. The idea is to use the approach of Section 2.3, building "slit covers" using formal patching rather than analytic patching (as was used in Section 2). This will be done by relying on Grothendieck's Existence Theorem, in the form of Theorem 3.2.4. (As will be discussed in Section 5, by using variants of Theorem 3.2.4, in particular Theorems 3.2.8 and 3.2.12, it is possible to make more general constructions as well. See also [Ha6], [St1], [HS1], and [Pr2] for other applications of those stronger patching results, concerning covers with given inertia groups over certain points, or even unramified covers of projective curves.)

The first key result is

Theorem 3.3.1. [Ha4, Theorem 2.3, Corollary 2.4] *Let R be a normal local domain other than a field, such that R is complete with respect to its maximal ideal. Let K be the fraction field of R, and let G be a finite group. Then G is the Galois group of a Galois field extension L of $K(x)$, which corresponds to a Galois branched cover of $\mathbb{P}^1_K$ with Galois group G. Moreover L can be chosen to be regular, in the sense that K is algebraically closed in L.*

Before discussing the proof, we give several examples:

Example 3.3.2. a) Let $K = \mathbb{Q}_p$, or a finite extension of $\mathbb{Q}_p$, for some prime p. Then every finite group is a Galois group over $\mathbb{P}^1_K$ (i.e. of some Galois branched cover of the K-line), and so is a Galois group over $K(x)$.

b) Let k be a field, let n be a positive integer, and let $K = k((t_1, \ldots, t_n))$, the fraction field of $k[[t_1, \ldots, t_n]]$. Then every finite group is a Galois group over $\mathbb{P}^1_K$, and so over $K(x)$.

c) If K is as in Example (b) above, and if $n > 1$, then every finite group is a Galois group over K (and not just over $K(x)$, as above). The reason is that K is separably Hilbertian, by Weissauer's Theorem [FJ, Theorem 14.17]. That is, every separable field extension of $K(x)$ specializes to a separable field extension of K, by setting $x = c$ for an appropriate choice of $c \in K$; such a specialization of a Galois field extension is then automatically Galois. (The condition of being separably Hilbertian is a bit weaker than being Hilbertian, but is sufficient for dealing with Galois extensions. See [FJ, Chapter 11], [Vö, Chapter 1], or [MM, Chapter IV, §1.1] for more about Hilbertian and separably Hilbertian fields.)

This example remains valid more generally, where the coefficient field k is replaced by any Noetherian normal domain A that is complete with respect to a prime ideal. Moreover if A is not a field, then the condition $n > 1$ can even be weakened to $n > 0$. In particular, if K is the fraction field of $\mathbb{Z}[[t]]$ (a field which is much smaller than $\mathbb{Q}((t))$), then every finite group is the Galois group of a regular cover of $\mathbb{P}^1_K$, and is a Galois group over K itself. The proof of this generalization uses formal A-schemes, and parallels the proof of Theorem 1; see [Le].

d) Let K be the ring of algebraic p-adics (i.e. the algebraic closure of $\mathbb{Q}$ in $\mathbb{Q}_p$), or alternatively the ring of algebraic Laurent series in n-variables over a field k (i.e. the algebraic closure of $k(t_1, \ldots, t_n)$ in $k((t_1, \ldots, t_n))$). Then every finite group is a Galois group over $\mathbb{P}^1_K$. More generally this holds if K is the fraction field of R, a normal henselian local domain other than a field. This follows by using Artin's Algebraization Theorem ([Ar3], a consequence of Artin's Approximation Theorem [Ar2]), in order to pass from formal elements to algebraic ones. See [Ha4, Corollary 2.11] for details. In the case of algebraic power series in $n > 1$ variables, Weissauer's Theorem then implies that every finite group is a Galois group over K, as in Example (c). □

Theorem 3.3.1 also implies that all finite groups are Galois groups over $K(x)$ for various other fields K, as discussed below (after the proof).

Theorem 3.3.1 can be proven by carrying over the slit cover construction of Section 2.3 to the context of formal schemes. Before doing so, it is first necessary to construct cyclic covers that can be patched together (as in Example 2.3.2). Rather than using complex discs as in §2.3, we will use "formal open subsets", i.e. we will take the formal completions of $\mathbb{P}^1_R$ along Zariski open subsets of the closed fibre $\mathbb{P}^1_k$ (where k is the residue field of R). In order to be able to use Grothedieck's Existence Theorem to patch these covers together, we will want the cyclic covers to agree on the "overlaps" of these formal completions — and this will be accomplished by having them be trivial on these overlaps (just as in Example 2.3.2).

In order to apply Grothedieck's Existence Theorem, we will use it in the case of Galois branched covers (rather than for modules), as in Theorem 3.2.4. There, it was stated just for two patches U_1, U_2 and their overlap U_0; but by induction, it holds as well for finitely many patches, provided that compatible isomorphisms are given on overlaps (and cf. the statement of Theorem 3.2.1).

Grothedieck's Existence Theorem will be applied to the following proposition, which yields the cyclic covers $Y \to \mathbb{P}^1$ that will be patched together in order to prove Theorem 3.3.1. The desired triviality on overlaps will be guaranteed by the requirement that the closed fibre $\phi_k : Y_k \to \mathbb{P}^1_k$ of the branched cover $\phi : Y \to \mathbb{P}^1_R$ be a *mock cover*; i.e. that the restriction of ϕ_k to each irreducible component of Y_k be an isomorphism. This condition guarantees that if $U \subset \mathbb{P}^1_k$ is the complement of the branch locus of ϕ_k, then the restriction of ϕ_k to U is trivial; i.e. $\phi_k^{-1}(U)$ just consists of a disjoint union of copies of U.

Proposition 3.3.3. [Ha4, Lemma 2.1] *Let $(R, \mathfrak{m})$ be a normal complete local domain other than a field, with fraction field K and residue field $k = R/\mathfrak{m}$. Let $S \subset \mathbb{P}^1_k$ be a finite set of closed points, and let $n > 1$. Then there is a cyclic field extension L of $K(x)$ of degree n, such that the normalization of $\mathbb{P}^1_R$ in L is an n-cyclic Galois branched cover $Y \to \mathbb{P}^1_R$ whose closed fibre $Y_k \to P^1_k$ is a mock cover that is unramified over S.*

Proof. We follow the proof in [Ha4], first observing that we are reduced to the situation that n is a prime power p^r. (Namely, if $n = \prod p_i^{r_i}$, and if $Y_i \to \mathbb{P}^1_R$ are $p_i^{r_i}$-cyclic covers, then we may take Y to be the fibre product of the Y_i's over $\mathbb{P}^1_R$.)

The easiest case is if the field K contains a primitive nth root of unity ζ_n. Then we may take L to be the field obtained by adjoining an nth root of $f(x)(f(x)-\alpha)^{n-1}$, where $f(x) \in R[x]$ does not vanish at any point of S, and where $\alpha \in \mathfrak{m} - \{0\}$. (For example, if k is infinite, we may choose $f(x) = x - c$ for some $c \in R$; compare Example 2.3.2.)

Next, suppose that K does not contain a primitive nth root of unity but that p is not equal to the characteristic of K. Then we can consider $K' = K[\zeta_n]$, and will construct an n-cyclic Kummer extension of $K'(x)$ which descends to a desired extension of $K(x)$. This will be done using constructions in [Slt] to find an element $g(x) \in R[\zeta_n, x]$ such that the extension $y^n - g(x)$ of $R[\zeta_n, x]$ descends to an n-cyclic extension of $R[x]$ whose closed fibre is a mock cover.

Specifically, first suppose that p is odd. Let s be the order of the cyclic group $\mathrm{Gal}(K'/K)$, with generator $\tau : \zeta_n \mapsto \zeta_n^m$. Choose $\alpha \in \mathfrak{m} - \{0\}$ and let $b = f(x)^n - \zeta_n p^2 \alpha$, for some $f(x) \in R[x]$ which does not vanish on S. Let L' be the n-cyclic field extension of $K'(x)$ given by adjoining an nth root of $M(b) = b^{m^{s-1}} \tau(b)^{m^{s-2}} \cdots \tau^{s-2}(b)^m \tau^{s-1}(b)$. Then $L' = L \otimes_K K'$ for some n-cyclic extension L of $K(x)$, by [Slt, Theorem 2.3]. (Note that the branch locus of the associated cover, which is given by $M(b) = 0$, is invariant under τ. Here the various powers of the factors of $M(b)$ are chosen so that τ will commute with the generator of $\mathrm{Gal}(L'/K'(x))$, given by $y \mapsto \zeta_n y$. These two facts enable the Kummer cover of the K'-line to descend to a cyclic cover of the K-line.)

On the other hand, suppose $p = 2$. If K contains a square root of -1 then $\mathrm{Gal}(K'/K)$ is again cyclic, so the same proof as in the odd case works. Otherwise, if $n = 2$ then take the extension of $K(x)$ given by adjoining a square root of $f(x)^2 - 4\alpha$. If $n = 4$, then adjoin a fourth root of $(f(x)^4 + 4i\alpha)^3(f(x)^4 - 4i\alpha)$ to $K'(x)$; this descends to a 4-cyclic extension of $K(x)$ by [Slt, Theorem 2.4]. If $n = 2^r$ with $r \geq 3$, then $\mathrm{Gal}(K'/K)$ is the product of a cyclic group of order 2 with generator $\kappa : \zeta_n \mapsto \zeta_n^{-1}$, and another of order $s \leq 2^{n-2}$ with generator $\zeta_n \mapsto \zeta_n^m$ for some $m \equiv 1 (\mathrm{mod}\, 4)$. Take $b = f(x)^n + 4\zeta_n\alpha$ and $a = b^{2^{n-1}+1}\kappa(b)^{2^{n-1}-1}$; and (in the notation of the odd case) consider the extension of $K'(x)$ given by adjoining an nth root of $M(a)$. By [Slt, Theorem 2.7], this descends to an n-cyclic extension of $K(x)$.

Finally, there is the case that p is equal to the characteristic of K. If $n = p$, we can adjoin a root of an Artin-Schreier polynomial $y^p - f(x)^{p-1}y - \alpha$, where $f(x) \in R[x]$ and $\alpha \in \mathfrak{m} - \{0\}$. More generally, with $n = p^r$, we can use Witt vectors, by adjoining the roots of the Witt coordinates of $\mathrm{Fr}(y) - f(x)^{p-1}y - \alpha$, where $f(x)$ and y denote the elements of the truncated Witt ring $W_r(R[x, y_0, \ldots, y_{r-1}])$ with Witt coordinates $(f(x), 0, \ldots, 0)$ and $(y_0, \ldots, y_n)$ respectively, and where Fr denotes Frobenius.

In each of these cases, one checks that the extension L of $K(x)$ has the desired properties. (See [Ha4, Lemma 2.1] for details.) □

Using this result together with Grothendieck's Existence Theorem (for covers), one easily obtains Theorem 3.3.1:

Proof of Theorem 3.3.1. Let G be a finite group, and let $g_1, \ldots, g_r$ be generators. Let H_i be the cyclic subgroup of G generated by g_i. By Proposition 3.3.3, for each i there is an irreducible normal H_i-Galois cover $Y_i \to \mathbb{P}^1_R$ whose closed fibre is a mock cover of $\mathbb{P}^1_k$; moreover these covers may be chosen inductively so as to have disjoint branch loci B_i (by choosing them so that the branch loci along the closed fibre are disjoint). For $i = 1, \ldots, r$, let $U_i = \mathbb{P}^1_R - \bigcup_{j \neq i} B_j$, let R_i be the ring of holomorphic functions on U_i along its closed fibre (i.e. the $\mathfrak{m}$-adic completion of the ring of functions on U_i), and let $\hat{U}_i = \mathrm{Spec}\, R_i$. Also let $U_0 = \mathbb{P}^1_R - \bigcup_{j=1}^r B_j$ (so that $U_0 = U_i \cap U_j$ for any $i \neq j$), let $H_0 = 1 \subset G$, and let $Y_0 = \mathbb{P}^1_R$. Then the restriction $\hat{Y}_i = Y_i \times_{\mathbb{P}^1_R} \hat{U}_i$ is an irreducible normal H_i-Galois cover, and we may identify the pullback $\hat{Y}_i \times_{\hat{U}_i} \hat{U}_0$ with the trivial cover $\hat{Y}_0 = \mathrm{Ind}_1^{H_i} \hat{U}_0$. Finally, let $\hat{Z}_i = \mathrm{Ind}_{H_i}^G \hat{Y}_i$; this is a (disconnected) G-Galois cover of $\hat{U}_i$, equipped with an isomorphism $\hat{Z}_i \times_{\hat{U}_i} \hat{U}_0 \xrightarrow{\sim} \hat{Z}_0$. By Grothendieck's Existence Theorem for covers (see Theorem 3.2.4), there is a unique G-Galois cover $Z \to \mathbb{P}^1_R$ whose restriction to $\hat{U}_i$ is $\hat{Z}_i$, compatibly. This cover is connected since its closed fibre is (because $H_1, \ldots, H_r$ generate G); it is normal since each $\hat{Z}_i$ is; and so it is irreducible (being connected and normal). The closed fibre of Z is a mock cover (and so reducible), since the same is true for each $\hat{Z}_i$; and so K is algebraically closed in the function field L of Z. So L is as desired in Theorem 3.3.1. □

Remark 3.3.4. A variant approach to Theorem 3.3.1 involves proving a modification of Proposition 3.3.3 — viz. requiring that Y_k contains a k-point that is not in the ramification locus of $Y_k \to \mathbb{P}^1_k$, rather than requiring that $Y_k \to \mathbb{P}^1_k$ is a mock cover. This turns out to be sufficient to obtain Theorem 3.3.1, e.g. by showing that after a birational change of variables on $\mathbb{P}^1$, the cover Y is taken to a cover whose closed fibre is a mock cover (and thereby recapturing the original proposition above). This modified version of the proposition can be proven by first showing that there is *some* n-cyclic extension of $K(x)$, e.g. as in [FJ, Lemma 24.46]; and then adjusting the extension by a "twist" in order to obtain an unramified rational point [HV, Lemma 4.2(a)]. (In general, this twisting method works for abelian covers, and so in particular for cyclic covers.) This modified proposition first appeared in [Li], where it was used to provide a proof of Theorem 3.3.1 using rigid analytic spaces, rather than formal schemes. See Theorem 4.3.1 below for a further discussion of this. □

As mentioned just after the statement of Theorem 3.3.1 above, that result can be used to deduce that many other fields K have the same inverse Galois property, even without being complete. In particular:

Corollary 3.3.5. [Ha3, Corollary 1.5] *Let k be an algebraically closed field. Then every finite group is a Galois group over $k(x)$; or equivalently, it is the Galois group of some branched cover of the k-line.*

In the case of $k = \mathbb{C}$, this result is classical, and was the subject of Section 2 above, where the proof involved topology and analytic patching. For a more general algebraically closed field, the proof uses Theorem 3.3.1 above and a trick that relies on the fact that every finite extension is given by finitely many polynomials (also used in the remark after Corollary 2.1.5):

Proof of Corollary 3.3.5. Let $R = k[[t]]$ and $K = k((t))$. Applying Theorem 3.3.1 to R and a given finite group G, we obtain an irreducible G-Galois branched cover $Y \to \mathbb{P}^1_K$ such that K is algebraically closed in its function field. This cover is of finite type, and so it is defined (as a G-Galois cover) over a k-subalgebra A of K of finite type; i.e. there is an irreducible G-Galois branched cover $Y_A \to \mathbb{P}^1_A$ such that $Y_A \times_A K \approx Y$ as G-Galois branched covers of $\mathbb{P}^1_K$. By the Bertini-Noether Theorem [FJ, Prop. 9.29], there is a non-zero element $\alpha \in A$ such that the specialization of Y_A to any k-point of $\operatorname{Spec} A[\alpha^{-1}]$ is (geometrically) irreducible. Any such specialization gives an irreducible G-Galois branched cover of $\mathbb{P}^1_k$. □

In fact, as F. Pop later observed [Po4], the proof of the corollary relied on k being algebraically closed only to know that every k-variety with a $k((t))$-point has a k-point. So for any field k with this more general property (a field k that is "existentially closed in $k((t))$"), the corollary holds as well. Moreover the resulting Galois extension of $k(x)$ can be chosen to be regular, i.e. with k algebraically closed in the extension, by the geometric irreducibility assertion in the Bertini-Noether Theorem. Pop proved [Po4, Proposition 1.1] that the fields k that are existentially closed in $k((t))$ can be characterized in another way: they are precisely those fields k with the property that every smooth k-curve with a k-rational point has infinitely many k-rational points. He called such fields "large", because they are sufficiently large within their algebraic closures in order to recapture the finite-type argument used in the above corollary. (In particular, if k is large, then any extension field of k, contained in the algebraic closure of k, is also large [Po4, Proposition 1.2].) Thus we obtain the following strengthening of the corollary:

Theorem 3.3.6 [Po4] *Let k be a large field, and let G be a finite group.*

a) Then G is the Galois group of a Galois field extension L of $k(x)$, and the extension may be chosen to be regular.

b) If k is (separably) Hilbertian, then G is a Galois group over k.

Here part (b) follows from part (a) as in Example 3.3.2(c).

Example 3.3.7. a) Let K be a complete valuation field. Then K is large by [Po4, Proposition 3.1], the basic idea being that K satisfies an Implicit Function Theorem (and so one may move a K-rational point a bit to obtain other K-rational points). So every finite group is a Galois group over $K(x)$, by Theorem 3.3.6. In particular, this is true for the fraction field K of a complete discrete valuation ring R — as was already shown in Theorem 3.3.1. On the other hand, Theorem 3.3.6 applies to complete valuation fields K that are not of that form.

b) More generally, a henselian valued field K (i.e. the fraction field of a henselian valuation ring) is large by [Po4, Proposition 3.1]. So again, every finite group is a Galois group over $K(x)$. If the valuation ring is a discrete valuation ring, then this conclusion can also be deduced using the Artin Algebraization Theorem, as in Example 3.3.2(d). But as in Example (a) above, K is large even if it is not discretely valued (in which case the earlier example does not apply).

c) It is immediate from the definition that a field k will be large if it is PAC (pseudo-algebraically closed); i.e. if every smooth geometrically integral k-variety has a k-point. Fields that are PRC (pseudo-real closed) or PpC (pseudo-p-adically closed) are also large. In particular, the field of all totally real algebraic numbers is large, and so is the field of totally p-adic algebraic numbers (i.e. algebraic numbers α such that $\mathbb{Q}(\alpha)$ splits completely over the prime p). Hence every finite group is a Galois group over $k(x)$, where k is any of the above fields. And if k is Hilbertian (as some PAC fields are), then every finite group is therefore a Galois group over k. See [Po4, Section 3] and [MB1, Thm. 1.3] for details.

d) Let K be a field that contains a large subfield K'. If K is algebraic over K' then K is automatically large [Po4, Proposition 1.2]; but otherwise K need not be large (e.g. $\mathbb{C}(t)$

is not large). Nevertheless, every finite group is the Galois group of a regular branched cover of $\mathbb{P}^1_K$. The reason is that this property holds for K'; and the function field F of the cover of $\mathbb{P}^1_{K'}$ is linearly disjoint from K over K', because K' is algebraically closed in F (by regularity). In particular, we may use this approach to deduce Theorem 3.3.1 from Theorem 3.3.2, since every normal complete local domain R other than a field must contain a complete discrete valuation ring R_0 — whose fraction field is large. (Namely, if R contains a field k, then take $R_0 = k[[t]]$ for some non-zero element t in the maximal ideal of R; otherwise, R contains $\mathbb{Z}_p$ for some p.) Similarly, we may recover Example 3.3.2(d) in this way (taking the algebraic Laurent series in $k((t_1)))$, even though it is not known whether $k((t_1, \ldots, t_n))$ and its subfield of algebraic Laurent series are large. (Note that $k((t_1, \ldots, t_n))$ is not a valuation field for $n > 1$, unlike the case of $n = 1$.) □

Remark 3.3.8. a) An arithmetic analog of Example 3.3.7(b) holds for the ring $T = \mathbb{Z}\{t\}$ of power series over $\mathbb{Z}$ convergent on the open unit disc. Namely, replacing Grothendieck's Existence Theorem by its arithmetic analog discussed in Remark 3.2.5(c) above, one obtains an analog of Theorem 3.3.1 above for $\mathbb{Z}\{t\}$ [Ha5, Theorem 3.7]; i.e. that every finite group is a Galois group over the fraction field of $\mathbb{Z}\{t\}$ (whose model over Spec $\mathbb{Z}\{t\}$ has a mock fibre modulo (t)). Moreover, the construction permits one to construct the desired Galois extension L of $\operatorname{frac} T$ so that it remains a Galois field extension, with the same Galois group, even after tensoring with the fraction field of $T_r = \mathbb{Z}_{r+}[[t]]$, the ring of power series over $\mathbb{Z}$ convergent on a neighborhood of the closed disc $|t| \leq r$. (Here $0 < r < 1$.) Even more is true: Using an arithmetic analog of Artin's Approximation Theorem (see [Ha5, Theorem 2.5]), it follows that these Galois extensions L_r of T_r can simultaneously be descended to a compatible system of Galois extensions L_r^{h} of $\operatorname{frac} T_r^{\mathrm{h}}$, where T_r^{h} is the ring of *algebraic* power series in T_r. Surprisingly, the intersection of the rings T_r^{h} has fraction field $\mathbb{Q}(t)$ [Ha2, Theorem 3.5] (i.e. every algebraic power series over $\mathbb{Z}$ that converges on the open unit disc is *rational*). So since the Galois extensions L, L_r, L_r^{h} (for $0 < r < 1$) are all compatible, this suggests that it should be possible to descend the system $\{L_r^{\mathrm{h}}\}$ to a Galois extension L^{h} of $\mathbb{Q}(t)$. If this could be done, it would follow that every finite group would be a Galois group over $\mathbb{Q}(t)$ and hence over $\mathbb{Q}$ (since $\mathbb{Q}$ is Hilbertian). See [Ha5, Section 4] for a further discussion of this (including examples that demonstrate pitfalls).

b) The field $\mathbb{Q}^{\mathrm{ab}}$ (the maximal abelian extension of $\mathbb{Q}$) is known to be Hilbertian [Vö, Corollary 1.28] (and in fact any abelian extension of a Hilbertian field is Hilbertian [FJ, Theorem 15.6]). It is *conjectured* that $\mathbb{Q}^{\mathrm{ab}}$ is large; and if it is, then Theorem 3.3.6(b) above would imply that every finite group is a Galois group over $\mathbb{Q}^{\mathrm{ab}}$. Much more is believed: The Shafarevich Conjecture asserts that the absolute Galois group of $\mathbb{Q}^{\mathrm{ab}}$ is a *free* profinite group on countably many generators. This conjecture has been posed more generally, to say that if K is a global field, then the absolute Galois group of K^{cycl} (the maximal cyclotomic extension of K) is a free profinite group on countably many generators. (Recall that $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}^{\mathrm{cycl}}$, by the Kronecker-Weber Theorem in number theory.) The Shafarevich Conjecture (along with its generalization to arbitrary number fields) *remains open* — though it too would follow from knowing that $\mathbb{Q}^{\mathrm{ab}}$ is large (see Section 5). On the other hand, the generalized Shafarevich Conjecture has been proven in the geometric case, i.e. for function fields of curves [Ha10] [Po1] [Po3]; see Section 5 for a further discussion of this. □

As another example of the above ideas, consider covers of the line over *finite* fields. Not surprisingly (from the terminology), finite fields $\mathbb{F}_q$ are not large. And it *is unknown* whether every finite group G is a Galois group over $k(x)$ for every finite field k. But it is known that every finite group G is a Galois group over $k(x)$ for *almost* every finite field k:

Proposition 3.3.9. (Fried-Völklein, Jarden, Pop) *Let G be a finite group. Then for all but finitely many finite fields k, there is a regular Galois field extension of $k(x)$ with Galois group G.*

Proof. First consider the case that k ranges just over prime fields $\mathbb{F}_p$. By Example 3.3.2(d) (or by Theorem 3.3.6 and Example 3.3.7(b) above), G is a regular Galois group over the field $\mathbb{Q}((t))^{\mathrm{h}}(x)$, where $\mathbb{Q}((t))^{\mathrm{h}}$ is the field of algebraic Laurent series over $\mathbb{Q}$ (the t-adic henselization of $\mathbb{Q}(t)$). Such a G-Galois field extension is finite, so it descends to a G-Galois field extension of $K(x)$, where K is a finite extension of $\mathbb{Q}(t)$ (in which $\mathbb{Q}$ is algebraically closed, since $K \subset \mathbb{Q}((t))$). This extension of $K(x)$ can be interpreted as the function field of a G-Galois branched cover $Z \to \mathbb{P}^1_V$; here V is a smooth projective curve over $\mathbb{Q}$ with function field K, viz. a finite branched cover of the t-line, say of genus g (see Figure 3.3.10). For all points $\nu \in V$ outside some finite set Σ, the fibre of Z over ν is an irreducible G-Galois cover of $\mathbb{P}^1_{k(\nu)}$, where $k(\nu)$ is the residue field at ν. By taking a normal model $\mathcal{Z} \to \mathcal{V}$ of $Z \to V$ over $\mathbb{Z}$, we may consider the reductions V_p and Z_p for any prime p. For all primes p outside some finite set S, the reduction V_p is a smooth connected curve over $\mathbb{F}_p$ of genus g; the reduction Z_p is an irreducible G-Galois branched cover of $\mathbb{P}^1_{V_p}$; and any specialization of this cover away from the reduction Σ_p of Σ is an irreducible G-Galois cover of the line. According to the Weil bound in the Riemann Hypothesis for curves over finite fields [FJ, Theorem 3.14], the number of k-points on a k-curve of genus g is at least $|k| + 1 - 2g\sqrt{|k|}$. So for all $p \notin S$ with $p > (2g + \deg(\Sigma))^2$, the curve Z_p has an $\mathbb{F}_p$-point that does not lie in the reduction of Σ. The specialization at that point is a regular G-Galois cover of $\mathbb{P}^1_{\mathbb{F}_p}$, corresponding to a regular G-Galois field extension of $\mathbb{F}_p(x)$.

For the general case, observe that if G is a regular Galois group over $\mathbb{F}_p(x)$, then it is also a regular Galois group over $\mathbb{F}_q(x)$ for every power q of p (by base change). Now consider the finitely many primes p such that G is not known to be a regular Galois group over $\mathbb{F}_p(x)$. Arguing as above (but using $\mathbb{F}_p((t))$ instead of $\mathbb{Q}((t))$), we obtain a geometrically irreducible G-Galois cover $Y_p \to \mathbb{P}^1_{W_p}$, for some $\mathbb{F}_p$-curve W_p. Again using

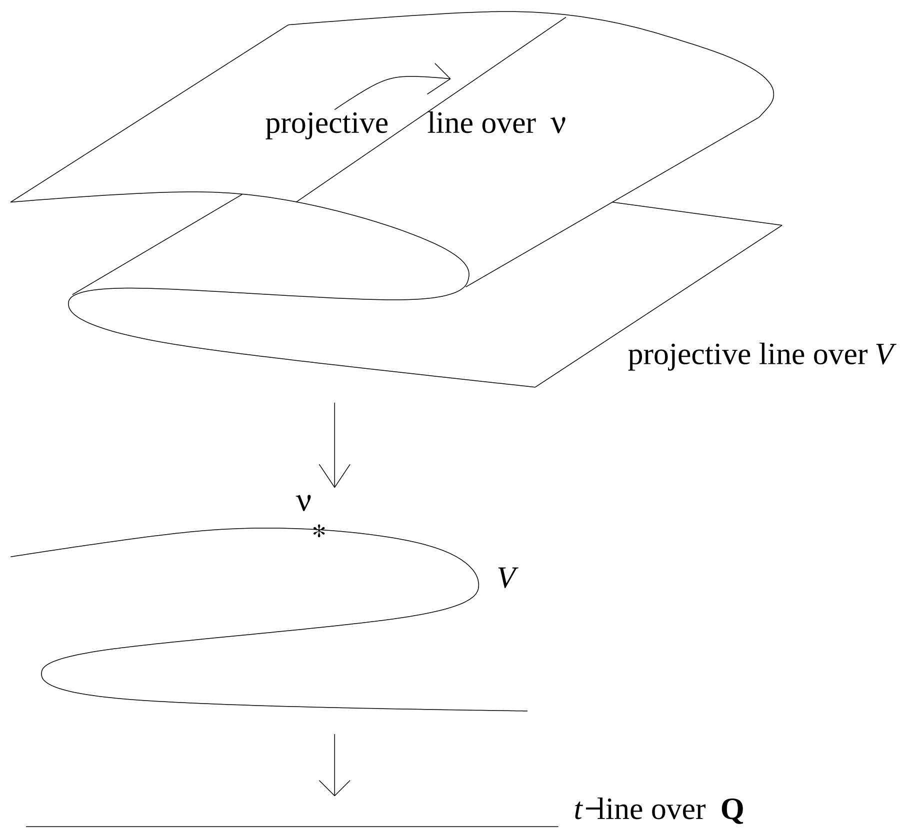

Figure 3.3.10: *Base of the Galois cover $Z \to \mathbb{P}^1_V$ in the first case of the proof of Proposition 3.3.9. For most choices of ν in V, the restriction of Z over ν is an irreducible cover of the projective line; and for most primes p, the same is true for its reduction mod p.*

the Weil bound, there is a constant c_p such that if q is a power of p and $q > c_p$, then W_p has an $\mathbb{F}_q$-point at which Y_p specializes to a regular G-Galois cover of $\mathbb{P}^1_{\mathbb{F}_q}$. So if c is chosen larger than each of the finitely many c_p's (as p ranges over the exceptional set of primes), then G is a regular Galois group over $k(x)$ for every finite field k of order $\geq c$. □

Remark 3.3.11. a) The above result can also be proven via ultraproducts, viz. using that a non-principal ultraproduct of the $\mathbb{F}_q$'s is large (and even PAC); see [FV1, §2.3, Cor. 2]. In [FV1], just the case of prime fields was shown. But Pop showed that the conclusion holds for general finite fields (as in the statement of Proposition 3.3.9), using ultraproducts.

b) It is *conjectured* that in fact there are *no* exceptional finite fields in the above result, i.e. that every finite group is a Galois group over each $\mathbb{F}_q(x)$. But at least, it would be desirable to have a better understanding of the possible exceptional set. For this, one could try to make more precise the sets S and Σ in the above proof, and also the bound on the exceptional primes. (The bound in the above proof is certainly not optimal.) □

Remark 3.3.12. a) The class of large fields also goes under several other names in the literature. Following the introduction of this notion by Pop in [Po4] under the name "large", D. Haran and M. Jarden referred to such fields as "ample" [HJ1]; P. Dèbes and B. Deschamps called them fields with "automatique multiplication des points lisses existants" (abbreviated AMPLE) [DD]; J.-L. Colliot-Thélène has referred to such fields as "epais" (thick); L. Moret-Bailly has called them "fertile" [MB2]; and the present author has even suggested that they be called "pop fields", since the presence of a single smooth rational point on a curve over such a field implies that infinitely many rational points will "pop up".

b) By whatever name, large fields form the natural context to generalize Corollary 3.3.5 above. As noted in Example 3.3.7(d), the class of fields K that *contain* large subfields also has the property that every finite group is a regular Galois group over $K(x)$; and this class is general enough to subsume Theorem 3.3.1, as well as Theorem 3.3.6. On the other hand, this Galois property holds for the fraction field of $\mathbb{Z}[[t]]$, as noted at the end of Example 3.3.2(c); but that field is not known to contain a large subfield. Conjecturally, *every* field K has the regular Galois realization property (see [Ha9, §4.5]; this conjecture has been referred to as the regular inverse Galois problem). But that degree of generality seems *very far from being proved* in the near future.

c) In addition to yielding regular Galois realizations, large fields have a stronger property: that *every finite split embedding problem is properly solvable* (Theorem 5.1.9 below). *Conjecturally*, all fields have this property (and this conjecture subsumes the one in Remark (b) above). See Section 5 for more about embedding problems, and for other results in Galois theory that go beyond Galois realizations over fields. The results there can be proven using patching theorems from Section 3.2 (including those at the end of §3.2, which are stronger than Grothendieck's Existence Theorem). □

We conclude this section with a reinterpretation of the above patching construction in terms of thickening and deformation. Namely, as discussed after Theorem 3.2.1 (Grothendieck's Existence Theorem), that earlier result can be interpreted either as a patching result or as a thickening result. Theorem 3.3.1 above, and its Corollary 3.3.5, relied on Grothendieck's Existence Theorem, and were presented above in terms of patching. It is instructive to reinterpret these results in terms of thickening, and to compare these results from that viewpoint with the slit cover construction of complex covers, discussed in Section 2.3.

Specifically, the proof of Theorem 3.3.1 above yields an irreducible normal G-Galois cover $Z \to \mathbb{P}^1_R$ whose closed fibre is a connected mock cover $Z_0 \to \mathbb{P}^1_k$. Viewing $\operatorname{Spec} R$ as a "small neighborhood" of $\operatorname{Spec} k$, we can regard $\mathbb{P}^1_R$ as a "tubular neighborhood" of $\mathbb{P}^1_k$; and the construction of $Z \to \mathbb{P}^1_R$ can be viewed as a thickening (or deformation) of $Z_0 \to \mathbb{P}^1_k$, built in such a way that it becomes irreducible (by making it locally irreducible near each of the branch points). Regarding formal schemes as thickenings of their closed fibres (given by a compatible sequence of schemes over the $R/\mathfrak{m}^i$), this construction be viewed as the result of infinitesimal thickenings (over each $R/\mathfrak{m}^i$) which in the limit give the desired cover of $\mathbb{P}^1_R$.

From this point of view, Corollary 3.3.5 above can be viewed as follows: As before, take $R = k[[t]]$ and as above obtain an irreducible normal G-Galois cover $Z \to \mathbb{P}^1_R$. Since this cover is of finite type, it is defined over a $k[t]$-subalgebra E of R of finite type (i.e. there is a normal irreducible G-Galois cover $Z_E \to \mathbb{P}^1_E$ that induces $Z \to \mathbb{P}^1_R$), such that there is a maximal ideal $\mathfrak{n}$ of E with the property that the fibre of $Z_E \to \mathbb{P}^1_E$ over the corresponding point $\xi_{\mathfrak{n}}$ is isomorphic to the closed fibre of $Z \to \mathbb{P}^1_R$ (viz. it is the mock cover $Z_0 \to \mathbb{P}^1_k$). The cover $Z_E \to \mathbb{P}^1_E$ can be viewed as a family of covers of $\mathbb{P}^1_k$, parametrized by the variety $V = \operatorname{Spec} E$, and which provides a deformation of $Z_0 \to \mathbb{P}^1_k$. A generically chosen member of this family will be an irreducible cover of $\mathbb{P}^1_k$, and this G-Galois cover is then as desired.

In the case that $k = \mathbb{C}$, we can be even more explicit. There, we are in the easy case of Proposition 3.3.3 above, where the field contains the roots of unity, ramification is cyclic, and cyclic extensions are Kummer. So choosing generators $g_1, \ldots, g_r$ of G of orders $n_1, \ldots, n_r$, and choosing corresponding branch points $x = a_1, \ldots, a_r$ for the mock cover $Z_0 \to \mathbb{P}^1_{\mathbb{C}}$, we may choose $Z \to \mathbb{P}^1_R$ so that it is given locally by the (normalization of the) equation $z_i^{n_i} = (x - a_i)(x - a_i - t)^{n_i - 1}$ in a neighborhood of a point over $x = a_i, t = 0$ (and so the mock cover is given locally by $z_i^{n_i} = (x - a_i)^{n_i}$). By Artin's Algebraization Theorem [Ar3] (cf. Example 3.3.2(d) above), this cover descends to a cover $Z \to \mathbb{P}^1_{R^{\mathrm{h}}}$, where $R^{\mathrm{h}} \subset R = \mathbb{C}[[t]]$ is the ring of algebraic power series. Since that cover is of finite type, it can be defined over a $\mathbb{C}[t]$-subalgebra of R^{h} of finite type; i.e. the cover further descends to a cover $Y_C \to \mathbb{P}^1_C$, where C is a complex curve together with a morphism $C \to \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$, and where the fibre of Y_C over some point $\xi \in C$ over $t = 0$ is the given mock cover $Z_0 \to \mathbb{P}^1_{\mathbb{C}}$. This family $Y_C \to \mathbb{P}^1_C$ can be viewed as a family of covers of $\mathbb{P}^1_{\mathbb{C}}$ deforming the mock cover; and this deformation takes place by allowing the positions of the branch points to move. By the choice of local equations, if we take a typical point on C near ξ, the corresponding cover has $2r$ branch points $x = a_1, a'_1, \ldots, a_r, a'_r$, with branch cycle description

$$(g_1, g_1^{-1}, \ldots, g_r, g_r^{-1}) \tag{$*$}$$

(see Section 2.1 and the beginning of Section 2.3 for a discussion of branch cycle descriptions). So this is a slit cover, in the sense of Example 2.3.2. See also the discussion following that example, concerning the role of the mock cover as a degeneration of the typical member of this family (in which a'_i is allowed to coalesce with a_i).

For more general fields k, we may not be in the easy case of Proposition 3.3.3, and so may have to use more complicated branching configurations. As a result, the deformed covers may have more than $2r$ branch points, and they may come in clusters rather than in pairs. Moreover, while the tamely ramified branch points will move in $\mathbb{P}^1$ as one deforms the cover, wildly ramified branch points can stay at the same location (with just the Artin-Schreier polynomial changing; see the last case in the proof of Proposition 3.3.3).

Still, in the tame case, by following this construction with a further doubling of branch points, it is possible to pair up the points of the resulting branch locus so that the resulting cover has "branch cycle description" of the form $(h_1, h_1^{-1}, \ldots, h_N, h_N^{-1})$, where each h_i is a power of some generator g_j. (Here, since we are not over $\mathbb{C}$, the notion of branch cycle description will be interpreted in the weak sense that the entries of the description are generators of inertia groups at some ramification points over the respective branch points.) This leads to a generalization of the "half Riemann Existence Theorem" (Theorem 2.3.5) from $\mathbb{C}$ to other fields. Such a result (though obtained using the rigid approach rather than the formal approach) was proven by Pop [Po2]; see Section 4.3 below.

The construction in the tame case can be made a bit more general by allowing the r branch points $x = a_i$ of the mock cover to be deformed with respect to independent variables. For example, in the case $k = \mathbb{C}$, we can replace the ring R by $k[[t_1, t_1', \ldots, t_r, t_r']]$ and use the (normalization of the) local equation $z_i^{n_i} = (x - a_i - t_i)(x - a_i - t_i')^{n_i - 1}$ in a neighborhood of a point over $x = a_i$ on the closed fibre $\underline{t} = \underline{t}' = 0$. Using Artin's Algebraization Theorem, we obtain a $2r$-dimensional family of covers that deform the given mock cover, with each of the r mock branch points splitting in two, each moving independently. The resulting family $Z \to \mathbb{P}^1_V$ is essentially a component of a *Hurwitz family* of covers (e.g. see [Fu1] and [Fr1]), which is by definition a total family $Y \to \mathbb{P}^1_H$ of covers of $\mathbb{P}^1$ over the moduli space H for branched covers with a given branch cycle description and variable branch points (the *Hurwitz space*). Here, however, a given cover is permitted to appear more than once in the family (though only finitely often), and part of the boundary of the Hurwitz space is included (in particular, the point of the parameter space V corresponding to the mock cover). That is, there is a finite-to-one morphism $V \to \bar{H}$, where $\bar{H}$ is the compactification of H. From this point of view, the desirability of using branch cycle descriptions of the form $(*)$ is that one can begin with an easily constructed mock cover, and use it to construct algebraically a component of a Hurwitz space with this branch cycle description. See [Fr3] for more about this point of view.

As mentioned above, still more general formal patching constructions of covers can be performed if one replaces Grothendieck's Existence Theorem by the variations at the end of Section 3.2. In particular, one can begin with a given irreducible cover, and then modify it near one point (e.g. by adding ramification there). Some constructions along these lines will be discussed in Section 5, in connection with the study of fundamental groups.