Fundamental groups of moduli and the Grothendieck-Teichmüller group

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Abstract. Let $\mathcal{M}_{0,n}$ denote the moduli space of Riemann spheres with n ordered marked points. In this article we define the group $\mathrm{Out}_n^\sharp$ of *quasi-special symmetric outer automorphisms* of the algebraic fundamental group $\widehat{\pi}_1(\mathcal{M}_{0,n})$ for all $n \geq 4$ to be the group of outer automorphisms respecting the conjugacy classes of the inertia subgroups of $\widehat{\pi}_1(\mathcal{M}_{0,n})$ and commuting with the group of outer automorphisms of $\widehat{\pi}_1(\mathcal{M}_{0,n})$ obtained by permuting the marked points. Our main result states that $\mathrm{Out}_n^\sharp$ is isomorphic to the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$ for all $n \geq 5$.

§0. Introduction

0.1. The main result

In this paper we prove an isomorphism of two groups that occur naturally in the study of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via the ideas laid out in Grothendieck's *Esquisse d'un Programme* [G1]. One of these is a certain subgroup $\mathrm{Out}_n^\sharp$ of the outer automorphism group of the fundamental group of the moduli space $\mathcal{M}_{0,n}$ of n-pointed curves of genus 0. The other group is $\widehat{\mathrm{GT}}$, the Grothendieck-Teichmüller group introduced by Drinfel'd. We give the definitions and state the main result in 0.1. In 0.2, we provide additional background and motivation concerning automorphism groups of fundamental groups of moduli spaces, and in 0.3 we discuss $\widehat{\mathrm{GT}}$-actions on various avatars of the Teichmüller tower, and give an overview of the paper.

For $n \geq 4$, the *pure mapping class group* $K(0,n)$ (cf. the Appendix) is the topological fundamental group of the moduli space $\mathcal{M}_{0,n}$ of Riemann spheres with n ordered marked points. It is generated by elements x_{ij} (for $1 \leq i < j \leq n$) corresponding to the i-th marked point winding once around the j-th point (these being the canonical generators of the inertia subgroups of $K(0,n)$). The symmetric group S_n acts on $\mathcal{M}_{0,n}$ by permuting the order of the marked points, and so induces outer automorphisms of $K(0,n)$ and its profinite completion $\widehat{K}(0,n)$.

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Definition. For $n \geq 4$, let $\mathrm{Out}_n^\sharp$ be the subgroup of outer automorphisms $\overline{F} \in \mathrm{Out}\big(\widehat{K}(0,n)\big)$ such that

(i) $\overline{F}$ is *quasi-special*, i.e. there exists $\lambda \in \mathbb{Z}^*$ such that $\overline{F}$ sends the conjugacy class of x_{ij} to the conjugacy class of x_{ij}^λ, for each i, j;

(ii) $\overline{F}$ is *symmetric*, i.e. $\overline{F}$ commutes with the image of S_n in $\mathrm{Out}\big(\widehat{K}(0,n)\big)$.

Here $\mathrm{Out}_n^\sharp$ contains (a copy of) $G_\mathbb{Q}$ as a subgroup, because the natural outer action of $G_\mathbb{Q}$ on $\widehat{K}(0,n)$ is faithful [B] and satisfies (i) and (ii) (cf. 0.2).

Let us recall the definition of the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$, which was introduced by Drinfel'd in [D,§4], in connection with the theory of Hopf algebras. Let $\widehat{\mathbb{Z}}$ and $\widehat{F}_2$ denote the profinite completions of $\mathbb{Z}$ and the free group $F_2 = \langle x, y\rangle$ respectively, and let $\widehat{F}_2'$ denote the derived subgroup of $\widehat{F}_2$. For all $f \in \widehat{F}_2'$ and a, b in a profinite group G, let $f(a,b)$ denote the image of f under the homomorphism $\widehat{F}_2 \to G$ sending $x \mapsto a$ and $y \mapsto b$. Consider the following three conditions on pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$:

$$f(y,x)f(x,y) = 1, \tag{I}$$

$$f(z,x)z^m f(y,z)y^m f(x,y)x^m = 1, \tag{II}$$

$$f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51}) = 1. \tag{III}$$

Here the first two relations take place in the free profinite group $\widehat{F}_2 = \langle x, y, z \,|\, xyz = 1\rangle$ with $m = (\lambda-1)/2$, and the third relation takes place in $\widehat{K}(0,5)$. Let $\widehat{\mathrm{GT}}_0$ (resp. $\widehat{\mathrm{GT}}$) be the set of pairs (λ, f) satisfying relations (I) and (II) (resp. (I), (II) and (III)), and such that the pair (λ, f) induces an automorphism F of $\widehat{F}_2$ via $x \mapsto x^\lambda$, $y \mapsto f^{-1}y^\lambda f$. Note that such an F determines (λ, f), since λ is recovered by $F(x) = x^\lambda$ and f is determined by $F(y)$ using that $f \in \widehat{F}_2'$. Considering elements of $\widehat{\mathrm{GT}}_0$ and $\widehat{\mathrm{GT}}$ as automorphisms of $\widehat{F}_2$ gives these sets a natural group structure [D]. The main result of this article is the following:

Main Theorem. *(a)* $\mathrm{Out}_4^\sharp \simeq \widehat{\mathrm{GT}}_0$*;*
(b) $\mathrm{Out}_n^\sharp \simeq \widehat{\mathrm{GT}}$ *for* $n \geq 5$.

This result has the following corollary, which is an analogue for profinite groups of results of Ihara [I2], [I3] on pro-ℓ groups and Lie algebras (cf. the end of 0.2).

Corollary. *The groups* $\mathrm{Out}_n^\sharp$ *are all isomorphic for* $n \geq 5$*, and there is an injection* $\mathrm{Out}_5^\sharp \hookrightarrow \mathrm{Out}_4^\sharp$.

This theorem is a strengthening of Drinfel'd's original observations in [D] about the $\widehat{\mathrm{GT}}$-actions on braid groups. Namely, in that paper Drinfel'd introduced not only the above profinite group $\widehat{\mathrm{GT}}$, but also (cf. [D,p.845-846]) a pro-unipotent version $\mathrm{GT}(k)$ for a characteristic 0 field k (as well as a pro-ℓ version of $\widehat{\mathrm{GT}}$ much studied by Ihara and others; cf. below). He showed that the group $\mathrm{GT}(k)$ acts on a k-pro-unipotent version of the Artin braid group B_n (cf. [D,4.13]). Specifically, if $\sigma_1, \ldots, \sigma_{n-1}$ denote the standard generators of B_n (cf. the Appendix), then under Drinfel'd's action, a pair (λ, f) sends

$$\sigma_1 \mapsto \sigma_1^{\lambda}, \quad \sigma_i \mapsto f(y_i, \sigma_i^2)^{-1} \sigma_i^{\lambda} f(y_i, \sigma_i^2), \text{ for } 2 \leq i \leq n-1, \tag{1}$$

where $y_i = \sigma_{i-1} \cdots \sigma_1 \cdot \sigma_1 \cdots \sigma_{i-1}$ for $2 \leq i \leq n$. Formula (1) gives an action also in the profinite and pro-ℓ contexts (cf. [IM, Appendix] or [LS]). Thus, $\widehat{\mathrm{GT}}$ acts on each profinite Artin braid group $\widehat{B}_n$; and this action induces a $\widehat{\mathrm{GT}}$-action on the subquotient $\widehat{K}(0,n)$ of $\widehat{B}_n$ (cf. the Appendix to this paper). Thus (1) induces natural homomorphisms $\widehat{\mathrm{GT}} \to \mathrm{Aut}\big(\widehat{K}(0,n)\big)$ and $\widehat{\mathrm{GT}} \to \mathrm{Out}\big(\widehat{K}(0,n)\big)$. Our Main Theorem (b) asserts that this latter map is an isomorphism of $\widehat{\mathrm{GT}}$ onto $\mathrm{Out}_n^\sharp$ for $n \geq 5$. Meanwhile, there is a natural action of $\widehat{\mathrm{GT}}_0$ on $\widehat{K}(0,4)$, via the identification of $\widehat{K}(0,4)$ with $\widehat{F}_2$ (cf. the Appendix); and our Main Theorem (a) asserts that the induced map $\widehat{\mathrm{GT}}_0 \to \mathrm{Out}\big(\widehat{K}(0,4)\big)$ is an isomorphism of $\widehat{\mathrm{GT}}_0$ onto $\mathrm{Out}_4^\sharp$.

Drinfel'd indicated, and Ihara showed (in [I4], [I5]) that there is an injective homomorphism $G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}$. Also, $\widehat{\mathrm{GT}} \subset \widehat{\mathrm{GT}}_0$. Thus the actions of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,5)$ and of $\widehat{\mathrm{GT}}_0$ on $\widehat{K}(0,4)$ restrict to actions of $G_{\mathbb{Q}}$ on $\widehat{K}(0,4)$ and $\widehat{K}(0,5)$. We show that these two actions of $G_{\mathbb{Q}}$ extend to actions of $\mathrm{Out}_4^\sharp$ and $\mathrm{Out}_5^\sharp$ respectively on $\widehat{K}(0,4)$ and $\widehat{K}(0,5)$, with respect to the natural inclusions of $G_{\mathbb{Q}}$ into $\mathrm{Out}_4^\sharp$ and $\mathrm{Out}_5^\sharp$. Moreover these latter two actions lift the tautological *outer* actions of $\mathrm{Out}_4^\sharp$ and $\mathrm{Out}_5^\sharp$, and the isomorphisms in our Main Theorem carry these two actions to the actions of $\widehat{\mathrm{GT}}_0$ and $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,4)$ and $\widehat{K}(0,5)$ induced by (1).

One application of the Main Theorem is that it permits (at least in principle) the determination of the $\widehat{\mathrm{GT}}_0$- or $\widehat{\mathrm{GT}}$-orbits of finite topological covers of $\mathbb{P}^1 - \{0,1,\infty\}$, or equivalently of dessins d'enfants (which can be identified with finite-index subgroups of $\widehat{K}(0,4)$ up to conjugacy). The procedure is described in [HS] (where $\mathrm{Out}_n^\sharp$ is denoted by $\mathcal{O}_n^\sharp$ for short, and the related automorphism group $\mathrm{Aut}_n^\sharp$ is abbreviated $\mathcal{A}_n^\sharp$). It gives an approach to studying $G_{\mathbb{Q}}$-orbits of dessins and their fields of moduli.

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0.2. Galois actions on fundamental groups

One of the principal ideas in Grothendieck's *Esquisse d'un Programme* [G1] is to study the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by considering its elements as automorphisms of algebraic fundamental groups of varieties defined over $\mathbb{Q}$ — especially the moduli spaces of curves with marked points. This idea has motivated research on automorphism groups of the profinite completions of certain familiar groups such as free groups and braid groups, which occur naturally as fundamental groups of various types of moduli and configuration spaces.

Specifically, if X is a variety defined over $\mathbb{Q}$, then there is an exact sequence

$$1 \to \widehat{\pi}_1(X_{\bar{\mathbb{Q}}}) \to \widehat{\pi}_1(X) \to G_{\mathbb{Q}} \to 1,$$

which induces an outer action of $G_{\mathbb{Q}}$ on $\widehat{\pi}_1(X_{\bar{\mathbb{Q}}})$. For $X = \mathbb{P}^1 - \{0, \infty\}$ this corresponds to the cyclotomic character $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^*$, while for $X = \mathbb{P}^1 - \{0, 1, \infty\}$ it yields an outer action of $G_{\mathbb{Q}}$ on the free profinite group of rank 2, viz. $\widehat{F}_2 \simeq \widehat{\pi}_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0, 1, \infty\})$. This latter outer action is faithful, by Belyi's result [B] that every $\overline{\mathbb{Q}}$-curve is a cover of $\mathbb{P}^1_{\bar{\mathbb{Q}}} - \{0, 1, \infty\}$. Thus, for example, one can try to understand $G_{\mathbb{Q}}$ by understanding its image in $\mathrm{Out}(\widehat{F}_2)$.

Unfortunately, the outer automorphism group of even quite a simple profinite group like $\widehat{F}_2$ can be so huge as to be essentially out of reach. To quote Grothendieck ([G2], p.164), "Il est possible qu'il soit un groupe à tel point démesuré et pathologique, qu'il ne pourra jamais être question de dire des choses raisonnables (et vraies) sur le groupe tout entier...et qu'on soit obligé de travailler avec des sous-groupes plus petits, qui restent proches du discret (avec quand-même des aspects supplémentaires 'arithmétiques', dûs au $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$)!" So in view of the goal of understanding $G_{\mathbb{Q}}$, one is led to restrict attention to a certain proper subgroup of $\mathrm{Out}(\widehat{F}_2)$ consisting of "special" outer automorphisms satisfying certain simple conditions of a geometric nature which are known to hold for the elements of $G_{\mathbb{Q}}$ (viewed as outer automorphisms of $\widehat{F}_2$).

Actually, in the case of the profinite group $\widehat{F}_2 \simeq \widehat{\pi}_1(\mathbb{P}^1 - \{0, 1, \infty\})$, it is possible to view $G_{\mathbb{Q}}$ as a subgroup of $\mathrm{Aut}(\widehat{F}_2)$, and not just of $\mathrm{Out}(\widehat{F}_2)$. This was observed by Belyi in 1980. Namely, let x and y denote the (topological) generators of $\widehat{F}_2$; set $z = (xy)^{-1}$; let $\widehat{F}_2'$ denote the derived subgroup; and let $a \sim b$ mean that a is conjugate to b. What Belyi showed [B] is that the canonical homomorphism

$$G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2)$$

can be lifted (uniquely) to an injective homomorphism

$$G_{\mathbb{Q}} \hookrightarrow A$$

where the *Belyi group* A is the subgroup of $\mathrm{Aut}(\widehat{F}_2)$ defined by

$$\begin{aligned} A = \big\{F \in \mathrm{Aut}(\widehat{F}_2) \mid \exists\ \lambda \in \widehat{\mathbb{Z}}^* \ \text{ and } \ f \in \widehat{F}_2' \ \text{ such that} \\ F(x) = x^\lambda,\ F(y) = f^{-1} y^\lambda f \text{ and } F(z) \sim z^\lambda\big\}. \end{aligned} \tag{2}$$

This lifting of the homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2)$ to $G_{\mathbb{Q}} \to A \subset \mathrm{Aut}(\widehat{F}_2)$ is known as the *Belyi lifting*; we study it further in Section 1. The Belyi group A helped motivate Drinfel'd's definition of the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$ (which can be identified with the subgroup of A satisfying conditions (I)-(III)), and the Belyi lift $G_{\mathbb{Q}} \to A$ helped suggest that there should be an injection $G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}$.

In the above situation, the existence of a homomorphism $G_{\mathbb{Q}} \to A$ comes from the fundamental fact that the action of the Galois group must preserve conjugacy classes of inertia subgroups, here represented by $\langle x\rangle$, $\langle y\rangle$ and $\langle z\rangle$. More precisely, $\sigma \in G_{\mathbb{Q}}$ maps the conjugacy classes of x, y, z to those of x^λ, y^λ, z^λ, where $\lambda = \chi(\sigma)$ and where $\chi : G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^*$ is the cyclotomic character. Cf. Fried's "branch cycle argument" in [F].

The above is the first example of what we mean by "restricting attention to automorphisms satisfying certain simple conditions of a geometric nature"; the geometric condition here is the preservation of inertia subgroups of a fundamental group. This is a key point in understanding the motivation behind the various definitions of particular automorphism groups of larger π_1's below (and above, i.e. property (i) in the definition of $\mathrm{Out}_n^\sharp$). Belyi's group A marks the first appearance of automorphism groups with this property, which are called "special".

Rather than generalizing the above to actions of $G_{\mathbb{Q}}$ on fundamental groups of $\mathbb{P}^1 - S$ for S a set of more than three points, Grothendieck [G1] suggested a different type of generalization. Namely, by identifying $\mathbb{P}^1 - \{0, 1, \infty\}$ with the moduli space $\mathcal{M}_{0,4}$ of Riemann spheres with four marked ordered points via the cross ratio (cf. the Appendix), one may consider it as the first non-trivial case in the study of fundamental groups of moduli spaces. From this point of view, the natural generalizations of $\widehat{F}_2 = \widehat{K}(0, 4)$ are the higher profinite pure mapping class groups $\widehat{K}(g, n) = \widehat{\pi}_1(\mathcal{M}_{g,n})$ rather than bigger free profinite groups; Grothendieck suggested trying to characterize $G_{\mathbb{Q}}$ as a subgroup of $\mathrm{Out}(\widehat{K}(g, n))$ of elements satisfying certain geometric properties.

In [N1, Appendix] Nakamura generalized Belyi's lifting to the case of $\widehat{K}(0, 5)$. Specifically, he showed that the canonical homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}\big(\widehat{K}(0, 5)\big)$ lifts uniquely to an injection $G_{\mathbb{Q}} \hookrightarrow A_5$, where (analogously to (2)) the group $A_5 \subset \mathrm{Aut}\big(\widehat{K}(0, 5)\big)$ is defined by:

$$
\begin{aligned}
A_5 := \big\{F \in \mathrm{Aut}(\widehat{K}(0, 5)) \mid \exists \lambda \in \widehat{\mathbb{Z}}^*, f \in \widehat{F}_2' \text{ with } F(x_{12}) = x_{12}^\lambda,\ F(x_{23}) = \tilde{f}^{-1} x_{23}^\lambda \tilde{f}, \\
F(x_{34}) = \hat{f}^{-1} x_{34}^\lambda \hat{f},\ F(x_{45}) = x_{45}^\lambda,\ F(x_{51}) \sim (x_{51})^\lambda\big\},
\end{aligned}
\tag{3}
$$

and where as before the x_{ij}'s are the standard generators of $\widehat{K}(0, 5)$ (cf. the Appendix). Here $\tilde{f}$ is the image of f under the injection $\widehat{F}_2 \hookrightarrow \widehat{K}(0, 5)$ given by $x \mapsto x_{12}$, $y \mapsto x_{23}$, and $\hat{f}$ is the image of f under the injection given by $x \mapsto x_{45}$, $y \mapsto x_{34}$. The above corresponds to a $G_{\mathbb{Q}}$-action on $\widehat{K}(0, 5)$, which turns out to be just the restriction to $G_{\mathbb{Q}}$ of the $\widehat{\mathrm{GT}}$-action on $\widehat{K}(0, 5)$ obtained via (1) — cf. the end of the Appendix. In other words, Nakamura's lifting can be constructed by using (1) to write down the $\widehat{\mathrm{GT}}$-action on $\widehat{K}(0,5)$ and applying Ihara's result that $G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}$. However, this is not the strategy used by Nakamura. Instead, he uses the fact that the group A_5 is a "special" automorphism group of $\widehat{K}(0,5)$ in the same sense as A is one for $\widehat{K}(0,4) \simeq \widehat{F}_2$; namely it consists of automorphisms that preserve the inertia subgroups generated by the x_{ij}. The strategy of our proof of the statement that $\widehat{\mathrm{GT}} \simeq \mathrm{Out}_5^\sharp$ directly follows Nakamura's strategy, but with essential differences to allow for the passage from $G_{\mathbb{Q}}$ to all of $\mathrm{Out}_5^\sharp$ (cf. the proof of Proposition 5). The proof that all the $\mathrm{Out}_n^\sharp$ are isomorphic for $n \geq 5$ is then remarkably simple, resulting from a combination of the result for $n=5$ with an injectivity lemma of Nakamura (cf. [N1, 3.2.2]); it is the subject of §3.

Let us review some other results closely related to ours. In earlier work, Ihara [I2] had considered what he called "special automorphisms" of pure braid groups, namely those fixing the conjugacy classes of the generators x_{ij}. Nakamura [N1] made the natural generalization to the group of *quasi-special* outer automorphisms $\mathrm{Out}^\flat\big(\widehat{K}(0,n)\big)$, which is the group of (continuous) outer automorphisms F for which there is some $\lambda \in \widehat{\mathbb{Z}}^*$ (not necessarily equal to 1) such that F sends the conjugacy class of x_{ij} to the conjugacy class of x_{ij}^λ. Thus his groups have property (i) of the groups $\mathrm{Out}_n^\sharp$ defined in 0.1, but not property (ii) on S_n-symmetry. (Our sharpening of his definition lies behind the musical notation $\sharp$.)

In the 1980's and 90's, Ihara and others extensively studied groups with properties like (i) and (ii) in the pro-ℓ context, coming from the study of Galois representations into automorphism groups of pro-ℓ rather than profinite completions of fundamental groups. In [I1] Ihara studied a pro-ℓ analogue of Belyi's group A (with $\lambda = 1$), namely the subgroup Φ of the group of outer automorphisms of the pro-ℓ completion of F_2 consisting of outer automorphisms preserving the conjugacy classes of x, y and z. An application of Grothendieck's comparison theorem shows that the representation $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \Phi$ is unramified outside ℓ; the paper is devoted to a detailed study of the properties of this representation (which, unlike what happens in the profinite case, is far from injective).

In [I2], Ihara considered the pure sphere braid groups P_n, which are closely related to the genus zero mapping class groups $K(0,n)$ (cf. the Appendix). Let $P_n^{(\ell)}$ denote the pro-ℓ completion of P_n. Ihara defined the *special automorphism group* $\mathrm{Aut}^*(P_n^{(\ell)})$ to be the subgroup of automorphisms fixing the conjugacy classes of each of the generators x_{ij}. There are natural "forgetful" homomorphisms $p_i : P_n \rightarrow P_{n-1}$ obtained by removing the i-th string for $1 \leq i \leq n$, which extend to the pro-ℓ completions. The kernels are stable under special automorphisms, so that there exist homomorphisms of the outer special automorphism group

$$\overline{q}_i : \mathrm{Out}^*(P_n^{(\ell)}) \rightarrow \mathrm{Out}^*(P_{n-1}^{(\ell)}).$$

The main result of [I2] states that these homomorphisms are injective for $n \geq 5$.

In [I3], Ihara proved a Lie algebra version of injectivity for $n \geq 5$, even obtaining bijectivity for $n > 5$, in the ℓ-adic case. There, the groups P_n were replaced by the graded Lie algebras $\mathcal{P}_n$ over $\mathbb{Q}$ associated with their lower central series, and the $\mathrm{Out}^*(P_n^{(\ell)})$ were replaced by the graded Lie algebras $\mathcal{D}_n$ over $\mathbb{Q}$ consisting of the S_n-invariant "special outer derivations" of the $\mathcal{P}_n$. This S_n-symmetry is a graded Lie version of the S_n-symmetry used in the definition of the groups $\mathrm{Out}_n^\sharp$ in 0.1. Although it is not the proof that Ihara uses in his paper, he notes in [I3] that the result, which comes down to proving the extendibility of each element of $\mathcal{D}_5$ to $\mathcal{D}_n$, can be proved using the fact that the graded Lie algebra of a certain subgroup $GT_1(k)$ of the k-pro-unipotent version $GT(k)$ of the Grothendieck-Teichmüller group is isomorphic to $\mathcal{D}_5 \otimes k$. The corollary to the Main Theorem of the present paper is a profinite version of this graded Lie result, and also of the pro-ℓ result in [I2].

0.3. The Teichmüller tower

Drinfel'd's observations about $\widehat{\mathrm{GT}}$-actions on fundamental groups (via (1) above), together with his suggestion that $G_{\mathbb{Q}} \hookrightarrow \widehat{\mathrm{GT}}$ (as later proved by Ihara), suggested a connection to observations of Grothendieck in [G1]. Namely, by considering fundamental groupoids (with more than one base point allowed) one obtains a "Teichmüller tower" of groupoids $\widehat{T}_{g,n}$ corresponding to the moduli spaces $\mathcal{M}_{g,n}$ of curves of genus g with n ordered marked points. In [G1,§2], Grothendieck had observed that there is an actual (as opposed to just an outer) action of $G_{\mathbb{Q}}$ on the $\widehat{T}_{g,n}$'s, if one takes sets of base points invariant under $G_{\mathbb{Q}}$. Drinfel'd suggested [D, p.847] that the Teichmüller tower forms an inverse system, and that $\widehat{\mathrm{GT}}$ is isomorphic to its automorphism group — or at least the automorphism group of the genus 0 Teichmüller tower of $\widehat{T}_{0,n}$'s. (This in turn raises the question of how close $G_{\mathbb{Q}}$ is to $\widehat{\mathrm{GT}}$; and this remains mysterious.)

A version of Drinfel'd's suggestion for groups instead of groupoids was proven in [LS], but using the profinite Artin braid groups $\widehat{B}_n$ rather than the profinite pure mapping class groups $\widehat{K}(0,n)$. Specifically, the purpose of [LS] was to realize a group-theoretic interpretation of $\widehat{\mathrm{GT}}$ as the set of compatible tuples (ϕ_n) of "special" automorphisms $\phi_n \in \mathrm{Aut}(\widehat{B}_n)$ (thus forming automorphisms of an appropriate "tower"). The term "special" here means automorphisms ϕ_n satisfying $\rho_n = \rho_n \circ \phi_n$, where $\rho_n : \widehat{B}_n \to S_n$ is the natural surjection (cf. 1.1); this is closely related to the use of the word "special" to indicate preservation of inertia subgroups as above. The main result of [LS] is the following: For $n \geq 3$ let $\widehat{A}_n$ denote the subgroup of $\widehat{B}_n$ generated by $\sigma_1^2, \sigma_2, \ldots, \sigma_{n-1}$, and define the tower T_N of braid groups to consist of the groups $\widehat{A}_n$ and $\widehat{B}_n$ for $1 \leq n \leq N$, together with the inclusions $\widehat{B}_n \hookrightarrow \widehat{B}_{n+1}$ via $\sigma_i \mapsto \sigma_i$ for $1 \leq i \leq n-2$ and the "string-doubling" homomorphisms $\widehat{A}_{n-1} \hookrightarrow \widehat{B}_n$ given by $\sigma_1^2 \mapsto \sigma_2\sigma_1^2\sigma_2$, $\sigma_i \mapsto \sigma_{i+1}$ for $2 \leq i \leq n-2$. Define the *special au-*

tomorphism group $\mathrm{Aut}^*(T_N)$ of the tower T_N to be the group of tuples $(\phi_n)_{2\le n\le N}$ where the ϕ_n are special automorphisms of the $\widehat{B}_n$ that commute with the inclusions and, when restricted to the subgroups $\widehat{A}_n$, with the string-doubling homomorphisms. Then the main result of [LS] states that $\mathrm{Aut}^*(T_3) \simeq \widehat{\mathrm{GT}}_0$ and $\mathrm{Aut}^*(T_N) \simeq \widehat{\mathrm{GT}}$ for $N \ge 4$; in particular the ϕ_n act according to Drinfel'd's formula (1).

A key difference between the situation of [LS] and the one in this paper is that there one considers automorphism groups, whereas here we use *outer* automorphism groups. (Recall that $G_{\mathbb{Q}}$ naturally has just an outer action on fundamental groups, and that a choice of splitting is needed to obtain a true action — unless one instead uses fundamental groupoids.) Indeed, the methods used in [LS] in conjunction with braid groups can be adapted to the situation of mapping class groups, if one merely wants results about automorphism groups. Namely, by those methods one can obtain the following (weaker) variant on our Main Theorem. Here, relations (I) - (III) are as in the definition of $\widehat{\mathrm{GT}}$ (cf. 0.1), and $\mathrm{inn}(f)$ denotes the inner automorphism $g \mapsto fgf^{-1}$; we use the commutator notation $[a,b] = aba^{-1}b^{-1}$. Also $\theta, \omega \in \mathrm{Aut}(\widehat{F}_2)$ and $\rho \in \mathrm{Aut}(\widehat{K}(0,5))$ are certain lifts of $(12), (123) \in S_3$ and $(12345)^3 \in S_5$ respectively (cf. 1.3, 2.2 below).

Theorem A. *(i) Let* $F \in \mathrm{Aut}(\widehat{F}_2)$ *be an automorphism of the form* $x \mapsto x^\lambda$, $y \mapsto f^{-1}y^\lambda f$ *for some* $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$. *Then* (λ, f) *satisfies relation (I) if and only if* $[\theta, F] = \mathrm{inn}(f)$ *in* $\mathrm{Aut}(\widehat{F}_2)$. *Furthermore, given such a pair* (λ, f) *satisfying (I), it satisfies relation (II) if and only if* $[\omega, F] = \mathrm{inn}(y^m f)$, *where* $m = (\lambda - 1)/2$.

(ii) Let (λ, f) *be a pair as above, satisfying (I) and (II) — i.e. an element of* $\widehat{\mathrm{GT}}_0$. *Then* (λ, f) *lies in* $\widehat{\mathrm{GT}}$ *if and only if there exists an automorphism* F *of* $\widehat{K}(0,5)$ *extending that of* $\widehat{F}_2$ *in the sense that* $F(x_{12}) = x_{12}^\lambda$ *and* $F(x_{23}) = f(x_{12}, x_{23})^{-1} x_{23}^\lambda f(x_{12}, x_{23})$, *and furthermore satisfying* $[\rho, F] = \mathrm{inn}(f(x_{12}, x_{23}))$. *Since this formula allows us to compute* F *successively on* x_{45}, x_{51} *and* x_{34}, *we see that if such an extension exists then it is unique.*

Theorem A is contained in the statements of Propositions 3, 4 and 7 below. But in order to pass to the (outer automorphic) Main Theorem from this automorphic version, it is necessary to have at our disposal a section from $\mathrm{Out}_n^\sharp$ to its preimage $\mathrm{Aut}_n^\sharp$ in $\mathrm{Aut}\big(\widehat{K}(0,n)\big)$. We construct such a section explicitly for $n = 4$ and 5 (and then deduce from these cases the result for $n > 5$; cf. §3). In doing so, we show that the group $\mathrm{Out}_n^\sharp$, like $\widehat{\mathrm{GT}}$ (or $\widehat{\mathrm{GT}}_0$, for $n = 4$), has an action on the group $\widehat{K}(0,n)$, and not just an outer action. In the case $n = 4$, the section that we construct extends the Belyi lifting $G_{\mathbb{Q}} \to A$ to an injection $\mathrm{Out}_4^\sharp \to A$; for $n = 5$, our section extends Nakamura's lifting to an injection $\mathrm{Out}_5^\sharp \to A_5$.

The structure of the rest of the paper is as follows: Section 1 considers the case of actions and outer actions on $\widehat{K}(0,4)$. We construct a section s of $\mathrm{Aut}_4^\sharp \to \mathrm{Out}_4^\sharp$ (Theorem 1 at the end of 1.2) and then show (Theorem 2 at the end of 1.3) that the image of s is in fact $\widehat{\mathrm{GT}}_0$ — so that $\mathrm{Out}_4^\sharp \simeq \widehat{\mathrm{GT}}_0$. Section 2 then considers the case of $\widehat{K}(0,5)$, and parallels

Section 1. Namely, we construct a section s_5 of $\mathrm{Aut}_5^\sharp \to \mathrm{Out}_5^\sharp$ (Theorem 3 at the end of 2.2) and then show (Theorem 4 at the end of 2.3) that the image of this section is in fact $\widehat{\mathrm{GT}}$ — so that $\mathrm{Out}_5^\sharp \simeq \widehat{\mathrm{GT}}$. Section 3 considers $\widehat{K}(0,n)$ for general n. It uses the results of Section 2 and a result of Nakamura to construct an isomorphism $e_n : \widehat{\mathrm{GT}} \xrightarrow{\sim} \mathrm{Out}_n^\sharp$ for $n \geq 5$ extending that of §2 such that the actions of $\widehat{\mathrm{GT}}$ and of $\mathrm{Out}_n^\sharp$ on $\widehat{K}(0,n)$ are carried to each other under e_n. This is done in Theorem 5 in 3.1. Finally, in 3.2, we pose several questions suggested by the results of this paper.

§1. Actions on four-point moduli

1.1. Fundamental groups

In this section, we consider the subgroup $\mathrm{Out}_4^\sharp$ of the outer automorphism group $\mathrm{Out}(\widehat{F}_2)$, where the free profinite group $\widehat{F}_2$ is regarded as the algebraic fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}$. This group contains a copy of $G_{\mathbb{Q}}$ (Proposition 1), and we show that Belyi's lifting $\beta : G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{F}_2)$ of the natural map $\alpha : G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2)$ can be extended from $G_{\mathbb{Q}}$ to a map defined on all of $\mathrm{Out}_4^\sharp$ (Proposition 2). This gives a section s of $\mathrm{Aut}_4^\sharp \to \mathrm{Out}_4^\sharp$, so that $\beta = s\alpha$; cf. Theorem 1. Now Belyi's lifting β also extends to a map defined on all of $\widehat{\mathrm{GT}}_0$, and in fact we show (Theorem 2) that the groups $\mathrm{Out}_4^\sharp$ and $\widehat{\mathrm{GT}}_0$ are isomorphic, compatibly with these maps to $\mathrm{Aut}(\widehat{F}_2)$ — thus proving part (a) of our Main Theorem (cf. 0.1) and providing an interpretation of $\widehat{\mathrm{GT}}_0$ that does not involve the usual cocycle conditions (I) and (II) of 0.1. This is done by showing (in Propositions 3 and 4) that conditions (I) and (II) are equivalent to the conditions of commuting with certain lifts of $(12), (123) \in S_3$ to $\mathrm{Out}(\widehat{F}_2)$. In a companion paper [HS], we show that β is effective in terms of α, and use this to obtain information about Galois orbits and fields of moduli of covers of $\mathbb{P}^1 - \{0,1,\infty\}$.

The space $\mathbb{P}^1 - \{0,1,\infty\}$ can be identified via the cross ratio with the moduli space $\mathcal{M}_{0,4}$, and its fundamental group F_2 with the pure mapping class group $K(0,4) = \pi_1(\mathcal{M}_{0,4})$. More generally, we can consider $K(0,n) = \pi_1(\mathcal{M}_{0,n})$ and its profinite completion $\widehat{K}(0,n)$ (the algebraic fundamental group of $\mathcal{M}_{0,n}$), having standard generators x_{ij}. (Cf. the Appendix for more details.) For each $i = 1, \ldots, n$, there is a natural surjective homomorphism $p_i : \widehat{K}(0,n) \to \widehat{K}(0,n-1)$ obtained by omitting the i-th entry. (Thus this copy of $\widehat{K}(0,n-1)$ is generated by the images of the elements x_{hj}, where $1 \leq h,j \leq n$ and $h,j \neq i$.) For each i, we have $\ker p_i = \langle x_{i1}, \ldots, x_{in} \mid x_{i1} \cdots x_{in} = 1 \rangle$. Hence we have an exact sequence

$$1 \to \widehat{F}_{n-2} \to \widehat{K}(0,n) \to \widehat{K}(0,n-1) \to 1.$$

Moreover this sequence is split; a (non-canonical) splitting $\iota_i : \widehat{K}(0,n-1) \hookrightarrow \widehat{K}(0,n)$ is given for example by $x_{hj} \mapsto x_{hj}$ for h,j unequal to i or $i-1$ and $\{h,j\} \neq \{i-3, i-2\}$ (cf. [LS, Appendix]). Thus we may write $\widehat{K}(0,n)$ as a semidirect product $\widehat{F}_{n-2} \rtimes \widehat{K}(0,n-1)$.

This provides the structure of $\widehat{K}(0,n)$ inductively, starting with $\widehat{K}(0,4) = \widehat{F}_2$ (where $x = x_{12}$ and $y = x_{23}$).

For each n, the symmetric group S_n acts on the moduli space $\mathcal{M}_{0,n}$ by permuting the order of the marked points. In particular, for $n = 4$, the automorphism group of $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$ is S_3, and the map $S_4 \to \mathrm{Aut}(\mathcal{M}_{0,4})$ is surjective with kernel equal to the even involutions in S_4 (which form a Klein four group). On the other hand, for $n > 4$, the map $S_n \to \mathrm{Aut}(\mathcal{M}_{0,n})$ is an isomorphism. For all n, the map $S_n \to \mathrm{Aut}(\mathcal{M}_{0,n})$ induces a homomorphism $\sigma^{(n)} : S_n \to \mathrm{Out}(\widehat{K}(0,n))$, which again is injective for $n > 4$ and has Klein four kernel if $n = 4$. (In fact, by a version of Grothendieck's anabelian conjecture [IN], the image of this homomorphism is exactly the subgroup of $\mathrm{Out}(\widehat{K}(0,n))$ that commutes with the natural outer action of $G_{\mathbb{Q}}$ on $\widehat{K}(0,n)$.) Let $\tilde{S}_n \subset \mathrm{Aut}(\widehat{K}(0,n))$ be the inverse image under $\mathrm{Aut}(\widehat{K}(0,n)) \to \mathrm{Out}(\widehat{K}(0,n))$ of the image of $\sigma^{(n)}$. For $n > 4$ (resp. $n = 4$) this is an extension of S_n (resp. S_3) by $\widehat{K}(0,n)$, and is isomorphic to the full mapping class group $\widehat{M}(0,n)$ (resp. to the quotient B_3 modulo center).

1.2. Outer automorphism groups.

For any group G, the outer automorphism group $\mathrm{Out}(G)$ acts on the set $[G]$ of conjugacy classes $[g]$ of elements of G in a well defined way. For every positive integer n, Nakamura [N1, §3.2] considered the subgroup

$$\mathrm{Out}^\flat(\widehat{K}(0,n)) \subset \mathrm{Out}(\widehat{K}(0,n))$$

consisting of the *quasi-special* elements $\overline{F}$, i.e. those satisfying

(i) $\overline{F}([x_{ij}]) = [x_{ij}^\lambda]$ for each i, j, for some $\lambda \in \widehat{\mathbb{Z}}^*$.

As noted in the introduction, here we focus on the subgroup

$$\mathrm{Out}^\sharp(\widehat{K}(0,n)) \subset \mathrm{Out}^\flat(\widehat{K}(0,n))$$

consisting of *symmetric* quasi-special elements $\overline{F}$, i.e. those also satisfying

(ii) $\overline{F}$ commutes with the image of S_n in $\mathrm{Out}(\widehat{K}(0,n))$.

Note that for $\overline{F} \in \mathrm{Out}^\sharp(\widehat{K}(0,n))$, the value of λ is independent of i, j by the symmetry condition (ii); so we may write $\lambda = \lambda(\overline{F})$.

Nakamura defined the subgroup $\mathrm{Aut}^\flat(\widehat{K}(0,n)) \subset \mathrm{Aut}(\widehat{K}(0,n))$ as the inverse image of $\mathrm{Out}^\flat(\widehat{K}(0,n))$ under $\mathrm{Aut}(\widehat{K}(0,n)) \to \mathrm{Out}(\widehat{K}(0,n))$. Similarly, we will let $\mathrm{Aut}^\sharp(\widehat{K}(0,n))$ be the inverse image of $\mathrm{Out}^\sharp(\widehat{K}(0,n))$ under $\mathrm{Aut}(\widehat{K}(0,n)) \to \mathrm{Out}(\widehat{K}(0,n))$, and write $\lambda(F) = \lambda(\overline{F})$ if $F \in \mathrm{Aut}^\sharp(\widehat{K}(0,n))$ maps to $\overline{F} \in \mathrm{Out}(\widehat{K}(0,n))$. Thus for $F \in \mathrm{Aut}(\widehat{K}(0,n))$, we have that F lies in $\mathrm{Aut}^\sharp(\widehat{K}(0,n))$ if and only if (i)′ $F(x_{ij}) \sim x_{ij}^{\lambda}$ for each i, j, where $\lambda = \lambda(F) \in \widehat{\mathbb{Z}}^*$, and

(ii)′ the commutator $[\phi, F] \in \widehat{K}(0,n)$ for all $\phi \in \tilde{S}_n$.

Here in (ii) we identify $h \in \widehat{K}(0,n)$ with its image $\mathrm{inn}(h) \in \mathrm{Inn}(\widehat{K}(0,n)) \subset \mathrm{Aut}(\widehat{K}(0,n))$, where $\mathrm{inn}(h)$ is defined by $\mathrm{inn}(h)(g) = hgh^{-1}$. For any $F \in \mathrm{Aut}(\widehat{K}(0,n))$ we have

$$\mathrm{inn}\big(F(h)\big)(g) = F(h)gF(h)^{-1} = F(hF^{-1}(g)h^{-1}) = F\big(\mathrm{inn}(h)\big)\big(F^{-1}(g)\big),$$

and so the under the above identification $F(h)$ becomes identified with FhF^{-1}.

For short, we will denote the groups $\mathrm{Out}^\sharp(\widehat{K}(0,n))$ and $\mathrm{Aut}^\sharp(\widehat{K}(0,n))$ by $\mathrm{Out}_n^\sharp$ and $\mathrm{Aut}_n^\sharp$ respectively. (In the companion paper [HS], the abbreviations $\mathcal{O}_n^\sharp = \mathrm{Out}_n^\sharp$ and $\mathcal{A}_n^\sharp = \mathrm{Aut}_n^\sharp$ are also used.) In the case of $n = 4$, we thus have $\mathrm{Out}_4^\sharp \subset \mathrm{Out}(\widehat{F}_2)$ and $\mathrm{Aut}_4^\sharp \subset \mathrm{Aut}(\widehat{F}_2)$, since $\widehat{K}(0,4) \simeq \widehat{F}_2 = \langle x, y \rangle$.

Proposition 1. *For any n, the image of the natural homomorphism $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{K}(0,n))$ is contained in $\mathrm{Out}_n^\sharp$.*

Proof. Condition (i) follows by Fried's branch cycle argument ([F], cf. also 0.2 above) — viz. that if $\sigma \mapsto \overline{F}$, then (i) holds with $\lambda(\overline{F}) = \chi(\sigma)$. Condition (ii) follows from the fact that the action of $G_{\mathbb{Q}}$ on $\mathcal{M}_{0,n}$ does not depend on the ordering of the marked points. $\diamondsuit$

Remark. Following Ihara, those (outer) automorphisms that satisfy (i) or (i)′ with $\lambda = 1$ are called *special*. By Proposition 1, we see that the image of $G_{\mathbb{Q}^{\mathrm{ab}}}$ is contained in the group of *symmetric special* outer automorphisms (i.e. elements of $\mathrm{Out}_n^\sharp$ with $\lambda = 1$).

In the case $n = 4$, denote the natural map in Proposition 1 by α. The Belyi lifting $\beta : G_{\mathbb{Q}} \to A \subset \mathrm{Aut}(\widehat{K}(0,4))$ chooses a certain element $F \in A$ over $\overline{F} \in \mathrm{Out}(\widehat{K}(0,4))$ for every $\overline{F}$ in the image of α. In particular, β takes complex conjugation to $\iota \in \mathrm{Aut}_4^\sharp \cap A$, where $\iota(x) = x^{-1}$, $\iota(y) = y^{-1}$, and $\iota(z) = x^{-1}z^{-1}x$. (Here, as before, we view $\widehat{F}_2 = \langle x, y, z \,|\, xyz = 1\rangle$.) The following result extends the Belyi lifting β from $G_{\mathbb{Q}}$ to all of $\mathrm{Out}_4^\sharp$ (with part (c) extending the corresponding formula for $G_{\mathbb{Q}}$ [N1, Appendix]):

Proposition 2. *Let $\overline{F} \in \mathrm{Out}_4^\sharp$ and let $\lambda = \lambda(\overline{F}) \in \widehat{\mathbb{Z}}^*$.*

(a) There exists $f \in \widehat{F}_2'$ such that some lifting $F \in \mathrm{Aut}_4^\sharp$ satisfies $F(x) = x^\lambda$ and $F(y) = f^{-1}y^\lambda f$.

(b) The f and F in (a) are unique.

(c) For f and F in (a), $F(z) = \mathrm{inn}(x^m f(x,z)^{-1})z^\lambda$, where $m = (\lambda - 1)/2$.

Proof. (a) First choose any lifting $F \in \mathrm{Aut}_4^\sharp$ of $\overline{F}$, so that $\lambda = \lambda(F)$. Thus $F(x) = \mathrm{inn}(h)(x^\lambda)$ and $F(y) = \mathrm{inn}(g)(y^\lambda)$ for some $h, g \in \widehat{F}_2$. Replacing F by $\mathrm{inn}(h^{-1})F$ (which also lies over $\overline{F}$), we may assume that $h = 1$. The image of g in $\widehat{F}_2/\widehat{F}_2'$ (the free abelian profinite group on generators x, y) is of the form $x^a y^b$ for some $a, b \in \widehat{\mathbb{Z}}$. Replacing g by gy^{-b} we may assume that $b = 0$, and then replacing F by $\mathrm{inn}(x^{-a})F$, we may assume that

$a = 0$. Thus for this lifting F the element g lies in $\widehat{F}_2'$ and g^{-1} has the desired property for f.

(b) Suppose that two elements $f, g \in \widehat{F}_2'$ both satisfy the desired property for f, say with respect to lifts F, G of $\overline{F}$. Then $G \circ F^{-1}$ maps to the identity in $\mathrm{Out}_4^\sharp$, and so $G \circ F^{-1} = \mathrm{inn}(h)$ for some $h \in \widehat{F}_2$. Thus $\mathrm{inn}(h)x = G \circ F^{-1}(x) = x$. So h lies in the centralizer of x, i.e. $h = x^c$ for some $c \in \widehat{\mathbb{Z}}$. Also, $\mathrm{inn}(hf^{-1})y^\lambda = G \circ F^{-1}\mathrm{inn}(f^{-1})(y^\lambda) = G(y) = \mathrm{inn}(g^{-1})(y^\lambda)$. Thus $ghf^{-1} = gx^cf^{-1}$ commutes with y^λ, and hence with $y = (y^\lambda)^\mu$, where $\mu = \lambda^{-1} \in \widehat{\mathbb{Z}}^*$. So gx^cf^{-1} is in the centralizer of y, and hence $gx^cf^{-1} = y^d$ for some $d \in \widehat{\mathbb{Z}}$. Since $f, g \in \widehat{F}_2'$, it follows that $c = d = 0$, and so $h = 1$, $F = G$, and $f = g$.

(c) Since $F \in \mathrm{Aut}_4^\sharp$, $F(z) = g(x,z)^{-1}z^\lambda g(x,z)$ for some $g(x,z) \in \widehat{F}_2$. Using the S_3-invariance of $\overline{F}$ and applying the automorphism $x \mapsto x$, $y \mapsto z$, $z \mapsto z^{-1}x^{-1} = xyx^{-1}$, we find that the automorphism

$$x \mapsto x^\lambda, \; y \mapsto \mathrm{inn}(g(x, xyx^{-1})^{-1}(xyx^{-1})^\lambda, \; z \mapsto \mathrm{inn}(f(x,z)^{-1})z^\lambda$$

also lies over $\overline{F}$. Hence this automorphism differs from F by an inner automorphism by some $\alpha \in \widehat{F}_2$. Thus α centralizes x^λ, so $\alpha = x^r$ for some $r \in \widehat{\mathbb{Z}}$. So $\mathrm{inn}(g(x,z)^{-1})z^\lambda = \mathrm{inn}(x^{-r}f(x,z)^{-1})z^\lambda$ and thus $g(x,z) = z^tf(x,z)x^r$ for some t. Since $g(x,z)$ is determined only up to left multiplication by a power of z, we can take $t = 0$ and so take $g(x,z) = f(x,z)x^r$. To prove (c) it remains to show that $-r = m = (\lambda - 1)/2$.

For this, we first show that the above value of r is unique, and is even uniquely determined just by λ (in the sense that if some $\overline{H} \in \mathrm{Out}_4^\sharp$ gives rise to the same λ, then only this value of r works for $\overline{H}$). To show this, observe that $1 = F(x)F(y)F(z) = x^\lambda[\mathrm{inn}(f(x,y)^{-1})y^\lambda][\mathrm{inn}(x^{-r}f(x,z)^{-1})z^\lambda]$. Since conjugation by f is trivial in $\widehat{F}_2/[\widehat{F}_2', \widehat{F}_2]$, we find that $x^\lambda y^\lambda x^{-r}z^\lambda x^r \in [\widehat{F}_2', \widehat{F}_2]$. Thus if there were two different r's, say r and r', that worked for the same λ (with different f's), then we could compare the two expressions and get $x^{r-r'}z^{-\lambda}x^{r'-r}z^\lambda \in [\widehat{F}_2', \widehat{F}_2]$. Since $\lambda \neq 0$, this forces $r = r'$. Thus for each value $\lambda \in \widehat{\mathbf{Z}}^*$ there is a unique r, say $r(\lambda)$, such that $g(x,z) = f(x,z)x^{r(\lambda)}$ for all pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ arising from elements of $\mathrm{Out}_4^\sharp$.

It remains to show that $r(\lambda) = (1-\lambda)/2 = -m$ for all λ. By the comments before the proposition, $\iota \in \mathrm{Aut}_4^\sharp$ is the lifting of $\bar{\iota} \in \mathrm{Out}_4^\sharp$ having the form asserted in (a), and it gives rise to $(-1, 1) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$. Here $\iota F \in \mathrm{Aut}_4^\sharp$ acts by

$$\begin{aligned} &x \mapsto x^{-\lambda} \\ &y \mapsto \mathrm{inn}(f(x,y)^{-1}y^{-\lambda} \\ &z \mapsto \mathrm{inn}(x^{-r(\lambda)-\lambda}f(x,z)^{-1})z^{-\lambda}. \end{aligned}$$

Thus $r(-\lambda) = r(\lambda) + \lambda$. Similarly $F\iota \in \mathrm{Aut}_4^\sharp$ takes

$$\begin{aligned} &x \mapsto x^{-\lambda} \\ &y \mapsto \mathrm{inn}(f(x^{-1}, y^{-1})^{-1} y^{-\lambda} \\ &z \mapsto \mathrm{inn}(x^{r(\lambda)} f(x^{-1}, x^{-1} z^{-1} x)^{-1} x^{-1}) z^{-\lambda}. \end{aligned}$$

Modding out by $[\widehat{F}_2', \widehat{F}_2]$, we get that $r(-\lambda) = 1 - r(\lambda)$. Since also $r(-\lambda) = r(\lambda) + \lambda$, we get that $r(\lambda) = (1-\lambda)/2 = -m$. $\diamondsuit$

Using this result, we obtain the desired result (where the Belyi group A is as in 0.2).

Theorem 1. *There is a unique section s of $\mathrm{Aut}_4^\sharp \to \mathrm{Out}_4^\sharp$ whose image lies in A. This section satisfies $\beta = s\alpha : G_{\mathbb{Q}} \to A \subset \mathrm{Aut}(\widehat{K}(0,4)) = \mathrm{Aut}(\widehat{F}_2)$.*

Proof. According to Proposition 2, over every element of $\mathrm{Out}_4^\sharp$ there is a unique element of $A^\sharp := A \cap \mathrm{Aut}_4^\sharp$. Thus there is a unique section s of $\mathrm{Aut}_4^\sharp \to \mathrm{Out}_4^\sharp$ whose image lies in A. For every $\omega \in G_{\mathbb{Q}}$, the elements $\beta(\omega)$ and $s\alpha(\omega)$ are each in A and both lie over $\alpha(\omega) \in \mathrm{Out}_4^\sharp$. So again by Proposition 2, they are equal. Thus $\beta = s\alpha$. $\diamondsuit$

1.3. Connection with $\widehat{\mathrm{GT}}_0$.

In Theorem 2 below we show, via the above section s, that $\mathrm{Out}_4^\sharp$ is isomorphic to $\widehat{\mathrm{GT}}_0$ (cf. 0.1 for the definition). This provides a rather natural way to view $\widehat{\mathrm{GT}}_0$ in terms of the perspective of [G1]. Let $\theta, \omega \in \mathrm{Aut}(\widehat{F}_2)$ be given respectively by $\theta(x) = y$, $\theta(y) = x$, $\theta(z) = x^{-1}zx$ and $\omega(x) = y$, $\omega(y) = z$, $\omega(z) = x$. Thus $\theta, \omega \in \mathrm{Aut}(\widehat{F}_2)$ respectively lie over the images of $(12), (123) \in S_3$ in $\mathrm{Out}(\widehat{F}_2)$ (which we also denote by θ, ω). Observe that conditions (I) and (II) can be rewritten as

$$\theta(f)f = 1, \quad \omega^2(fx^m)\omega(fx^m)fx^m = 1.$$

Proposition 3. *Let $F \in \mathrm{Aut}(\widehat{F}_2)$ be an automorphism of the form $x \mapsto x^\lambda$, $y \mapsto f^{-1}y^\lambda f$ for some $\lambda \in \widehat{\mathbb{Z}}^*$ and some $f \in \widehat{F}_2'$. Let $\overline{F} \in \mathrm{Out}(\widehat{F}_2)$ be the image of F. Then the following are equivalent:*

(i) (λ, f) *satisfies condition (I).*

(ii) $[\theta, \overline{F}] = 1$ *in* $\mathrm{Out}(\widehat{F}_2)$.

(iii) $[\theta, F] = \mathrm{inn}(f)$ *in* $\mathrm{Aut}(\widehat{F}_2)$.

Proof. (i) $\Rightarrow$ (iii): If we suppose that (I) holds, i.e. that $\theta(f) = f^{-1}$, then checking $\mathrm{inn}(f)F\theta$ and θF on x and y, we see immediately that they are equal in $\mathrm{Aut}(\widehat{F}_2)$.

(iii) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (i): Suppose that $[\theta, \overline{F}] = 1$ in $\mathrm{Out}(\widehat{F}_2)$. Then $[\theta, F] = \mathrm{inn}(\gamma)$ for some $\gamma \in \widehat{F}_2$. That is, $\mathrm{inn}(\gamma)F\theta = \theta F$. Comparing $F\theta(x) = F(y) = f^{-1}y^\lambda f$ with $\theta F(x) = y^\lambda$, we see that $\gamma = y^k f$ for some $k \in \widehat{\mathbb{Z}}$, because the centralizer of y in $\widehat{F}_2$ is the subgroup $\langle y \rangle$ generated by y. But comparing $F\theta(y) = x^\lambda$ with $\theta F(y) = \theta(f)^{-1} x^\lambda \theta(f)$, we also see that $\theta(f) = x^m \gamma$ for some $m \in \widehat{\mathbb{Z}}$. Reducing the equation $x^{-m}\theta(f) = \gamma = f^{-1}y^{-k}$ modulo $\widehat{F}_2'$ and using the fact that $f \in \widehat{F}_2'$, we immediately see that $k = m = 0$. Thus $\theta(f) = f^{-1}$, proving (i). $\diamondsuit$

Proposition 4. *In the situation of Proposition 3, assume that the three equivalent conditions of that proposition hold. Then the following are equivalent:*

(i) (λ, f) *satisfies condition (II).*

(ii) $[\omega, \overline{F}] = 1$ *in* $\mathrm{Out}(\widehat{F}_2)$.

(iii) $[\omega, F] = \mathrm{inn}(y^m f)$ *in* $\mathrm{Aut}(\widehat{F}_2)$, *where* $m = (\lambda - 1)/2$.

Proof. (i) $\Rightarrow$ (iii): Consider the profinite Artin braid group $\widehat{B}_3$, which has generators σ_1, σ_2 subject to the single relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Its center is generated by $c = (\sigma_1\sigma_2)^3$, and we may embed $\widehat{F}_2 = \langle x, y \rangle = \langle x, y, z \,|\, xyz = 1 \rangle$ in $\widehat{B}_3$ by $x \mapsto \sigma_1^2$, $y \mapsto \sigma_2^2$. Note also that the abelianization $\widehat{B}_3^{\mathrm{ab}} = \widehat{B}_3/\widehat{B}_3'$ is a free procyclic group, whose generator g is the image of σ_1 and of σ_2. Here $c \mapsto g^6$ under $\widehat{B}_3 \to \widehat{B}_3^{\mathrm{ab}}$.

By [LS], since (I) and (II) hold it follows that F extends to an automorphism of $\widehat{B}_3$, given by $\sigma_1 \mapsto \sigma_1^\lambda$ and $\sigma_2 \mapsto f^{-1}\sigma_2^\lambda f$. (This is stated and proved in [LS, Lemma 5] for the pro-ℓ braid group, and afterwards it is observed that it carries over to the full $\widehat{B}_3$.) In $\widehat{B}_3$, using the identity $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, we have that

$$z = (xy)^{-1} = (\sigma_1^2\sigma_2^2)^{-1} = z'c^{-1},$$

where $z' = \sigma_2^{-1}\sigma_1^2\sigma_2$ and where c is as above. Since c generates the center of $\widehat{B}_3$, its image $F(c)$ is of the form c^s for some $s \in \widehat{\mathbb{Z}}^*$. But computing $F(c)$ directly and using that $f \in \widehat{F}_2' \subset \widehat{B}_3'$, we find that $F(c)$ is congruent to c^λ modulo $\widehat{B}_3'$. Hence $c^{\lambda - s} \in \widehat{B}_3'$, and so its image in $\widehat{B}_3^{\mathrm{ab}}$ is trivial. Since this image is $g^{6(\lambda - s)}$, we conclude that $\lambda = s$ and so $F(c) = c^\lambda$.

Since conjugation commutes with taking products, and since the same holds under inverse limits, we have that

$$\sigma_2^{-1} f(\sigma_2^2, \sigma_1^2)\sigma_2 = f(\sigma_2^2, \sigma_2^{-1}\sigma_1^2\sigma_2),$$

or equivalently

$$f(y, x) = \sigma_2 f(y, z')\sigma_2^{-1}. \tag{4}$$

Also, since $c \in \widehat{B}_3$ is central and since f is a commutator, we have

$$f(y, z') = f(y, cz) = f(y, z). \tag{5}$$

Meanwhile, according to relation (II) (and also using (I)) we have

$$f(y,z)y^m f(x,y) = z^{-m} f(x,z)x^{-m}. \tag{6}$$

So

$$\begin{aligned} F(\sigma_2) &= f(y,x)\sigma_2^\lambda f(x,y) \\ &= \sigma_2 f(y,z')\sigma_2^{\lambda-1} f(x,y) && \text{by (4)} \\ &= \sigma_2 f(y,z)y^m f(x,y) && \text{by (5)} \\ &= \sigma_2 z^{-m} f(x,z)x^m. && \text{by (6)} \end{aligned}$$

Thus

$$\begin{aligned} F(z') &= F(\sigma_2)^{-1}F(\sigma_1)^2 F(\sigma_2) \\ &= x^m f(z,x)z^m(\sigma_2^{-1}\sigma_1^{2\lambda}\sigma_2)z^{-m} f(x,z)x^{-m} \\ &= x^m f(z,x)z^m(z')^\lambda z^{-m} f(x,z)x^{-m}. \end{aligned}$$

So since $z = z'c$, we obtain

$$\begin{aligned} F(z) &= F(z')F(c) \\ &= x^m f(z,x)z^m(z')^\lambda z^{-m} f(x,z)x^{-m}c^\lambda \\ &= x^m f(z,x)z^m(z'c)^\lambda z^{-m} f(x,z)x^{-m} \\ &= x^m f(z,x)z^\lambda f(x,z)x^{-m}. \end{aligned}$$

That is,

$$F(z) = x^m\omega^2(f)z^\lambda\omega^2(f)^{-1}x^{-m}. \tag{7}$$

Now let us compare $F\omega$ and $\mathrm{inn}(y^m f)^{-1}\omega F$ on the generators x and y of $\widehat{F}_2$. To start with, these two automorphisms agree on x, both sending it to $f^{-1}y^\lambda f$. Their values on y are respectively given by $F\omega(y) = F(z)$ and by

$$\mathrm{inn}(y^m f(x,y))^{-1}\omega F(y) = \mathrm{inn}\big(f(y,z)y^m f(x,y)\big)^{-1} z^\lambda.$$

By (6), the right hand side is equal to

$$x^m f(z,x)z^\lambda f(z,x)^{-1}x^{-m}$$

which is indeed the same as $F\omega(y) = F(z)$, by (7). Thus $F\omega$ and $\mathrm{inn}(y^m f(x,y))^{-1}\omega F$ agree on x and y, so they are equal. This yields (iii).

(iii) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (i): By (ii), there is a $\gamma \in \widehat{F}_2$ such that $[\omega, F] = \mathrm{inn}(\gamma)$; i.e. $F\omega = \mathrm{inn}(\gamma)^{-1}\omega F$. Equating these two automorphisms on x gives $\gamma = y^k f$ for some $k \in \widehat{\mathbb{Z}}$ (since $Z_{\widehat{F}_2}(y^\lambda) =$

$Z_{\widehat{F}_2}(y) = \langle y \rangle$). Using this, and equating the automorphisms $F\omega$ and $(\mathrm{inn}(\gamma)^{-1}\omega F$ on y, we obtain

$$F(z) = \Big(\mathrm{inn}\big(\omega(f) y^k f\big)\Big)^{-1} z^\lambda. \tag{8}$$

But by Proposition 2(c) we also have

$$F(z) = \big(\mathrm{inn}(x^m f(x,z)^{-1})\big) z^\lambda = \big(\mathrm{inn}(x^m \omega^2(f))\big) z^\lambda, \tag{9}$$

using that (i) of Proposition 3 gives us $f(x,z)^{-1} = f(z,x)$. Since the centralizer of z^λ is $\langle z \rangle$, we obtain from (8) and (9) that $\omega(f) y^k f x^m \omega^2(f) z^n = 1$ for some $n \in \widehat{\mathbb{Z}}$. In $\widehat{F}_2^{\mathrm{ab}} \approx \widehat{\mathbb{Z}}^3/\langle(1,1,1)\rangle$ this yields $y^k x^m z^n = 1$, and so $k = m = n$. Thus $\omega(f) y^m f x^m \omega^2(f) z^m = 1$, which gives (II). $\diamondsuit$

Using the above results, we obtain the main result of this section (where as before, β is the Belyi lifting of the natural map $\alpha : G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{F}_2)$).

Theorem 2. *Let $s : \mathrm{Out}_4^\sharp \to A^\sharp = A \cap \mathrm{Aut}_4^\sharp$ be as in Theorem 1. Then the image of s is equal to $\widehat{\mathrm{GT}}_0$. Thus $\widehat{\mathrm{GT}}_0$ is isomorphic to $\mathrm{Out}_4^\sharp$, compatibly with the actions of these two groups on $\widehat{F}_2$.*

Proof. Recall that $\widehat{\mathrm{GT}}_0$ consists of the elements F of $A \subset \mathrm{Aut}(\widehat{F}_2)$ for which the corresponding pair (λ, f) satisfies conditions (I) and (II). Now if $\overline{F} \in \mathrm{Out}_4^\sharp$ then $F := s(\overline{F})$ lies both in A and in $\mathrm{Aut}_4^\sharp$. Since $\overline{F}$ commutes with the elements (12) and (123) of S_3, Propositions 3 and 4 imply that F satisfies (I) and (II), and so is in $\widehat{\mathrm{GT}}_0$. Thus the image of s is contained in $\widehat{\mathrm{GT}}_0$. For the other containment, say $F \in \widehat{\mathrm{GT}}_0$. Thus $F \in A$, and by Propositions 3 and 4 its image $\overline{F} \in \mathrm{Out}(\widehat{F}_2)$ commutes with $(12), (123) \in S_3$. So $\overline{F} \in \mathrm{Out}_4^\sharp$. By the uniqueness part of Proposition 2 it follows that $F = s(\overline{F})$. So $\widehat{\mathrm{GT}}_0$ is contained in the image of s.

The final assertion is then immediate, since the action of $\widehat{\mathrm{GT}}_0$ on $\widehat{F}_2$ is simply the restriction of the action of $\mathrm{Aut}(\widehat{F}_2)$, and since the action of $\mathrm{Out}_4^\sharp$ on $\widehat{F}_2$ is the pullback under s of the action of $\mathrm{Aut}(\widehat{F}_2)$. $\diamondsuit$

Remark. This theorem provides an independent proof that $\widehat{\mathrm{GT}}_0$ is a group. It also shows that $\widehat{\mathrm{GT}}_0 \subset A^\sharp := A \cap \mathrm{Aut}_4^\sharp$, and so $\widehat{\mathrm{GT}}_0$ can instead be defined as the (*a priori* smaller) set of $F \in A^\sharp$ satisfying conditions (I) and (II). In the other direction, it shows that in the definition of $\widehat{\mathrm{GT}}_0$, one may drop the requirement that $F(xy) \sim (xy)^\lambda$ (i.e. that $F(z) \sim z^\lambda$) from the condition that $F \in A$. That is, $\widehat{\mathrm{GT}}_0$ is equal to the (*a priori* larger) set of automorphisms $F \in \mathrm{Aut}(\widehat{F}_2)$ such that $F(x) = x^\lambda$ and $F(y) = f^{-1} y^\lambda f$ for some $\lambda \in \mathbb{Z}^*$ and $f \in \widehat{F}_2'$, and which satisfy (I) and (II).

§2. Actions on five-point moduli

2.1. Fundamental groups

This section parallels Section 1, but for $\widehat{K}(0,5)$ and $\widehat{\mathrm{GT}}$ rather than for $\widehat{K}(0,4) = \widehat{F}_2$ and $\widehat{\mathrm{GT}}_0$, and proves part (b) of our Main Theorem for $n = 5$. Section 1 considered Belyi's lift $\beta : G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{K}(0,4))$ of the natural map $\alpha : G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{K}(0,4))$, having image in the Belyi group $A \subset \mathrm{Aut}(\widehat{K}(0,4))$ (cf. (2) of 0.1). Here we consider Nakamura's lift $\nu : G_{\mathbb{Q}} \to \mathrm{Aut}(\widehat{K}(0,5))$ of the natural map $\mu : G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{K}(0,5))$, with image in Nakamura's group $A_5 \subset \mathrm{Aut}(\widehat{K}(0,5))$ (cf. (3) of 0.1). As in the case of $\mathcal{M}_{0,4}$, an element of A_5 determines the pair $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$. Moreover there is agreement with the Belyi lift: The lifts β and ν associate the same pair (λ, f) to a given element of $G_{\mathbb{Q}}$.

In Proposition 5, we show that ν extends to a map on all of $\mathrm{Out}_5^\sharp$, with image in A_5 (analogously to Proposition 2 of §1). The proof of this result parallels Nakamura's strategy in constructing his lift [Na, Theorem A20], but with differences to allow for the extension to $\mathrm{Out}_5^\sharp$. As Theorem 3 then shows (analogously to Theorem 1 of 1.2), this extension of ν gives a section s_5 of $\mathrm{Aut}_5^\sharp \to \mathrm{Out}_5^\sharp$, so that $\nu = s_5\mu$. In the companion paper [HS], we show that ν is effective in terms of μ, using the corollary to Proposition 5; and this yields additional information about Galois orbits and fields of moduli of covers of $\mathbb{P}^1 - \{0, 1, \infty\}$.

The case $n = 5$ of our Main Theorem (Theorem 4, the analogue of Theorem 2 of 1.3) provides an interpretation of $\widehat{\mathrm{GT}}$ without the cocycle conditions (I)-(III). This uses Proposition 7, which shows that condition (III) is equivalent to commutation with a certain lift to $\mathrm{Aut}(\widehat{K}(0,5))$ of $(14253) \in S_5$, along with Propositions 3 and 4, the corresponding results (in §1) for (I) and (II). Using these, we show that image of s_5 is isomorphic to $\widehat{\mathrm{GT}}$ — and hence so is $\mathrm{Out}_5^\sharp$ (using an injectivity result, Proposition 6). This last fact is related to Grothendieck's suggestion that $G_{\mathbb{Q}}$ be studied by examining it as a group of outer actions on the $\widehat{K}(g,n)$'s, and to the fact ([LS], cf. 0.3 above) that $\widehat{\mathrm{GT}}$ is the group of "special automorphisms" of the tower of braid groups $\widehat{B}_n = \pi_1(\mathrm{Sym}^n(\mathbb{C}) - \Delta)$. Indeed, steps from that proof are used here in deducing one direction of the isomorphism $\mathrm{Out}_5^\sharp \xrightarrow{\sim} \widehat{\mathrm{GT}}$.

2.2. Outer automorphism groups

We retain the notation of §1. In particular, $\mathrm{Out}_n^\sharp = \mathrm{Out}^\sharp(\widehat{K}(0,n))$ is the subgroup of $\mathrm{Out}(\widehat{K}(0,n))$ consisting of the outer automorphisms that are symmetric (i.e. commute with the action of S_n) and quasi-special (i.e. take each conjugacy class $[x_{ij}]$ to a power of itself). Also, $\mathrm{Aut}_n^\sharp = \mathrm{Aut}^\sharp(\widehat{K}(0,n))$ is the inverse image of $\mathrm{Out}_n^\sharp$ under the map o : $\mathrm{Aut}(\widehat{K}(0,n)) \twoheadrightarrow \mathrm{Out}(\widehat{K}(0,n))$, and $\tilde{S}_n$ denotes the inverse image of S_n in $\mathrm{Aut}(\widehat{K}(0,n))$. By Proposition 1 (in 1.2), the image of the natural map $G_{\mathbb{Q}} \to \mathrm{Out}(\widehat{K}(0,n))$ is contained in $\mathrm{Out}_n^\sharp$. For $n = 4$ we denote this map by α, and for $n = 5$ we denote it by μ.

Lemma 1. *Let $\eta \in \tilde{S}_n \subset \mathrm{Aut}(\widehat{K}(0,n))$ be an element of finite order d. Let $F \in \mathrm{Aut}_n^\sharp$, so the commutator $h := [\eta, F] \in \widehat{K}(0,n)$. Then the product $\eta^{d-1}(h) \cdot \eta^{d-2}(h) \cdots \eta(h)h = 1$.*

Proof. Note that $\eta(h) = \eta h \eta^{-1}$ and that $\eta F \eta^{-1} = hF$. So by induction we obtain for each i that $\eta^i F \eta^{-i} = \eta^{i-1}(h) \cdot \eta^{i-2}(h) \cdots \eta(h)hF$. The result now follows by taking $i = d$. $\diamondsuit$

As in 1.1, let $p_i : \widehat{K}(0,n) \to \widehat{K}(0,n-1)$ be the natural surjective homomorphism obtained by omitting the ith entry, and whose kernel is generated by $\{x_{ij} \mid j \neq i\}$ subject to the single relation $\prod_{j\neq i} x_{ij} = 1$. Define an induced map $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}(\widehat{K}(0,n-1))$ as follows: Given $F \in \mathrm{Aut}_n^\sharp$, take any $\overline{f} \in \widehat{K}(0,n-1)$. Since $p_i : \widehat{K}(0,n) \to \widehat{K}(0,n-1)$ is surjective, there exists $f \in \widehat{K}(0,n)$ lying over $\overline{f}$. Define $q_i(F) \in \mathrm{Aut}(\widehat{K}(0,n-1))$ by $\overline{f} \mapsto p_i(F(f)) \in \widehat{K}(0,n-1)$. The fact that this is well defined (i.e. that $(q_i(F))(\overline{f})$ is independent of the choice of f over $\overline{f}$) follows from the fact that the kernel of $p_i : \widehat{K}(0,n) \to \widehat{K}(0,n-1)$ is generated by the conjugates of the elements x_{ij}, for $j \neq i$, and thus $\ker(p_i)$ is invariant under $F \in \mathrm{Aut}_n^\sharp$. (Alternatively, one may define the map $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}(\widehat{K}(0,n-1))$ by $g \mapsto p_i \circ g \circ \iota_i$, where $\iota_i : \widehat{K}(0,n-1) \hookrightarrow \widehat{K}(0,n)$ is the splitting given at the beginning of 1.2. Since the previous construction of $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}(\widehat{K}(0,n-1))$ was independent of the choice of f over $\overline{f}$, this construction agrees with that one.)

It is then straightforward to check that in fact we have actually constructed a homomorphism $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}_{n-1}^\sharp$. This in turn descends to a homomorphism $\overline{q}_i : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ such that $oq_i = \overline{q}_i o : \mathrm{Aut}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$, where $o : \mathrm{Aut}(\widehat{K}(0,n)) \twoheadrightarrow \mathrm{Out}(\widehat{K}(0,n))$ is as above.

For any element $g \in \widehat{K}(0,n)$, conjugation by g defines an element of $\mathrm{Aut}_n^\sharp$. Thus the image of $\widehat{K}(0,n) \to \mathrm{Aut}(\widehat{K}(0,n))$ lies in $\mathrm{Aut}_n^\sharp$. Since $\widehat{K}(0,n) \approx \widehat{F}_{n-2} \rtimes \widehat{K}(0,n-1)$ for $n \geq 4$ (cf. 1.2), and since $\widehat{K}(0,3)$ is trivial, it follows by induction that $\widehat{K}(0,n)$ has trivial center. (Cf. also [N1, p.104].) So we obtain the exact sequence

$$1 \to \widehat{K}(0,n) \to \mathrm{Aut}_n^\sharp \to \mathrm{Out}_n^\sharp \to 1. \tag{10}$$

Moreover, the sequences for n and $n-1$ are compatible via the maps q_i and $\overline{q}_i$.

We now restrict attention to the case of $n = 5$. Following the strategy of [N1, proof of Theorem A20], let

$$L = \{F \in \mathrm{Aut}_5^\sharp \,|\, F(x_{12}) = x_{12}^{\lambda(F)},\ \exists t \in (\ker p_2)'\langle x_{24}\rangle : F(x_{23}) = t x_{23}^{\lambda(F)} t^{-1}\}$$

(As before, we denote the commutator subgroup of a group G by G'.) Let $L_0 = \widehat{K}(0,5) \cap L$ and let $\overline{L}$ be the image of L in $\mathrm{Out}_5^\sharp$. Thus we have

$$1 \to L_0 \to L \to \overline{L} \to 1,$$

where the inclusion $L_0 \hookrightarrow L$ is via the identification of an element $g \in \widehat{K}(0,5)$ with $\mathrm{inn}\, g \in \mathrm{Aut}_5^\sharp$. Also, the maps $q_2 : \mathrm{Aut}_5^\sharp \to \mathrm{Aut}_4^\sharp$ and $\overline{q}_2 : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp$ restrict to compatible maps $q_2 : L \to \mathrm{Aut}_4^\sharp$ and $\overline{q}_2 : \overline{L} \to \mathrm{Out}_4^\sharp$, and the former map restricts to a map $p_2 : L_0 \to \widehat{K}(0,4) = \langle x_{34}, x_{45}\rangle$. Thus we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_0 & \longrightarrow & L & \longrightarrow & \overline{L} & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle q_2} & & \Big\downarrow{\scriptstyle \overline{q}_2} & & \\ 1 & \longrightarrow & \widehat{K}(0,4) & \longrightarrow & \mathrm{Aut}_4^\sharp & \longrightarrow & \mathrm{Out}_4^\sharp & \longrightarrow & 1 \end{array} \qquad (11)$$

The proof of the following lemma is adapted from an argument due to H. Nakamura (in a slightly different context; cf. the remark after the proof):

Lemma 2. *With the above notation, we have:*

(a) $L_0 = \langle x_{34}, x_{45}\rangle \subset \widehat{K}(0,5)$.

(b) The natural map $\overline{L} \to \mathrm{Out}_5^\sharp$ *is an isomorphism.*

(c) Let $F \in L$ *be an element that satisfies* $F(x_{45}) = x_{45}^{\lambda(F)}$. *Then* $F(x_{34}) = s x_{34}^{\lambda(F)} s^{-1}$ *for some* $s \in \langle x_{34}, x_{45}\rangle'$ *if and only if* $F(x_{34}) = s x_{34}^{\lambda(F)} s^{-1}$ *for some* $s \in (\ker p_4)'\langle x_{24}\rangle$.

(d) Over each element of $\mathrm{Out}_5^\sharp$ *there is a unique* $F \in \mathrm{Aut}_5^\sharp$ *such that* $F(x_{12}) = x_{12}^{\lambda(F)}$, $F(x_{23}) = t x_{23}^{\lambda(F)} t^{-1}$ *for some* $t \in \langle x_{12}, x_{23}\rangle'$, $F(x_{34}) = s x_{34}^{\lambda(F)} s^{-1}$ *for some* $s \in \langle x_{34}, x_{45}\rangle'$, *and* $F(x_{45}) = x_{45}^{\lambda(F)}$.

Proof. (a) View $L_0 \subset \widehat{K}(0,5)$. Since x_{45} commutes with x_{12} and x_{23} (because the subscripts are disjoint), we have that $x_{45} \in L_0$. Moreover, since $\prod_{i\neq 4} x_{i,4} = 1$, and since x_{14}, x_{54} commute with x_{23}, we have that $x_{24}x_{34}$ commutes with x_{23} and hence that $x_{34}x_{23}x_{34}^{-1} = x_{24}^{-1}x_{23}x_{24}$. Since we also have that x_{34} commutes with x_{12}, it follows that $x_{34} \in L_0$. Thus $\langle x_{34}, x_{45}\rangle \subset L_0$. Meanwhile, $\widehat{K}(0,5) = \ker(p_2) \rtimes \langle x_{34}, x_{45}\rangle$. So to prove (a), it suffices to show that $p_2 : L_0 \to \widehat{K}(0,4)$ has trivial kernel.

Now the kernel of $p_2 : \widehat{K}(0,5) \to \widehat{K}(0,4)$ is generated by the elements $x_{12}, x_{23}, x_{24}, x_{25}$ subject to the single relation $x_{12}x_{23}x_{24}x_{25} = 1$, and hence is the free profinite group on x_{12}, x_{23}, x_{24}. So if $g \in \ker(p_2 : L_0 \to \widehat{K}(0,4))$ commutes with x_{12}, then g is a (profinite) power of x_{12}.

Next, observe that if $g \in L_0$, then $x_{12} \sim g x_{12} g^{-1} = x_{12}^\lambda$ with $\lambda \in \widehat{\mathbb{Z}}^*$. Thus $\lambda = 1$ and so g commutes with x_{12}.

Now suppose that $g \in L_0 \subset L$ lies in $\ker p_2$; we wish to show that $g = 1$. By the previous two paragraphs, g commutes with x_{12} and hence is of the form $g = x_{12}^a$, with $a \in \widehat{\mathbb{Z}}$. Moreover $\lambda = 1$. Using the definition of L_0, we have that $g x_{23} g^{-1} = t x_{23} t^{-1}$ for some $t \in (\ker p_2)'\langle x_{24}\rangle$. Thus $t = g x_{23}^b = x_{12}^a x_{23}^b$ for some $b \in \widehat{\mathbb{Z}}$.

Let X_{ij} be the image of x_{ij} in the abelianization of $\ker p_2$. By the structure of $\ker p_2$, it follows that this abelianization is the free profinite abelian group on X_{12}, X_{23}, X_{24}. Since

$t \in (\ker p_2)'\langle x_{24}\rangle$, its image in the abelianization is a multiple of X_{24}. But since $t = x_{12}^a x_{23}^b$, its image is $aX_{12} + bX_{23}$. Thus $a = b = 0$, and so $g = 1$, as desired.

(b) Since $L_0 = \widehat{K}(0,5) \cap L$ and since sequence (10) is exact, the homomorphism $\overline{L} \to \mathrm{Out}_5^\sharp$ is injective, so it suffices to prove surjectivity. In doing so, we will construct an element of $\mathrm{Aut}_5^\sharp$, and then show that the element lies in L using the fact that

$$\begin{aligned} \ker(p_2)'\langle x_{24}\rangle = \{t \in \ker(p_2) \mid \\ \exists C \in \widehat{\mathbb{Z}} \text{ such that } t \mapsto CX_{24} \text{ under } \ker(p_2) \twoheadrightarrow \ker(p_2)/\ker(p_2)'\}, \end{aligned} \tag{12}$$

where $X_{ij} \in \ker(p_2)^{\mathrm{ab}}$ is as in (a) above.

So let $\overline{g} \in \mathrm{Out}_5^\sharp$, and let $g \in \mathrm{Aut}_5^\sharp$ lie over $\overline{g}$. Thus $g(x_{12}) = ux_{12}^\lambda u^{-1}$ for some (non-unique) $u \in \widehat{K}(0,5)$, where $\lambda = \lambda(g) \in \widehat{\mathbb{Z}}^*$. Since $\widehat{K}(0,5) = \ker(p_2) \rtimes \langle x_{34}, x_{45}\rangle$ and since $\langle x_{34}, x_{45}\rangle$ is contained in the centralizer of x_{12}, it follows that the above u may be chosen to lie in $\ker p_2$. Now the element $(\mathrm{inn}\, u^{-1})g \in \mathrm{Aut}_5^\sharp$ also lies over $\overline{g}$, and it takes x_{12} to x_{12}^λ. So replacing g by $(\mathrm{inn}\, u^{-1})g$, we may assume that g maps x_{12} to x_{12}^λ. Meanwhile, we similarly have $g(x_{23}) = tx_{23}^\lambda t^{-1}$ for some $t \in \ker p_2$ (using the decomposition $\widehat{K}(0,5) = \ker(p_2) \rtimes \langle x_{45}, x_{51}\rangle$). Here t is also non-unique, and in particular any element of the form tx_{23}^ν (with $\nu \in \widehat{\mathbb{Z}}$) is another possible choice.

As in the proof of (a), the abelianization of $\ker p_2$ is the free profinite abelian group on generators X_{12}, X_{23}, X_{24} (the images of x_{12}, x_{23}, x_{24}), and so the image of t in this abelianization is of the form $aX_{12} + bX_{23} + CX_{24}$. We may replace g by $(\mathrm{inn}\, x_{12}^{-a})g$, since that element also lies over $\overline{g}$ and also maps x_{12} to x_{12}^λ; thus we obtain a new choice of t for which $a = 0$. Next, replacing this t by tx_{23}^{-b} (which is another possible choice), we may assume that $b = 0$. Thus with these choices, the image of t in the abelianization of $\ker p_2$ lies in the image of $\langle x_{24}\rangle$, so $t \in \ker(p_2)'\langle x_{24}\rangle$ by (11). That is, this choice of g lies in L. Hence $\overline{g}$ lies in (the image of) $\overline{L}$, proving surjectivity.

(c) Let B_1 (resp. B) be the set of elements $F \in L$ such that $F(x_{45}) = x_{45}^{\lambda(F)}$ and such that $F(x_{34}) = sx_{34}^{\lambda(F)}s^{-1}$ for some $s \in (\ker p_4)'\langle x_{24}\rangle$ (resp. for some $s \in \langle x_{34}, x_{45}\rangle'$). We wish to show that $B = B_1$.

Since display (12) in (b) remains true with p_2 replaced by p_4, it follows that $\langle x_{34}, x_{45}\rangle' = \langle x_{34}, x_{45}\rangle \cap (\ker p_4)'\langle x_{24}\rangle$ and hence that $B \subset B_1$. To show the other containment, it suffices to show that the map $B_1 \to \mathrm{Out}_5^\sharp$ is injective and that its restriction to $B \subset B_1$ is surjective onto $\mathrm{Out}_5^\sharp$.

First we show injectivity of $B_1 \to \mathrm{Out}_5^\sharp$, as a map of sets (since we have not shown that B_1 is a group). So suppose that $g, h \in B_1$ have the same image in $\mathrm{Out}_5^\sharp$. Then $h = (\mathrm{inn}\, u)g$ for some $u \in \langle x_{34}, x_{45}\rangle$, and $\lambda(g) = \lambda(h)$ $(= \lambda$, say). We wish to show that $u = 1$. Since $g, h \in B_1 \subset L$, we have that $x_{45}^\lambda = h(x_{45}) = (\mathrm{inn}\, u)g(x_{45}) = (\mathrm{inn}\, u)x_{45}^\lambda$. So $u \in Z_{\widehat{K}(0,5)}(x_{45}^\lambda) = Z_{\widehat{K}(0,5)}(x_{45}) = \langle x_{45}\rangle$, since $\lambda \in \widehat{\mathbb{Z}}^*$. Thus $u = x_{45}^a$ for some $a \in \widehat{\mathbb{Z}}$.

Since $g, h \in B_1$, there exist $r, s \in (\ker p_4)'\langle x_{24}\rangle$ such that $g(x_{34}) = s x_{34}^{\lambda} s^{-1}$ and $h(x_{34}) = r x_{34}^{\lambda} r^{-1}$. Since also $h(x_{34}) = u g(x_{34}) u^{-1}$, it follows that $r^{-1}us$ commutes with x_{34}. Now $r^{-1}us \in \ker(p_4) = \langle x_{24}, x_{34}, x_{45}\rangle$, which is a free profinite group on those three generators, so $r^{-1}us = x_{34}^b$ for some $b \in \widehat{\mathbb{Z}}$. Now in the abelianization of $\langle x_{24}, x_{34}, x_{45}\rangle$, the images of r, s each lie in $\langle X_{24}\rangle$, so the equation $r^{-1}us = x_{34}^b$ implies that $aX_{45} - bX_{34} \in \langle X_{24}\rangle$. Since $\langle x_{24}, x_{34}, x_{45}\rangle^{\mathrm{ab}}$ is the free profinite abelian group on X_{23}, X_{34}, X_{45}, it follows that $a = b = 0$, so $u = 1$. This proves the desired injectivity.

For surjectivity of $B \to \mathrm{Out}_5^\sharp$, take any element $\overline{F} \in \mathrm{Out}_5^\sharp$, and any element $F \in L$ over $\overline{F}$ (which exists by part (b)). By (a), the map $q_2 : L \to \mathrm{Aut}_4^\sharp$ restricts to the identity on $L_0 = \langle x_{34}, x_{45}\rangle$, which is viewed as a group of inner automorphisms. In particular, since $x_{34}, F(x_{34}) \in L_0 = \langle x_{34}, x_{45}\rangle$, we have that $p_2(x_{34}) = x_{34}$ and $p_2(F(x_{34})) = F(x_{34})$. Since the diagram (11) commutes, we obtain $F(x_{34}) = p_2(F(x_{34})) = \big(q_2(F)\big)\big(p_2(x_{34})\big) = \big(q_2(F)\big)(x_{34}) = f x_{34}^{\lambda(F)} f^{-1}$ for some $f \in \langle x_{34}, x_{45}\rangle$, since $q_2(F) \in \mathrm{Aut}_4^\sharp$. Similarly, $F(x_{45}) = h x_{45}^{\lambda(F)} h^{-1}$ for some $h \in \langle x_{34}, x_{45}\rangle$. Replacing F by its conjugate by h^{-1}, we may assume that $h = 1$. Since L_0 is the free profinite group on x_{34}, x_{45}, there exist $a, b \in \widehat{\mathbb{Z}}$ such that $x_{45}^a f x_{34}^b \in L_0'$. By replacing F by its conjugate by x_{45}^{-a}, we may assume that $a = 0$ (and still that $h = 1$). Thus this F satisfies $F(x_{45}) = x_{45}^{\lambda(F)}$ and $F(x_{34}) = s x_{34}^{\lambda(F)} s^{-1}$, where $s = f x_{34}^b \in \langle x_{34}, x_{45}\rangle'$. So $F \in B$, proving surjectivity.

(d) Let B_2 be the set of $F \in \mathrm{Aut}_5^\sharp$ such that $F(x_{12}), F(x_{23}), F(x_{34}), F(x_{45})$ are as in the assertion of (d). We wish to show that $o : B_2 \to \mathrm{Out}_5^\sharp$ is bijective.

We preserve the notation of the proof of part (c). Let $\phi \in \mathrm{Aut}(\widehat{K}(0,5))$ be given by $x_{i,j} \mapsto x_{6-i,6-j}$ (where as usual $x_{i,j}$ denotes the same element as $x_{j,i}$). Let $\phi(B)$ denote $\{\phi f \phi^{-1} \mid f \in B\}$, and similarly for B_1. Now $B = B_1$ by part (c), and so $\phi(B) = \phi(B_1)$. But $\phi(B_1) = B_1$. So every element $F \in B_1$ lies in $\phi(B)$, and hence such an F satisfies $F(x_{23}) = t x_{23}^{\lambda(F)} t^{-1}$ for some $t \in \langle x_{12}, x_{23}\rangle'$. That is, $F \in B_2$. This shows that $B_1 \subset B_2$, and hence $B_1 = B_2$ (since the opposite inclusion is immediate).

In the proof of (c) it was shown that $B_1 \to \mathrm{Out}_5^\sharp$ is injective, that $B \to \mathrm{Out}_5^\sharp$ is surjective, and that $B = B_1$. Since $B_1 = B_2$, we have that $B_2 \to \mathrm{Out}_5^\sharp$ is bijective, as desired. $\diamondsuit$

Remark. The above argument partly employs the first part of the proof of [N1, Theorem (A20)]. There, the role of $\mathrm{Out}_n^\sharp$ is played by G_k (the absolute Galois group of a number field k) and the role of $\mathrm{Aut}_n^\sharp$ is played by the algebraic fundamental group of the moduli space $M_{0,n}$ of ordered n-tuples of distinct points of $\mathbb{P}^1$. The role of L above was played by a subgroup $\mathcal{L} \subset \pi_1(M_{0,5})$ of the algebraic fundamental group. The analogues of the arguments in parts (a) and (b) above were implicitly used to show the injectivity and surjectivity of the "second projection map" $\mathcal{L} \to \pi_1(M_{0,4})$, in order to obtain the desired exact sequence $1 \to \langle x_{34}, x_{45}\rangle \to \mathcal{L} \to G_k \to 1$; and afterwards the analogues of (c) and (d) were used. Note that the proofs in both contexts use a pair of short exact sequences with kernels $\widehat{K}(0,5)$ and $\widehat{K}(0,4)$ respectively, related by second projection maps. But in [N1], the cokernels are known to be equal (viz. to G_k), whereas here the map on cokernels $\overline{q}_2 : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp$ is not surjective (although in retrospect it is injective; cf. Proposition 6 of 2.2 below). This is why we need to proceed as in the proof of Proposition 5 below.

Lemma 3. *Let $e \in \widehat{K}(0,5)$, let $f, g \in \langle x_{12}, x_{23}\rangle'$, and suppose that the inner automorphism $\mathrm{inn}\, e$ takes $(\mathrm{inn}\, f)x_{12} \mapsto (\mathrm{inn}\, g)x_{12}$, $x_{23} \mapsto x_{23}$, $x_{45} \mapsto x_{45}$. Then $e \in \langle x_{45}\rangle$.*

Proof. The group $\langle x_{23}, x_{45}\rangle$ is a free abelian profinite group of rank 2, and it is self-centralizing in $\widehat{K}(0,5)$ (as can be seen via the decomposition $\widehat{K}(0,5) \xrightarrow{\sim} \ker(p_4) \rtimes \widehat{K}(0,4) \approx \widehat{F}_3 \rtimes \widehat{F}_2$). So $e \in \langle x_{23}, x_{45}\rangle$, and we may write $e = x_{23}^a x_{45}^b$ for some $a, b \in \widehat{\mathbb{Z}}$. Thus $x_{12} = (\mathrm{inn}\, g^{-1}ef)x_{12} = (\mathrm{inn}\, g^{-1}x_{23}^a f)x_{12}$, since $(\mathrm{inn}\, ef)x_{12} = (\mathrm{inn}\, g)x_{12}$ and since x_{45} commutes with x_{12} and x_{23}. Hence $g^{-1}x_{23}^a f \in \langle x_{12}, x_{23}\rangle$ commutes with x_{12} and thus is of the form x_{12}^c for some $c \in \widehat{\mathbb{Z}}$. Since $f, g \in \langle x_{12}, x_{23}\rangle'$, it follows that x_{23}^a and x_{12}^c have the same image in the free abelian profinite group $\langle x_{12}, x_{23}\rangle^{\mathrm{ab}}$. Hence $a = c = 0$, and so actually $e = x_{45}^b \in \langle x_{45}\rangle$. ♢

Let $\rho \in \mathrm{Aut}(\widehat{K}(0,5))$ be defined by $\rho(x_{i,j}) = x_{i+3,j+3}$ (indices modulo 5). Thus ρ is an automorphism of order 5.

Lemma 4. *Let $f \in \widehat{K}(0,5)'$, let $a \in \widehat{\mathbb{Z}}$, and suppose that the element $h = x_{45}^a f \in \widehat{K}(0,5)$ satisfies $\rho^4(h)\rho^3(h)\rho^2(h)\rho(h)h = 1$. Then $a = 0$ and $h = f$.*

Proof. The group $\widehat{K}(0,5)$ is generated by the five elements $x_{i,i+1}$ (with i modulo 5). Let G be the free abelian profinite group on those five generators, and let $\pi : \widehat{K}(0,5) \to G$ be the surjective homomorphism taking $x_{i,i+1} \mapsto x_{i,i+1}$. (In fact, $G = \widehat{K}(0,5)^{\mathrm{ab}}$, and this is the reduction map.) Then $1 = \pi\big(\rho^4(h)\rho^3(h)\rho^2(h)\rho(h)h\big) = \prod_i x_{i,i+1}^a$, so $a = 0$ and $h = f$. ♢

The following result is an analogue of Proposition 2 of §1 for $\mathrm{Out}_5^\sharp$. That earlier result considered the Belyi lift $\beta : G_{\mathbb{Q}} \to A^\sharp = A \cap \mathrm{Aut}_4^\sharp \subset \mathrm{Aut}(K(0,4))$ of the natural map $\alpha : G_{\mathbb{Q}} \to \mathrm{Out}_4^\sharp \subset \mathrm{Out}(\widehat{K}(0,4))$, and extended β to all of $\mathrm{Out}_4^\sharp$. That was then interpreted (in Theorem 1) as providing a section s of $\mathrm{Aut}_4^\sharp \twoheadrightarrow \mathrm{Out}_4^\sharp$ such that $\beta = s\alpha$. Here we consider Nakamura's lift $\nu : G_{\mathbb{Q}} \to A_5^\sharp = A_5 \cap \mathrm{Aut}_5^\sharp \subset \mathrm{Aut}(K(0,5))$ of the natural map $\mu : G_{\mathbb{Q}} \to \mathrm{Out}_5^\sharp \subset \mathrm{Out}(\widehat{K}(0,5))$, and we extend ν to all of $\mathrm{Out}_5^\sharp$. Afterwards, in Theorem 3, we interpret this as providing a section s_5 of $\mathrm{Aut}_5^\sharp \twoheadrightarrow \mathrm{Out}_5^\sharp$ such that $\nu = s\mu$.

Proposition 5. *Let $\overline{F} \in \mathrm{Out}_5^\sharp$ and let $\lambda = \lambda(F) \in \widehat{\mathbb{Z}}^*$.*

(a) There is an $f \in \widehat{F}_2'$ such that some lift $F \in \mathrm{Aut}_5^\sharp$ of $\overline{F}$ satisfies $F(x_{12}) = x_{12}^\lambda$, $F(x_{23}) = (\mathrm{inn}\, f(x_{12}, x_{23})^{-1})x_{23}^\lambda$, $F(x_{34}) = (\mathrm{inn}\, f(x_{45}, x_{34})^{-1})x_{34}^\lambda$, and $F(x_{45}) = x_{45}^\lambda$.

(b) The f and F in part (a) are unique, and satisfy $[\rho, F] = \mathrm{inn}\, f$.

(c) The element f in part (a) is equal to the f obtained by applying Proposition 2 to $\overline{q}_4(\overline{F}) \in \mathrm{Out}_4^\sharp \approx \mathrm{Out}(\widehat{F}_2)$, with respect to the map $x_{12} \mapsto x \in \widehat{F}_2$, $x_{23} \mapsto y \in \widehat{F}_2$. The same

is true if one instead applies Proposition 2 to $\overline{q}_2(\overline{F}) \in \mathrm{Out}_4^\sharp \approx \mathrm{Out}(\widehat{F}_2)$, *with respect to the map* $x_{45} \mapsto x \in \widehat{F}_2$, $x_{34} \mapsto y \in \widehat{F}_2$.

(d) *For* f *and* F *in (a),* $F(x_{51}) = (\mathrm{inn}\,(f(x_{45}, x_{51})f(x_{12}, x_{23}))^{-1})x_{51}^\lambda$.

Proof. By Lemma 2(d), there is a unique $F \in \mathrm{Aut}_5^\sharp$ lying over $\overline{F} \in \mathrm{Out}_5^\sharp$ such that

$$F(x_{12}) = x_{12}^\lambda;\ F(x_{23}) = (\mathrm{inn}\,t)x_{23}^\lambda;\ F(x_{34}) = (\mathrm{inn}\,s)x_{34}^\lambda;\ F(x_{45}) = x_{45}^\lambda$$

for some $t \in \langle x_{12}, x_{23}\rangle'$ and $s \in \langle x_{34}, x_{45}\rangle'$. Let $\rho \in \tilde{S}_5 \subset \mathrm{Aut}(\widehat{K}(0,5))$ be as in Lemma 3. Since $F \in \mathrm{Aut}_5^\sharp$, we have that the commutator $[\rho, F] = \mathrm{inn}\,f$ for some $f \in \widehat{K}(0,5)$ (as in condition (ii) prior to Proposition 1 of §1). Now the automorphism $[\rho, F]$ takes

$$x_{12}^\lambda \mapsto (\mathrm{inn}\,s(x_{12}, x_{23}))x_{12}^\lambda;\ (\mathrm{inn}\,t(x_{12}, x_{23}))x_{23}^\lambda \mapsto x_{23}^\lambda;\ x_{45}^\lambda \mapsto x_{45}^\lambda.$$

So the inner automorphism $\mathrm{inn}\,ft(x_{12}, x_{23})$ takes

$$(\mathrm{inn}\,t(x_{12}, x_{23})^{-1})x_{12}^\lambda \mapsto (\mathrm{inn}\,s(x_{12}, x_{23}))x_{12}^\lambda;\ x_{23}^\lambda \mapsto x_{23}^\lambda;\ x_{45}^\lambda \mapsto x_{45}^\lambda.$$

Raising each of these expressions to the λ^{-1}th power, we see $\mathrm{inn}\,ft(x_{12}, x_{23})$ acts by

$$(\mathrm{inn}\,t(x_{12}, x_{23})^{-1})x_{12} \mapsto (\mathrm{inn}\,s(x_{12}, x_{23}))x_{12};\ x_{23} \mapsto x_{23};\ x_{45} \mapsto x_{45}.$$

So by Lemma 3 we have $ft(x_{12}, x_{23}) \in \langle x_{45}\rangle$. Thus $f = x_{45}^a t(x_{12}, x_{23})^{-1}$ for some $a \in \widehat{\mathbb{Z}}$. Also $t(x_{12}, x_{23})^{-1} \in \widehat{K}(0,5)'$, and $\rho^4(f)\rho^3(f)\rho^2(f)\rho(f)f = 1$ by Lemma 1. So Lemma 4 implies that $a = 0$ and $f = t(x_{12}, x_{23})^{-1}$. So we may write $f = f(x_{12}, x_{23}) \in \langle x_{12}, x_{23}\rangle' \subset \widehat{K}(0,5)$, and regard $f \in \widehat{F}_2'$, where we identify the free profinite group $\widehat{F}_2$ with $\langle x_{12}, x_{23}\rangle$. Thus we have that $F(x_{23}) = (\mathrm{inn}\,f(x_{12}, x_{23})^{-1})x_{23}^\lambda$. Moreover we have $\rho F \rho^{-1} F^{-1} = [\rho, F] = \mathrm{inn}\,f = \mathrm{inn}\,t^{-1}$, so $F\rho^{-1} = \rho^{-1}(\mathrm{inn}\,t^{-1})F$. Evaluating both sides on x_{12}, we obtain

$$\begin{aligned} F(x_{34}) &= \rho^{-1}(\mathrm{inn}\,t(x_{12}, x_{23})^{-1})x_{12}^\lambda \\ &= (\mathrm{inn}\,t(x_{34}, x_{45})^{-1})x_{34}^\lambda = (\mathrm{inn}\,f(x_{34}, x_{45}))x_{34}^\lambda. \end{aligned}$$

In order to complete the proof of part (a), it suffices to show that $f(x,y) = f(y,x)^{-1} \in \widehat{F}_2$.

Now consider the map $\overline{q}_4 : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp = \mathrm{Out}(\widehat{K}(0,4))$, and identify $\widehat{K}(0,4) \approx \widehat{F}_2$ by $x_{12} \mapsto x,\ x_{23} \mapsto y$. Then $q_4(F)$ is a lift of $\overline{q}_4(\overline{F})$, and satisfies the condition of Proposition 2 (in 1.2) with respect to the above $f \in \widehat{F}_2$. Thus our f is equal to the f of Proposition 2, by the uniqueness assertion of that proposition. This proves the first part of (c). Since the f of Proposition 2 satisfies $f(x,y) = f(y,x)^{-1}$ (by Proposition 3 (ii) $\Rightarrow$ (i)), the proof of part (a) is complete. Finally, we may repeat the above argument with $\overline{q}_2 : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp$ replacing $\overline{q}_4$, and with the identification $x_{45} \mapsto x,\ x_{34} \mapsto y$. This yields the second part of (c). In the first part of (b), the uniqueness of F follows by again invoking Lemma 2(d), and the uniqueness of f follows from part (c). We have already seen the second half of (b), which may be rewritten as $\rho F\rho^{-1} = (\mathrm{inn} f(x_{12}, x_{23}))F$. Applying both sides to x_{51} proves part (d) and completes the proof. $\diamondsuit$

Since $\widehat{K}(0,4)$ is generated by $x = x_{12}$ and $y = x_{23}$, any element $F \in \mathrm{Aut}_4^\sharp$ is determined by $F(x)$ and $F(y)$. So if $\overline{F}$ is the image of F in $\mathrm{Out}_4^\sharp$, then $\overline{F}$ determines the pair $(F(x), F(y))$ up to uniform conjugacy, and is conversely determined by the uniform conjugacy class of this pair. Using Proposition 5 above, we obtain an analogue of this fact for $\mathrm{Out}_5^\sharp$:

Corollary. *For each $F \in \mathrm{Aut}_5^\sharp$, let $\overline{F}$ be the image of F in $\mathrm{Out}_5^\sharp$ and let $\xi(\overline{F})$ be the uniform conjugacy class of $(F(x_{12}), F(x_{23}))$. Then $\xi : \mathrm{Out}_5^\sharp \to \widehat{K}(0,5) \times \widehat{K}(0,5)/\sim$ is a well-de fined injection, where $\sim$ denotes the equivalence relation of uniform conjugacy on the set of pairs $\widehat{K}(0,5) \times \widehat{K}(0,5)$.*

Proof. The map ξ is well defined, since another choice of F over $\overline{F}$ would differ by an inner automorphism, and so the two pairs would be uniformly conjugate.

To show injectivity, observe that if $\xi(\overline{F}) = \xi(\overline{G})$ then for appropriate lifts F, G of $\overline{F}, \overline{G}$ we have $F(x_{12}) = G(x_{12})$ and $F(x_{23}) = G(x_{23})$. Thus $H = G^{-1} \circ F \in \mathrm{Aut}_5^\sharp$ fixes both x_{12} and x_{23}. Le t $\overline{H}$ be the image of H in $\mathrm{Out}_5^\sharp$. Now $q_4(H) \in \mathrm{Aut}_4^\sharp$ lifts $\overline{q}_4(\overline{H}) \in \mathrm{Out}_4^\sharp$, and is equal to the identity (since it fixes $x = x_{12}$ and $y = x_{23}$). Thus $q_4(H)$ is the element of $\mathrm{Aut}_4^\sharp$ that is associate d to $\overline{q}_4(\overline{H})$ in Proposition 2, and the corresponding element of $\widehat{F}_2'$ there is the identity. Meanwhile, $\lambda(\overline{H}) = \lambda(H) = 1$, and so by Proposition 5(a) there is an element $f \in \widehat{F}_2'$ and a lift $H' \in \mathrm{Aut}_5^\sharp$ of $\overline{H}$ such that each $H(x_{i,i+1})$ is as given there. In particular, $H'(x_{12}) = x_{12}$ and $H'(x_{23}) = f^{-1}x_{23}f$. By Proposition 5(c), f is equal to the element of $\widehat{F}_2'$ that is associated to $\overline{q}_4(\overline{H})$ in Proposition 2; i.e. $f = 1$. Hence by the formulas for $H(x_{34})$ and $H(x_{45})$ in Proposition 5(a) and by the formula for $H(x_{51})$ in Proposition 5(d), we have that H' is the identity. Thus $\overline{H} = 1$ and hence $\overline{F} = \overline{G}$. $\diamondsuit$

Finally, using Proposition 5 we obtain the following analogue of Theorem 1 for $\widehat{K}(0,5)$:

Theorem 3. *There is a unique section s_5 of $\mathrm{Aut}_5^\sharp \to \mathrm{Out}_5^\sharp$ whose image lies in A_5. This section satisfies $\nu = s_5\mu : G_{\mathbb{Q}} \to A_5 \subset \mathrm{Aut}(\widehat{K}(0,5))$.*

Proof. According to Proposition 5, over every element of $\mathrm{Out}_5^\sharp$ there is a unique element of $A_5^\sharp = A \cap \mathrm{Aut}_5^\sharp$. Thus there is a unique section s_5 of $\mathrm{Aut}_5^\sharp \to \mathrm{Out}_5^\sharp$ whose image lies in A_5. For every $\omega \in G_{\mathbb{Q}}$, the elements $\nu(\omega)$ and $s_5\mu(\omega)$ are each in A_5 and each lies over $\mu(\omega) \in \mathrm{Out}(\widehat{K}(0,5))$. By Proposition 1 of §1.2, $\mu(\omega) \in \mathrm{Out}_5^\sharp$. So by the uniqueness assertion in Proposition 5, we have that $\nu(\omega) = s_5\mu(\omega)$. Thus $\nu = s_5\mu$. $\diamondsuit$

Remark. By the results of 1.2 concerning $\widehat{K}(0,4)$, the image of $G_{\mathbb{Q}}$ under the Belyi lift $\beta : G_{\mathbb{Q}} \to \mathrm{Out}_4^\sharp$ actually lies in $A^\sharp := A \cap \mathrm{Aut}_4^\sharp$, and the same is true for the image of the section s of $\mathrm{Aut}_4^\sharp \twoheadrightarrow \mathrm{Out}_4^\sharp$. Here, the above results on $\widehat{K}(0,5)$ show that the image of $G_{\mathbb{Q}}$

under Nakamura's lift $\nu : G_{\mathbb{Q}} \to \mathrm{Out}_5^\sharp$ actually lies in $A_5 := A \cap \mathrm{Aut}_5^\sharp$, and the same is true for the image of the above section s_5 of $\mathrm{Aut}_5^\sharp \twoheadrightarrow \mathrm{Out}_5^\sharp$. The sets $A^\sharp$ and $A_5^\sharp$ can be related as follows. As in the proof of Lemma 2(d) above, define $\phi \in \mathrm{Aut}(\widehat{K}(0,5))$ by $x_{i,j} \mapsto x_{6-j,6-i}$ and as before let $q_5 : \mathrm{Aut}_5^\sharp \twoheadrightarrow \mathrm{Aut}_4^\sharp$ be the map obtained by suppressing the x_{ij}'s with i or j equal to 5. Then $A_5^\sharp$ is the set of elements of $q_5^{-1}(A_4^\sharp)$ that commute with ϕ.

2.3. Connection to $\widehat{\mathrm{GT}}$.

Below we show Theorem 4, an analogue of Theorem 2 (of 1.3) for the case $n = 5$. It shows, via the above section s_5, that $\mathrm{Out}_5^\sharp$ is isomorphic to the Grothendieck-Teichmüller group $\widehat{\mathrm{GT}}$ (cf. 0.3). This provides a way of viewing $\widehat{\mathrm{GT}}$ without the three usual cycle relations (I) - (III), and proves part (b) of our Main Theorem in the case $n = 5$. As before, we will view elements of $\widehat{\mathrm{GT}}$ either as elements F of the Belyi group A (cf. 0.2) such that the associated pair (λ, f) satisfies conditions (I) - (III), or as the group of such invertible pairs (λ, f).

Recall from 2.2 that for each $i = 1, \ldots, 5$, there is a homomorphism $q_i : \mathrm{Aut}_5^\sharp \to \mathrm{Aut}_4^\sharp$ that is compatible with the surjection $p_i : \widehat{K}(0,5) \twoheadrightarrow \widehat{K}(0,4)$. Moreover, $q_i : \mathrm{Aut}_5^\sharp \to \mathrm{Aut}_4^\sharp$ descends to a compatible map $\overline{q}_i : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp$; i.e. $oq_i = \overline{q}_i o$, where $o : \mathrm{Aut}(\widehat{K}(0,n)) \twoheadrightarrow \mathrm{Out}(\widehat{K}(0,n))$ is the natural map as before.

Proposition 6. *For any $i = 1, \ldots, 5$, the homomorphism $\overline{q}_i : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp$ is injective.*

Proof. By symmetry (i.e. by applying an appropriate power of ρ) we may assume that $i = 4$. Let $\overline{F} \in \ker \overline{q}_4$. Then $\lambda(\overline{F}) = \lambda(\overline{q}_4(\overline{F})) = \lambda(1) = 0 \in \widehat{\mathbb{Z}}^*$. Also, since $\overline{q}_4(\overline{F}) = 1$, the element $f \in \widehat{F}_2$ that is associated to $\overline{q}_4(\overline{F})$ by Proposition 2 of 1.2 must also equal 1. So by Proposition 5, there is a lift F of $\overline{F}$ to $\mathrm{Aut}_5^\sharp$ such that $F(x_{i,i+1}) = x_{i,i+1}$ for all i (modulo 5). Since the elements $x_{i,i+1}$ generate $\widehat{K}(0,5)$, we have that $F = 1$ and so $\overline{F} = 1$. $\diamondsuit$

Let $j : \widehat{F}_2 = \widehat{K}(0,4) \to \widehat{K}(0,5)$ be given by $x = x_{12} \mapsto x_{12}$, $y = x_{23} \mapsto x_{23}$. If $F \in \mathrm{Aut}(\widehat{K}(0,4))$ and $\tilde{F} \in \mathrm{Aut}(\widehat{K}(0,5))$, we will say that $\tilde{F}$ *extends* F if $\tilde{F}j = jF$. We will also say that $\overline{\tilde{F}} \in \mathrm{Out}(\widehat{K}(0,5))$ *extends* $\overline{F} \in \mathrm{Out}(\widehat{K}(0,4))$ if some $\tilde{F} \in \mathrm{Aut}(\widehat{K}(0,5))$ in the class of $\overline{\tilde{F}}$ extends some F in the class of $\overline{F}$. Thus, for example, if $\tilde{F}$ is any element of $\mathrm{Out}_5^\sharp$, then $\tilde{F}$ extends $p_4(\tilde{F})$.

Similarly, consider the profinite mapping class group $\widehat{M}(0,5) \simeq \pi_1(\mathcal{M}_{0,5}/S_5)$, where $\mathcal{M}_{0,5}/S_5$ is the moduli space of genus 0 curves together with five unordered marked points. Then $\widehat{M}(0,5)$ is a quotient of the profinite braid group $\widehat{B}_5 = \pi_1(\mathbf{A}^n - D)$, where D is the discriminant locus, and it is generated by elements $\sigma_i = \sigma_{i,i+1}$ for i modulo 5. (The element $\sigma_5 = \sigma_{5,1}$ can also be expressed in terms of $\sigma_1, \ldots, \sigma_4$.) Here $\widehat{K}(0,5) = \pi_1(\mathcal{M}_{0,5})$ is the kernel of the natural map $\widehat{M}(0,5) \twoheadrightarrow S_5$, and is a characteristic subgroup of $\widehat{M}(0,5)$. We will say that $G \in \mathrm{Aut}(\widehat{M}(0,5))$ *extends* $F \in \mathrm{Aut}(\widehat{K}(0,4))$ if the restriction $\tilde{F} \in \mathrm{Aut}(\widehat{K}(0,5))$ of G extends F.

As before, let $\rho \in \mathrm{Aut}(\widehat{K}(0,5))$ be given by $\rho(x_{i,j}) = x_{i+3,j+3}$ (indices modulo 5). Here $\rho \in \tilde{S}_5$ (cf. 1.2) lies over the permutation $(14253) \in S_5$. Recall that $A_5^\sharp := A_5 \cap \mathrm{Aut}_5^\sharp$.

The following result is analogous to Propositions 3 and 4 of 1.3:

Proposition 7. *Let $F \in A$, corresponding to $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$. Suppose that (λ, f) satisfies conditions (I) and (II) (or equivalently, that $F \in A^\sharp = A \cap \mathrm{Aut}_4^\sharp$). Let $\overline{F} \in \mathrm{Out}_4^\sharp$ be the image of F. Consider the homomorphism $p_i : \mathrm{Out}_5^\sharp \to \mathrm{Out}_4^\sharp$ for some $i = 1, \dots, 5$ and the injection $j : \widehat{F}_2 = \widehat{K}(0,4) \to \widehat{K}(0,5)$ given by $x = x_{12} \mapsto x_{12}$, $y = x_{23} \mapsto x_{23}$.*

The following are equivalent:

(i) (λ, f) *satisfies condition (III).*

(ii) $\overline{F} = p_i(\overline{\tilde{F}})$ *for some* $\overline{\tilde{F}} \in \mathrm{Out}_5^\sharp$ *such that* $[\rho, \overline{\tilde{F}}] = 1$ *in* $\mathrm{Out}(\widehat{K}(0,5))$.

(ii)′ $\overline{F}$ *extends to some* $\overline{\tilde{F}} \in \mathrm{Out}_5^\sharp$ *such that* $[\rho, \overline{\tilde{F}}] = 1$ *in* $\mathrm{Out}(\widehat{K}(0,5))$.

(iii) $F = p_i(\tilde{F})$ *for some* $\tilde{F} \in A_5^\sharp$ *such that* $[\rho, \tilde{F}] = \mathrm{inn}\, f$ *in* $\mathrm{Aut}(\widehat{K}(0,5))$.

(iii)′ F *extends to some* $\tilde{F} \in A_5^\sharp$ *such that* $[\rho, \tilde{F}] = \mathrm{inn}\, f$ *in* $\mathrm{Aut}(\widehat{K}(0,5))$.

Proof. (i) $\Rightarrow$ (iii)′: By hypothesis, $F \in \widehat{\mathrm{GT}}$. So by [LS, Lemma 7], F extends to an automorphism G of $\widehat{M}(0,5)$ given by

$$
\begin{aligned}
G(\sigma_1) &= \sigma_1^\lambda \\
G(\sigma_2) &= \big(\mathrm{inn} f(x_{12}, x_{23})^{-1}\big)\sigma_2^\lambda \\
G(\sigma_3) &= \big(\mathrm{inn} f(x_{45}, x_{34})^{-1}\big)\sigma_3^\lambda \\
G(\sigma_4) &= \sigma_4^\lambda \\
G(\sigma_5) &= \Big(\mathrm{inn}\big(f(x_{45}, x_{51}) f(x_{12}, x_{23})\big)^{-1}\Big)\sigma_{51}^\lambda.
\end{aligned}
$$

Viewing $\widehat{K}(0,5) \subset \widehat{M}(0,5)$ by $x_{i,i+1} = \sigma_i^2$, we obtain a restriction $\tilde{F} \in \mathrm{Aut}(\widehat{K}(0,5))$ of $G \in \mathrm{Aut}(\widehat{M}(0,5))$, which extends $F \in \mathrm{Aut}(\widehat{K}(0,4))$. Namely, on the generators $x_{i,i+1}$ of $\widehat{K}(0,5)$ (i modulo 5), we have that $\tilde{F}(x_{12}) = x_{12}^\lambda$, $\tilde{F}(x_{23}) = (\mathrm{inn}\, f(x_{12}, x_{23})^{-1}) x_{23}^\lambda$, $\tilde{F}(x_{34}) = (\mathrm{inn}\, f(x_{45}, x_{34})^{-1}) x_{34}^\lambda$, $\tilde{F}(x_{45}) = x_{45}^\lambda$, and $\tilde{F}(x_{51}) = (\mathrm{inn}\,(f(x_{45}, x_{51}) f(x_{12}, x_{23}))^{-1}) x_{51}^\lambda$. In particular, we see that $\tilde{F} \in A_5$ and that $\tilde{F}(x_{51})$ is given as in Proposition 5(d).

Now under the surjection $\widehat{M}(0,5) \twoheadrightarrow S_5$, the element $\sigma_i \in \widehat{M}(0,5)$ maps to the transposition $(i, i+1)$ for each i modulo 5. The above formulas for $G(\sigma_i)$ show that $G(\sigma_i)$ also maps to $(i, i+1) \in S_5$. Thus for each i, σ_i and $G(\sigma_i)$ differ by an element in the kernel of $\widehat{M}(0,5) \twoheadrightarrow S_5$, i.e. in $\widehat{K}(0,5)$. Writing $G(\sigma_i) = g_i \sigma_i$ with $g_i \in \widehat{K}(0,5)$, and using that $\tilde{F}$ is the restriction of G to $\widehat{K}(0,5)$, we have for all $\alpha \in \widehat{K}(0,5)$ that $\tilde{F} \circ (\mathrm{inn}\, \sigma_i)(\alpha) = G(\sigma_i \alpha \sigma_i^{-1}) = G(\sigma_i) G(\alpha) G(\sigma_i)^{-1} = g_i \sigma_i G(\alpha) \sigma_i^{-1} g_i^{-1} = (\mathrm{inn}\, g_i)(\mathrm{inn}\, \sigma_i) \circ \tilde{F}(\alpha)$. Hence the image $\overline{\tilde{F}}$ of $\tilde{F}$ in $\mathrm{Out}(\widehat{K}(0,5))$ commutes with the image of each $\mathrm{inn}\, \sigma_i$ in $\mathrm{Out}(\widehat{K}(0,5))$, i.e. with the action of each transposition $(i, i+1)$. But these transpositions generate the symmetric group S_5. Since $\tilde{F}(x_{i,i+1}) \sim x_{i,i+1}^\lambda$ for each i modulo 5, we thus have that $\overline{\tilde{F}} \in \mathrm{Out}_5^\sharp$ and $\tilde{F} \in \mathrm{Aut}_5^\sharp$. So in fact $\tilde{F} \in A_5^\sharp = A_5 \cap \mathrm{Aut}_5^\sharp$, as desired.

In order to complete the proof of (iii)$'$, it remains to show that $[\rho, \tilde{F}] = \operatorname{inn} f$. This is equivalent to showing that the maps $\rho\tilde{F}$ and $\big(\operatorname{inn} f(x_{12}, x_{23})\big)\tilde{F}\rho$ are equal. So consider the actions of these two maps on the five generators $x_{i,i+1}$ of $\widehat{K}(0,5)$. Direct computation shows that both maps take x_{12} to $(\operatorname{inn} f(x_{12}, x_{23}))x_{45}^{\lambda}$; x_{23} to $\big(\operatorname{inn} f(x_{45}, x_{51})^{-1}\big)x_{51}^{\lambda}$; and x_{45} to x_{23}^{λ}. Moreover, using condition (I), we obtain that both maps take x_{34} to $\big(\operatorname{inn} f(x_{12}, x_{23})\big)x_{12}^{\lambda}$. Finally, using both (I) and (III), we obtain that both maps take x_{51} $\big(\operatorname{inn} f(x_{12}, x_{23}) f(x_{45}, x_{34})^{-1}\big)x_{34}^{\lambda}$. So these two maps agree on the generators of $\widehat{K}(0,5)$ and thus are equal. Hence indeed $[\rho, \tilde{F}] = \operatorname{inn} f$.

(iii)$'$ $\Rightarrow$ (iii): If F extends to $\tilde{F} \in \operatorname{Aut}(\widehat{K}(0,5))$ then $F = p_i(\tilde{F})$.

(iii)$'$ $\Rightarrow$ (ii)$'$: This is trivial.

(iii) $\Rightarrow$ (ii): This is also trivial.

(ii)$'$ $\Rightarrow$ (ii): If $\overline{F}$ extends to $\tilde{\overline{F}} \in \operatorname{Out}(\widehat{K}(0,5))$ then $\overline{F} = p_i(\tilde{\overline{F}})$.

(ii) $\Rightarrow$ (i): By the invariance of condition (III) under ρ, we may assume $i = 4$. By applying part (b) of Proposition 5 to a lift of $\overline{F}$ to $\operatorname{Out}_5^\sharp$ and then applying Lemma 1 of 2.2 with $\eta = \rho$, the assertion follows. $\diamondsuit$

Remark. Proposition 7 shows the equivalence of (III) and the commutation with ρ, under the hypothesis that (I) and (II) are satisfied. This hypothesis is in fact necessary. Namely, Ihara has shown [I3] that properties (I) and (III) d o not imply property (II). So take $F \in A$ such that corresponding pair (λ, f) satisfies (I) and (III) but not (II). By Proposition 3, $\overline{F}$ commutes with θ, corresponding to a 2-cycle in S_5. If (III) always implies commutation with ρ, then $\overline{F}$ also commutes with a 5-cycle in S_5. But these two cycles generate all of S_5, and so $\overline{F}$ also commutes with ω (which corresponds to a 3-cycle). By Proposition 4, it follows that (λ, f) also satisfies condition (II), a contradiction.

Using Proposition 7, we obtain the main result of this section, which parallels Theorem 2 of section 1.3. As before, ν is Nakamura's lift of the natural map $\mu : G_{\mathbb{Q}} \to \operatorname{Out}(\widehat{K}(0,5))$.

Theorem 4. *Let $s_5 : \operatorname{Out}_5^\sharp \to A_5^\sharp = A_5 \cap \operatorname{Aut}_5^\sharp$ be as in Theorem 3. Then the image of $p_4 s_5$ is equal to $\widehat{\mathrm{GT}}$. Thus $\widehat{\mathrm{GT}}$ is isomorphic to $\operatorname{Out}_5^\sharp$.*

Remark. In fact, even more is true. Namely, the analogue of the final assertion in Theorem 2 holds here. That is, the section s_5 induces an action of $\operatorname{Out}_5^\sharp$ on $\widehat{K}(0,5)$, and this agrees with the (known) action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,5)$ via the above isomorphism. See Proposition 9 in 3.1 below.

Proof of Theorem 4. Recall that $\widehat{\mathrm{GT}}$ consists of the elements F of $A \subset \operatorname{Aut}(\widehat{F}_2)$ for which the corresponding pair $(\lambda, f) \in \widehat{\mathbb{Z}} \times \widehat{F}_2'$ satisfies conditions (I) - (III). Now if $\overline{F} \in \operatorname{Out}_5^\sharp$ then $\tilde{F} := s_5(\overline{F})$ lies both in A_5 and in $\operatorname{Aut}_5^\sharp$. Thus $\tilde{\overline{F}} = \pi(\tilde{F})$ commutes with the elements (12), (123) and (14253) of S_5. Let $F = p_4(\tilde{F}) \in \operatorname{Aut}_4^\sharp$ and let $\overline{F} = p_4(\tilde{\overline{F}}) \in \operatorname{Out}_4^\sharp$. Then $\overline{F}$ commutes with $(12), (123) \in S_3$, and so Propositions 3 and 4 of 1.3 imply that the pair

(λ, f) corresponding to F satisfies (I) and (II). Also, since $\tilde{\overline{F}}$ commutes with $(14253) \in S_5$, Proposition 7 implies that (λ, f) satisfies (III) (via the implication (ii) $\Rightarrow$ (i)). So $F \in \widehat{\mathrm{GT}}$. This shows that the image of $p_4 s_5$ is contained in $\widehat{\mathrm{GT}}$.

For the other containment, say $F \in \widehat{\mathrm{GT}}$. In particular $F \in \widehat{\mathrm{GT}}_0$, and so $F \in A^\sharp \subset \mathrm{Aut}_4^\sharp$ by Theorem 2. Thus the image $\overline{F}$ of F in $\mathrm{Out}_4^\sharp$ commutes with $(12), (123) \in S_3$. Since the corresponding pair (λ, f) satisfies (III), Proposition 8 ((i) $\Rightarrow$ (iii)) implies that $F = p_4(\tilde{F})$ for some $\tilde{F} \in A_5^\sharp$. Since Proposition 5(a,b) says that there is a unique element of $A_5^\sharp$ over each element of $\mathrm{Out}_5^\sharp$, by the definition of s_5 (cf. Theorem 3) we have that $\tilde{F} = s_5(\tilde{\overline{F}})$. So $F = p_4 s_5(\tilde{\overline{F}})$. This shows that $\widehat{\mathrm{GT}}$ is contained in the image of $p_4 s_5$.

Now p_4 is injective by Proposition 6, and s_5 is injective since it is a section of $\pi : \mathrm{Aut}_5^\sharp \twoheadrightarrow \mathrm{Out}_5^\sharp$. So the last assertion follows. $\diamondsuit$

Remark. In analogy with the remark after Theorem 2, we have that Theorem 4 has other consequences regarding $\widehat{\mathrm{GT}}$. In particular, the above theorem provides an independent proof that $\widehat{\mathrm{GT}}$ is a group. It also shows that $\widehat{\mathrm{GT}} \subset p_4(A_5^\sharp)$, and so $\widehat{\mathrm{GT}}$ can instead be defined as the set of $F \in A^\sharp = A \cap \mathrm{Aut}_4^\sharp$ that satisfy conditions (I) - (III).

§3 Actions on general moduli.

3.1. The groups $\mathrm{Out}_n^\sharp$.

In this section we complete the proof of part (b) of our Main Theorem. That is, we show that there is an isomorphism $\widehat{\mathrm{GT}} \simeq \mathrm{Out}_n^\sharp$ for all $n \geq 5$, thus extending Theorem 4. Furthermore, we show that the isomorphism we construct carries the action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0, n)$, derived from equation (1) in 0.2, to that of $\mathrm{Out}_n^\sharp$ on $\widehat{K}(0, n)$. Because the isomorphisms are compatible as n varies, this shows that $\widehat{\mathrm{GT}}$ is the automorphism group of the inverse system of fundamental groups $\widehat{K}(0, n)$.

We first prove an injectivity result (Proposition 8) that generalizes Proposition 6 to the case $n \geq 5$, using a result of Nakamura [N1]. We then show (Proposition 9) that the isomorphism $\mathrm{Out}_5^\sharp \xrightarrow{\sim} \widehat{\mathrm{GT}}$ of Theorem 4 agrees with the maps of each of these two groups into $\mathrm{Aut}(\widehat{K}(0, 5))$. Finally, in Theorem 5, we use these to finish the proof of the Main Theorem. Afterwards, in 3.2, we pose several open questions suggested by this result.

Recall (cf. 0.2 and 1.1) that the projection map $p_i : \widehat{K}(0, n) \to \widehat{K}(0, n-1)$ is obtained by omitting the ith entry (viewing $\widehat{K}(0, n) = \widehat{\pi}_1(\mathcal{M}_{0,n})$). The induced map $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}_{n-1}^\sharp$ was defined (in 2.2) in such a way that if $F \in \mathrm{Aut}_n^\sharp$, then $q_i(F)$ is the unique element F' of $\mathrm{Aut}_{n-1}^\sharp$ such that $F' p_i = p_i F$. The map $\overline{q}_i : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ is the unique descent of $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}_{n-1}^\sharp$ to the corresponding outer automorphism groups. In exactly the same way, the groups $\mathrm{Aut}_n^\flat$ and $\mathrm{Out}_n^\flat$ (cf. 1.2) have projection maps $q_i : \mathrm{Aut}_n^\flat \to \mathrm{Aut}_{n-1}^\flat$ and $\overline{q}_i : \mathrm{Out}_n^\flat \to \mathrm{Out}_{n-1}^\flat$, and these extend the corresponding maps on $\mathrm{Aut}_n^\sharp$ and

$\mathrm{Out}_n^\sharp$. In Theorem 5 below we show that the projection maps $\overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ are isomorphisms, which reduces the proof to the case of $n = 5$.

The injectivity of $\overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ would follow from that of $\overline{q}_i : \mathrm{Out}^\flat\big(\widehat{K}(0,n)\big) \to \mathrm{Out}^\flat\big(\widehat{K}(0,n-1)\big)$. This latter injectivity might be expected to hold in light of Ihara's analogous result for $P_n^{(\ell)}$ (cf. the end of 0.2); but unfortunately it remains unknown. Still, Nakamura showed in [N1, Lemma 3.2.2] the following weaker result: The homomorphism

$$(\overline{q}_i, \overline{q}_j) : \mathrm{Out}^\flat\big(\widehat{K}(0,n)\big) \to \mathrm{Out}^\flat(\widehat{K}(0,n-1)) \times \mathrm{Out}^\flat(\widehat{K}(0,n-1))$$

is injective whenever $1 \le i \ne j \le n$, $n \ge 5$. We show in Proposition 8 that this result is enough to imply injectivity for our groups $\mathrm{Out}^\sharp(\widehat{K}(0,n))$, thus generalizing Proposition 6 above to all n.

Proposition 8. *If $n \ge 5$ and $1 \le i \le n$, then $\overline{q}_i : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ is injective.*

Proof. We wish to show that $\ker(\overline{q}_i : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp)$ is trivial. So consider an element $\overline{F}$ in this kernel, choose $F \in \mathrm{Aut}_n^\sharp$ over $\overline{F}$, and let $F' = q_i(F) \in \mathrm{Aut}_{n-1}^\sharp$. Then $p_i F = F' p_i : \widehat{K}(0,n) \to \widehat{K}(0,n-1)$, by definition of the map $q_i : \mathrm{Aut}_n^\sharp \to \mathrm{Aut}_{n-1}^\sharp$. Since $\overline{F} \in \ker \overline{q}_i$, the image of $F' \in \mathrm{Aut}_{n-1}^\sharp$ in $\mathrm{Out}_{n-1}^\sharp$ is trivial.

Choose $j \ne i$ in $\{1, \ldots, n\}$. Consider the image of the transposition (i,j) under the map $\sigma^{(n)} : S_n \to \mathrm{Out}(\widehat{K}(0,n))$ defined at the end of 1.1, and let $\tau \in \mathrm{Aut}(\widehat{K}(0,n))$ be an element that lifts this image. Then there exists some $a \in \widehat{K}(0,n-1)$ such that $p_j = \mathrm{inn}(a) p_i \tau$. Now, since $F \in \mathrm{Aut}_n^\sharp$ and τ lifts an element of S_n, the automorphism $F^{-1}\tau F \tau^{-1}$ must lie above the trivial class in $\mathrm{Out}(\widehat{K}(0,n))$; so there exists $b \in \widehat{K}(0,n)$ such that $\mathrm{inn}(b)F = \tau F \tau^{-1}$. Set $c = a p_i(b)$ and $d = cF'(a^{-1})$ in $\widehat{K}(0,n-1)$. We obtain

$$\begin{aligned} p_j F &= \mathrm{inn}(a) p_i \tau F = \mathrm{inn}(a) p_i (\tau F \tau^{-1}) \tau = \mathrm{inn}(a) p_i \mathrm{inn}(b) F \tau \\ &= \mathrm{inn}(c) p_i F \tau = \mathrm{inn}(c) F' p_i \tau = \mathrm{inn}(c) F' \mathrm{inn}(a^{-1}) p_j = \mathrm{inn}(d) F' p_j. \end{aligned}$$

Thus $q_j(F) = \mathrm{inn}(d)F' \in \mathrm{Aut}(\widehat{K}(0,n-1))$, by definition of q_j. But then $\overline{q}_j(\overline{F})$ is the image in $\mathrm{Out}_{n-1}^\sharp$ of $\mathrm{inn}(d)F'$, and this image is trivial since it is the same as the image of F'. So $\overline{F}$ lies in the kernels of each of the two maps $\overline{q}_i, \overline{q}_j : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ (and thus in the kernels of the extensions of these two maps to $\mathrm{Out}_n^\flat$). But by [N1, Lemma 3.2.2], since $i \ne j$, the map $(\overline{q}_i, \overline{q}_j) : \mathrm{Out}_5^\flat \to \mathrm{Out}_4^\flat \times \mathrm{Out}_4^\flat$ is injective, and thus $\ker \overline{q}_i \cap \ker \overline{q}_j = 1$ in $\mathrm{Out}_n^\flat$. Hence $\overline{F} = 1$, as desired. $\diamondsuit$

In order to prove the main result of this section, we will use Drinfel'd's action of $\widehat{\mathrm{GT}}$ on the profinite braid group $\widehat{B}_n$ (as in expression (1) of 0.1) and the induced action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,n)$. In particular, for $n = 5$, this latter action is given as in expression (13) of the Appendix.

Proposition 9. *Under the isomorphism* $\mathrm{Out}_5^\sharp \xrightarrow{\sim} \widehat{\mathrm{GT}}$ *of Theorem 4, the above action of* $\widehat{\mathrm{GT}}$ *on* $\widehat{K}(0,5)$ *agrees with the action of* $\mathrm{Out}_5^\sharp$ *on* $\widehat{K}(0,5)$ *that is given in Theorem 3.*

Proof. The isomorphism $\mathrm{Out}_5^\sharp \xrightarrow{\sim} \widehat{\mathrm{GT}}$ in Theorem 4 is given by $q_4 s_5$, where $q_4 : \mathrm{Aut}_5^\sharp \to \mathrm{Aut}_4^\sharp$ is the fourth projection map, and where $s_5 : \mathrm{Out}_5^\sharp \to A_5 \cap \mathrm{Aut}_5^\sharp$ is the unique section of $\mathrm{Aut}_5^\sharp \to \mathrm{Out}_5^\sharp$ having image in A_5 (cf. Theorem 3). By Proposition 5(a,b,d), if $\overline{F} \in \mathrm{Out}_5^\sharp$, then $s_5(\overline{F})$ acts on the elements $x_{12}, x_{23}, x_{34}, x_{45}, x_{51}$ of $\widehat{K}(0,5)$ by the same formulas as in expression (13) of the Appendix, where $\lambda = \lambda(\overline{F})$, and for some (unique) $f \in \widehat{F}_2'$. Since these five elements generate $\widehat{K}(0,5)$ (cf. the Appendix), it suffices to verify that $q_4 s_5(\overline{F}) \in \widehat{\mathrm{GT}}$ corresponds to this pair (λ, f). But this follows from Proposition 5(c), together with the fact that an element $F \in \widehat{\mathrm{GT}}$ corresponds to $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$ if $F(x) = x^\lambda$ and $F(y) = f^{-1} y^\lambda f$ (as in 1.2). $\diamondsuit$

For more general $\widehat{K}(0,n)$, we have the following:

Lemma 1. *(a) For each* $n \geq 4$*, the action of* $\widehat{\mathrm{GT}}$ *on* $\widehat{K}(0,n)$ *induces a homomorphism* $e_n : \widehat{\mathrm{GT}} \to \mathrm{Out}_n^\sharp$*.*
(b) The map e_5 *is an isomorphism, and is the inverse of the isomorphism* $q_4 s_5 : \mathrm{Out}_5^\sharp \xrightarrow{\sim} \widehat{\mathrm{GT}}$ *of Theorem 4.*
(c) For each $n \geq 5$*,* $e_{n-1} = \overline{q}_n e_n : \widehat{\mathrm{GT}} \to \mathrm{Out}_{n-1}^\sharp$*.*

Proof. (a) The above action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,n)$ corresponds to a homomorphism $\widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{K}(0,n))$, which induces a homomorphism $\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{K}(0,n))$.

As observed in [IM, A.3(d)], formula (1) of 0.1 above implies that under the above action, any element $(\lambda, f) \in \widehat{\mathrm{GT}}$ takes each $x_{ij} \in \widehat{K}(0,n)$ to a $\widehat{K}(0,n)$-conjugate of x_{ij}^λ. Thus $F = (\lambda, f)$ actually induces an element of $\mathrm{Out}^\flat(\widehat{K}(0,n))$ (cf. 1.2). Moreover, as observed in [IM, A.3(c)], the image of $\widehat{\mathrm{GT}}$ in $\mathrm{Out}(\widehat{K}(0,n))$ commutes with the action of the symmetric group S_n. Hence the image of $\widehat{\mathrm{GT}} \to \mathrm{Out}(\widehat{K}(0,n))$ is contained in $\mathrm{Out}_n^\sharp = \mathrm{Out}^\sharp(\widehat{K}(0,n))$.

(b) This follows from Proposition 9 above.

(c) This is immediate from formula (1) of 0.1. $\diamondsuit$

For every $n > 5$, we may successively compose the projection maps $\overline{q}_i : \mathrm{Out}_i^\sharp \to \mathrm{Out}_{i-1}^\sharp$, for $5 < i < n$, and obtain a projection $\overline{q}_6 \overline{q}_7 \cdots \overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_5^\sharp$.

Lemma 2. *Let* $n > 5$*. Then:*
(a) The composition $\overline{q}_6 \overline{q}_7 \cdots \overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_5^\sharp$ *is surjective.*
(b) The map $e_n : \widehat{\mathrm{GT}} \to \mathrm{Out}_n^\sharp$ *is injective.*

Proof. By Lemma 1, $(\overline{q}_6 \overline{q}_7 \cdots \overline{q}_n) \circ e_n = e_5 : \widehat{\mathrm{GT}} \xrightarrow{\sim} \mathrm{Out}_5^\sharp$. So both assertions follow. $\diamondsuit$

Using the above results, we obtain the main result of this section:

Theorem 5. *(a) For $n \geq 5$, the map $e_n : \widehat{\mathrm{GT}} \to \mathrm{Out}_n^\sharp$ is an isomorphism.*
(b) For $n > 5$, the map $\overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ is an isomorphism, and $e_{n-1} = \overline{q}_n e_n$.

Proof. The map e_5 is an isomorphism by Lemma 1(b) above. By Proposition 8 and Lemma 2 above, the composition $\overline{q}_6 \overline{q}_7 \cdots \overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_5^\sharp$ is both injective and surjective, hence an isomorphism. This is true for all $n > 5$; so $\overline{q}_n : \mathrm{Out}_n^\sharp \to \mathrm{Out}_{n-1}^\sharp$ is itself an isomorphism. By Lemma 1, $e_{n-1} = \overline{q}_n e_n$, and e_5 is an isomorphism. So by induction together with the fact that p_n is an isomorphism, it follows that e_n is an isomorphism. $\diamondsuit$

Remark. In Theorem 2 (in 1.3), we showed that our action of $\mathrm{Out}_4^\sharp$ on $\widehat{F}_2 = \widehat{K}(0,4)$ is compatible with the action of $\widehat{\mathrm{GT}}_0$. We showed the corresponding fact for the actions of $\mathrm{Out}_5^\sharp$ and $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,5)$ in Proposition 9 above. For $n > 5$, the above isomorphism $\mathrm{Out}_n^\sharp \xrightarrow{\sim} \widehat{\mathrm{GT}}$ yields actions of $\mathrm{Out}_n^\sharp$ on $\widehat{K}(0,n)$ in retrospect, which are automatically compatible with the action of $\widehat{\mathrm{GT}}$.

3.2. Open questions.

We conclude with several open questions that are suggested by the above.

Question 1. Let ${\mathrm{Out}_5^\sharp}^1$ be the subgroup of $\mathrm{Out}_5^\sharp$ with $\lambda = 1$. Is ${\mathrm{Out}_5^\sharp}^1$ equal to the commutator subgroup of $\mathrm{Out}_5^\sharp$, and is it a free profinite group?

By the isomorphism in Theorem 4, this question is equivalent to the analogous one for $\widehat{\mathrm{GT}}$, viz. whether $\widehat{\mathrm{GT}}^1 := \{F \in \widehat{\mathrm{GT}} \,|\, \lambda(F) = 1\}$ is equal to the commutator subgroup of $\widehat{\mathrm{GT}}$ and is free (cf. questions 2 and 7 in 1.4 of [S]). Possibly the geometric information contained in the definition of $\mathrm{Out}_5^\sharp$ may provide a method that could be used to obtain a solution to the equivalent version stated above, perhaps by seeking an infinite system of independent generators.

The first part of this question is suggested by the corresponding property for $G_{\mathbb{Q}}$, i.e. that its commutator subgroup consists of the elements σ for which the value $\lambda = \chi(\sigma)$ of the cyclotomic character is equal to 1. And if it were to turn out that the inclusion $G_{\mathbb{Q}} \hookrightarrow \mathrm{Out}_5^\sharp \approx \widehat{\mathrm{GT}}$ is an isomorphism, then the answer to this part of the question would necessarily be "yes".

The second part of the question is related to the Shafarevich Conjecture, that if K is a global field and $\tilde{K}$ its maximal cyclotomic extension, then the absolute Galois group $G_{\tilde{K}}$ is a free profinite group. This has been proven in the function field case ([H], [P]), but remains open in the (original) number field case, even when $K = \mathbb{Q}$. But if $G_{\mathbb{Q}}$ is an open subgroup of $\widehat{\mathrm{GT}}$ (e.g. if they are equal), and if ${\mathrm{Out}_5^\sharp}^1$ is free, then so is $G_{\mathbb{Q}^{\mathrm{ab}}}$. Namely, $G_{\mathbb{Q}^{\mathrm{ab}}} = G_{\mathbb{Q}} \cap {\mathrm{Out}_5^\sharp}^1$ would then be open in ${\mathrm{Out}_5^\sharp}^1$, and an open subgroup of a free profinite group is free (cf. [FJ, 15.20]). Thus if $G_{\mathbb{Q}}$ is open in $\widehat{\mathrm{GT}}$, then an affirmative answer to the second part of the above question would prove the Shafarevich Conjecture for $K = \mathbb{Q}$.

Question 2. Is there a "Galois theory" for the group $\mathrm{Out}_5^\sharp \approx \widehat{\mathrm{GT}}$ that extends that of $G_{\mathbb{Q}}$?

Namely, just as $G_{\mathbb{Q}}$ injects into $\widehat{\mathrm{GT}}$ and $G_{\mathbb{Q}^{\mathrm{ab}}}$ injects into $\widehat{\mathrm{GT}}^1$, is there, for every subfield $K \subset \overline{\mathbb{Q}}$, a naturally associated subgroup $\widehat{\mathrm{GT}}_K$ of $\widehat{\mathrm{GT}}$ containing G_K? This association should be compatible with field inclusions $K \subset K'$, behave well with respect to Galois theory, and generalize $\widehat{\mathrm{GT}}$ and $\widehat{\mathrm{GT}}^1$. In particular, since $\widehat{\mathrm{GT}} = \mathrm{Out}_n^\sharp = \mathrm{Out}^\flat(\widehat{K}(0,n))^{S_n}$ (i.e. the elements of $\mathrm{Out}^\flat(\widehat{K}(0,n))$ that commute with S_n), and since $S_n = \mathrm{Out}_{G_{\mathbb{Q}}}(\widehat{K}(0,n))$ ([IN]), can we take $\widehat{\mathrm{GT}}_K = \mathrm{Out}^\flat(\widehat{K}(0,n))^{\mathrm{Sym}_{n,K}}$, where $\mathrm{Sym}_{n,K} := \mathrm{Out}_{G_K}(\widehat{K}(0,n))$? And is there a natural action of $\widehat{\mathrm{GT}}$ on (the set) $\overline{\mathbb{Q}}$ extending that of $G_{\mathbb{Q}}$, such that K is the set of elements fixed under the restriction of the action to $\widehat{\mathrm{GT}}_K$? In the other direction, does $\widehat{\mathrm{GT}}_K$ consist of all the elements of $\widehat{\mathrm{GT}}$ that fix K?

Of course, if the inclusion $G_{\mathbb{Q}} \hookrightarrow \mathrm{Out}_5^\sharp \simeq \widehat{\mathrm{GT}}$ is an isomorphism, then one could take $\widehat{\mathrm{GT}}_K$ to be equal to $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. But in that case there is still the question of whether G_K is equal to $\mathrm{Out}^\flat(\widehat{K}(0,n))^{\mathrm{Sym}_{n,K}}$, which would be a kind of duality. On the other hand, the discussion in the previous paragraph still makes sense even if $\widehat{\mathrm{GT}}$ is strictly larger than $G_{\mathbb{Q}}$.

Question 3. Can the main result of this paper be generalized to the case of the moduli space $\mathcal{M}_{g,n}$?

Specifically, for any pair (g,n) with $g \geq 0$, $n > 0$, let $\mathrm{Sym}(g,n) = \mathrm{Out}_{G_{\mathbb{Q}}}(\widehat{K}(g,n))$ (thus $\mathrm{Sym}(0,n) = S_n$). Let $\mathrm{Out}^\flat(\widehat{K}(g,n))$ consist of the elements that take each conjugacy class $[x]$ to a power of itself, where x ranges over the set of Dehn twists along loops on a topological surface of genus g with n punctures (or boundary components). Consider the group $\mathrm{Out}_{g,n}^\sharp = \mathrm{Out}^\sharp(\widehat{K}(g,n)) := \mathrm{Out}^\flat(\widehat{K}(g,n))^{\mathrm{Sym}(g,n)}$. This generalizes $\mathrm{Out}_n^\sharp = \mathrm{Out}^\sharp(\widehat{K}(0,n))$ to $g > 0$. What is the relationship of $\mathrm{Out}_{g,n}^\sharp$ to $\widehat{\mathrm{GT}}$?

A comment of Drinfel'd [D, p.847] suggests that the groups $\mathrm{Out}^\sharp(\widehat{K}(g,n))$ should form an inverse system whose inverse limit may equal $\widehat{\mathrm{GT}}$. A weaker (but perhaps more likely) possibility is that this inverse limit is isomorphic to a subgroup $\widehat{\mathrm{GT}}^* \subset \widehat{\mathrm{GT}}$ that can be defined by a finite set of additional conditions (besides (I) - (III)). Indeed, Grothendieck suggested in [G1] that groups similar to the $\mathrm{Out}^\sharp(\widehat{K}(g,n))$ should be stable for pairs (g,n) with $3g-3+n \geq 2$; and this is is borne out in the genus zero case, where $\mathrm{Out}^\sharp(\widehat{K}(0,n))$ is stable for $n \geq 5$ (by Theorem 5).

In [G1], Grothendieck proposed describing $G_{\mathbb{Q}}$ as a subgroup of the automorphism group of a "Teichmüller tower" consisting of fundamental groupoids of the spaces $\mathcal{M}_{g,n}$ generalizing the fundamental groups $K(g,n)$. Thus if there is such a $\widehat{\mathrm{GT}}^*$ (either $\widehat{\mathrm{GT}}$ itself or a subgroup), it would be a natural candidate for $G_{\mathbb{Q}}$.

Appendix: Fundamental groups of configuration spaces

This appendix concerns fundamental groups of certain moduli spaces, and their profinite completions (which are the algebraic fundamental groups of those spaces). We focus especially on the case of configuration spaces, which parametrize r-tuples of points on complex 1-space. Depending on how one makes precise sense of this, there are several different spaces and groups that can be considered. These arise in the consideration of the group $\widehat{\mathrm{GT}}$ and the Grothendieck program, in this paper and elsewhere. In particular, they generalize the space $\mathbb{P}^1 - \{0, 1, \infty\}$ and its fundamental group $F_2 = \langle x, y \rangle$ (and algebraic fundamental group $\widehat{F}_2$). Here we summarize the basic properties without proof; for further detail cf. [LS, Appendix] or other items referred to there.

For any $g, n \geq 0$, let $\mathcal{M}_{g,n}$ denote the moduli space of (isomorphism classes of) complex curves of genus g with n distinct ordered marked points. So in particular, $\mathcal{M}_{0,n}$ parametrizes isomorphism classes of Riemann spheres with n ordered marked points. If instead we consider n distinct ordered marked points on a *fixed* copy of the Riemann sphere (so that the automorphisms of $\mathbb{P}^1$ are no longer taken into consideration) then the corresponding moduli space is $(\mathbb{P}^1)^n - \Delta$, where Δ denotes the n-tuples in which two or more entries are equal. Similarly, $\mathbb{C}^n - \Delta$ is the moduli space of n distinct ordered marked points on a fixed copy of complex affine 1-space. There are also the related *symmetrized* variants on these three spaces, where instead *unordered* n-tuples of distinct marked points are considered: $\mathcal{M}_{0,n}^{\mathrm{sym}}$, $\mathrm{Sym}^n(\mathbb{P}^1) - \Delta$, and $\mathrm{Sym}^n(\mathbb{C}) - \Delta$, respectively.

The fundamental groups of these six spaces are denoted as follows:

- The *Artin braid group* $B_n := \pi_1(\mathrm{Sym}^n(\mathbb{C}) - \Delta)$.
- The *pure Artin braid group* $K_n := \pi_1(\mathbb{C}^n - \Delta)$.
- The *sphere* (or *Hurwitz*) *braid group* $H_n := \pi_1(\mathrm{Sym}^n(\mathbb{P}^1) - \Delta)$.
- The *pure sphere braid group* $P_n := \pi_1((\mathbb{P}^1)^n - \Delta)$.
- The *mapping class group* (or *modular group*) $M(0, n) := \pi_1(\mathcal{M}_{0,n}^{\mathrm{sym}})$.
- The *pure mapping class group* $K(0, n) := \pi_1(\mathcal{M}_{0,n})$.

More generally one can also consider $M(g, n) := \pi_1(\mathcal{M}_{g,n}^{\mathrm{sym}})$ and $K(g, n) := \pi_1(\mathcal{M}_{g,n})$. (The groups $M(g, n)$ and $K(g, n)$ are also denoted $\Gamma_{g,[n]}$ and $\Gamma_{g,n}$ by many authors.) The algebraic fundamental groups of these various spaces are the profinite completions $\widehat{B}_n$, $\widehat{K}_n$, etc.

There is a natural forgetful map $(\mathbb{P}^1_{\mathbb{C}})^n - \Delta \to \mathcal{M}_{0,n}$, which induces a homomorphism $\widehat{\pi}_1((\mathbb{P}^1_{\mathbb{C}})^n - \Delta, \xi) \to \widehat{K}(0, n)$, where ξ is the base point $(1, \zeta_n, \ldots, \zeta_n^{n-1})$. Since $\mathrm{Aut}(\mathbb{P}^1)$ is triply transitive, the moduli space $\mathcal{M}_{0,3}$ is a point, $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$, and in general $\mathcal{M}_{0,n} \simeq (\mathbb{P}^1 - \{0, 1, \infty\})^{n-3} - \Delta = (\mathbb{P}^1)^{n-3} - D$ (where D is the closed subset where either two entries are equal or some entry is equal to 0, 1, or ∞). In particular, we may identify

$\widehat{K}(0,4)$ with the free profinite group $\widehat{F}_2$ on generators x, y, where x, y, z correspond to counterclockwise loops around $0, 1, \infty$ respectively satisfying $xyz = 1 \in \pi_1(\mathbb{P}^1 - \{0,1,\infty\})$.

The above fundamental groups can also be described purely group-theoretically. For $n \geq 1$, B_n has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ \text{ for } \ 1 \leq i \leq n-2 \ \text{ and } \ \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ for } \ |i-j| \geq 2.$$

Topologically, if the π_1 is taken at the base point $\xi = \{1, \zeta_n, \ldots, \zeta_n^{n-1}\} \in \mathrm{Sym}^n(\mathbb{C}) - \Delta$ (with $\zeta_n = e^{2\pi i/n}$), then σ_i corresponds to a loop in $\mathrm{Sym}^n(\mathbb{C}) - \Delta$ given by counterclockwise arcs from ζ_n^i to ζ_n^{i+1} and from ζ_n^{i+1} to ζ_n^i, with the other points remaining fixed. The center of B_n is cyclic, generated by the element $\omega_n = (\sigma_1 \cdots \sigma_{n-1})^n = y_1 \cdots y_n$, where $y_i = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_1 \cdot \sigma_1 \cdots \sigma_{i-2}\sigma_{i-1}$ as in (1) of 0.1. The sphere braid group H_n is the quotient of B_n by the relation $y_n = 1$ (called the *sphere relation*), and the mapping class group $M(0,n)$ is the quotient of H_n by the further relation $\omega_n = 1$ (called the *center relation*).

Each of these three groups has a natural surjection to the symmetric group S_n, corresponding to considering the permutation of the n marked points induced by a given braid. Group-theoretically, $\rho_n : B_n \twoheadrightarrow S_n$ takes $\sigma_i \mapsto (i, i+1)$, and S_n is the quotient of B_n by the elements σ_i^2. The other two surjections $H_n \twoheadrightarrow S_n$ and $M(0,n) \twoheadrightarrow S_n$ are induced by this quotient map. The three "pure" groups $K_n \subset B_n$, $P_n \subset H_n$ and $K(0,n) \subset M(0,n)$ are the kernels of these surjections to S_n. All three kernels are generated by the elements $x_{ij} := \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ for $1 \leq i < j \leq n$, corresponding to the i-th point winding counterclockwise around the j-th point (but around no other point). Here $x_{ij} = x_{ji}$, $x_{ii} = 1$, and indices may be considered modulo n. Also, $y_i = x_{1i}x_{2i} \cdots x_{i-1,i}$ for each i. The group P_n is the image of K_n under the quotient map $B_n \to H_n$, and is the quotient of K_n by the normal subgroup generated by the elements $\alpha_i := x_{1i} \cdots x_{ni}$. Also, $K(0,n)$ is the quotient of P_n by its unique order 2 central element (which is the image of $\omega_n \in H_n$).

In fact, the groups P_n and $K(0,n)$ are each generated just by the set of elements $E := \{x_{ij} \,|\, 1 \leq i < j \leq n-1\}$. This is because one can solve for each x_{in} in terms of these elements, using that $\alpha_i = 1$ in P_n and $K(0,n)$. Moreover, using the fact that the image of

$$\omega_n = y_1 \cdots y_n = x_{12}(x_{13}x_{23})(x_{14}x_{24}x_{34}) \cdots (x_{1n} \cdots x_{n-1,n})$$

vanishes in $K(0,n)$, as does the last factor $y_n = x_{1n} \cdots x_{n-1,n} = \alpha_n$, we see that in $K(0,n)$ it is possible to solve for any element of E in terms of the others. So excluding any one element from E still gives a generating set for $K(0,n)$.

In particular, excluding x_{14} from E, we find that the group $K(0,5)$ is generated by the five elements $x_{12}, x_{13}, x_{23}, x_{24}, x_{34}$. Also, $x_{14}x_{24}x_{34} = x_{45}^{-1} = x_{15}x_{25}x_{35}$ in $K(0,5)$ since $\alpha_4 = 1 = \alpha_5$ there. So

$$1 = \omega_5 = (x_{12}x_{13}x_{23})(x_{14}x_{24}x_{34})(x_{15}x_{25}x_{35})x_{45} = (x_{12}x_{13}x_{23})x_{45}^{-1}$$

in $K(0,5)$, i.e. $x_{45} = x_{12}x_{13}x_{23}$ there. Similarly $x_{51} = x_{23}x_{24}x_{34}$. So in the above set of five elements, we may replace x_{13}, x_{24} by x_{45}, x_{51}. That is, $K(0,5)$ is generated by the elements $x_{12}, x_{23}, x_{34}, x_{45}, x_{51}$.

Using the relationships among the above groups, one can show that the action of $\widehat{\mathrm{GT}}$ on $\widehat{B}_n$, given by expression (1) in 0.1, induces an action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,n)$. First, one observes (cf. [LS]) that under the action of $\widehat{\mathrm{GT}}$ on $\widehat{B}_n$, $F = (\lambda, f)$ takes y_i to y_i^λ, where y_i is as above. In particular, $F(y_n) = y_n^\lambda$. Since the center of $\widehat{B}_n$ is cyclic with generator $\omega_n = (\sigma_1 \cdots \sigma_{n-1})^n$, it follows that $F(\omega_n)$ must be of the form ω_n^μ. The abelianization $\widehat{B}_n^{\mathrm{ab}}$ is cyclic, and by examining the induced action there one obtains $\mu = \lambda$, i.e. $F(\omega_n) = \omega_n^\lambda$. By the discussion above, the mapping class group $\widehat{M}(0,n)$ is the quotient of $\widehat{B}_n$ by the normal subgroup generated by ω_n and y_n. So the above $\widehat{\mathrm{GT}}$-action passes from $\widehat{B}_n$ to $\widehat{M}(0,n)$. Since $\widehat{K}(0,n)$ is a characteristic subgroup of $\widehat{M}(0,n)$, this in turn induces an action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,n)$ for all n. (This conclusion can also be seen another way: one checks that the $\widehat{\mathrm{GT}}$-action on $\widehat{B}_n$ restricts to the pure braid groups $\widehat{K}_n \subset \widehat{B}_n$, and then that it passes to the quotients $\widehat{K}(0,n)$.)

This action of $\widehat{\mathrm{GT}}$ on $\widehat{K}(0,n)$ can then be computed explicitly, using (1) and the above. In particular (cf. [LS, Lemma 7]), for $n = 5$ we obtain the following (where as before, we write $f(a,b) \in \widehat{K}(0,5)$ for the image of $f \in \widehat{F}_2$ under the map $\widehat{F}_2 \to \widehat{K}(0,5)$ given by $x \mapsto a$, $y \mapsto b$; and we write $\operatorname{inn} f$ for the inner automorphism map $g \mapsto fgf^{-1}$):

$$
\begin{aligned}
&x_{12} \mapsto x_{12}^\lambda,\ x_{23} \mapsto (\operatorname{inn} f(x_{12}, x_{23})^{-1})x_{23}^\lambda,\ x_{34} \mapsto (\operatorname{inn} f(x_{45}, x_{34})^{-1})x_{34}^\lambda \\
&x_{45} \mapsto x_{45}^\lambda,\ x_{51} \mapsto (\operatorname{inn}(f(x_{45}, x_{51})f(x_{12}, x_{23}))^{-1})x_{51}^\lambda.
\end{aligned}
\tag{13}
$$

Namely, [LS, Lemma 7] derives the above action of $\widehat{\mathrm{GT}}$ on $\widehat{M}(0,5)$, giving the action on the generators $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \widehat{M}(0,5)$, and on the element $\sigma_{15} := \sigma_4\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1} \in \widehat{M}(0,5)$; and (13) follows from these by the two identities $x_{i,i+1} = \sigma_i^2$ for $1 \le i < 5$ and $x_{51} = x_{15} = \sigma_{15}^2$. In fact, all but the last expression of (13) follow immediately from (1) and the first of these two identities in $\widehat{M}(0,5)$. Moreover, since $x_{12}, x_{23}, x_{34}, x_{45}, x_{51}$ generate $\widehat{K}(0,5)$ (as seen above), these expressions determine the resulting automorphism of $\widehat{K}(0,5)$ uniquely.

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