Local Galois theory in dimension two

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Abstract. This paper proves a generalization of Shafarevich's Conjecture, for fields of Laurent series in two variables over an arbitrary field. This result says that the absolute Galois group G_K of such a field K is *quasi-free* of rank equal to the cardinality of K, i.e. every non-trivial finite split embedding problem for G_K has exactly $\operatorname{card} K$ proper solutions. We also strengthen a result of Pop and Haran-Jarden on the existence of proper regular solutions to split embedding problems for curves over large fields; our strengthening concerns integral models of curves, which are two-dimensional.

Section 1. Introduction.

This paper concerns the absolute Galois group of a field of the form $K = k((x,t))$, where k is an arbitrary field. Here K is by definition the fraction field of the power series ring $k[[x,t]]$, the complete local ring of a point on a smooth surface. The absolute Galois group G_K of K can never be free, but we show that it is *quasi-free* of rank equal to the cardinality of K. This is quite different from the one-variable Laurent series case, where the absolute Galois groups are very far from being free.

Specifically, we prove the following result (see Theorem 5.1):

Theorem 1.1. *If k is a field, then the absolute Galois group of $K := k((x,t))$ is quasi-free of rank* $\operatorname{card} K$.

This theorem relies on a generalization of a result of Pop and Haran-Jarden on split embedding problems for curves over large fields (see Theorem 4.1 and the discussion below). The notion of "quasi-free" is introduced and discussed in Section 2, where we show in particular (Theorem 2.1) that a profinite group of infinite rank if free if and only if it is projective and quasi-free.

The condition of being quasi-free says in particular that every finite split embedding problem has a proper solution. (In fact, the definition of quasi-free is that there are as many proper solutions as the rank, for any non-trivial finite split embedding problem.) Thus Theorem 1.1 provides evidence for

Conjecture 1.2. (See [DD, §2.1.2].) *If F is any Hilbertian field, then its absolute Galois group G_F has the property that every finite split embedding problem has a proper solution.*

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Note that $K = k((x,t))$ is Hilbertian by a result of Weissauer [FJ, Theorem 14.17].) Conjecture 1.2 is already known for Hilbertian fields that are large, by a result of Pop [Po96, Main Theorem B]. But it is not known whether or not our field K is large (and we suspect that it is not).

For $K = k((x,t))$, the absolute Galois group G_K cannot be free because it is not even projective (having cohomological dimension > 1; cf. [AGV, Exp. X, Cor. 2.4] and [Se, I, 3.4, Proposition 16]). But by Theorem 2.1, our result says that G_K is "as close as possible to being free," given that it is not projective. The lack of projectivity means that not every finite embedding problem has even a *weak* solution. So in Theorem 1.1, the restriction to *split* embedding problems is essential. Of course every split embedding problem has a weak solution, induced by the splitting; and our result asserts that there are *proper* solutions. (See Section 2 for a review of definitions of these terms.)

Theorem 1.1 can be regarded as a higher dimensional local version of Shafarevich's Conjecture (cf. [Ha02]). That conjecture says that the absolute Galois group of $\mathbb{Q}^{\mathrm{ab}}$ is a free profinite group of countable rank, and more generally that the absolute Galois group of the maximal cyclotomic extension of a global field is free of countable rank. This assertion is known in the geometric case, where more generally it has been shown that the absolute Galois group of the function field of a curve over an algebraically closed field is free ([Ha95], [Po95]). This absolute Galois group was previously known to be projective; so in the terminology of the current paper, it sufficed to show that it was also quasi-free. Thus we may regard a more general form of Shafarevich's Conjecture (even in higher dimensions, where projectivity fails) as saying that the absolute Galois groups of the function fields of certain schemes are quasi-free. This remains open in the global case in dimension > 1, but here we show it in the smooth equidimensional local case in dimension 2.

Our result implies in particular that every finite group is a Galois group over $K = k((x,t))$. This consequence was previously proven by T. Lefcourt [Le], using that every finite group is the Galois group of a regular cover of the K-line and the fact that K is Hilbertian. Also, in the special case that $k = \mathbb{C}$, Theorem 5.3.9 of [Ha03] proved that every finite split embedding problem for K has a proper solution (though the number of solutions was not considered there).

One may wish to consider $K = k((x,t))$ as a "two-dimensional local field", but the situation differs from the one-variable case. In dimension one, the notion of being local is essentially unambiguous, indicating the fraction field of a complete discrete valuation ring. In dimension two, we can correspondingly consider the fraction field of a two-dimensional complete regular local ring, in particular the fraction field K of $k[[x,t]]$. But K is also the fraction field of non-local rings, e.g. of $k[[x,t]][1/t]$, a Dedekind domain with infinitely many maximal ideals (corresponding to the height one primes of $k[[x,t]]$ other than (t)). Similarly, K is the function field of the blow-up of $\operatorname{Spec} k[[x,t]]$ at the closed point; and this blow-up is highly non-local, containing a copy of the projective line as the exceptional divisor. This aspect makes the higher dimensional situation different from the one-dimensional theory. At the same time, there exists a "more highly local" two dimensional field, viz.

the iterated Laurent series field $k((x))((t))$, which strictly contains $K = k((x,t))$. But this is a discrete valuation field over $k((x))$, and its Galois theory is better understood (e.g. see [HP] in the case that k is algebraically closed), with its absolute Galois group being very far from free. In still higher dimensions, there is a whole "zoo" of fields between $k((x_1, \ldots, x_n))$ and $k((x_1)) \cdots ((x_n))$ that are increasingly "local" and having increasingly simpler absolute Galois groups. The case we consider here is thus the first of a much larger class.

In the process of proving Theorem 1.1, we also establish a strengthening of a result of Pop and Haran-Jarden. The result of Pop (see [Po96, Main Theorem A] and [Ha03, Theorem 5.1.9]) says that for any large field F, every finite split embedding problem for a one-variable function field over F has a proper regular solution. It is this result of Pop that implied Conjecture 1.2 in the case of large Hilbertian fields. Haran and Jarden [HJ, Theorem 6.4] strengthened Pop's result to say that the proper regular solution may be chosen to be totally split over a given unramified point. Here we show (Theorems 4.1 and 4.3) that there are infinitely many such solutions, which can be chosen to be totally split over any given finite set of (possibly ramified) points. Moreover we show that if $F = k((t))$ and if we are given a smooth model for the curve over $k[[t]]$, then the solution may be chosen to be totally ramified over a given point on the closed fibre of the given cover, and there are exactly $\operatorname{card} F$ such solutions. In doing so, it suffices to show that there are at least $\operatorname{card} F$ such solutions, since the other inequality follows from the fact that there are at most $\operatorname{card} F$ covers of the give F-curve.

The proof of Theorem 1.1, like that of the special case of $k = \mathbb{C}$ in [Ha03], uses formal patching and blowing up. But while the proof in the case $k = \mathbb{C}$ could rely on the fact that $\mathbb{C}$ is algebraically closed of characteristic 0, here the proof must proceed differently, since k can now be an arbitrary field. Instead, we use an argument that relies on the strengthened version of the theorem of Pop and Haran-Jarden described above.

The structure of this paper is as follows: In Section 2, which contains foundational material about embedding problems, we define and study the notion of a "quasi-free" profinite group of infinite rank. In particular, we show that being free is equivalent to being quasi-free and projective; and we also prove results in the countably generated case (Proposition 2.7, Corollary 2.8) which are related to Iwasawa's theorem [Iw, p.567]. Section 4 contains the strengthened version of the result of Pop and Haran-Jarden, with Section 3 containing preliminary results for that proof. Finally, in Section 5 we prove Theorem 1.1 above (Theorem 5.1 there), that G_K is quasi-free of rank equal to $\operatorname{card} K$.

The techniques in this paper include the deformation of degenerate covers — an approach that was used in previous work of the present authors, including their Ph.D. theses (written under the supervision of M. Artin and the first author, respectively). The authors wish to acknowledge Michael Artin's inspiration for this circle of ideas.

Terminology: A (branched) *cover* of schemes $f : Y \to X$ is a finite generically separable morphism. If $f : Y \to X$ is a cover, then the *Galois group* $\operatorname{Gal}(Y/X)$ consists of the automorphisms ϕ of Y satisfying $f\phi = f$. If G is a finite group, then a *G-Galois cover* is a cover $Y \to X$ together with an injection $G \hookrightarrow \mathrm{Gal}(Y/X)$ such that $\mathcal{O}_Y^G = f^*(\mathcal{O}_X)$ (where the left side denotes the sheaf of G-invariants). If X is a normal scheme, this condition is equivalent to saying that G acts simply transitively on a generic geometric fibre of $Y \to X$. A cover $Y \to X$ is *connected* [resp. *irreducible, normal*], if the schemes X and Y are.

Let $f : Y \to X$ be a G-Galois cover, let Q be a point of Y and let $P = f(Q) \in X$. The *decomposition group* at Q is the subgroup of G consisting of elements that fix the point Q. The *inertia group* at Q is the subset of the decomposition group that induces the identity automorphism on the residue field at Q. The cover is *ramified* at Q if the inertia group is non-trivial, and it is *totally ramified* at Q if the inertia group is all of G (in which case Q is the only point in its fibre over P). The cover is *totally split* (or *splits completely*) over a point $P \in X$ if each decomposition group over P is trivial.

Let k be an arbitrary field and let X be an integral k-scheme such that k is algebraically closed in its function field. Let $Y \to X$ be a connected cover. We say that $Y \to X$ is a *regular* cover of k-schemes if Y is geometrically connected as a k-scheme (or equivalently, if k is algebraically closed in the function field of Y). We say that $Y \to X$ is a *purely arithmetic* cover of k-schemes if Y is isomorphic to $X \times_k \ell$ as a cover of X, where ℓ is a finite (necessarily separable) field extension of k. Thus for any cover $Y \to X$, if ℓ is the algebraic closure of k in the function field of Y, then $Y \to X$ factors as $Y \to X_\ell \to X$, where $X_\ell = X \times_k \ell$; here $Y \to X_\ell$ is regular and $X_\ell \to X$ is purely arithmetic. (Note that this notion of "regular cover" is unrelated to the notion of a "regular scheme" in the sense of having regular local rings.)

If $H \subset G$ and if $Y \to X$ is an H-Galois cover, then the *induced* G-Galois cover $\mathrm{Ind}_H^G Y \to X$ is the G-Galois cover of X obtained by taking a disjoint union of copies of $Y \to X$, indexed by the left cosets of H in G. Here the stabilizer of the "identity copy" of Y (corresponding to the identity coset) is H, and the stabilizers of the other copies are the conjugates of H in G. The *trivial* G-Galois cover of X is $\mathrm{Ind}_1^G X \to X$; this consists of disjoint copies of X that are indexed by the elements of G and are permuted according to the regular representation.

If X is irreducible, then a cover $f : Y \to X$ is a *mock cover* if the restriction of f to each irreducible component of Y is an isomorphism to X.

If p is a prime number, then a *cyclic-by-p* group is a semi-direct product $P \rtimes C$, where P is a finite p-group and C is a cyclic group of order prime to p. If $p = 0$, then by a *p-group* we will mean the trivial group, and by a *cyclic-by-p* group we will mean a cyclic group. Thus if k is an algebraically closed field of characteristic $p \geq 0$, and if $\hat{\ell}$ is a finite Galois field extension of $\hat{k} = k((t))$, then the associated Galois group $\mathrm{Gal}(\hat{\ell}/\hat{k})$ is cyclic-by-p.

Section 2. Embedding Problems and Quasi-Free Profinite Groups

In this section we define the notion of "quasi-free" profinite groups, and prove that a profinite group of infinite rank is free if and only if it is quasi-free and projective. This notion is defined in terms of embedding problems for profinite groups. We also prove related results in the countably generated case (Proposition 2.7 and Corollary 2.8). In addition, we review basic definitions, discuss the relationship of embedding problems to Galois theory, and prove a technical result (Proposition 2.9) which will be useful in Section 4. (See also Chapters 24 and 25 of the 2005 second edition of [FJ] for a further discussion along these lines.)

If Π is a profinite group and m is an infinite cardinal number, then we will say that Π is *quasi-free* of *rank* m if every non-trivial finite split embedding problem for Π has exactly m proper solutions. (See below for definitions concerning profinite groups and embedding problems.) We will prove:

Theorem 2.1. *Let* Π *be a profinite group and let* m *be an infinite cardinal. Then* Π *is a free profinite group of rank* m *if and only if the following conditions are satisfied:*

(i) Π *is projective.*

(ii) Π *is quasi-free of rank* m*.*

Thus the condition of being quasi-free of rank m generalizes the condition of being free of rank m, to the class of profinite groups that are not necessarily projective.

Remark 2.2. This theorem is a variant on a result of Melnikov and Chatzidakis [Ja, Lemma 2.1]. That result says that a profinite group Π is free of rank m if and only if every (not necessarily split) non-trivial finite embedding problem for Π has exactly m proper solutions. Like the theorem above, the key direction is the forward implication. Melnikov had proved that direction of the earlier result under the additional assumption that Π has rank m (see [FJ, Prop. 24.18]); and Chatzidakis later showed that in fact the rank is automatically m.

Before proving the theorem, we recall some basic definitions and facts about profinite groups and embedding problems (cf. [FJ]), and prove some preliminary results.

In the category of profinite groups, all *homomorphisms* are required to be continuous, and *generating sets* are taken in the profinite sense (i.e. the generating condition is that there are no proper closed subgroups containing the set). A generating set S of a profinite group Π *converges to* 1 if every open normal subgroup of Π contains all but finitely many elements of S. Such a generating set always exists, by a result of Douady [FJ, Prop. 15.11]. The minimal cardinality of such a generating set is called the *rank* of Π. In the case that Π is finitely generated (as a profinite group), the rank is the minimal number of generators of Π. If Π is not finitely generated, then the rank is the cardinality of any generating set S that converges to 1; this is independent of the choice of S, by [FJ, Supplement 15.12]. In fact that result says that $\operatorname{card} S$ (and hence the rank of Π) is equal to the cardinality of the set of all open normal subgroups of Π.

A profinite group Π is *free* if there is a generating set S that converges to 1 and that has the following additional property: for every profinite group Δ and every map $\phi_0 : S \to \Delta$ such that $\phi_0(S) \subset \Delta$ converges to 1, there exists an extension of ϕ_0 to a homomorphism $\phi : \Pi \to \Delta$. For every cardinal m, there is (up to isomorphism) a unique free profinite group of rank m, denoted by $\hat{F}_m$ [FJ, p.191].

An *embedding problem* $\mathcal{E}$ is a pair $(\alpha : \Pi \to G, f : \Gamma \to G)$ of epimorphisms of profinite groups. The *kernel* of $\mathcal{E}$ is $\ker(f)$. We say that $\mathcal{E}$ is *finite* if Γ is finite; it is *non-trivial* if its kernel is non-trivial; and it is split if f has a section. A *weak solution* to $\mathcal{E}$ is a homomorphism $\lambda : \Pi \to \Gamma$ such that $f\lambda = \alpha$. A *proper solution* to $\mathcal{E}$ is a weak solution in which λ is surjective. A profinite group Π is *projective* if every finite embedding problem has a weak solution. (Note that for any profinite group Π, every finite split embedding problem has a weak solution.)

In fact, a profinite group Π is projective if and only if it is isomorphic to a closed subgroup of a free profinite group [FJ, Cor. 20.14]; in particular, every free profinite group is projective. Also, a profinite group Π is projective if and only if it has cohomological dimension ≤ 1, by [Gru, Theorem 4] (or by [Se, I, §3.4 Prop. 16 and §5.9 Prop. 45], using that $\mathrm{cd} = \max \mathrm{cd}_p$).

For any two profinite groups Π, Δ, let $\mathrm{Epi}(\Pi, \Delta)$ be the set of epimorphisms $\Pi \to \Delta$.

Lemma 2.3. *Let Π be a profinite group. Suppose that $\mathrm{Epi}(\Pi, G)$ is non-empty for every finite group G. Then the rank of Π is the sum of the cardinalities of the sets $\mathrm{Epi}(\Pi, G)$, where G ranges over isomorphism classes of finite groups.*

Proof. First observe that Π is not a finitely generated profinite group. Namely, if it were generated by a set S of n elements, then $\mathrm{Epi}(\Pi, G)$ would be empty if G is the product of $n+1$ copies of the cyclic group of two elements.

So [FJ, Supplement 15.12] applies, and says that the rank of Π is equal to the cardinality of the set of open normal subgroups of Π. But this is the same as the cardinality of the set of epimorphisms from Π to finite groups G (up to isomorphism), i.e. the sum of the cardinalities of $\mathrm{Epi}(\Pi, G)$, where G ranges over isomorphism classes of finite groups. □

Let Π be a profinite group and let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be a finite embedding problem for Π. Suppose that the epimorphism $\alpha : \Pi \to G$ factors as $r\alpha'$, where $\alpha' : \Pi \to G'$ and $r : G' \to G$ are epimorphisms, for some finite group G'. We consider the *induced embedding problem* $\mathcal{E}_{\alpha'} = (\alpha' : \Pi \to G', f' : \Gamma' \to G')$ by taking $\Gamma' = \Gamma \times_G G'$ and letting $f' : \Gamma' \to G'$ be the second projection map. Here f' is surjective because f is; and so $\mathcal{E}_{\alpha'}$ is a finite embedding problem. Here $\mathcal{E}$ and $\mathcal{E}_{\alpha'}$ have isomorphic kernels; indeed $\ker(f') = \ker(f) \times 1 \subset \Gamma \times_G G' = \Gamma'$. Note also that the first projection map $q : \Gamma' = \Gamma \times_G G' \to \Gamma$ is surjective since $r : G' \to G$ is surjective; and $fq = rf'$.

In this situation, every proper solution $\lambda' : \Pi \to \Gamma'$ of $\mathcal{E}_{\alpha'}$ induces a proper solution $\lambda := q\lambda' : \Pi \to \Gamma$ of $\mathcal{E}$; viz. $f\lambda = fq\lambda' = rf'\lambda' = r\alpha' = \alpha$, and λ is surjective because q and λ' are. So we obtain a map $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$, where PS denotes the set of proper solutions to the embedding problem.

Lemma 2.4. *In the above situation, the map $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$ is injective.*

Proof. We have $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ and $\mathcal{E}_{\alpha'} = (\alpha' : \Pi \to G', f' : \Gamma' \to G')$. Say $\lambda_1', \lambda_2' : \Pi \to \Gamma'$ are proper solutions to $\mathcal{E}_{\alpha'}$. So $f'\lambda_1' = \alpha' = f'\lambda_2'$. If λ_1' and λ_2' have the same image under $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$, then $q\lambda_1' = q\lambda_2'$. So λ_1', λ_2' have the same composition with $(q, f') : \Gamma' \to \Gamma \times_G G' = \Gamma'$. But (q, f') is the identity map on Γ'. So $\lambda_1' = \lambda_2'$. □

Example 2.5. In certain key cases the above map $r : G' \to G$ factors as $f\phi$ for some $\phi : G' \to \Gamma$:

(a) If $G' \subset \Gamma$ and $r = f|_{G'}$, then α' is just a weak solution to $\mathcal{E}$. (Conversely, any weak solution α' to $\mathcal{E}$ induces such a $G' := \mathrm{image}(\alpha')$ and an $r = f|_{G'}$ with $\alpha = r\alpha'$.) In this case we may take ϕ to be the inclusion $G' \hookrightarrow \Gamma$.

(b) In the general situation considered above, if $\mathcal{E}$ is a split embedding problem and if $s : G \to \Gamma$ is a section for f, then we may take $\phi = sr$.

Lemma 2.6. *In the general situation above, suppose that $r : G' \to G$ factors through $f : \Gamma \to G$, say as $r = f\phi$ with $\phi : G' \to \Gamma$. Then the induced embedding problem $\mathcal{E}_{\alpha'}$ has a splitting $s' : G' \to \Gamma'$, given by $s' = (\phi, \mathrm{id}_{G'})$. In particular, $\mathcal{E}_{\alpha'}$ is split in the situations of Examples 2.5 (a) and (b) above.*

Proof. With s' as above, $f's' = \mathrm{id}_{G'}$, so s' is a splitting. □

Note that in the situation of Example 2.5(a), we may identify the splitting s' with the diagonal map $G' \to G' \times_G G' \subset \Gamma \times_G G' = \Gamma'$. In the situation of Example 2.5(b), s' lifts s in the sense that $qs' = sr$, because $qs' = q \circ (\phi, \mathrm{id}_{G'}) = q \circ (sr, \mathrm{id}_{G'}) = sr$.

We can now prove Theorem 2.1 above:

Proof of Theorem 2.1. The forward direction follows from [FJ, Cor. 20.14] (for (i)) and [FJ, Lemma 24.14] (for (ii)).

For the reverse direction, let G be a non-trivial finite group, and consider the non-trivial finite split embedding problem $\mathcal{E}_G := (\Pi \to 1, G \to 1)$. By (ii), this has exactly m proper solutions. That is, there are exactly m epimorphisms $\Pi \to G$. So by Lemma 2.3 above, the profinite group Π has rank m (using that m is infinite and that there are countably many isomorphism classes of finite groups G). It remains to show that Π is *free* of rank m.

Let $\mathcal{E}$ be any non-trivial finite embedding problem for Π. Since Π is projective by (i), there is a weak solution α' to $\mathcal{E}$, and hence an induced non-trivial finite split embedding problem $\mathcal{E}_{\alpha'}$ for Π, as in Example 2.5(a). By (ii), $\mathcal{E}_{\alpha'}$ has m proper solutions. So by Lemma 2.4, $\mathcal{E}$ has at least m proper solutions. But by Lemma 2.3, $\mathcal{E}$ has at most m proper solutions, since Π has rank m, and since every proper solution is an epimorphism from Π to a fixed finite group. So $\mathcal{E}$ has exactly m proper solutions.

We now conclude the proof using Melnikov's theorem [FJ, Prop. 24.18]. Namely, that result says that two profinite groups of the same infinite rank m must be isomorphic provided that each has the property that every non-trivial finite embedding problem has exactly m solutions. As just shown, the profinite group Π has this property. But so does the free profinite group of rank m, by [FJ, Lemma 24.14]. So Π is isomorphic to this free profinite group. □

In the case that m is countable, the quasi-free condition becomes a bit simpler:

Proposition 2.7. *A profinite group Π is quasi-free of countably infinite rank if and only if every finite split embedding problem for Π has a proper solution.*

Proof. The forward direction is trivial. For the reverse direction, consider a non-trivial finite split embedding problem $\mathcal{E} = (\alpha : \Pi \to G,\ f : \Gamma \to G)$ for Π. For any positive integer n, let Γ^n_G denote the n^{th} fibre power of Γ over G, and let $f_n : \Gamma^n_G \to G$ be the natural projection map (i.e. the composition of f with the i^{th} projection map $\Gamma^n_G \to G$, for any i). Then $\mathcal{E}_n = (\alpha : \Pi \to G,\ f_n : \Gamma^n_G \to G)$ is also a non-trivial finite embedding problem, having a splitting given by composing the splitting of $\mathcal{E}$ with the diagonal embedding $\Gamma \hookrightarrow \Gamma^n_G$. So by hypothesis, $\mathcal{E}_n$ has a proper solution λ_n. Composing λ_n with the n projection maps $\Gamma^n_G \to \Gamma$ (in turn) yields n proper solutions to $\mathcal{E}$; and these solutions are distinct because $\mathcal{E}_n$ is non-trivial and its solution λ_n is proper and not contained in the diagonal of Γ^n_G. Since this holds for every n, it follows that the set of proper solutions to $\mathcal{E}$ is infinite. But Π has countable rank. So the set of epimorphisms $\Pi \to \Gamma$ is at most countable. So in fact the set of proper solutions to $\mathcal{E}$ is countably infinite, by Lemma 2.3 above. Since this is true for all $\mathcal{E}$, the profinite group Π is quasi-free of countably infinite rank. □

As a consequence, we obtain the following result, which is related to Iwasawa's theorem ([Iw, p.567]; see below):

Corollary 2.8. *Let Π be a profinite group of countably infinite rank. Then Π is a free profinite group (of countable rank) if and only if the following conditions are satisfied:*

(i) Π is projective.

(ii) Every finite split embedding problem for Π has a proper solution.

Proof. By Proposition 2.7, condition (ii) in the corollary is equivalent to condition (ii) of Theorem 2.1, in this situation. So the corollary follows from Theorem 2.1. □

Note that the argument proving Proposition 2.7 remains valid if one instead considers *all* finite embedding problems, rather than just split ones. That is, if a profinite group Π has the property that every finite embedding problem for Π has a proper solution, then every non-trivial finite embedding problem for Π must have infinitely many proper solutions. And so if Π has countable rank, then each such embedding problem has exactly countably infinite proper solutions.

This shows that the result of Melnikov and Chatzidakis generalizes the earlier related result of Iwasawa ([Iw, p.567]; cf. also [FJ, Cor. 24.2]) in the countable rank case. Iwasawa's result says that a profinite group of countable rank is free if and only if every (not necessarily split) finite embedding problem has a proper solution.

On the other hand, by relying on Iwasawa's result instead of using Theorem 2.1, we obtain another proof of the above corollary:

Alternative proof of Corollary 2.8. The forward direction follows (as in the proof of Theorem 2.1) from [FJ, Cor. 20.14 and Lemma 24.14]. For the reverse direction, by Iwasawa's result we are reduced to proving that every finite embedding problem $\mathcal{E}$ for Π has a proper solution. For this, first note that $\mathcal{E}$ has a weak solution α' because Π is projective. The induced finite split embedding problem $\mathcal{E}_{\alpha'}$ has a proper solution λ' by (ii). The image of λ' under $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$ is then a proper solution to $\mathcal{E}$. $\square$.

Embedding problems arise in Galois theory by taking the profinite group Π to be the absolute Galois group G_K of some field K. If K is a field, then by a *finite embedding problem* $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ *for* K we will mean such an embedding problem for G_K. Giving such a problem corresponds to giving a G-Galois field extension L of K, where G is a quotient of Γ; and a proper solution to $\mathcal{E}$ corresponds to giving a Γ-Galois field extension M of K that contains L. That is, giving a proper solution is equivalent to embedding the given G-Galois field extension of K into a Γ-Galois field extension of K, compatibly with the quotient map $f : \Gamma \to G$ (and this is the origin of the terminology "embedding problem"). If k is a subfield of K and we regard K as a k-algebra, then we say that a proper solution as above is *regular* (over k) if the algebraic closures of k in L and in M are the same (viewing $L \subset M$).

In particular, given a field k and a connected normal k-scheme X with function field K, a finite embedding problem $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ for K corresponds to giving a G-Galois normal connected (branched) cover $Y \to X$. A weak solution to $\mathcal{E}$ corresponds to giving a Γ-Galois normal cover $Z \to X$ that dominates $Y \to X$; such a solution is proper if and only if Z is also connected. A *proper regular solution* to $\mathcal{E}$ (over k) is a proper solution such that the algebraic closures of k in the function fields of Y and Z are the same; here we regard the function field of Y as contained in that of Z. (This notion generalizes the notion of a "regular cover" of k-schemes, by considering the case in which G is trivial.)

With notation as before we have the following result:

Proposition 2.9. *Let K be a field and let $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ be a finite embedding problem for K. Let L be the G-Galois field extension of K corresponding to α. Let L' be a finite Galois extension of K that contains L, say with Galois group G' over K, and let $\alpha' : G_K \to G'$ be the corresponding epimorphism. Consider the induced embedding problem $\mathcal{E}_{\alpha'} = (\alpha' : G_K \to G', f' : \Gamma' \to G')$, where $\Gamma' = \Gamma \times_G G'$. Let $\lambda' : G_K \to \Gamma'$ be a proper solution to $\mathcal{E}_{\alpha'}$, let $\lambda : G_K \to \Gamma$ be the image of λ' under $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$, and let M' and M be the corresponding Galois field extensions of K, with groups Γ' and Γ respectively. Then*

a) $M' = ML'$.

b) $M \cap L' = L$ in M', and the natural map $M \otimes_L L' \to M'$ is an isomorphism.

c) If λ' is a regular solution to $\mathcal{E}_{\alpha'}$ over k, then λ is a regular solution to $\mathcal{E}$ over k.

Proof. (a) Since $\Gamma' = \Gamma \times_G G'$, we have that $G_{M'} = \ker(\lambda') = \ker(\lambda) \cap \ker(\alpha') = G_M \cap G_{L'}$. So $M' = ML'$.

(b) The natural map $M \otimes_L L' \to M'$ is surjective by (a). But $[M : L] = [M' : L']$ since the kernels of $\mathcal{E}$ and $\mathcal{E}_{\alpha'}$ are isomorphic. So $[M \otimes_L L' : L] = [M : L][L' : L] = [M' : L]$ and hence the map is an isomorphism. Thus $M \cap L' = L$ in M', since this holds in the tensor product.

(c) By hypothesis, the algebraic closures of k in L' and in M' are equal. We wish to show that the algebraic closures of k in L and in M are equal. So suppose that $u \in M$ is algebraic over k. Then $u \in M \subset M'$; so by hypothesis we also have $u \in L'$. But by part (b), $L' \cap M = L$. So actually $u \in L$. This shows that the algebraic closure of k in L contains the algebraic closure of k in M; and the other containment is trivial. □

Section 3. Models of covers and blowing up

This section contains several results about models of covers of curves over complete discrete valuation rings R (especially $R = k[[t]]$), for use in Section 4. In particular we consider the effect on covers of blowing up at points on the closed fibre. This will be useful later, in constructing R-models of covers with reducible closed fibres.

The reader may wish to skip this section initially, and to refer back to it as needed in Section 4.

Proposition 3.1. *Let R be a complete d.v.r. with fraction field F. Let X be a smooth projective connected curve over F.*

a) Then there is a normal proper model $\bar{X}$ for X over R together with a finite R-morphism $\phi : \bar{X} \to \mathbb{P}^1_R$ that is unramified at the generic point of $\bar{X}$.

b) Suppose that $\bar{X}$ is a smooth proper model for X over R. Then there is a finite R-morphism $\phi : \bar{X} \to \mathbb{P}^1_R$ that is unramified at the generic point of the special fibre.

Proof. Let K be the function field of X and let k be the residue field of R.

(a) Since X is smooth over F, it follows that K is separably generated over F; let $\{f\}$ be a separating transcendence basis for K over F. This yields a branched covering morphism $\phi_F : X \to \mathbb{P}^1_F$. Here we view $\mathbb{P}^1_F$ as the generic fibre of $\mathbb{P}^1_R$, and we let $\bar{X}$ be the normalization of $\mathbb{P}^1_R$ in X. So ϕ_F extends to a morphism $\phi : \bar{X} \to \mathbb{P}^1_R$. Since K is finite and separable over $F(f)$, the morphism ϕ has the desired property.

(b) The closed fibre $X_0 = \bar{X} \times_R k$ of $\bar{X}$ is connected, by Zariski's Connectedness Theorem [Hrt, III, Cor. 11.3]. Since X_0 is smooth over k, it is irreducible, with one generic point η. Note that a uniformizer for R is also a uniformizer for the local ring $\mathcal{O}_{\bar{X},\eta}$, since X_0 is reduced. Also by smoothness, the function field K_0 of X_0 has a separating transcendance basis $\{f_0\}$ over k.

Let C_0, D_0 be the zero and pole loci of the rational function f_0 on the smooth proper k-curve X_0. Write $D_0 = \sum_{i=1}^r a_i P_{i,0}$ with $a_i > 0$, where the $P_{i,0}$'s are distinct closed points on X_0, say with local uniformizers $\pi_{i,0} \in \mathcal{O}_{X_0,P_{i,0}}$ on X_0. Lifting $\pi_{i,0}$ to an element $\pi_i \in \mathcal{O}_{\bar{X},P_{i,0}}$, we obtain an effective divisor $\bar{P}_i$ on $\bar{X}$ whose restriction to X_0 is $P_{i,0}$. So $\bar{D} := \sum_{i=1}^r a_i \bar{P}_i$ is an effective divisor on $\bar{X}$ whose restriction to X_0 is D_0.

We claim that the canonical homomorphism $\mathrm{H}^0(\bar{X}, \mathcal{O}(n\bar{D})) \to \mathrm{H}^0(X_0, \mathcal{O}(nD_0))$ is surjective for all sufficiently large n. To see this, let $\hat{X}$ be the formal scheme associated to $\bar{X}$, let $\hat{D}$ be the divisor on $\hat{X}$ associated to $\bar{D}$, and let $\mathcal{O}_0$ be the reduction of

$\mathcal{O} := \mathcal{O}_{\hat{X}}$ modulo the maximal ideal of R. If $n \gg 0$, then the canonical homomorphism $\mathrm{H}^0(\hat{X}, \mathcal{O}(n\hat{D})) \to \mathrm{H}^0(\hat{X}, \mathcal{O}_0(n\hat{D}))$ is a surjection [Gr61, Prop. 5.2.3]. The claim then follows by applying the canonical isomorphisms $\mathrm{H}^0(\hat{X}, \mathcal{O}(n\hat{D})) = \mathrm{H}^0(\bar{X}, \mathcal{O}(n\bar{D}))$ [Gr61, Prop. 5.1.2] and $\mathrm{H}^0(\hat{X}, \mathcal{O}_0(n\hat{D})) = \mathrm{H}^0(X_0, \mathcal{O}(nD_0))$.

Choose such a sufficiently large n that is not divisible by the characteristic of k. So the rational function f_0^n on the smooth k-curve X_0 still gives a separating transcendence basis for K_0 over k. Thus the corresponding morphism $X_0 \to \mathbb{P}^1_k$ is finite and generically separable.

The divisor of f_0^n on X_0 is $nC_0 - nD_0$; so there is some $g \in \mathrm{H}^0(\bar{X}, \mathcal{O}(n\bar{D}))$ that maps to $f_0^n \in \mathrm{H}^0(X_0, \mathcal{O}(nD_0))$ under the above surjective map. Let $(g)_0$, $(g)_\infty$ be the divisors of zero and poles of g on $\bar{X}$. Thus $(g)_0 - (g)_\infty$ restricts to $nC_0 - nD_0$, the divisor of f_0^n, on X_0. Since C_0, D_0 have disjoint support, it follows that the restriction of $(g)_0$ [resp. $(g)_\infty$] to X_0 is at least nC_0 [resp. nD_0]. But by definition of g, its divisor of poles $(g)_\infty$ on $\bar{X}$ is at most $n\bar{D}$, whose restriction to X_0 is nD_0. So in fact $(g)_\infty = n\bar{D}$, with restriction nD_0. Thus the restriction of $(g)_0$ to X_0 is exactly nC_0. Hence the supports of $(g)_0, (g)_\infty$ are disjoint on $\bar{X}$; and so the rational function g on the smooth R-curve $\bar{X}$ has no locus of indeterminacy. Thus g defines a morphism $\phi : \bar{X} \to \mathbb{P}^1_R$ over R. Its restriction to the closed fibre is the morphism given by f_0^n, which is generically separable and hence unramified at the generic point η of X_0 (and so also at the generic point of $\bar{X}$).

The R-morphism ϕ is finite-to-one on the closed fibre since the fibre is irreducible and since the morphism is unramified at the generic point of the fibre; and similarly it is finite-to-one on the generic fibre. It is proper since $\bar{X}$ is proper over R. Hence ϕ is finite. So the morphism is as asserted. □

Remark. A related result appears at [GMP, Theorem 3.1].

Let R be a complete d.v.r. and let $\bar{Y}$ be an irreducible normal R-curve. Let Y be the general fibre of $\bar{Y}$, let B be a reduced proper closed subset of Y, and let Σ be the specialization of B to the closed fibre of $\bar{Y}$ (i.e. the reduced intersection of the closure of B in $\bar{Y}$ with the closed fibre of $\bar{Y}$). Write $\bar{Y}_0 = \bar{Y}$ and $\Sigma_0 = \Sigma$, and let $\bar{Y}_1$ be the normalization of the blow-up of $\bar{Y}_0$ at the points of Σ_0. So there is a birational morphism $\bar{Y}_1 \to \bar{Y}_0$ which induces an isomorphism between their general fibres (and so identifies Y with the general fibre of $\bar{Y}_1$). Regarding $B \subset Y \subset \bar{Y}_1$, let Σ_1 be the specialization of B to the closed fibre of $\bar{Y}_1$. We call $(\bar{Y}_1, B, \Sigma_1)$ the *blow-up* of $(\bar{Y}, B, \Sigma)$. Inductively, define the n^{th} *blow-up* $(\bar{Y}_n, B, \Sigma_n)$ of $(\bar{Y}, B, \Sigma)$ to be the blow-up of $(\bar{Y}_{n-1}, B, \Sigma_{n-1})$. So $\bar{Y}_n$ is an irreducible normal R-curve that is birational to $\bar{Y}$ and whose general fibre is equipped with an isomorphism to Y. For short, we will say that $\bar{Y}_n$ is the n^{th} blow-up of $\bar{Y}$ with respect to B. (This is well-defined since $\bar{Y}$ and B determine Σ.)

Lemma 3.2. *Let R be a complete d.v.r. and let $\bar{Y}$ be a projective normal connected R-curve, with general fibre Y. Let B be a proper closed subset of Y. Let $(\bar{Y}_n, B, \Sigma_n)$ be the n^{th} blow-up of $(\bar{Y}, B, \Sigma)$, where Σ is the intersection of the closure of B in $\bar{Y}$ with the closed fibre of $\bar{Y}$. Let C be a proper closed subset of Y that is disjoint from B. Then for* *all sufficiently large n, the following conditions hold:*

(i) the closures of the points of B are each regular 1-dimensional subschemes of $\bar{Y}_n$ that meet the closed fibre of $\bar{Y}_n$ at distinct points;

(ii) the closure of C in $\bar{Y}_n$ is disjoint from Σ_n;

(iii) the reduced closed fibre of $\bar{Y}_n$ is regular at the points of Σ_n.

Proof. For sufficiently large n, the closure of B in $\bar{Y}_n$ is a regular 1-dimensional scheme (of relative dimension 0 over R), and so a disjoint union of irreducible regular components. So (i) holds.

Let f be a rational function on Y that vanishes on C and has value 1 on B. Then for n sufficiently large, f is defined at the points of Σ_n and the order of vanishing of f is 0 there. So (ii) holds for such n.

By resolution of singularities for surfaces, after finitely many blow-ups the scheme $\bar{Y}$ will become regular. So for sufficiently large n, the n^{th} blow-up $\bar{Y}_n$ of $\bar{Y}$ with respect to B will be regular at each point $\sigma \in \Sigma_n$. So the closure of B is a Cartier divisor, given near σ by some local defining function f. Blowing up $\bar{Y}_n$ further with respect to B will reduce the multiplicity of f at the points of Σ_m over σ (for $m > n$) until it becomes 1 at Σ_N, for some $N > n$. At that point, the closure of B will meet the closed fibre of $\bar{Y}_N$ over σ only at a point of the last exceptional divisor, which is isomorphic to a projective line (since $\bar{Y}_m$ is regular over σ for $m \geq n$, and in particular for $m = N - 1$). Hence after sufficiently many blow-ups, the reduced closed fibre will be regular where it meets the closure of B, giving (iii). □

Lemma 3.3. *Let R be a complete d.v.r., let G be a finite group, and suppose that $\psi : \bar{Y} \to \bar{X}$ is a G-Galois cover of projective normal connected R-curves, with general fibre $Y \to X$. Suppose also that B is the (reduced) inverse image of some proper closed subset $A \subset X$. Let $\bar{Y}_n$ be the n^{th} blow-up of $\bar{Y}$ with respect to B.*

a) Then for every $n \geq 0$, the G-action on $\bar{Y}$ lifts to a G-action on $\bar{Y}_n$, whose quotient $\bar{X}_n$ is equipped with a birational proper morphism to $\bar{X}$.

b) If n is sufficiently large, then distinct points of A have the property that their closures in $\bar{X}_n$ meet the closed fibre of $\bar{X}_n$ at distinct points, and this closed fibre is locally irreducible at each of these points.

Proof. (a) The set B is G-invariant, and so the action of G on $\bar{Y}$ extends inductively to each $\bar{Y}_n$. The quotient $\bar{X}_n = \bar{Y}_n/G$ is proper over R because $\bar{Y}_n$ is. By the universal property of quotients, the composition $\bar{Y}_n \to \bar{Y} \to \bar{X}$ factors through $\bar{X}_n$; this gives a morphism $\bar{X}_n \to \bar{X}$ which is proper because $\bar{X}_n$ is proper over R. This morphism is an isomorphism on the general fibre, because this property holds for the morphism $\bar{X}_n \to \bar{X}$. So it is birational.

(b) If n is sufficiently large, then the closures of the points of B in $\bar{Y}_n$ meet the closed fibre of $\bar{Y}_n$ at distinct points, by Lemma 3.2(i). So the closures of the points of A in $\bar{X}_n$ meet that closed fibre at distinct points (viz. the points in $\psi(\Sigma_n)$). Also, by Lemma 3.2(iii), the closed fibre of $\bar{Y}_n$ is locally irreducible at the points of Σ_n. But if there were distinct branches in the complete local ring of the closed fibre of $\bar{X}_n$ at some point ξ in $\psi(\Sigma_n)$, then the closed fibre of $\bar{Y}_n$ would similarly have distinct branches locally at any point η of Σ_n over ξ (since $\bar{Y}_n \to \bar{X}_n$ is a Galois branched cover). So in fact the closed fibre of $\bar{X}_n$ is locally irreducible at each point of $\psi(\Sigma_n)$, i.e. at the points where the closure of A meets the closed fibre of $\bar{X}_n$. $\square$

Let R be a complete d.v.r. with fraction field $\hat{k}$ and uniformizer π, let $\bar{Y}$ be a proper normal R-curve, and let η, η' be closed points of the general fibre Y of $\bar{Y}$. For a natural number r, we will say that the points η, η' are *congruent modulo* π^r if their closures in $\bar{Y}$ have the same pullbacks via $\operatorname{Spec}(R/\pi^r R) \to \operatorname{Spec} R$. (Note that this depends on the model $\bar{Y}$ of Y, not just on the isomorphism class of the $\hat{k}$-curve Y.)

In particular, consider the projective u-line $\mathbb{P}^1_R$, and a closed point η of the general fibre $\mathbb{P}^1_{\hat{k}}$. Suppose that the closure of η in $\mathbb{P}^1_R$ does not pass through the point at infinity. Then η is dominated by an S-point $(u = f)$ of $\mathbb{P}^1_R$ for some finite extension S of R and some $f \in S$. If $g \in S$, then the closed point η' that is dominated by $(u = g)$ will be congruent to η modulo π^r if f, g are congruent modulo $\pi^r S$. Since $\pi^r S$ is infinite, there are infinitely many possibilities for such g; so for every r there are infinitely many points η' with the same residue field that are congruent to η modulo π^r.

Lemma 3.4. *Let k be a field, let $\hat{k} = k((t))$, and let $R = k[[t]]$. Let Δ be a finite closed subset of the general fibre $\mathbb{P}^1_{\hat{k}}$ of $\mathbb{P}^1_R$.*

a) Let P be a closed point of $\mathbb{P}^1_R$ whose residue field is separable over k. Then there is a closed point $\hat{P}$ of $\mathbb{P}^1_{\hat{k}}$ whose residue field is separable over $\hat{k}$; whose closure in $\mathbb{P}^1_R$ meets the closed fibre exactly at P; and such that no closed point of $\mathbb{P}^1_{\hat{k}}$ that is congruent to $\hat{P}$ $(\operatorname{mod} t^2)$ is contained in Δ.

b) Let $\bar{Y} \to \mathbb{P}^1_R$ be a Galois branched cover of projective normal R-curves, with general fibre $Y \to \mathbb{P}^1_{\hat{k}}$. Let $\hat{\ell}$ be a finite extension of $\hat{k}$ that is contained in the function field of Y. Assume that $\hat{P}'$ is a rational point of $\mathbb{P}^1_{\hat{\ell}}$ that lies over a point $\hat{P}$ as in part (a). Suppose moreover that $\hat{P}'$ splits completely under $Y \to \mathbb{P}^1_{\hat{\ell}}$. Let h be a positive integer, and let Π_h be the set of rational points of $\mathbb{P}^1_{\hat{\ell}}$ that are congruent to $\hat{P}'$ modulo t^h, are totally split under $Y \to \mathbb{P}^1_{\hat{\ell}}$, have trivial decomposition group over $\mathbb{P}^1_{\hat{k}}$, and have image in $\mathbb{P}^1_{\hat{k}}$ that is not contained in Δ. Then the cardinality of Π_h is equal to that of $\hat{k}$.

Proof. (a) Let k^s and $\hat{k}^s$ be the separable closures of k and $\hat{k}$. The point P lies on the closed fibre $\mathbb{P}^1_k \subset \mathbb{P}^1_R$, which is the u-line over k. Pick a closed point μ_0 on $\mathbb{P}^1_{k^s}$ over P; this is of the form $(u = c_0)$ for some $c_0 \in k^s$, and P is given by the minimal polynomial of c_0 over k.

The closure $\bar{\Delta}$ of Δ in $\mathbb{P}^1_R$ is a proper closed subset of $\mathbb{P}^1_R$, and the reduction Δ_2 of $\bar{\Delta}$ modulo t^2 is a proper closed subset of $\mathbb{P}^1_R/(t^2)$. Since k^s is infinite, we may choose $c_1 \in k^s$ such that the locus of $(u = c_0 + c_1 t)$ in $\mathbb{P}^1_R/(t^2)$ is not contained in Δ_2. Let $c = c_0 + c_1 t \in \hat{k}^s$ and let $\mu \in \mathbb{P}^1_{\hat{k}^s}$ be the $\hat{k}^s$-point given by $(u = c)$. Thus no $\hat{k}^s$-point of $\mathbb{P}^1_{\hat{k}^s}$ that is congruent to μ modulo t^2 is contained in Δ. Also, P is the closed point of $\mathbb{P}^1_R$ where the closures of

μ and its $\hat{k}$-conjugates each meet the closed fibre $\mathbb{P}^1_k$. So we may take $\hat{P}$ to be the closed point of $\mathbb{P}^1_{\hat{k}}$ over which the $\hat{k}^s$-point μ lies.

(b) Since $\hat{k}$ is infinite, and since $\hat{\ell}$ is a finite extension of $\hat{k}$, these two fields have the same cardinality, which is equal to that of $\operatorname{card}\mathbb{P}^1(\hat{\ell})$. So $\operatorname{card}\Pi_h$ is bounded above by $\operatorname{card}\hat{k}$. Also, $\Pi_h \supset \Pi_{h'}$ for $h < h'$. So it suffices to show that $\operatorname{card}\Pi_h \geq \operatorname{card}\hat{k}$ for all sufficiently large h.

Let S be the integral closure of R in $\hat{\ell}$. The closure of $\hat{P}'$ in $\mathbb{P}^1_S$ meets the closed fibre at a unique point of $\mathbb{P}^1_\ell$, where ℓ is the residue field of S. After a change of variables, we may assume that this is not the point at infinity on $\mathbb{P}^1_\ell$; and hence that $\hat{P}'$ is given by $(u = c)$ for some $c \in S$.

Since the point $\hat{P}'$ splits completely, the strong form of Hensel's Lemma [La, II §2, Prop. 2] implies that for all sufficiently large integers $h > 1$ the cover $Y \to \mathbb{P}^1_{\hat{\ell}}$ is also totally split over any rational point of $\mathbb{P}^1_{\hat{\ell}}$ that is congruent modulo t^h to $\hat{P}'$. Since $\hat{\ell}$ is a finite separable extension of $\hat{k}$, there is a primitive element $d \in \hat{\ell}$ over $\hat{k}$. For each non-zero element $a \in \hat{k}$, the element $ad \in \hat{\ell}$ is also a primitive element over $\hat{k}$, and the cover $Y \to \mathbb{P}^1_{\hat{\ell}}$ is totally split over the $\hat{\ell}$-rational point $(u = c + adt^h)$ in $\mathbb{P}^1_{\hat{\ell}}$. Moreover this point has trivial decomposition group over its image in $\mathbb{P}^1_{\hat{k}}$, because ad is a primitive element for $\hat{\ell}$ over $\hat{k}$. By part (a), this image cannot be contained in Δ, since it is congruent to $\hat{P}$ modulo t^h and $h > 1$. So this point lies in Π_h. So $\operatorname{card}\Pi_h \geq \operatorname{card}\hat{k}$ for sufficiently large h, as needed. □

The statement and proof of the next result are similar to those of [Ha87, Theorem 2.3].

Proposition 3.5. *Let k be a field of characteristic $p \geq 0$, and let $R = k[[t]]$. Let Γ be a finite group with a normal subgroup N, such that the quotient map $\Gamma \to G := \Gamma/N$ has a section σ. Let $\tilde{Y} \to \tilde{X}$ be a G-Galois cover of normal proper R-curves, with general fibre $Y \to X$. Let $n_1, \dots, n_r, m_1, \dots, m_s \neq 1$ be generators for N such that p does not divide the order $o(n_i)$ of any n_i, and such that the order $o(m_j)$ of each m_j is a power of p. Suppose that the function field of $\tilde{Y}$ contains a primitive $o(n_i)^{\text{th}}$ root of unity for each i.*

Let $\Phi_1, \dots, \Phi_r, \Psi_1, \dots, \Psi_s$ be rational functions on $\tilde{X}$ that restrict to the constant function 1 at each generic point of the closed fibre, and whose zero and pole divisors are reduced. Suppose that the supports of these functions on $\tilde{X}$ are pairwise disjoint and contained in the smooth locus over R; that $\tilde{Y} \to \tilde{X}$ splits completely over each closed point in these supports; and that their supports are disjoint from some proper closed subset $D \subset Y$ containing the ramification locus of $Y \to X$. Then there is an irreducible normal N-Galois cover $\tilde{Z} \to \tilde{Y}$ such that the following conditions hold:

(i) $\tilde{Z} \to \tilde{Y}$ is branched precisely over the supports of the divisors of the Φ_i's and the zero divisors of the Ψ_j's, with ramification indices $o(n_i)$ and $o(m_j)$ respectively;
(ii) the composition $\tilde{Z} \to \tilde{X}$ is Γ-Galois;
(iii) the field k has the same algebraic closures in the function fields of $\tilde{Y}$ and $\tilde{Z}$;
(iv) the closed fibre of $\tilde{Z} \to \tilde{Y}$ is an N-Galois mock cover, whose inertia group at any

point of the identity sheet lying over the closure of the divisor of Φ_i [resp. the zero divisor of Ψ_j] is generated by n_i [resp. m_j].

(v) the points δ of D are totally split under $\tilde{Z} \to \tilde{Y}$;

(vi) the decomposition groups of $\tilde{Z} \to \tilde{X}$ at the points of $\tilde{Z}$ over any $\delta \in D \subset \tilde{Y}$ are the conjugates of $\sigma(G_\delta)$, where G_δ is the decomposition group of $\tilde{Y} \to \tilde{X}$ at δ.

Proof. For each $i = 1, \ldots, r$, and each closed point $P \in \tilde{X}$ in the support of the divisor of Φ_i, pick a closed point $Q \in \tilde{Y}$ over P. Thus each point in the fibre of $\tilde{Y} \to \tilde{X}$ over P is of the form $g(Q)$ for some unique $g \in G$, because $\tilde{Y} \to \tilde{X}$ is totally split over P.

For each $g \in G$, let $\mathcal{L}_{g(Q)}$ be the fraction field of the complete local ring $\hat{\mathcal{O}}_{\tilde{Y},g(Q)}$. Also, let $\mathcal{A}_{g(Q)}$ be the normalization of $\hat{\mathcal{O}}_{\tilde{Y},g(Q)}$ in the field extension of $\mathcal{L}_{g(Q)}$ given by $z^{o(n_i)} = \Phi_i$. Since $\zeta_{o(n_i)}$ is contained in the function field of $\tilde{Y}$ and hence in $\mathcal{L}_{g(Q)}$, this is a cyclic extension of degree $o(n_i)$. Moreover, its Galois group may be identified with gN_ig^{-1}, where N_i is the subgroup of N generated by n_i. Since Φ_i restricts to the constant function 1 at the generic point $g(Q)^\circ$ of the closed fibre of $\operatorname{Spec} \hat{\mathcal{O}}_{\tilde{Y},g(Q)}$, the pullback of the cyclic cover $\operatorname{Spec} \mathcal{A}_{g(Q)} \to \operatorname{Spec} \hat{\mathcal{O}}_{\tilde{Y},g(Q)}$ to the complete local ring $\hat{\mathcal{O}}_{g(Q)^\circ}$ at this generic point is trivial. This cover is equipped with an induced indexing, by the elements of gN_ig^{-1}, of the components of the restriction to $\operatorname{Spec} \hat{\mathcal{O}}_{g(Q)^\circ}$.

Taking a disjoint union of copies of this cover, indexed by the cosets of gN_ig^{-1} in N, we obtain an induced (disconnected) N-Galois cover of $\operatorname{Spec} \mathcal{A}_{g(Q)}$, together with an indexing by N of the components over $\hat{\mathcal{O}}_{g(Q)^\circ}$. As Q varies over the points in the fibre of $\tilde{Y}$ over P, these indexings are compatible with the action of G on $\tilde{Y}$; and so we obtain a (disconnected) Γ-Galois cover of $\operatorname{Spec} \hat{\mathcal{O}}_{\tilde{X},P}$, viz. $\operatorname{Ind}_{N_i}^{\Gamma} \operatorname{Spec} \mathcal{A}_Q$, which is ramified precisely over the support of the divisor of Φ_i on $\operatorname{Spec} \hat{\mathcal{O}}_{\tilde{X},P}$ (using that the zero and pole loci of Φ_i are reduced).

Similarly, for each $j = 1, \ldots, s$, pick a closed point $Q \in \tilde{Y}$ over each closed point $P \in \tilde{X}$ in the support of the zero divisor of Ψ_j, and again consider the field $\mathcal{L}_{g(Q)}$ for each $g \in G$. We have that $o(m_j) = p^{\beta_j}$, for some positive integer β_j. In the β_j^{th} truncated Witt vector ring $W_{\beta_j}(\hat{\mathcal{O}}_{\tilde{Y},g(Q)}[y_1, \ldots, y_{\beta_j}])$, let $\underline{\Psi}_j$ and $\underline{y}$ denote the elements with Witt coordinates $(\Psi_j, 0, \ldots, 0)$ and $(y_1, \ldots, y_{\beta_j})$ respectively, and let Fr denote Frobenius. Consider the field extension of $\mathcal{L}_{g(Q)}$ given by the Witt coordinates of $\operatorname{Fr}(\underline{y}) - \underline{\Psi}_j^{p-1}\underline{y} = t$, and let $\mathcal{A}_{g(Q)}$ be the normalization of $\hat{\mathcal{O}}_{\tilde{Y},g(Q)}$ in this extension. This is a cyclic extension whose Galois group may be identified with gN'_jg^{-1}, where N'_j is the subgroup of N generated by m_j. Proceeding as before, we obtain an induced (disconnected) N-Galois cover of $\operatorname{Spec} \mathcal{A}_{g(Q)}$, whose restriction to $\hat{\mathcal{O}}_{g(Q)^\circ}$ is trivial, with components indexed by the elements of N. Letting g vary and taking the union, we obtain a Γ-Galois cover of $\operatorname{Spec} \hat{\mathcal{O}}_{\tilde{X},P}$, viz. $\operatorname{Ind}_{N'_j}^{\Gamma} \operatorname{Spec} \mathcal{A}_Q$, which is ramified precisely over the zero locus of Ψ_j on $\operatorname{Spec} \hat{\mathcal{O}}_{\tilde{X},P}$ (using that this locus is reduced).

Let $\tilde{Y}_0$ be the closed fibre of $\tilde{Y}$; let Δ be the set of points where the support of some Φ_i or Ψ_j meets $\tilde{Y}_0$; let U be the complement of Δ in $\tilde{Y}_0$; and let $\mathcal{U}$ be the completion of $\tilde{Y}$ along U. (That is, $\mathcal{U} = \operatorname{Spec} \mathcal{O}_{\mathcal{U}}$, where $\mathcal{O}_{\mathcal{U}}$ is the t-adic completion of the ring of functions on an affine open subset $\tilde{U} \subset \tilde{Y}$ such that $\tilde{U} \cap \tilde{Y}_0 = U$.) Consider the trivial N-Galois cover of $\mathcal{U}$, consisting of $|N|$ disjoint copies of $\mathcal{U}$ that are indexed by the elements of N, and on which N acts by the regular representation. By formal patching (e.g. [HS99, §1, Cor. to Thm. 1], [Pr, Thm. 3.4]), there is an N-Galois cover $\tilde{Z} \to \tilde{Y}$ whose pullback to the spectrum of each $\hat{\mathcal{O}}_{\tilde{Y},g(Q)} = g(\hat{\mathcal{O}}_{\tilde{Y},Q})$ is isomorphic to the cover described above; whose pullback to $\mathcal{U}$ is trivial; and such that over each $g(Q)$, the two isomorphisms with the trivial cover (induced from pulling back the isomorphism over $\mathcal{U}$ and the one over $\hat{\mathcal{O}}_{\tilde{Y},g(Q)}$) give the same indexing of components. Since these indexings were compatible (for various $g \in G$), the action of $G = \operatorname{Gal}(\tilde{Y}/\tilde{X})$ lifts to an action on $\tilde{Z}$, compatibly with the conjugation action of G on N in Γ; and so the composition $\tilde{Z} \to \tilde{X}$ is Galois with group Γ. Since the local N-covers given above are normal, so is $\tilde{Z}$ (as normality is a local property).

The closed fibre of $\tilde{Z} \to \tilde{Y}$ is an N-Galois mock cover, consisting of a union of copies of $\tilde{Y}_0$, indexed by the elements of N. For each point $P \in \tilde{X}$ in the closure of the support of the divisor of some Φ_i [resp. the zero divisor of some Ψ_j], consider the chosen point $Q \in \tilde{Y}$ over P. The closed fibre $\tilde{Z}_0$ of $\tilde{Z}$ is a union of (intersecting) copies of $\tilde{Y}_0$ indexed by N; let $\tilde{Q} \in \tilde{Z}_0$ be the closed point on the identity copy of $\tilde{Y}_0$ corresponding to Q. The inertia group of $\tilde{Z} \to \tilde{Y}$ at the point $\tilde{Q}$ is generated by n_i [resp. m_j]. Since the n_i's and m_j's generate N, it follows that the closed fibre of $\tilde{Z}$ is connected. Hence $\tilde{Z}$ is connected and thus irreducible (since $\tilde{Z}$ is normal). Moreover, if $\hat{\ell}$ is the algebraic closure of $\hat{k}$ in the function field of $\tilde{Y}$, then $\hat{\ell}$ is algebraically closed in the function field of $\tilde{Z}$, since the closed fibre $\tilde{Z}_0 \to \tilde{Y}_0$ is a mock cover.

It remains to verify (v) and (vi). The cover $\tilde{Z} \to \tilde{Y}$ is totally split over each point δ of D, since those points are in the image of $\mathcal{U} \to \tilde{Y}$ and since the pullback of $\tilde{Z} \to \tilde{Y}$ to $\mathcal{U}$ is trivial. Also, since that pullback is trivial, it consists of a disjoint union of copies of $\mathcal{U}$, indexed by the cosets of N in Γ, and acted upon by Γ. The identity copy and $\mathcal{U}$ itself are isomorphic, together with their G-actions (if we identify G with $\sigma(G) \subset \Gamma$). The other copies are isomorphic to $\mathcal{U}$ as schemes, but their group actions are conjugated by the corresponding elements of N. So if we denote by δ_n the point of $\tilde{Z}$ over δ on the copy of $\mathcal{U}$ indexed by $n \in N$, then the decomposition group of $\tilde{Z} \to \tilde{X}$ at the point δ_n is the conjugate of $\sigma(G_\delta)$ by n. Since Γ is generated by N and $\sigma(G)$, the decomposition groups of $\tilde{Z} \to \tilde{X}$ at the points of $\tilde{Z}$ over $\delta \in \tilde{Y}$ are precisely the Γ-conjugates of G_δ. So (v) and (vi) also hold. □

Section 4. Split embedding problems over curves

In this section we extend Pop's result on solving split embedding problems for curves over large fields. That extension (Theorem 4.1 below) will be used in next section in the proof of Theorem 1.1 (which appears there as Theorem 5.1).

Pop's result [Po96, Main Theorem A] showed that such embedding problems have proper regular solutions; and a later result of Haran and Jarden [HJ, Theorem 6.4] showed that solutions can be chosen so as to split completely over a specified unramified point of the given cover. (Actually, the results proven in [Po96] and [HJ] and in unpublished manuscripts of those authors were somewhat less general than this; see [Ha03, Remark 5.1.11].)

Here we show that more is true: that the solutions to the embedding problem can be chosen to be totally split over any finite set of (possibly ramified) points; that there are "many" such solutions; and for a large field of the form $k((t))$, that the solution to the embedding problem can be chosen to be totally ramified over a specified closed point on a model over $k[[t]]$. These assertions are contained in Theorem 4.1 in the case of $k((t))$. Afterwards, in Theorem 4.3, we obtain a more general result for curves over large fields that need not be of the form $k((t))$ (though at the expense of no longer being able to speak of an integral model). Then in Corollary 4.4, we deduce a result about absolute Galois groups of function fields over "very large" fields being quasi-free.

Theorem 4.1. *Let k be an arbitrary field and let X be a smooth connected projective curve over $\hat{k} = k((t))$, with function field K.*

a) Then every finite split embedding problem $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ for K has a proper regular solution $\beta : G_K \to \Gamma$.

b) For such an $\mathcal{E}$, with section σ of f, let $\pi : Y \to X$ be the G-Galois branched cover of normal curves corresponding to α, and let $D \subset Y$ be a finite set of closed points. Then we may choose the proper regular solution in (a) so that the corresponding normal cover $Z \to Y$ is totally split over the points of D, and the decomposition groups of $Z \to X$ at the points of Z over $\delta \in D$ are the conjugates of $\sigma(G_\delta)$, where G_δ is the decomposition group of $Y \to X$ at δ.

c) If $\mathcal{E}$ is non-trivial then there are $\mathrm{card}(\hat{k})$ *distinct proper regular solutions $Z \to Y \to X$ to $\mathcal{E}$ satisfying (b).*

d) Suppose that $\bar{X}$ is a smooth projective model for X over R; and let $\bar{Y}$ be the normalization of $\bar{X}$ in Y. Suppose also that Q is a closed point of $\bar{Y}$ such that the residue field of its image P in $\bar{X}$ is separable over k. If $\mathcal{E}$ is non-trivial then there are $\mathrm{card}(\hat{k})$ *distinct proper regular solutions $Z \to Y \to X$ in (c) such that the normalization $\bar{Z}$ of $\bar{Y}$ in Z is totally ramified over the closed point $Q \in \bar{Y}$. The pullbacks $Z^* \to Y^*$ via $Y^* :=$* $\mathrm{Spec}\,\hat{\mathcal{O}}_{\bar{Y},Q} \to \bar{Y}$ *of the corresponding* $\ker(f)$*-Galois covers $\bar{Z} \to \bar{Y}$ are each irreducible and together have cardinality equal to* $\mathrm{card}(\hat{k})$*. Moreover, if Q is the only closed point of $\bar{Y}$ over $P \in \bar{X}$, then the pullbacks of the covers $\bar{Z} \to \bar{X}$ under $X^* :=$* $\mathrm{Spec}\,\hat{\mathcal{O}}_{\bar{X},P} \to \bar{X}$ *form a set of irreducible Γ-Galois covers $Z^* \to X^*$ having the same cardinality.*

In using Theorem 4.1 to prove Theorem 5.1, we will start with a G-Galois cover of $\mathrm{Spec}\,k[[x,t]]$; extend it to a cover of x-line over $k[[t]]$; use Theorem 4.1 to dominate that by a Γ-Galois cover; and then restrict that back to $\mathrm{Spec}\,k[[x,t]]$. The condition on total ramification over a closed point (in part (d) of Theorem 4.1 above) is then used to conclude that the restricted cover remains connected. In the proof of Theorem 4.1, the condition on total ramification is obtained by blowing up and down in such a way that the branch components are all forced to pass through the designated point. The assertion in part (b) of Theorem 4.1 strengthens the splitting condition of Haran-Jarden; and the "many solutions" assertion on part (c) allows us to conclude in Section 5 that the absolute Galois group of $k((x,t))$ is quasi-free.

Theorem 4.1 is proved with the aid of Proposition 4.2, which reduces the theorem to a more manageable special case (and whose proof uses results from Section 3).

Proposition 4.2. *Let* $p = \mathrm{char}\ k$. *In order to prove Theorem 4.1, it suffices to prove it in the special case that the G-Galois cover* $\pi : Y \to X$ *corresponding to* α *has the following properties:*

(i) the function field of Y *contains a primitive* d^{th} *root of unity, where* d *is the maximal prime-to-p factor of* $|\ker f|$;

(ii) there is a finite generically unramified $\hat{k}$*-morphism* $\phi : X \to \mathbb{P}^1_{\hat{k}}$ *such that the composition* $\phi\pi : Y \to \mathbb{P}^1_{\hat{k}}$ *is Galois, and in part (d) also such that* ϕ *extends to a finite* R*-morphism* $\bar{X} \to \mathbb{P}^1_R$ *(where in parts (a) - (c),* $\bar{X}$ *denotes the normalization of* $\mathbb{P}^1_R$ *in* X*, viewing* $\mathbb{P}^1_{\hat{k}}$ *as the generic fibre of* $\mathbb{P}^1_R$*);*

(iii) there is a closed point $\hat{P} \in X$ *such that the points in the fibre of* $Y \to X$ *over* $\hat{P}$ *are rational over the algebraic closure of* $\hat{k}$ *in the function field of* Y*; no closed point of* X *that is congruent to* $\hat{P}$ *modulo* t^2 *(with respect to* $\bar{X}$*) is contained in the branch locus of* $Y \to X$ *or in the image of* D*;* $\hat{P}$ *specializes at the closed fibre to a closed point* $P \in \bar{X}$ *whose residue field is separable over* k*; and in part (d),* P *is the image of the given point* $Q \in \bar{Y}$.

Proof. Suppose that Theorem 4.1 is proven in the above special case. For part (d) of Theorem 4.1, this special case includes the additional assumptions that ϕ extends to the given model $\bar{X}$, and that the specialization P of $\hat{P}$ is the image of the given point Q.

Now assume that we are in the general case of Theorem 4.1, which we wish to prove. Let $N = \ker f$. In part (b), after enlarging D, we may assume that D contains the ramification locus of $Y \to X$. (In part (a), we let D denote the ramification locus of $Y \to X$.)

According to Proposition 3.1(a), there is a finite generically unramified R-morphism $\bar{X} \to \mathbb{P}^1_R$ for some proper normal model $\bar{X}$ for X over R. By Proposition 3.1(b), in part (d) we may choose $\bar{X} \to \mathbb{P}^1_R$ with respect to the given R-model $\bar{X}$ of X.

Let $P_0 \in \mathbb{P}^1_R$ be a closed point whose residue field is separable over k; in part (d) we may choose this point to be the image of $Q \in \bar{Y}$ in $\mathbb{P}^1_R$ (since the residue field at Q is separable over k). Also let Δ be the union of the branch locus of $X \to \mathbb{P}^1_{\hat{k}}$ with the image of D in $\mathbb{P}^1_{\hat{k}}$. Applying Lemma 3.4(a) to P_0 and Δ, we obtain a closed point $\hat{P}_0$ of $\mathbb{P}^1_{\hat{k}}$ whose residue field is separable over $\hat{k}$; whose closure in $\mathbb{P}^1_R$ meets the closed fibre exactly at P_0; and such that no closed point of $\mathbb{P}^1_{\hat{k}}$ that is congruent to $\hat{P}_0$ modulo t^2 (with respect to $\mathbb{P}^1_R$) is contained in Δ. In particular, $\hat{P}_0$ is not in the branch locus of $Y \to \mathbb{P}^1_{\hat{k}}$; hence the residue fields at the points of X and Y over $\hat{P}_0$ are separable over $\hat{k}$.

Let $\tilde{Y} \to \mathbb{P}^1_{\hat{k}}$ be the Galois closure of $Y \to \mathbb{P}^1_{\hat{k}}$. So its function field $\tilde{L}$ is Galois over $\hat{k}(u)$, the function field of $\mathbb{P}^1_{\hat{k}}$. Let $\hat{\ell}$ be the algebraic closure of $\hat{k}$ in $\tilde{L}$. Let $\hat{\ell}'$ be a finite field extension of $\hat{\ell}$ that is Galois over $\hat{k}$ and that contains a primitive d^{th} root of unity, and such that every closed point of Y lying over $\hat{P}_0$ has residue field contained in $\hat{\ell}'$.

Let $Y' = \tilde{Y} \times_{\hat{\ell}} \hat{\ell}'$, with function field equal to the compositum $\tilde{L}\hat{\ell}'$ in a separable closure of K. Since $\tilde{L}/\hat{k}(u)$ and $\hat{\ell}'/\hat{k}$ are Galois, so is $\tilde{L}\hat{\ell}'/\hat{k}(u)$; i.e. $Y' \to \mathbb{P}^1_{\hat{k}}$ is Galois. Moreover this morphism extends to a finite morphism $\bar{Y}' \to \mathbb{P}^1_R$, where $\bar{Y}'$ is the normalization of $\bar{X}$ in Y'. Note that $\bar{Y}' \to \bar{X}$ factors through $\bar{Y}$, the normalization of $\bar{X}$ in Y.

The intermediate cover $Y' \to X$ is necessarily also Galois, with Galois group $G' := \mathrm{Gal}(\tilde{L}\hat{\ell}'/K)$; this is an extension of $G = \mathrm{Gal}(L/K)$ through which $\alpha : G_K \to G$ factors, via an epimorphism $\alpha' : G_K \to G'$. Let $\mathcal{E}_{\alpha'} = (\alpha' : G_K \to G', f' : \Gamma' \to G')$ be the induced embedding problem, where $\Gamma' = \Gamma \times_G G'$ (see Section 2). This is also a finite embedding problem, and it is split by Lemma 2.6, in the context of Example 2.5(b) there. Also, $\ker f' = N \times 1 \approx N = \ker f$. So the G'-Galois cover $Y' \to X$ satisfies (i) and (ii), with respect to the embedding problem $\mathcal{E}_{\alpha'}$, which has the same kernel as $\mathcal{E}$.

Choose a closed point $P \in \bar{X}$ lying over $P_0 \in \mathbb{P}^1_R$; in part (d) we may take P to be the image of $Q \in \bar{Y}$. Let $\hat{P}$ be a closed point of X that specializes to P and that lies over $\hat{P}_0$; this exists by the going-down theorem applied to $\bar{X} \to \mathbb{P}^1_R$. So $\hat{P}$ satisfies (iii).

Thus the finite split embedding problem $\mathcal{E}_{\alpha'} = (\alpha' : G_K \to G', f' : \Gamma' \to G')$, with α' corresponding to $Y' \to X$, satisfies the three conditions of Proposition 4.2. So by assumption, the conclusion of Theorem 4.1 holds for this embedding problem.

By (a) of Theorem 4.1, there is a proper regular solution to $\mathcal{E}_{\alpha'}$, corresponding to an N-Galois cover $Z' \to Y'$. By Lemma 2.4 and Proposition 2.9, every proper regular solution to $\mathcal{E}_{\alpha'}$ yields a proper regular solution to $\mathcal{E}$, via the map $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$. So we obtain a proper regular solution to $\mathcal{E}$, corresponding to a cover $Z \to Y$. This proves (a) for $\mathcal{E}$.

For (b), let $D' \subset Y'$ be the inverse image of $D \subset Y$, and let $Z' \to Y'$ correspond to a proper regular solution to $\mathcal{E}_{\alpha'}$ over which D' splits completely, which exists by the special case. Then the decomposition groups of $Z' \to X$ at the points of Z' over $\delta' \in D'$ are the conjugates of $\sigma'(G'_{\delta'})$, where $G'_{\delta'}$ is the decomposition group of $Y' \to X$ at δ' and where σ' is the splitting of f' given by Lemma 2.6. So for the cover $Z \to Y$ corresponding to the proper regular solution obtained for $\mathcal{E}$, the decomposition groups of $Z \to X$ at the points of Z over $\delta \in D$ are the conjugates of $\sigma(G_\delta)$. So the decomposition groups of $Z \to X$ at these points are mapped isomorphically onto that of $Y \to X$ at δ; hence $Z \to Y$ is totally split over δ. So (b) holds for the given embedding problem $\mathcal{E}$.

For (c), by the special case applied to $\mathcal{E}_{\alpha'}$, we have that $\mathcal{E}_{\alpha'}$ has $\mathrm{card}(\hat{k})$ solutions to that embedding problem satisfying (a) and (b) there. But by Lemma 2.4, the map $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$ is injective. So there are at least $\mathrm{card}(\hat{k})$ solutions to $\mathcal{E}$ above, corresponding to connected covers $Z \to Y \to X$ of $\hat{k}$-curves, and to irreducible normal R-models $\bar{Z} \to \bar{Y} \to \bar{X}$. But the cardinality of the function field of X is equal to $\mathrm{card}(\hat{k})$; so in fact the cardinality of this solution set is at most $\mathrm{card}(\hat{k})$ and hence exactly $\mathrm{card}(\hat{k})$, as desired.

For (d), let $Q' \in \bar{Y}'$ be a closed point over $Q \in \bar{Y}$. Then Q, Q' have the same image P

in $\bar{X}$, which has separable residue field. So by part (d) of the special case of Theorem 4.1, the proper regular solution $Z' \to X'$ to $\mathcal{E}_{\alpha'}$ can be chosen, in $\mathrm{card}(\hat{k})$ many ways, so that the normalization $\bar{Z}'$ of $\bar{X}$ in Z' is totally ramified over Q'. For each of these solutions, the above cover $\bar{Z} \to \bar{Y}$ is totally ramified over $Q \in \bar{Y}$, since $\bar{Z}' \to \bar{Y}'$ is the normalized pullback of $\bar{Z} \to \bar{Y}$ by Lemma 2.9(b). So the first part of (d) holds, again using the injectivity of $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$.

Next, we consider the cardinality of the pullbacks $Z^* \to Y^*$ of $\bar{Z} \to \bar{Y}$. First, by this part of (d) in the above special case, there are $\mathrm{card}(\hat{k})$ distinct pullbacks $Z'^* \to Y'^*$ of the above solutions $\bar{Z}' \to \bar{Y}'$ for $\mathcal{E}_{\alpha'}$, where $Y'^* = \mathrm{Spec}\,\hat{\mathcal{O}}_{\bar{Y}',Q'}$ and where $Q' \in \bar{Y}'$ is a point over $Q \in \bar{Y}$. Let $\mathcal{F}$ be the embedding problem $(G_{L^*} \to 1, N \to 1)$, where L^* is the function field of $\hat{\mathcal{O}}_{\bar{Y},Q}$; let J be the Galois group of the function field of Y'^* over that of Y^*; let $\beta : G_{L^*} \to J$ be the corresponding surjection; and let $\mathcal{F}_\beta$ be the induced embedding problem with respect to $\beta : G_{L^*} \to J$. (So $\mathcal{F}, \beta, J$ here play the roles of the objects $\mathcal{E}, \alpha', G'$ in the definition of induced embedding problem in Section 2.) The above pullbacks $Z'^* \to Y'^*$ form a subset of $\mathrm{PS}(\mathcal{F}_\beta)$; and their images $Z^* \to Y^*$ under the injection $\mathrm{PS}(\mathcal{F}_\beta) \to \mathrm{PS}(\mathcal{F})$ are precisely the pullbacks to Y^* of the above solutions $\bar{Z} \to \bar{Y}$ to $\mathcal{E}$. So the cardinality of these pullbacks $Z^* \to Y^*$ is at least $\mathrm{card}(\hat{k})$. Since the cardinality of the function field of Y^* is equal to $\mathrm{card}(\hat{k})$, this inequality is actually an equality. So this part of (d) is verified.

Finally, if two N-Galois covers $Z^* \to Y^*$ are non-isomorphic, then their compositions with $Y^* \to X^*$ are also non-isomorphic as (possibly disconnected) Γ-Galois covers. But these compositions are irreducible Γ-Galois covers if Q is the unique point of $\bar{Y}$ over $P \in \bar{X}$, so the last part of (d) follows. □

Using the above reduction result, we now prove Theorem 4.1. That is, given a finite G-Galois cover $Y \to X$, where $G = \Gamma/N$, we want to construct a Γ-Galois cover $Z \to X$ that dominates the given G-Galois cover and has certain additional properties. The proof will use Proposition 3.5; and for this, we need to construct rational functions Φ_i, Ψ_j corresponding to generators of G, as in the proposition. Using Proposition 4.2, it will be sufficient to construct these rational functions on $\mathbb{P}^1_{\hat{k}}$ rather than on X (which maps to $\mathbb{P}^1_{\hat{k}}$). These functions will be obtained as norms (from $\hat{\ell}$ to $\hat{k}$, where $\hat{\ell}$ is the algebraic closure of $\hat{k}$ in the function field of Y) of rational functions φ_i, ψ_j on $\mathbb{P}^1_{\hat{\ell}}$. The divisors of these functions will have the form $\hat{P}'_i - \hat{P}'^*_i$ and $\hat{P}''_j - \hat{P}''^*_j$ respectively, for some $\hat{\ell}$-points $\hat{P}'_i, \hat{P}'^*_i, \hat{P}''_j, \hat{P}''^*_j$ on the projective line over which $Y \to \mathbb{P}^1_{\hat{\ell}}$ is totally split. These points are constructed with the aid of Lemma 3.4(b), and then yield the functions φ_i, ψ_j and Φ_i, Ψ_j.

Proof of Theorem 4.1. We proceed in several steps, following the strategy outlined above.

Step 1. Construction of the points $\hat{P}'_i, \hat{P}''_j$:

By Proposition 4.2, we may assume that the hypothesis (i)-(iii) there hold; and we preserve the notation used there. So we have a morphism $\phi : X \to \mathbb{P}^1_{\hat{k}}$ and a closed point $\hat{P} \in X$ specializing to a point P on the closed fibre, having the properties stated in Proposition 4.2. In particular, in part (d) of the theorem, P is the image of the given closed point $Q \in \bar{Y}$. Let E be the Galois group of the cover $Y \to \mathbb{P}^1_{\hat{k}}$ given by (ii) of Proposition 4.2. Note that $Y \to \mathbb{P}^1_{\hat{k}}$ is the generic fibre of $\bar{Y} \to \mathbb{P}^1_R$, and that the latter cover is also Galois with group E.

Let N be the kernel of the surjection $\Gamma \to G$. We may choose non-trivial generators $n_1, \dots, n_r, m_1, \dots, m_s$ for N such that $p = \operatorname{char} k$ does not divide the order $o(n_i)$ of any n_i, and such that the order $o(m_j)$ of each m_j is a power of p. Here $r, s \geq 0$. (In parts (c) and (d), N is assumed non-trivial and so r, s are not both 0.) By hypothesis (i) of Proposition 4.2, the function field of Y contains a primitive $o(n_i)^{\text{th}}$ root of unity for each i.

Let $\hat{\ell}$ be the algebraic closure of $\hat{k}$ in the function field of Y and let S be the integral closure of R in $\hat{\ell}$. So $\hat{\ell}$ is separable over $\hat{k}$ since $Y \to X$ is generically separable. By (iii) of Proposition 4.2, the points in the fibre of $Y \to X$ over $\hat{P} \in X$ are $\hat{\ell}$-rational. Pick such a point $\hat{Q} \in Y$ in this fibre; in part (d) we may choose it so that it specializes to the given point Q on the closed fibre of $\bar{Y}$. (In parts (a)-(c)), we choose $\hat{Q}$ arbitrarily in this fibre, and we let Q denote the specialization of $\hat{Q}$ to the closed fibre of $\bar{Y}$. Its residue field need not be separable over k.)

Let P_0 be the image of P in $\mathbb{P}^1_R$ and let P_0' be the image of Q in $\mathbb{P}^1_S$. So P_0' lies over P_0. Similarly, let $\hat{P}_0$ be the image of $\hat{P}$ in $\mathbb{P}^1_{\hat{k}}$ and let $\hat{P}_0'$ be the image of $\hat{Q}$ in $\mathbb{P}^1_{\hat{\ell}}$. Thus $\hat{P}_0'$ is a rational point of $\mathbb{P}^1_{\hat{\ell}}$ that lies over $\hat{P}_0$. Let $\Delta \subset \mathbb{P}^1_{\hat{k}}$ be a finite closed subset containing the branch locus of the composition $Y \to X \to \mathbb{P}^1_{\hat{k}}$, together with the image of D under this composition (in parts (b)-(d)). Choose an integer $h > 1$. By Lemma 3.4(b), there are infinitely many rational points of $\mathbb{P}^1_{\hat{\ell}}$ that are congruent to $\hat{P}_0'$ modulo t^h, are totally split under $Y \to \mathbb{P}^1_{\hat{\ell}}$, have trivial decomposition group over $\mathbb{P}^1_{\hat{k}}$, and have image in $\mathbb{P}^1_{\hat{k}}$ that is not contained in Δ (in fact, there are as many as the cardinality of $\hat{k}$). Pick such points $\hat{P}_i'$ and $\hat{P}_j''$ for $1 \leq i \leq r$ and $1 \leq j \leq s$, no two of which are conjugate over $\hat{k}$. The closures of these points, and their $\hat{k}$-conjugates, meet the closed fibre of $\mathbb{P}^1_S$ at P_0' and its k-conjugates.

Step 2. Blowing up to separate the closures of the points $\hat{P}_i', \hat{P}_j''$:

Let $A_0 \subset \mathbb{P}^1_{\hat{\ell}}$ be the set consisting of all the $\hat{\ell}$-points $\hat{P}_i'$ and $\hat{P}_j''$, and let $A \subset \mathbb{P}^1_{\hat{\ell}}$ consist of the elements of A_0 and their $\hat{k}$-conjugates. Let $B_0, B \subset Y$ be the inverse images of A_0, A respectively; these are sets of $\hat{\ell}$-points of Y, totally split over $\mathbb{P}^1_{\hat{\ell}}$. So the fibre of $Y \to \mathbb{P}^1_{\hat{\ell}}$ over any $\hat{P}_i'$ or $\hat{P}_j''$ consists of $|H|$ distinct $\hat{\ell}$-points, where H is the Galois group of Y over $\mathbb{P}^1_{\hat{\ell}}$. Note that the points in B have the property that the closures of their images under $\bar{Y} \to \mathbb{P}^1_R$ each meet the closed fibre $\mathbb{P}^1_k$ precisely at the point P_0. Similarly, the points of B_0 have the property that the closures of their images under $\bar{Y} \to \mathbb{P}^1_S$ each meet the closed fibre of $\mathbb{P}^1_S$ precisely at the point P_0', over the closed point P_0 of $\mathbb{P}^1_R$. So we may index the points in the fibres of $Y \to \mathbb{P}^1_{\hat{\ell}}$ over $\hat{P}_i'$ and $\hat{P}_j''$ as $\hat{Q}_{i,\delta}'$ and $\hat{Q}_{j,\delta}''$ for $1 \leq \delta \leq |H|$, such that the closures of $\hat{Q}_{i,1}'$ and $\hat{Q}_{j,1}''$ in $\bar{Y}$ pass through Q.

Let Σ be the reduced intersection of the closure of B in $\bar{Y}$ with the closed fibre of $\bar{Y}$. For each non-negative integer ι, consider the ι^{th} blow-up $(\bar{Y}_\iota, B, \Sigma_\iota)$ of $(\bar{Y}, B, \Sigma)$.

Lemma 3.2 applies to $\bar{Y}$ and Lemma 3.3 applies to the E-Galois cover $\bar{Y} \to \mathbb{P}^1_R$; so the conclusions of these lemmas hold for sufficiently large $\iota > h$. Fix such a value $\iota = \lambda$, write $\tilde{Y}$ for $\bar{Y}_\lambda$, and write Δ' for Δ_λ. Thus the action of E on $\bar{Y}$ lifts to an action of E on $\tilde{Y}$. So the Galois groups $G \subset E$ of $\bar{Y} \to \bar{X}$ and $H \subset E$ of $\bar{Y} \to \mathbb{P}^1_S$ also lift to actions on $\tilde{Y}$.

Let $\tilde{X} = \tilde{Y}/G$, $\tilde{\mathbb{P}}_R = \tilde{Y}/E$, and $\tilde{\mathbb{P}}_S = \tilde{Y}/H$. Thus $\tilde{Y} \to \tilde{\mathbb{P}}_S$ is an H-Galois cover, and there is a birational proper morphism $\omega : \tilde{\mathbb{P}}_S \to \mathbb{P}^1_S$ such that the composition $\tilde{Y} \to \bar{Y} \to \mathbb{P}^1_S$ factors as $\tilde{Y} \to \tilde{\mathbb{P}}_S \to \mathbb{P}^1_S$. Similarly, the composition $\tilde{Y} \to \bar{Y} \to \mathbb{P}^1_R$ factors as $\tilde{Y} \to \tilde{\mathbb{P}}_R \to \mathbb{P}^1_R$, where $\tilde{\mathbb{P}}_R \to \mathbb{P}^1_R$ is a birational proper morphism. Also, the general fibre of $\tilde{\mathbb{P}}_S$ is $\mathbb{P}^1_{\hat{\ell}}$ and that of $\tilde{\mathbb{P}}_R$ is $\mathbb{P}^1_{\hat{k}}$. Let P'_i, P''_j be the closed points of $\tilde{\mathbb{P}}_S$ where the closures $\bar{P}'_i, \bar{P}''_j$ of $\hat{P}'_i, \hat{P}''_j$ respectively meet the closed fibre of $\tilde{\mathbb{P}}_S$. Then these $r+s$ closed points, and their k-conjugates, are all distinct, because the fibres of $\tilde{Y} \to \tilde{\mathbb{P}}_S$ over $\hat{P}'_i, \hat{P}''_j$ and their $\hat{k}$-conjugates have disjoint closures. Using Lemma 3.2(iii), observe that the closed fibre $(t = 0)$ of $\tilde{\mathbb{P}}_S$ is locally irreducible at each of the points P'_i, P''_j, since the corresponding assertion holds for the points of $\tilde{Y}$ over P'_i, P''_j (these being points of Δ').

Step 3. Construction of the points $\hat{P}'^*_i, \hat{P}''^*_j$:

Since the points of Y over a given $\hat{P}'_i$ (which form a subset of $B_0 \subset B$) have disjoint closures in $\tilde{Y}$, it follows that the fibre of $\tilde{Y} \to \tilde{\mathbb{P}}_S$ over P'_i consists of $|H|$ distinct closed points; we may label these as $Q'_{i,\delta}$, for $1 \le \delta \le |H|$, where $Q'_{i,\delta}$ is the point of $\tilde{Y}$ over P'_i that is in the closure of $\hat{Q}'_{i,\delta} \in Y$ in $\tilde{Y}$. Similarly, the fibre of $\tilde{Y} \to \tilde{\mathbb{P}}_S$ over each P''_j consists of $|H|$ distinct closed points; and we may index these as $Q''_{j,\delta}$, for $1 \le \delta \le |H|$, such that the closure of $\hat{Q}''_{j,\delta}$ in $\tilde{Y}$ meets the closed fibre at $Q''_{j,\delta}$. Since the $\hat{\ell}$-points $\hat{Q}'_{i,\delta}, \hat{Q}''_{j,\delta}$ and their $\hat{k}$-conjugates have disjoint closures in $\tilde{Y}$, it follows that the closed points $e(Q'_{i,\delta})$ and $e(Q''_{j,\delta})$ are all distinct, where $1 \le i \le r$, $1 \le j \le s$, $1 \le \delta \le |H|$, and where $e \in E = \mathrm{Gal}(Y/\mathbb{P}^1_{\hat{k}})$ ranges over a set of coset representatives of E/H.

If $h' > \lambda$ is sufficiently large, then any $\hat{\ell}$-point $\hat{P}'^*_i$ [resp. $\hat{P}''^*_j$] of $\mathbb{P}^1_{\hat{\ell}}$ that is congruent to $\hat{P}'_i$ [resp. $\hat{P}''_j$] modulo $t^{h'}$ (relative to $\mathbb{P}^1_S$) will have the property that its closure in $\tilde{\mathbb{P}}_S$ will contain P'_i [resp. P''_j], and the two Cartier divisors $\omega^*(\bar{P}'_i)$ and $\omega^*(\bar{P}'^*_i)$ [resp. $\omega^*(\bar{P}''_j)$ and $\omega^*(\bar{P}''^*_j)$] have the same multiplicities at each generic point of the closed fibre of $\tilde{\mathbb{P}}_S$. (Here $\bar{P}'^*_i$ and $\bar{P}''^*_j$ are the closures of $\hat{P}'^*_i$ and $\hat{P}''^*_j$ in $\mathbb{P}^1_S$.) Applying Lemma 3.4(b) to the points P'_i and P''_j for $1 \le i \le r$ and $1 \le j \le s$ (with h' playing the role of the h in the lemma), we obtain $\hat{\ell}$-points $\hat{P}'^*_i, \hat{P}''^*_j$ congruent to $\hat{P}'_i, \hat{P}''_j$ modulo $t^{h'}$ and satisfying the conclusions there. Since there are infinitely many choices for these points (in fact, card $\hat{k}$ of them), they may be chosen so that they and their $\hat{k}$-conjugates are distinct from the points $\hat{P}'_i, \hat{P}''_j$ and their $\hat{k}$-conjugates.

Step 4. Construction of the rational functions φ_i, ψ_j:

According to the conclusion of Lemma 3.4(b), $\hat{P}'^*_i$ totally splits in the Galois cover $Y \to \mathbb{P}^1_{\hat{\ell}}$; so each point of $\tilde{Y}$ over P'_i is in the closure of a unique point of Y over $\hat{P}'^*_i$. Thus we may index the points of Y over $\hat{P}'^*_i$ as $\hat{Q}'^*_{i,\delta}$ for $1 \le \delta \le |H|$, such that the closure of

$\hat{Q}'^*_{i,\delta}$ in $\tilde{Y}$ meets the closed fibre at $Q'_{i,\delta}$ (and thus meets the closure of $\hat{Q}'_{i,\delta}$ there). The corresponding assertions hold for $\hat{P}''^*_j$, $\hat{P}''_j$, $\hat{Q}''^*_{j,\delta}$, $\hat{Q}''_{j,\delta}$.

For $1 \le i \le r$, the two S-points $\bar{P}'_i, \bar{P}'^*_i$ are linearly equivalent on $\mathbb{P}^1_S$. So there is a rational function φ_i on $\mathbb{P}^1_S$ whose divisor is $\bar{P}'_i - \bar{P}'^*_i$. Viewing φ_i as a rational function on $\tilde{\mathbb{P}}_S$ via $\omega : \tilde{\mathbb{P}}_S \to \mathbb{P}^1_S$, its divisor is $\omega^*(\bar{P}'_i) - \omega^*(\bar{P}'^*_i)$; so its zero and pole divisors on $\tilde{\mathbb{P}}_S$ are the closures of $\hat{P}'_i$ and $\hat{P}'^*_i$ in $\tilde{\mathbb{P}}_S$ (since the supports of $\omega^*(\bar{P}'_i)$ and $\omega^*(\bar{P}'^*_i)$ on the closed fibre are equal and hence cancel). Hence as a rational function on $\tilde{\mathbb{P}}_S$, φ_i is defined and invertible on the closed fibre except possibly at the one point P'_i. But the closed fibre is a connected projective curve. So actually φ_i defines a non-zero constant function on this closed fibre; and after multiplying it by a unit in S, we may assume that this constant is 1. Thus φ_i restricts to the constant function 1 at each generic point of the closed fibre of $\tilde{\mathbb{P}}_S$.

Similarly, for $1 \le j \le s$ there is a rational function ψ_j on $\tilde{\mathbb{P}}_S$ restricting to the constant function 1 at each generic point of the closed fibre, with zero and pole divisors being the closures of $\hat{P}''_j$ and $\hat{P}''^*_j$ respectively.

Step 5. Construction the rational functions Φ_i, Ψ_j and the cover $Z \to Y \to X$:

Taking norms, let $\Phi_i = N_{\hat{\ell}/\hat{k}}\, \varphi_i$; this is the product of φ_i and its $\hat{k}$-conjugates. Thus Φ_i defines a rational function on $\mathbb{P}^1_R$ and hence on $\tilde{\mathbb{P}}_R$, where its zero divisor is the closure of $\hat{P}'_i$ and its conjugates, and where its pole divisor is the closure of $\hat{P}'^*_i$ and its conjugates. Moreover Φ_i restricts to the constant function 1 at each generic point of the closed fibre (since this holds for its pullback to $\tilde{\mathbb{P}}_S$). Similarly, for $1 \le j \le s$ we let Ψ_j be the rational function on $\tilde{\mathbb{P}}_R$ obtained by taking the norm of ψ_j. This has the corresponding properties, with respect to the points $\hat{P}''_j, \hat{P}''^*_j$.

The functions Φ_i, Ψ_j on $\tilde{\mathbb{P}}_R$ pull back to rational functions on $\tilde{X}$, which we again denote by Φ_i and Ψ_j. These functions again restrict to the constant function 1 at the generic points of the closed fibre. Their zero and pole divisors are the inverse images of those on $\tilde{\mathbb{P}}_R$, and are therefore reduced, since the supports of those divisors do not lie in the branch locus of $X \to \mathbb{P}^1_{\hat{\ell}}$.

So Proposition 3.5 applies, yielding an irreducible normal N-Galois cover $\tilde{Z} \to \tilde{Y}$ such that $\tilde{Z} \to \tilde{X}$ is Γ-Galois. The generic fibre $Z \to Y$ of this cover is branched precisely at the supports on Y of the divisors of the Φ_i's and the zero divisors of the Ψ_j's; i.e. at the points of Y over $\hat{P}'_i, \hat{P}'^*_i, \hat{P}''_j \in \mathbb{P}^1_{\hat{\ell}}$. Moreover $\hat{\ell}$ is the algebraic closure of $\hat{k}$ in the function field of Z. So $Z \to Y$ provides a proper regular solution to the given embedding problem, proving (a) of the theorem in the general case.

Step 6. Verification of properties (b)-(d).

The cover $\tilde{Z} \to \tilde{Y}$ provided by Proposition 3.5 has the property that the points of $D \subset Y \subset \tilde{Y}$ are totally split, and that the decomposition groups at the points of Z over $\delta \in D$ are the conjugates of $\sigma(G_\delta)$, where σ is the given splitting of the exact sequence and G_δ is the decomposition group of $Y \to X$ at δ. So (b) of the theorem holds.

For (c), we want to show that the number of proper regular solutions constructed above to the given embedding problem $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ is equal to the cardinality of $\hat{k} = k((t))$. As before, it suffices to show that the cardinality of the set of proper regular solutions to $\mathcal{E}$ is at least $\operatorname{card} \hat{k}$.

The above construction of a solution $Z \to Y$ depended on a choice of $2r + s$ points, $\hat{P}'_i, \hat{P}'^*_i, \hat{P}''_j$ (where $1 \le i \le r$ and $1 \le j \le s$). By Lemma 3.4(b), there are $\operatorname{card} \hat{k}$ choices for these points (using also that r and s are not both 0 in part (c)). Distinct choices for this set of points will yield non-isomorphic solutions $Z \to Y$, since the branch locus of $Z \to Y$ is precisely the inverse image of these points $\hat{P}'_i, \hat{P}'^*_i, \hat{P}''_j \in \mathbb{P}^1_{\hat{\ell}}$ under $Y \to \mathbb{P}^1_{\hat{\ell}}$ (using also that the chosen generators of N are all non-trivial). So there are at least $\operatorname{card} \hat{k}$ choices for the cover $Z \to Y$, proving (c).

To verify the conclusion of part (d), let $\bar{Z}$ be the normalization of $\bar{Y}$ in Z. So the composition $\tilde{Z} \to \tilde{Y} \to \bar{Y}$ factors as $\tilde{Z} \to \bar{Z} \to \bar{Y}$, where $\tilde{Z} \to \bar{Z}$ is a proper birational morphism (contracting components of the closed fibre). Since the closure of $\hat{Q}'_{i,1}$ [resp. $\hat{Q}''_{j,1}$] in $\tilde{Y}$ meets the closed fibre at $Q'_{i,1}$ [resp. $Q''_{j,1}$], and since the corresponding closure in $\bar{Y}$ meets the closed fibre just at Q, it follows that each $Q'_{i,1}$ and each $Q''_{j,1}$ must map to Q under $\tilde{Y} \to \bar{Y}$. Let $\bar{Y}_0$, $\tilde{Y}_0$ be the closed fibres of $\bar{Y}$, $\tilde{Y}$. Since the closed fibre of $\tilde{Z} \to \tilde{Y}$ is a mock cover by Proposition 3.5, the same is true for the closed fibre of $\bar{Z} \to \bar{Y}$, with the identity copy of $\tilde{Y}_0$ in the closed fibre of $\tilde{Z}$ mapping to the identity copy of $\bar{Y}_0$ in the closed fibre of $\bar{Z}$. So the points $Q'_{i,1}$ and $Q''_{j,1}$ on the identity copy of $\tilde{Y}_0$ must map to the point Q on the identity copy of $\bar{Y}_0$. But the inertia groups of $\tilde{Z} \to \tilde{Y}$ at the points $Q'_{i,1}$ and $Q''_{j,1}$ on the identity sheet are respectively generated by n_i and m_j. So the inertia group of $\bar{Z} \to \bar{Y}$ at the point Q on the identity sheet contains every n_i and m_j. But these elements generate $N = \operatorname{Gal}(\bar{Z}/\bar{Y})$. So $\bar{Z} \to \bar{Y}$ is totally ramified over Q. Thus each of the $\operatorname{card} \hat{k}$ solutions constructed above have this property, in addition to (a) and (b). So the first part of (d) of the theorem holds.

For the local part of (d), consider a global solution $Z \to Y$, and its pullback $Z^* \to Y^* = \operatorname{Spec} \hat{\mathcal{O}}_{\bar{Y},Q} \to \bar{Y}$. The cover $Z \to Y$ is branched non-trivially at the points of Y lying over $\hat{P}'_i, \hat{P}'^*_i, \hat{P}''_j \in \mathbb{P}^1_{\hat{\ell}}$; in particular, at the $\hat{\ell}$-points $\hat{Q}'_{i,1}, \hat{Q}'^*_{i,1}, \hat{Q}''_{j,1} \in Y$. But the closures of those latter points in $\bar{Y}$ contain the closed point $Q \in \bar{Y}$. Hence the branch locus of $Z^* \to Y^*$ contains the closures of the (distinct) $\hat{\ell}$-points $\hat{Q}'_{i,1}, \hat{Q}'^*_{i,1}, \hat{Q}''_{j,1} \in Y^*$. Since there are $\operatorname{card} \hat{k}$ choices for $\hat{P}'_i, \hat{P}''_j$, the same is true for $\hat{Q}'_{i,1}, \hat{Q}'^*_{i,1}, \hat{Q}''_{j,1}$ and hence for $Z^* \to Y^*$. So there are at least (and hence exactly) $\operatorname{card} \hat{k}$ such pullbacks. Each of them is connected because $\bar{Z}$ and hence Z^* has only one closed point over Q, and each connected component of Z^* must contain a point over Q (as $Z^* \to Y^*$ is finite). Since $\bar{Z}$ and hence Z^* is normal, it follows that Z^* is irreducible.

Similarly, if Q is the only point of $\bar{Y}$ over $P \in \bar{X}$, then there is just one point of $\bar{Z}$ over P, and so the pullbacks of $\bar{Y} \to \bar{X}$ and of $\bar{Z} \to \bar{X}$ under $X^* = \operatorname{Spec} \hat{\mathcal{O}}_{\bar{X},P} \to \bar{X}$ are also irreducible. So in that case these pullbacks are the same as the above Y^* and Z^*. Since distinct covers $Z^* \to Y^*$ remain distinct upon composition with $Y^* \to X^*$, it follows that there are exactly card $\hat{k}$ distinct covers $Z^* \to X^*$ arising from these pullbacks. This proves the local part of (d). □

Next, we extend Theorem 4.1 to arbitrary large fields, not necessarily of the form $k((t))$. Recall from [Po96] that a field F is called *large* if it has the property that every smooth F curve with an F-point has infinitely many F-points. According to [Po96, Proposition 1.1] this is equivalent to each of the following two conditions: For every smooth integral F-variety with an F-point, the set of such points is dense. Every F-variety with an $F((t))$-point has an F-point.

Large fields are infinite, and examples of large fields include algebraically closed fields and fields of the form $\hat{k} = k((t))$. (The case of $\hat{k}$ follows by choosing a covering map from a given $\hat{k}$-curve to the line, such that the given $\hat{k}$-point is unramified; and then using Hensel's Lemma to choose infinitely many t-adically nearby $\hat{k}$-points.) In fact these examples satisfy a stronger condition:

We will say that a field F is *very large* if it has the property that for every smooth F-curve with an F-point, the set of F-points has cardinality equal to card F. Trivially, every very large field is large. Also, algebraically closed fields and fields of the form $\hat{k} = k((t))$ are very large (by the same arguments as for large).

We then have the following generalization of parts (a)-(c) of Theorem 4.1, which also extends the results of Pop [Po96, Main Theorem A] and Haran-Jarden [HJ, Theorem 6.4]. Those results assert the existence of proper regular solutions for embedding problems for curves over large fields, and assert that such solutions can be chosen to be totally split over one given unramified point. (See also [Ha03, Remark 5.1.11].)

Theorem 4.3. *Let F be a large field and let X be a smooth projective connected F-curve, with function field K. Let $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ be a non-trivial finite split embedding problem with section σ, let $\pi : Y \to X$ be the G-Galois branched cover of normal curves corresponding to α, and let $D \subset Y$ be a finite set of closed points.*

a) Then there are infinitely many distinct proper regular solutions to $\mathcal{E}$ such that the corresponding cover $Z \to Y$ is totally split over the points of D, and the decomposition groups of $Z \to X$ at the points of Z over $\delta \in D$ are the conjugates of $\sigma(G_\delta)$, where G_δ is the decomposition group of $Y \to X$ at δ.

b) Moreover, if F is very large, then the cardinality of the set of such solutions is equal to card F.

Proof. Taking a separating transcendence basis for K over F, we obtain a cover $X \to \mathbb{P}^1_F$. The field F is infinite, since it is large; so there is an F-point of $\mathbb{P}^1_F$ that is not in the branch locus of the composition $Y \to X \to \mathbb{P}^1_F$. Let Q be a point of Y lying over this point of $\mathbb{P}^1_F$, and consider its image in X. So the residue field at this image is separable over F.

Let $R = F[[t]]$ and $\hat{F} = F((t))$, and let $\hat{X} = X \times_F \hat{F}$ and $\hat{Y} = Y \times_F \hat{F}$. So the function field of $\hat{X}$ is $\hat{K} := K \otimes_F \hat{F}$. The cover $Y \to X$ induces a G-Galois connected normal cover $\hat{Y} \to \hat{X}$, and this in turn corresponds to an epimorphism $\hat{\alpha} : G_{\hat{K}} \to G$. We thus obtain a split embedding problem $\hat{\mathcal{E}} = (\hat{\alpha} : G_{\hat{K}} \to G, f : \Gamma \to G)$ for $\hat{K}$. For every closed point P of Y, we may consider the closed point $\hat{P} = P \times_F \hat{F}$ of $\hat{Y}$. Let $\hat{D} \subset \hat{Y}$ be the finite subset consisting of the points $\hat{P}$ for each $P \in D$ together with the point $\hat{Q}$. Let $\bar{X} = X \times_F R$ and let $\bar{Y} = Y \times_F R$. Since X is smooth over F, it follows that $\bar{X}$ is smooth over R. Similarly, Y is a regular scheme (being a normal curve), and hence so is $\bar{Y}$. Regarding Y as the closed fibre of $\bar{Y}$, we may view Q as a point of that closed fibre.

We may now apply Theorem 4.1 to the $\hat{F}$-curve $\hat{X}$, the finite split embedding problem $\hat{\mathcal{E}}$, the set $\hat{D} \subset \hat{Y}$, the R-model $\bar{Y} \to \bar{X}$, and the closed point $Q \in \bar{Y}$.

By Theorem 4.1, there is a proper regular solution $\hat{\beta} : G_{\hat{K}} \to \Gamma$ such that the corresponding Γ-Galois cover $\hat{Z} \to \hat{X}$ dominates $\hat{Y} \to \hat{X}$ and satisfies conditions (b) and (d) there (with respect to $\hat{F}$, $\hat{D}$, $\bar{Y} \to \bar{X}$, and $Q \in \bar{Y}$). Let $\bar{Z}$ be the normalization of $\bar{X}$ in $\hat{Z}$.

The scheme $\bar{Y}$ is regular and the scheme $\bar{Z}$ is normal; so Purity of Branch Locus applies to $\bar{Z} \to \bar{Y}$. Since $Q \in \bar{Y}$ is totally ramified in this cover, it lies on a codimension 1 branch divisor. This divisor does not contain the closed fibre of $\bar{Z} \to \bar{Y}$, since that fibre is generically étale. It also does not contain $\bar{Q} := Q \times_F R$, since $\bar{Z} \to \bar{Y}$ is totally split over the generic point of $\bar{Q}$. So the branch locus of $\bar{Z} \to \bar{Y}$ is not induced from its closed fibre by base change from F to R. Hence the branch locus of $\hat{Z} \to \hat{Y}$ is also not induced from F to $\hat{F}$. Let r be the degree of this branch locus.

The cover $\hat{Z} \to \hat{Y}$ descends from $\hat{F}$ to some finite type F-subalgebra A of $\hat{F}$. Thus for some such A, if we let $X_A = X \times_F A$ and $Y_A = Y \times_F A$, there is a connected projective normal A-curve Z_A and a covering morphism $Z_A \to Y_A$ such that the composition $Z_A \to X_A$ is Γ-Galois; the cover $Z_A \to Y_A$ is totally split over every point of $D_A := D \times_F A$, with $\sigma(G_\delta)$ a decomposition group of $Z_A \to X_A$ over any point of $\delta \times_F A$ for $\delta \in D$; and the pullback $Z_A \times_A \hat{F} \to \hat{Y}$ is isomorphic to $\hat{Z} \to \hat{Y}$. Necessarily, the branch locus of $Z_A \to Y_A$ is not induced from F, because of this property for $\hat{Z} \to \hat{Y}$. That is, the branch loci of the fibres of $Z_A \to Y_A$ over the points of $\operatorname{Spec} A$ are non-constant. After replacing A by a basic affine open subset, we may assume that $\operatorname{Spec} A$ is smooth over F. Also $\operatorname{Spec} A$ is irreducible, since $A \subset \hat{F}$ is a domain.

The curve $\hat{Z}$ is geometrically irreducible as a scheme over the algebraic closure of $\hat{F}$ in the function field of $\hat{Y}$, since $\hat{Z} \to \hat{X}$ corresponds to a proper regular solution to the embedding problem $\hat{\mathcal{E}}$. So the general fibre of Z_A is geometrically irreducible as a scheme over the algebraic closure of $\operatorname{frac}(A)$ in the function field of Y_A. But the set of points of $\operatorname{Spec} A$ at which the fibre of Z_A is geometrically irreducible is a constructible set [Gr66, 9.7.7]; so after replacing A by a basic affine open subset, we may assume that every closed fibre of $Z_A \to \operatorname{Spec} A$ is geometrically irreducible. We may also assume that each such fibre is generically étale, and the normalization of each fibre has exactly r geometric branch points (each branched non-trivially). So for every F-point of $\operatorname{Spec} A$, the normalization of the fibre of $Z_A \to Y_A$ over this point will be a proper regular solution to the given embedding problem with desired properties.

It remains to show that the set of such solutions has the asserted cardinality in parts (a) and (b) above.

Let $Y^{(r)}$ be the r^{th} symmetric power of Y, and let $b : \operatorname{Spec} A \to Y^{(r)}$ be the morphism assigning to each point of $\operatorname{Spec} A$ the branch locus of the corresponding normalized geometric fibre of $Z_A \to Y_A$. Thus b is non-constant, and so its image B has dimension ≥ 1. Let S be the set of F-points of $\operatorname{Spec} A$. Since F is large and $\operatorname{Spec} A$ contains a $\hat{F}$-point (viz. the point corresponding to the inclusion $A \hookrightarrow \hat{F}$), the set S is dense in $\operatorname{Spec} A$. Hence $b(S)$ is dense in B, and is thus infinite. Since covers with distinct branch loci are non-isomorphic, it follows that the set of proper regular solutions as above is infinite. This proves part (a).

For part (b), we wish to show in the very large case that the set of such solutions has cardinality equal to $\operatorname{card} F$. As before, it suffices to show that there are at least that many. Proceeding as above, take an F-point P on $\operatorname{Spec} A$ (of which there are infinitely many since F is large). Then $b^{-1}(b(P))$ is closed and is strictly contained in $\operatorname{Spec} A$, because b is non-constant. Since $\operatorname{Spec} A$ is smooth and connected, there is a smooth curve C on $\operatorname{Spec} A$ that passes through P and is not contained in $b^{-1}(b(P))$. Hence b is non-constant on the curve C, and thus $b|_C$ is finite-to-one. Since F is very large, the set of F-points on C has cardinality $\operatorname{card} F$. Hence the set of proper regular solutions induced by the F-points of C has cardinality equal to $\operatorname{card} F$; and as above, these induced solutions each have the desired properties. This proves part (b). $\square$

Corollary 4.4. *If K is the function field of a smooth projective curve over a very large field k, then the absolute Galois group of K is quasi-free.*

Remark. If k is also perfect, then every field extension K of transcendence degree 1 over k is the function field of a smooth projective k-curve, and hence Corollary 4.4 applies.

Because of Corollary 4.4, it would be interesting to know the answer to

Question 4.5. Is every large field very large?

Section 5. Main Result

In this section we prove the main result of the paper, stated as Theorem 1.1 in the introduction.

Theorem 5.1. (Main theorem) *Let k be a field. Then the absolute Galois group of the field $K := k((x,t))$ is quasi-free of rank $\operatorname{card} K$. Equivalently, every non-trivial finite split embedding problem for K has $\operatorname{card} K$ proper solutions.*

This result generalizes Theorem 5.3.9 of [Ha03], which showed the existence of proper solutions in the case $k = \mathbb{C}$, but did not consider the number of such solutions. The proof of that earlier result relied heavily on the fact that $\mathbb{C}$ is algebraically closed and of characteristic 0, in order to know that every finite extension of $\mathbb{C}((x))$ is obtained by adjoining some n^{th} root of x, and in order to be able to use Abhyankar's Lemma. Here, under the more general hypothesis that k is an arbitrary field, the proof of Theorem 5.1 is completely different, and relies on Theorem 4.1.

Theorem 5.1 follows immediately from Proposition 5.3 below, a geometric version of the main theorem stated in terms of covers. Proposition 5.3 considers a G-Galois branched cover $Y^* \to X^* = \operatorname{Spec} k[[x,t]]$ and a split short exact sequence $1 \to N \to \Gamma \xrightarrow{f} G \to 1$. This proposition asserts that there exist "many" Γ-Galois covers $Z^* \to X^*$ dominating $Y^* \to X^*$ and having an additional splitting property. The proof of Proposition 5.3 uses Theorem 4.1 to solve an embedding problem for the $k((t))$-line X; then we close that up over the $k[[t]]$-line $\bar{X}$ (i.e. normalizing $\bar{X}$ in these covers of X); and finally we restrict from $\bar{X}$ to X^*, which we regard as the complete local neighborhood of a closed point of $\bar{X}$. In the process, we rely on a preliminary result, Lemma 5.2, which permits us to pass from an embedding problem over X^* to a more global embedding problem in Proposition 5.3.

In the following result, we consider the morphism from $X^* = \operatorname{Spec} k[[x,t]]$ to the affine x-line $\mathbb{A}^1_{k[[t]]}$ that corresponds to the inclusion of rings $k[[t]][x] \hookrightarrow k[[x,t]]$. Composing with the inclusion $\mathbb{A}^1_{k[[t]]} \hookrightarrow \bar{X} := \mathbb{P}^1_{k[[t]]}$, we obtain a morphism $X^* \to \bar{X}$.

Lemma 5.2. *Let k be a field of characteristic $p \geq 0$, let G be a finite group, and let $Y^* \to X^* = \operatorname{Spec} k[[x,t]]$ be a connected normal G-Galois cover that is unramified over the generic point of $(t=0)$. Then there is a connected normal generically smooth G-Galois cover $\bar{Y} \to \bar{X}$ of $k[[t]]$-curves whose pullback via $X^* \to \bar{X}$ is the given cover $Y^* \to X^*$.*

Proof. Let $Y_0^* \to X_0^* = \operatorname{Spec} k[[x]]$ be the reduction modulo t of $Y^* \to X^*$. By hypothesis, this finite morphism is generically étale. Let $Y_0^\circ \to X_0^\circ = \operatorname{Spec} k((x))$ be the generic fibre of $Y_0^* \to X_0^*$, and let Z_0° be a connected component of Y_0° (corresponding to a finite field extension of $k((x))$). So $Z_0^\circ \to X_0^\circ$ is H-Galois for some $H \subset G$, and $Y_0^\circ = \operatorname{Ind}_H^G Z_0^\circ$. Applying the theorem of Katz-Gabber [Ka, Theorem 1.4.1], there is an étale H-Galois cover $Z_0'' \to X_0'' := \mathbb{P}^1_k - \{0, \infty\} = \operatorname{Spec} k[x, x^{-1}]$ whose pullback to X_0° is $Z_0^\circ \to X_0^\circ$. Let $Z_0' \to X_0' := \mathbb{P}^1_k - \{0\} = \operatorname{Spec} k[x^{-1}]$ be the normalization of X_0' in Z_0''. Thus the pullback of $Z_0' \to X_0'$ to X_0° is again $Z_0^\circ \to X_0^\circ$; and so we may identify the pullback of $Y_0' := \operatorname{Ind}_H^G Z_0' \to X_0'$ to X_0° with $Y_0^\circ \to X_0^\circ$ as a (disconnected) G-Galois cover.

Meanwhile, let $\bar{X}^\circ = \operatorname{Spec} k((x))[[t]]$ and let $\bar{Y}^\circ = Y^* \times_{X^*} \bar{X}^\circ$. Since $Y^* \to X^*$ is unramified at the generic point of $(t=0)$, its pullback $\bar{Y}^\circ \to \bar{X}^\circ$ under $\bar{X}^\circ \to X^*$ is unramified at the closed point $(t=0)$ of $\bar{X}^\circ$ and is therefore étale. So by Hensel's Lemma, we have that $\bar{Y}^\circ = Y_0^\circ \times_{X_0^\circ} \bar{X}^\circ$ canonically, as G-Galois covers (compatibly with the restriction of $\bar{Y}^\circ$ to Y_0°).

Let $\bar{X}' = \operatorname{Spec} k[x^{-1}][[t]]$ and let $\bar{Y}' = Y_0' \times_{X_0'} \bar{X}'$. So we have identifications of the G-Galois covers $\bar{Y}' \times_{\bar{X}'} \bar{X}^\circ = Y_0' \times_{X_0'} \bar{X}^\circ = Y_0^\circ \times_{X_0^\circ} \bar{X}^\circ = \bar{Y}^\circ$. Thus the pullbacks of $Y^* \to X^*$ and of $\bar{Y}' \to \bar{X}'$ to $\bar{X}^\circ$ are each identified with $\bar{Y}^\circ \to \bar{X}^\circ$.

So by formal patching ([HS99, §1, Cor. to Thm. 1], [Pr, Thm. 3.4]), there is a unique G-Galois cover $\bar{Y} \to \bar{X}$ whose pullbacks to $X^*, \bar{X}', \bar{X}^\circ$ respectively agree with $Y^* \to X^*$, $\bar{Y}' \to \bar{X}'$, and $\bar{Y}^\circ \to \bar{X}^\circ$, compatibly with the above identifications. Now Y^* is connected; Y^* and $\bar{Y}'$ are normal; and $Y_0' = \bar{Y} \times_{\bar{X}} X_0'$ is generically smooth over k. So $\bar{Y}$ is connected and normal, and is generically smooth over $k[[t]]$. □

We now prove the geometric form of our main result:

Proposition 5.3. *Let k be a field, let $1 \to N \to \Gamma \xrightarrow{f} G \to 1$ be a split short exact sequence of finite groups, and let $Y^* \to X^* = \operatorname{Spec} k[[x,t]]$ be a G-Galois connected normal branched cover.*

a) Then there is a Γ-Galois connected normal branched cover $Z^* \to X^*$ dominating $Y^* \to X^*$.

b) The cover $Z^* \to X^*$ may be chosen such that $Z^* \to Y^*$ is totally split at the generic points of the ramification locus of $Y^* \to X^*$.

c) The set of isomorphism classes of such covers $Z^* \to X^*$ has cardinality equal to that of $k((x,t))$ (whether or not condition (b) is required).

Proof. After a change of variables of the form $x' = x, t' = t + x^n$ for some $n > 0$, we may assume that $Y^* \to X^*$ is unramified over the generic point of $(t = 0)$.

By Lemma 5.2, we may extend $Y^* \to X^*$ to a cover $\bar{Y} \to \bar{X}$ of the x-line $\bar{X} = \mathbb{P}^1_{k[[t]]}$ over $k[[t]]$. Let η be the unique closed point of Y^*, and also write η for the image of this point in $\bar{Y}$; this is the unique closed point of $\bar{Y}$ where $x = 0$. In order to prove the proposition, we will use Theorem 4.1 to construct an appropriate Γ-Galois cover of $\bar{X}$ dominating $\bar{Y}$, and then restrict that to obtain the desired Γ-Galois cover of X^* dominating Y^*.

Let $Y \to X$ be the general fibre of $\bar{Y} \to \bar{X}$, and let K be the function field of X (or equivalently, of $\bar{X}$). Let $\alpha : G_K \to G$ be the surjection corresponding to the G-Galois cover $Y \to X$, and consider the finite split embedding problem $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$. To give a proper solution $\beta : G_K \to \Gamma$ to $\mathcal{E}$ is equivalent to giving a connected normal Γ-Galois cover $Z \to X$ dominating $Y \to X$; and for such a cover, we may consider the normalization $\bar{Z}$ of $\bar{X}$ in Z. In that situation, there are then intermediate N-Galois covers $Z \to Y$ and $\bar{Z} \to \bar{Y}$ of the Γ-Galois covers $Z \to X$ and $\bar{Z} \to \bar{X}$.

Since $\bar{X} = \mathbb{P}^1_{k[[t]]}$ is smooth over $k[[t]]$, we may apply Theorem 4.1, taking $D \subset Y$ to be the ramification locus of $Y \to X$. By that result, we know in fact that there exists a proper solution $\beta : G_K \to \Gamma$ to $\mathcal{E}$ such that the intermediate cover $Z \to Y$ is totally split over the ramified points of $Y \to X$ (by part (b)), and such that $\bar{Z}$ is totally ramified at the unique point of $\bar{Y}$ over η (by part (d) of Theorem 4.1; there $\eta = Q$).

Let $Z^* \to X^*$ be the pullback of $\bar{Z} \to \bar{X}$ via $X^* \to \bar{X}$; this is a Γ-Galois cover. Since $\bar{Z} \to \bar{X}$ dominates $\bar{Y} \to \bar{X}$, we see that $Z^* \to X^*$ dominates $Y^* \to X^*$. Since $\bar{Z} \to \bar{Y}$ is totally ramified over η, the same is true for $Z^* \to Y^*$. But η is the unique closed point of Y^*. So Z^* has only one closed point, and is therefore connected. Being normal, it is irreducible. Here $Z^* \to Y^*$ splits completely over the generic points of the ramification locus of $Y^* \to X^*$, because this is a local condition and this splitting holds on the generic fibre $Z \to Y$ of $\bar{Z} \to \bar{Y}$. So $Z^* \to X^*$ is as asserted.

By Theorem 4.1, parts (c) and (d), there were $\operatorname{card} k((t))$ non-isomorphic choices above for $Z \to X$, and there were the same number of pullbacks $Z^* \to X^*$. But $\operatorname{card} k((t)) = \operatorname{card} k[[t]] = (\operatorname{card} k)^{\aleph_0} = \operatorname{card} k[[x,t]] = \operatorname{card} k((x,t))$ (where, as usual, $\aleph_0$ denotes $\operatorname{card} \mathbb{Z}$). So there is the asserted cardinality for the covers $Z^* \to X^*$ satisfying (a) and (b) of the proposition. The same is true for covers $Z^* \to X^*$ that are just required to satisfy (a) of the proposition, by applying Theorem 4.1 with $D = \emptyset$ there, instead of taking D to be the ramification locus of $Y \to X$ as before. □

Restating Proposition 5.3, we obtain our main result, Theorem 5.1:

Proof of 5.1. Given any finite split embedding problem $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$ for G_K, consider the G-Galois connected normal cover $Y^* \to X^* = \operatorname{Spec} k[[x,t]]$ corresponding to f. By Proposition 5.3, there are precisely $\operatorname{card} k[[x,t]] = \operatorname{card} K$ non-isomorphic Γ-Galois covers $Z^* \to X^*$ that dominate $Y^* \to X^*$. These covers correspond in turn to $\operatorname{card} K$ distinct proper solutions to the embedding problem $\mathcal{E}$. □

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