

Local-global Galois theory of arithmetic function fields

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Abstract

We study the relationship between the local and global Galois theory of function fields over a complete discretely valued field. We give necessary and sufficient conditions for local separable extensions to descend to global extensions, and for the local absolute Galois group to inject into the global absolute Galois group. As an application we obtain a local-global principle for the index of a variety over such a function field. In this context we also study algebraic versions of van Kampen's theorem, describing the global absolute Galois group as a direct limit of local absolute Galois groups.

1 Introduction

In this paper we relate the local and the global Galois theory of function fields F of curves over a complete discretely valued field K . Each such curve has a normal projective model \mathcal{X} over the valuation ring T of K . Given a closed point P on \mathcal{X} , one can compare the Galois theory of F to that of the fraction field F_P of the complete local ring \widehat{R}_P of \mathcal{X} at P . In particular, is every finite separable extension of F_P induced by a finite separable extension of F ? That is, for every finite separable field extension E_P/F_P , is there a finite separable field extension E/F such that $E \otimes_F F_P$ is isomorphic to E_P ? As a related question, is the homomorphism of absolute Galois groups $\text{Gal}(F_P) \rightarrow \text{Gal}(F)$ an inclusion? The answers turn out to depend on the situation.

We show that the answer to the first question is yes if and only if the residue field k of T has characteristic zero and the closed fiber X of \mathcal{X} is unbranched at P ; and that the answer to the second question is yes if and only if $\text{char}(k) = 0$. (See the paragraph preceding Theorem 2.6 for the terminology.) In [CHHKPS17], we considered a related question: in the situation as above, let U be a nonempty connected affine open subset of X , and let F_U be the fraction field of the ring \widehat{R}_U of formal functions along U . (See Notation 2.2.) Then is every finite separable extension of F_U induced by a finite separable extension of F ? There we showed that the answer to that question is always yes, regardless of characteristic. In the

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current paper, we use that to show that the homomorphism $\mathrm{Gal}(F_U) \rightarrow \mathrm{Gal}(F)$ is always an inclusion.

These results raise the question of how $\mathrm{Gal}(F)$ is related to the groups $\mathrm{Gal}(F_P)$ and $\mathrm{Gal}(F_U)$, if we pick a finite set \mathcal{P} of closed points P and let \mathcal{U} be the set of connected components of the complement of \mathcal{P} in X , such that each element of \mathcal{U} is affine. An important situation is if the reduction graph of \mathcal{X} is a tree (see the sentence before Theorem 3.7 for the definition). Under this hypothesis, we show that $\mathrm{Gal}(F)$ is a direct limit of local Galois groups. In particular, in the special case of one unibranched point P and its complement U , this gives an analog of van Kampen's theorem in topology. More generally, without the tree hypothesis, we obtain a description of $\mathrm{Gal}(F)$ in terms of groupoids, as well as a description (via a result of J. Stix) in terms of a maximal subtree of the reduction graph.

As an application of our descent results, we obtain a more explicit version of a local-global principle that appeared in [CHHKPS17]. That result concerned the index of a variety over F , and related it to its index over the fields F_P and F_U . The version that we prove here is in the equal characteristic zero case, and it relies on the descent results mentioned above.

The structure of the paper is as follows: Section 2 concerns the question of descent of finite separable extensions from F_P to F . In Section 2.1 we provide a positive answer if $\mathrm{char}(k) = 0$ and X is unibranched at P (Theorem 2.6), but show that there are always counterexamples if X is not unibranched at P (Remark 2.7(b)). In Section 2.2 we combine Theorem 2.6 and a result from [CHHKPS17] to obtain an explicit local-global principle for zero-cycles on varieties over F under the characteristic zero hypothesis (Corollary 2.10). We show in Section 2.3 that if $\mathrm{char}(k) = p > 0$, then there are always degree p separable extensions of F_P that do not descend to extensions of F (Proposition 2.15).

Section 3 concerns the implications for absolute Galois groups. In Section 3.1 we show that $\mathrm{Gal}(F_P) \rightarrow \mathrm{Gal}(F)$ is injective if and only if $\mathrm{char}(k) = 0$, and that $\mathrm{Gal}(F_U) \rightarrow \mathrm{Gal}(F)$ is always injective (Theorem 3.2). Section 3.2 obtains a van Kampen theorem in a simple case (Theorem 3.4, for diamonds), with generalizations given in Section 3.3 (Theorem 3.7, for trees; and Theorem 3.9, in terms of groupoids).

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2 Descent of extensions from local to global

2.1 Descent in characteristic zero

As in [HH10], [HHK09], and [HHK15b], we consider the following situation:

Definition 2.1. Let K be a complete discretely valued field with valuation ring T , uniformizer t , and residue field k . A *semi-global field* is a one-variable function field over such a field K . A *normal model* of a semi-global field F is a T -scheme \mathcal{X} with function field F

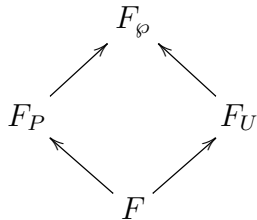
that is flat and projective over T of relative dimension one, and that is normal as a variety. The *closed fiber* of \mathcal{X} is $X := \mathcal{X}_k$.

The following notation will be used throughout this manuscript.

Notation 2.2. Let \mathcal{X} be a normal model for a semi-global field F . If P is a (not necessarily closed) point of the closed fiber X , let R_P be the local ring of \mathcal{X} at P ; let \widehat{R}_P be its completion with respect to the maximal ideal \mathfrak{m}_P ; and let F_P be the fraction field of \widehat{R}_P . In the case that P is a closed point of X , the *branches* of X at P are the height one prime ideals of \widehat{R}_P that contain t , which we can also regard as the codimension one points of $\text{Spec}(\widehat{R}_P)$ that lie on the closed fiber. The localization R_φ of \widehat{R}_P at a branch φ is a discrete valuation ring; we write \widehat{R}_φ for its completion, and F_φ for the fraction field of \widehat{R}_φ . The contraction of φ to F_P determines an irreducible component X_0 of X , whose generic point η has the property that $F_\eta \subset F_\varphi$. We then say that φ *lies on* X_0 . Note that the residue field $k(\varphi) := \widehat{R}_\varphi/\varphi\widehat{R}_\varphi = R_\varphi/\varphi R_\varphi$ is isomorphic to the fraction field of \widehat{R}_P/φ , and hence is a complete discretely valued field.

If U is a nonempty connected affine open subset of X , then we write R_U for the subring of F consisting of rational functions that are regular at each point of U . We let \widehat{R}_U be the t -adic completion of R_U ; this completion is the *ring of formal functions along* U . This is an integral domain by [HHK15b, Proposition 3.4], and we let F_U be the fraction field of \widehat{R}_U . If $P \in U \subseteq U'$, then $\widehat{R}_{U'} \subseteq \widehat{R}_U \subseteq \widehat{R}_P$ and $F_{U'} \subseteq F_U \subseteq F_P$. We say that a branch φ of X at a closed point $P \in X$ *lies on* U if it lies on a component of the closure \bar{U} . The field F_U is then contained in F_φ . If U as above is affine, then the absolute value on the complete discretely valued field $k(\varphi)$ restricts to an absolute value on $k(\bar{U}^{\text{red}})$, where the superscript indicates the underlying reduced scheme. Here $k(\varphi) = \text{frac}(\widehat{\mathcal{O}}_{\bar{U}^{\text{red}},P})$ while $k(U^{\text{red}}) = k(\bar{U}^{\text{red}}) = \text{frac}(\mathcal{O}_{\bar{U}^{\text{red}},P})$, and so $k(U^{\text{red}})$ is dense in $k(\varphi)$ under the above absolute value.

To illustrate the above in a simple case, consider a smooth T -curve \mathcal{X} , such as the projective line over T , and pick a closed point $P \in X$. Let $U \subset X$ be the complement of P in X , and let φ be the unique branch of X at P . We then have containments of fields $F \subset F_P, F_U \subset F_\varphi$, forming a diamond



with $F = F_P \cap F_U$ (see [HH10, Proposition 6.3]). Given a finite separable field extension of one of these four fields, we can ask whether it is induced by base change from an extension of the smaller fields. More generally, we may have more complicated configurations of fields (see Notation 2.8 below), but we can still ask this question. Note that if a field extension E

of a larger field is shown to be induced by an étale algebra A over a smaller field, then A is automatically a (separable) field extension of the smaller field, because it is contained in E .

Along these lines, the following results were proven in [CHHKPS17] (Propositions 2.3 and 2.4 there), in connection with obtaining local-global principles for zero-cycles on varieties over F . These results in particular concern two of the four edges of the above diamond in the special case of the above simple example.

Proposition 2.3. *Let F be a semi-global field with normal model \mathcal{X} and closed fiber X . Let P be a closed point of X , let \wp be a branch of X at P , and let E_\wp be a finite separable field extension of F_\wp , say of degree n . Then there exists a finite separable field extension E_P of F_P such that $E_P \otimes_{F_P} F_\wp \cong E_\wp$ as extensions of F_\wp , and such that E_P induces the trivial étale algebra $F_{\wp'}^{\oplus n}$ over $F_{\wp'}$ for every other branch \wp' at P .*

Proposition 2.4. *Let F be a semi-global field with normal model \mathcal{X} and closed fiber X . Let U be a nonempty connected affine open subset of X , and let E_U be a finite separable field extension of F_U . Then there is a finite separable field extension E of F such that $E \otimes_F F_U \cong E_U$ as extensions of F_U .*

These results raise the question of whether there are analogs with the roles of P and U interchanged; these would in particular treat the other two edges of the above diamond in that example. Such analogs of Propositions 2.3 and 2.4 do not hold in general; see Section 2.3 below for counterexamples. But analogs do hold if $\text{char}(k) = 0$, where as above k is the residue field of the complete discretely valued field K :

Proposition 2.5. *Let F be a semi-global field over a complete discrete valuation ring with residue field k , and assume that $\text{char}(k) = 0$. Let \mathcal{X} be a normal model for F with closed fiber X . Let $U \subset X$ be a nonempty connected affine open subset, and let \wp be a branch of X lying on U . Then for every finite separable field extension E_\wp of F_\wp there is a finite separable field extension E_U of F_U such that $E_U \otimes_{F_U} F_\wp \cong E_\wp$ as F_\wp -algebras. In fact there is a finite separable field extension E of F such that $E \otimes_F F_\wp \cong E_\wp$ as F_\wp -algebras.*

Proof. Since $F \subset F_U \subset F_\wp$, the first assertion follows from the second, by taking $E_U = E \otimes_F F_U$. So it suffices to prove the second assertion. By Proposition 2.4, to prove that assertion it suffices to prove the first assertion for *some* choice of U on which \wp lies.

We first deal with the unramified part of the extension. Since F_\wp is a complete discretely valued field, the maximal unramified extension F'_\wp of F_\wp contained in E_\wp is also a complete discretely valued field. Moreover E_\wp is totally ramified over F'_\wp . Let \widehat{R}'_\wp be the integral closure of \widehat{R}_\wp in F'_\wp , say with maximal ideal \wp' . The reduction $k(\wp') = \widehat{R}'_\wp/\wp'$ is then a finite separable field extension of the complete discretely valued field $k(\wp) = \widehat{R}_\wp/\wp$.

Let X_0 be the irreducible component of X on which \wp lies, and let U be a nonempty affine open subset of X_0 that does not meet any other irreducible component of X and does not contain the point P at which \wp is a branch, and such that U^{red} is regular. Since $k(U^{\text{red}})$ is dense in $k(\wp)$, by Krasner's Lemma there is a finite separable field extension L of $k(U^{\text{red}})$ that induces $k(\wp')$ over $k(\wp)$. Thus $L = k(U')$, where U' is a finite branched cover

of U^{red} . After shrinking U , we may assume that $U' \rightarrow U^{\text{red}}$ is étale; i.e., $U' = \text{Spec}(B)$ for some étale algebra B over $k[U^{\text{red}}]$ whose fraction field is L . By [Gr71, I, Corollaire 8.4], up to isomorphism there is a unique étale \widehat{R}_U -algebra \widehat{R}'_U that lifts the étale $k[U^{\text{red}}]$ -algebra $B = k[U']$.

The base change of \widehat{R}'_U to \widehat{R}_φ lifts the residue field extension $k(\varphi')$ of $k(\varphi)$. But by [Gr71, I, Théorème 6.1], such a lift to a complete local ring is unique. Hence the \widehat{R}_φ -algebra $\widehat{R}'_U \otimes_{\widehat{R}_U} \widehat{R}_\varphi$ is isomorphic to \widehat{R}'_φ , and so \widehat{R}'_U is a domain. Thus $F'_U := \widehat{R}'_U \otimes_{\widehat{R}_U} F_U$ is a field that satisfies $F'_U \otimes_{F_U} F_\varphi \cong F'_\varphi$. This completes the unramified step.

We now turn to the totally ramified part, and work explicitly. With notation as above, let η' be the generic point of U' , let $R_{\eta'}$ be the local ring of \widehat{R}'_U at η' , and let $\tau \in R_{\eta'} \subset F'_U$ be a uniformizer for the discrete valuation ring $R_{\eta'}$. Thus τ is also a uniformizer for the complete discrete valuation ring \widehat{R}'_φ , which contains $R_{\eta'}$. Let \widehat{S}_φ be the integral closure of \widehat{R}'_φ in E_φ , or equivalently of \widehat{R}_φ in E_φ . Let $\tilde{\varphi}$ be the maximal ideal of the complete discrete valuation ring \widehat{S}_φ , and let $\sigma \in \widehat{S}_\varphi$ be a uniformizer at $\tilde{\varphi}$.

Now E_φ is totally ramified over F'_φ along φ' , say of degree $n = [E_\varphi : F'_\varphi]$. By the characteristic zero hypothesis, $\tau = \sigma^n v$ for some unit $v \in \widehat{S}_\varphi$; and the extension of residue fields $k(\varphi') = \widehat{R}'_\varphi/\varphi' \subseteq \widehat{S}_\varphi/\tilde{\varphi}$ is an isomorphism. So the image $\bar{v} \in \widehat{S}_\varphi/\tilde{\varphi}$ of $v \in \widehat{S}_\varphi$ may be regarded as a non-zero element in the complete discretely valued field $k(\varphi')$.

Let C, C' be the regular connected projective curves containing U^{red}, U' , respectively. Thus C' is a branched cover of C ; and there is a birational map $C \rightarrow X_0^{\text{red}}$ which is an isomorphism over U^{red} . Since φ is a branch of X at P , it is also a branch of C at a closed point $Q \in C$ that lies over $P \in \bar{U}$ on X . Moreover φ' is a branch of C' at a point Q' that maps to Q . Also, $k(\varphi)$ is the fraction field of the complete local ring $\widehat{\mathcal{O}}_{C,Q}$ of C at Q , and $k(\varphi')$ is the fraction field of $\widehat{\mathcal{O}}_{C',Q'}$.

Let $\bar{\pi} \in \mathcal{O}_{C',Q'} \subset k(C')$ be a uniformizer of the local ring of C' at Q' . Thus $\bar{\pi}$ is also a uniformizer of the complete local ring $\widehat{\mathcal{O}}_{C',Q'}$, which is the valuation ring of $k(\varphi')$. So we may write $\bar{v} = \bar{\pi}^r \bar{u}$ for some integer r and some unit $\bar{u} \in \widehat{\mathcal{O}}_{C',Q'}$. Since $\mathcal{O}_{C',Q'}$ is $\bar{\pi}$ -adically dense in $\widehat{\mathcal{O}}_{C',Q'}$, there is a unit $\bar{w} \in \mathcal{O}_{C',Q'} \subset k(C')$ such that $\bar{w} \equiv \bar{u}$ modulo $\bar{\pi}$. Thus $\bar{u} = \bar{a}\bar{w}$ for some $\bar{a} \equiv 1 \pmod{\bar{\pi}}$ in $\widehat{\mathcal{O}}_{C',Q'}$.

Since the residue field of the complete discrete valuation ring $\widehat{\mathcal{O}}_{C',Q'}$ has characteristic zero, by Hensel's Lemma there is a unit $\bar{b} \in \widehat{\mathcal{O}}_{C',Q'} \subset k(\varphi')$ such that $\bar{b}^n = \bar{a}$. So $\bar{v} = \bar{\pi}^r \bar{w}\bar{b}^n$. Let $\pi, w \in R_{\eta'} \subset F'_U$ be lifts of $\bar{\pi}, \bar{w} \in k(C') = R_{\eta'}/\eta'R_{\eta'}$, and let $b \in \widehat{R}'_\varphi$ be a lift of $\bar{b} \in k(\varphi') = \widehat{R}'_\varphi/\varphi'$. Thus $v = \pi^r w b^n c \in \widehat{S}_\varphi$, for some $c \equiv 1 \pmod{\sigma}$ in \widehat{S}_φ . Again by Hensel's Lemma, there exists $d \in \widehat{S}_\varphi$ such that $c = d^n$. Note that π, w are units in $R_{\eta'}$ and hence in \widehat{R}'_φ ; and that b, d are units in \widehat{S}_φ .

Thus $\tau \pi^{-r} w^{-1} \in R_{\eta'} \subset \widehat{R}'_\varphi$ is a uniformizer for \widehat{R}'_φ ; $bd\sigma$ is a uniformizer for \widehat{S}_φ ; and $(bd\sigma)^n = \tau \pi^{-r} w^{-1}$. Since $E_\varphi = \text{frac}(\widehat{S}_\varphi)$ is of degree n over $F'_\varphi = \text{frac}(\widehat{R}'_\varphi)$, it follows that E_φ is generated over F'_φ by $bd\sigma$; i.e., $E_\varphi \cong F'_\varphi[Y]/(Y^n - \tau \pi^{-r} w^{-1})$. Since $\tau \pi^{-r} w^{-1} \in R_{\eta'} \subset F'_U$, we may consider the finite separable F'_U -algebra $E_U = F'_U[Y]/(Y^n - \tau \pi^{-r} w^{-1})$. We then have $E_U \otimes_{F'_U} F'_\varphi \cong E_\varphi$, and hence E_U is a field. Since $F'_\varphi \cong F'_U \otimes_{F_U} F_\varphi$, it follows that

$E_U \otimes_{F_U} F_\varphi \cong E_\varphi$, as desired. \square

The key ingredient in the proof of the next theorem, in addition to the propositions above, is patching over fields, as in [HH10] and [HHK09]. Recall that a scheme V is *unibranched* at a point P if the complete local ring $\widehat{\mathcal{O}}_{V,P}$ has a unique minimal prime; or equivalently, $\text{Spec}(\widehat{\mathcal{O}}_{V,P})$ is irreducible.

Theorem 2.6. *Let F be a semi-global field over a complete discrete valuation ring with residue field k , and assume that $\text{char}(k) = 0$. Let \mathcal{X} be a normal model of F with closed fiber X . Let P be a closed point of X , and assume that every irreducible component of X that contains P is unibranched at P . If E_P is a finite separable field extension of F_P , then there is a finite separable field extension E of F such that $E \otimes_F F_P \cong E_P$ as extensions of F_P .*

Proof. Choose a finite set \mathcal{P} of closed points of X that contains P and also all the points where distinct irreducible components of X meet; and let \mathcal{U} be the set of connected components of the complement of \mathcal{P} in X . Thus each $U \in \mathcal{U}$ is affine and meets just one irreducible component of X . Let \mathcal{U}_P be the subset of \mathcal{U} consisting of those $U \in \mathcal{U}$ whose closures contain P ; and let \mathcal{P}_P be the subset of \mathcal{P} consisting of points other than P that lie in the closure of some element of \mathcal{U}_P . Blowing up \mathcal{X} at the points of \mathcal{P}_P produces a new model, but does not change F or F_P . After doing so (possibly several times), we may assume that the following two conditions hold:

- (i) If $P' \in \mathcal{P}_P$ lies on the closure of $U \in \mathcal{U}_P$, then \bar{U} is unibranched at P' .
- (ii) For each $P' \in \mathcal{P}_P$, there is a unique $U \in \mathcal{U}_P$ whose closure contains P' .

For each branch φ at P , $E_\varphi := E_P \otimes_{F_P} F_\varphi$ is a finite direct product of finite separable field extensions $E_{\varphi,i}$ of F_φ . By Proposition 2.5, for each branch φ at P and each i , there is a finite separable field extension $E_{U,i}$ of F_U such that $E_{U,i} \otimes_{F_U} F_\varphi \cong E_{\varphi,i}$, where φ lies on $U \in \mathcal{U}$. For each $U \in \mathcal{U}$ whose closure contains P , let E_U be the direct product of the fields $E_{U,i}$, ranging over i . This is well defined, for each $U \in \mathcal{U}_P$, because of the assumption on being unibranched at P . For each branch φ along any $U \in \mathcal{U}_P$, let $E_\varphi = E_U \otimes_{F_U} F_\varphi$. (For the branches at P , this agrees with the previous definition of E_φ .)

By conditions (i) and (ii), for each $P' \in \mathcal{P}_P$ there is a unique $U \in \mathcal{U}_P$ whose closure contains P' ; and there is a unique branch φ at P' along U . For such a triple P', U, φ , applying Proposition 2.3 to each direct factor of E_φ provides a finite étale $F_{P'}$ -algebra $E_{P'}$ such that $E_{P'} \otimes_{F_{P'}} F_\varphi \cong E_\varphi$ and such that $E_{P'} \otimes_{F_{P'}} F_{\varphi'}$ is the trivial étale $F_{\varphi'}$ -algebra of degree $n := [E_P : F_P]$ for every branch φ' at P' other than φ . For every $U \in \mathcal{U}$ that is not in \mathcal{U}_P , let E_U be the trivial étale F_U -algebra of degree n . Similarly, for every $P' \in \mathcal{P}$ that does not lie in $\mathcal{P}_P \cup \{P\}$, let $E_{P'}$ be the trivial étale $F_{P'}$ -algebra of degree n ; and for every branch φ_B at a point of \mathcal{P} that is not in $\mathcal{P}_P \cup \{P\}$, let E_φ be the trivial étale F_φ -algebra of degree n . Then for every branch φ at a point $P' \in \mathcal{P}$ lying on some $U \in \mathcal{U}$, we have isomorphisms $E_{P'} \otimes_{F_{P'}} F_\varphi \cong E_\varphi \cong E_U \otimes_{F_U} F_\varphi$. We then conclude by Theorem 7.1 of [HH10]. \square

Remark 2.7. (a) The hypothesis that $\text{char}(k) = 0$ in Proposition 2.5 was used in order to avoid wild ramification and inseparable residue field extensions; and it was used in Theorem 2.6 in order to be able to rely on Proposition 2.5. Otherwise the proofs carry over to characteristic $p > 0$. For example, if E_φ/F_φ is a Galois field extension of degree prime to p , then all ramification is tame, and the conclusion of Proposition 2.5 holds. Similarly, the conclusion of Theorem 2.6 holds for a Galois field extension E_P/F_P of degree prime to p , provided that the condition on being unbranched at P is satisfied. But these two propositions fail in general for wildly ramified extensions, as shown in Section 2.3.

(b) In Theorem 2.6, the hypothesis on being unbranched is essential. Namely, suppose instead that φ_1, φ_2 are distinct branches of an irreducible component X_0 of X at P ; these are height one primes in the Noetherian normal domain \widehat{R}_P . By the corollary in [Bo72, Section VII.3], \widehat{R}_P is a Krull domain; so by [Bo72, Proposition VII.5.9 and Theorem VII.6.3], there exists $f \in \widehat{R}_P$ that is a uniformizer at φ_1 but does not lie in φ_2 . Let ℓ be a prime unequal to $\text{char } k$; let E_P be the finite separable extension of F_P given by adjoining an ℓ -th root of f ; this is ramified over φ_1 but not over φ_2 . Write $E_{\varphi_i} = E_P \otimes_{F_P} F_{\varphi_i}$. Then $E_{\varphi_1}/F_{\varphi_1}$ is ramified over φ_1 but $E_{\varphi_2}/F_{\varphi_2}$ is not ramified over φ_2 . Let η be the generic point of X_0 . A uniformizer of R_η is also a uniformizer of R_{φ_i} for $i = 1, 2$. Thus if E/F is a degree ℓ field extension, then $E_i := E \otimes_F F_{\varphi_i}$ is ramified over φ_i if and only if E/F is ramified over η . Thus E/F cannot induce both $E_{\varphi_1}/F_{\varphi_1}$ and $E_{\varphi_2}/F_{\varphi_2}$. But E_P/F_P induces $E_{\varphi_1}/F_{\varphi_1}$ and $E_{\varphi_2}/F_{\varphi_2}$. Then E/F cannot induce E_P/F_P .

(c) A result related to Theorem 2.6 was proven in [HS05, Lemma 5.2]. That assertion was stated in the equal characteristic case, and it permitted the characteristic to be non-zero. By that result, if a finite Galois extension of $k((x, t))$ is unramified over the ideal (t) of $k[[x, t]]$, then it is induced by a finite Galois extension of the function field $F = k((t))(x) \subset k((x, t))$ of the $k((t))$ -line. Moreover, by a change of variables in $k((x, t))$, the condition on being unramified over (t) can be dropped (see also [HHK13, Lemma 3.8]); but this makes it impossible to specify F in advance as a subfield of $k((x, t))$, unlike in the above results that restrict to characteristic zero.

2.2 Application to a local-global principle for zero-cycles

As in [HHK09], we also use the following notation:

Notation 2.8. Let F be a semi-global field with normal model \mathcal{X} , and let X denote the closed fiber. Let \mathcal{P} be a finite set of closed points of X that meets each irreducible component of X . We then let \mathcal{U} be the set of connected components of the complement of \mathcal{P} in X , and we let \mathcal{B} be the set of branches of X at points of \mathcal{P} .

Recall that given a variety V over a field k , the *index* (resp. *separable index*) of V is the greatest common divisor of the degrees of the finite (resp. finite separable) field extensions

of k over which V has a rational point. This is the same as the smallest positive degree of a zero-cycle (resp. separable zero cycle) on V .

As in [CHHKPS17], given a collection Ω of overfields of F and an F -scheme Z , we say that (Z, Ω) *satisfies a local-global principle for rational points* if the following holds: $Z(F) \neq \emptyset$ if and only if $Z(L) \neq \emptyset$ for all $L \in \Omega$. In particular, we will consider the collection of overfields $\Omega_{\mathcal{X}, \mathcal{P}}$ consisting of the overfields $F_{\mathcal{P}}, F_{\mathcal{U}}$ for \mathcal{P} and \mathcal{U} as in Notation 2.8.

If Z is a torsor under a connected and rational linear algebraic group over F , then $(Z, \Omega_{\mathcal{X}, \mathcal{P}})$ satisfies a local-global principle for rational points, for any choice of \mathcal{P} and \mathcal{U} as above by [HHK09, Theorem 3.7]; see [CHHKPS17, Corollary 3.10] for a further discussion.

In Corollaries 3.5 and 3.6 of [CHHKPS17], we showed:

Proposition 2.9. *Let F be a semi-global field with normal model \mathcal{X} , and let X be the closed fiber. Let Z be an F -scheme of finite type, and choose \mathcal{P} and \mathcal{U} as in Notation 2.8. Assume that for every finite separable extension E/F , $(Z_E, \Omega_{\mathcal{X}_E, \mathcal{P}_E})$ satisfies a local-global principle for rational points, where \mathcal{X}_E denotes the normalization of \mathcal{X} in E and \mathcal{P}_E denotes the preimage of \mathcal{P} under the normalization map. Then*

- (a) *The prime numbers that divide the separable index of Z are precisely those that divide the separable index of some Z_{F_ξ} , where ξ ranges over $\mathcal{P} \cup \mathcal{U}$. In particular, the separable index of Z is equal to one if and only if the separable index of each Z_{F_ξ} is equal to one.*
- (b) *If $\text{char } K = 0$, or if Z is regular and generically smooth, then the assertion also holds with the separable index replaced by the index.*

Proposition 2.4 and Theorem 2.6 together yield a strengthening of Proposition 2.9 in the equal characteristic zero case:

Corollary 2.10. *In the situation of Proposition 2.9, suppose that $\text{char}(k) = 0$ and that each irreducible component X_0 of the closed fiber X of \mathcal{X} is unbranched at each point $P \in \mathcal{P}$ on X_0 . Then the index of Z divides the product of the indices of Z_{F_ξ} , for $\xi \in \mathcal{P} \cup \mathcal{U}$.*

Proof. First consider field extensions E_ξ/F_ξ for $\xi \in \mathcal{P} \cup \mathcal{U}$, say of degree d_ξ , such that Z_{F_ξ} has an E_ξ -point. By Proposition 2.4 and Theorem 2.6, there are finite field extensions A_ξ/F of degree d_ξ that induce E_ξ/F_ξ , for each ξ . Thus $Z(A_\xi \otimes_F F_\xi) = Z(E_\xi)$ is nonempty for each ξ . Let $A = \bigotimes_F A_\xi$, where the tensor product is taken over all $\xi \in \mathcal{P} \cup \mathcal{U}$. Then for each ξ , $Z(A \otimes_F F_\xi)$ is nonempty. Now A is an étale algebra over F , and so it is the direct product of finite field extensions A_i/F . Note that $\sum_i [A_i : F] = \dim_F(A) = \prod d_\xi$. Also, for each i , $Z(A_i \otimes_F F_\xi)$ is nonempty for every ξ .

Let \mathcal{X}_i be the normalization of \mathcal{X} in A_i ; this is a normal model of the semi-global field A_i , equipped with a finite morphism $\mathcal{X}_i \rightarrow \mathcal{X}$. Let $\mathcal{P}_i, \mathcal{U}_i$ consist of the inverse images of the elements of \mathcal{P}, \mathcal{U} under this morphism, respectively. As in Notation 2.5, we then have associated fields $(A_i)_{P'}, (A_i)_{U'}$ for $P' \in \mathcal{P}_i$ and $U' \in \mathcal{U}_i$. For each $P \in \mathcal{P}$, $A_i \otimes_F F_P$ is the direct product of the fields $(A_i)_{P'}$, where P' runs over the points of \mathcal{P}_i that lie over P ; and similarly for each $U \in \mathcal{U}$. Hence for each $\xi' \in \mathcal{P}_i \cup \mathcal{U}_i$, $Z((A_i)_{\xi'})$ is nonempty. By the local-global assumption, this implies $Z(A_i)$ is nonempty, for each i .

Let I (resp. I_ξ) be the ideal in \mathbb{Z} generated by the degrees of closed points on Z (resp. on Z_{F_ξ}); or equivalently, generated by the index of Z (resp. of Z_{F_ξ}). Since $\prod d_\xi = \sum_i [A_i : F]$ above, it follows that $\prod d_\xi \in I$; i.e. $\prod I_\xi \subseteq I$. The asserted conclusion follows. \square

Note that the above bound is not sharp, since by enlarging the set \mathcal{P} we can in general increase the product of the local indices. It seems reasonable to ask whether, under the above hypotheses, the index of Z is equal to the least common multiple of the indices of Z_{F_ξ} , for $\xi \in \mathcal{P} \cup \mathcal{U}$. We do not know of any counterexamples.

2.3 Failure of descent in characteristic p

Proposition 2.5 and Theorem 2.6 fail if $\text{char}(k) \neq 0$, as shown in Proposition 2.15 below. First we state a lemma.

Lemma 2.11. *Let D/L be a finite field extension, and let $\Lambda \supseteq L$ be a field in which L is algebraically closed. If $D \otimes_L \Lambda$ is a Galois field extension of Λ , then D/L is Galois.*

Proof. First note that if $D \otimes_L \Lambda$ is Galois, and hence separable, then D/L is separable. Let \widehat{D} be the Galois closure of D/L , and write $G = \text{Gal}(\widehat{D}/L)$ and $H = \text{Gal}(\widehat{D}/D)$. Since L is algebraically closed in Λ whereas D and \widehat{D} are each separable over L , it follows that Λ is linearly disjoint over L from both D and \widehat{D} . Let $\Delta = D \otimes_L \Lambda$ and $\widehat{\Delta} = \widehat{D} \otimes_L \Lambda$. Thus $\widehat{\Delta}$ is a Galois field extension of Λ with $\text{Gal}(\widehat{\Delta}/\Lambda) = G$, and $\text{Gal}(\widehat{\Delta}/\Delta) = H$. Since Δ/Λ is Galois, it follows that H is normal in G ; hence D/L is Galois. \square

Lemma 2.12. *Let k_2/k_1 be a field extension in characteristic $p > 0$ such that the algebraic closure of k_1 in k_2 is separable over k_1 , and write $F_i = k_i((t))$ for $i = 1, 2$. Let $\alpha \in k_2^\times$, and let $E := F_2[Y]/(Y^p - Y - \alpha/t)$. Then E is a degree p Galois field extension of F_2 . Moreover:*

- (a) *If α is not in k_1 , then E is not induced by a degree p Galois field extension of F_1 .*
- (b) *If α is not algebraic over k_1 , then E is not induced by any degree p field extension of F_1 .*

Proof. Since F_2 is of characteristic p , and since α/t is not of the form $c^p - c$ for any $c \in F_2$, it follows that E is a degree p Galois field extension of F_2 .

For part (a), assume $\alpha \notin k_1$, and suppose that E is induced by a degree p Galois field extension of F_1 . We may then write that extension of F_1 as $F_1[W]/(W^p - W - \beta)$ for some $\beta \in F_1$, with $\alpha t^{-1} - \beta = \gamma^p - \gamma$ for some $\gamma \in F_2$. Projecting both sides of this equality onto the k_2 -vector subspace of F_2 spanned by $t^{-1}, t^{-p}, t^{-p^2}, \dots$, we may assume that β and γ lie in that subspace. Write $\beta = \sum_{i=0}^n a_i t^{-p^i}$ with $a_i \in k_1$. Then $\alpha^{p^n} t^{-p^n} - (\sum_{i=0}^n a_i^{p^{n-i}}) t^{-p^n} = \delta^p - \delta$ for some $\delta \in F_2$; so $\alpha^{p^n} = \sum_{i=0}^n a_i^{p^{n-i}} \in k_1$. Thus $\alpha \in k_2$ is purely inseparable over k_1 , and hence lies in k_1 ; a contradiction.

For part (b), assume that α is not algebraic over k_1 . Let k'_1 be the algebraic closure of k_1 in k_2 ; this is a separable extension of k_1 not containing α . Let $F'_1 = k'_1((t))$. The algebraic closure of F'_1 in F_2 is an unramified extension of complete discretely valued fields with trivial

residue field extension, and so this extension of F'_1 is trivial; i.e., F'_1 is the algebraic closure of F_1 in F_2 . If E_1/F_1 is a degree p field extension such that $E_1 \otimes_{F_1} F_2 = E$, then Lemma 2.11 asserts that $E_1 \otimes_{F_1} F'_1$ is a Galois field extension of F'_1 . But this field extension induces E/F_2 . By applying part (a) to the extension k_2/k'_1 , we obtain a contradiction. \square

Example 2.13. The transcendental condition in part (b) of Lemma 2.12 is necessary. For example, let κ be a field of characteristic three; let $k_1 = \kappa(x)$; and let $k_2 = \kappa(u)$ where $u^2 = x$. Let $\alpha = u$, which lies in k_2 and is algebraic over k_1 but does not lie in k_1 . Write $F_i = k_i((t))$ for $i = 1, 2$, and let E/F_2 be the 3-cyclic Galois extension given by $E = F_2[Y]/(Y^3 - Y - u/t)$. Then E is not induced by a 3-cyclic Galois extension of F_1 , but it is induced by the degree three non-Galois extension E_0/F_1 given by $E_0 = F_1[W]/(W^3 + W^2 + W - x/t^2)$. Here E/F_1 is an S_3 -Galois field extension, and E_0 is the fixed field of the order two subgroup of S_3 generated by the involution taking u to $-u$ and taking Y to $-Y$. As an extension of F_1 , the field E_0 is generated by the element $W = Y^2$.

We will apply Lemma 2.12 in the situation in which k_1 is the fraction field of a Dedekind domain D , and k_2 is the fraction field of the completion of D at a maximal ideal. By Artin's Approximation Theorem (Theorem 1.10 of [Art69]), the separability hypothesis in Lemma 2.12 will hold if D is excellent; e.g., if it is the coordinate ring of a curve over a field of characteristic p . In that same situation, we next prove a mixed characteristic analog of Lemma 2.12.

Lemma 2.14. *Let k_1 be the fraction field of a characteristic p excellent Dedekind domain D ; let k_2 be the fraction field of the completion \hat{D} of D at a maximal ideal \mathfrak{m} , and let D' be the algebraic closure of D in \hat{D} . Let $R_1 \subseteq R_2$ be an extension of complete discrete valuation rings of mixed characteristic $(0, p)$ having residue field extension k_2/k_1 , and let F_i be the fraction field of R_i . Let $x \in R_1$ be an element whose reduction $\bar{x} \in k_1$ is a uniformizer for D at \mathfrak{m} , and let $g \in R_2$ be an element whose reduction $\bar{g} \in k_2$ lies in $\hat{D} \setminus D'$. Then $E = F_2[Y]/(Y^p - g^p - x)$ is a degree p field extension of F_2 that is not induced by any degree p field extension of F_1 , nor by any degree p Galois extension of the algebraic closure F'_1 of F_1 in F_2 .*

Proof. It suffices to show that the extension $E(\zeta_p)/F_2(\zeta_p)$ is not induced by any degree p field extension of $F_1(\zeta_p)$, nor by any degree p Galois extension of $F'_1(\zeta_p)$. So after replacing F_i by $F_i(\zeta_p)$ for $i = 1, 2$ (which does not change k_i), we may assume that $\zeta_p \in R_i \subset F_i$. Now the residue class $\bar{f} \in k_2$ of $f := g^p + x \in R_2$ is not a p -th power because the residue class of x is not a p -th power there. So f is also not a p -th power in F_2 , and E is a degree p Galois field extension of F_2 (viz. a Kummer extension).

We claim that for every $e \in k_2^\times$, $\bar{f}e^p$ does not lie in the algebraic closure k'_1 of k_1 in k_2 . Since the residue field of F'_1 is k'_1 , the claim implies that $f\tilde{e}^p \notin F'_1$ for every $\tilde{e} \in F_2^\times$. Hence $f \in F_2^\times$ is not in the same p -th power class as any element of F'_1 . Thus E is not induced by any degree p Galois field extension of F'_1 . By Lemma 2.11, E is also not induced by any degree p field extension of F_1 .

To prove the claim (and therefore the assertion), suppose otherwise; i.e., $\bar{f}e^p \in k'_1$ for some $e \in k_2^\times$. After multiplying e by some non-zero element of D' , we may assume that $\bar{f}e^p$ is equal

to some element $h \in D'$. Since $f = g^p + x$, the elements $\bar{g}e, e$ satisfy the polynomial equation $Y_1^p + \bar{x}Y_2^p - h = 0$ over D' . By Artin's Approximation Theorem, D' is the henselization of D at \mathfrak{m} , and there exist elements $\bar{g}', e' \in D'$ such that $\bar{g}'e', e'$ are solutions to the above equation, with $e' \neq 0$. So $(\bar{g}e)^p + \bar{x}e^p = (\bar{g}'e')^p + \bar{x}e'^p$. If $e \neq e'$ then $\bar{x} = (\bar{g}e - \bar{g}'e')^p / (e' - e)^p$, which is a contradiction since \bar{x} is not a p -th power in D . So $e = e' \in D'$, and $\bar{g}^p = (h - \bar{x}e^p) / e^p \in D'$, using that $(\bar{g}e)^p + \bar{x}e^p = h$. Since D' is algebraically closed in \hat{D} , it follows that $\bar{g} \in D'$, which is a contradiction. \square

Proposition 2.15. *Let F be a semi-global field over a complete discrete valuation ring with residue field k , and assume that $\text{char } k = p > 0$. Let \mathcal{X} be a normal model for F , and let X denote its closed fiber. Let P be a closed point of X lying on an irreducible component X_0 , let \wp be a branch of X_0 at P , and let U be a nonempty connected affine open subset of X that meets X_0 but does not contain P .*

- (a) *There is a degree p Galois field extension E_\wp of F_\wp that is not induced by any degree p field extension of F_U , or even a degree p field extension of the algebraic closure of F_U in F_\wp .*
- (b) *There is a degree p Galois field extension E_P of F_P that is not induced by any degree p field extension of F , or even a degree p field extension of the algebraic closure of F in F_P .*

Proof. Let η be the generic point of X_0 , and let F'_η be the algebraic closure of F_η in F_\wp . Note that F_η contains F_U since $\eta \in U$, and so F'_η contains the algebraic closure of F_U in F_\wp . We will show the following statement, which implies both parts of the proposition: There is a degree p Galois field extension E_P of F_P such that $E_\wp := E_P \otimes_{F_P} F_\wp$ is a degree p Galois field extension of F_\wp that is not induced by any degree p field extension of F'_η .

Let \tilde{X}_0 be the normalization of X_0 and let $\tilde{\wp}$ be the branch on \tilde{X}_0 lying over \wp ; this is the unique branch of \tilde{X}_0 at some point \tilde{P} of \tilde{X}_0 lying over P . The residue field $k(\tilde{\wp})$ of $\hat{R}_{\tilde{\wp}}$ at $\tilde{\wp}$ is also the residue field of $R_\wp \subset \hat{R}_P$ at $\tilde{\wp}$; it is also the fraction field of the complete local ring $\hat{\mathcal{O}}_{\tilde{X}_0, \tilde{P}}$ of \tilde{X}_0 at \tilde{P} .

By Artin's Approximation Theorem (Theorem 1.10 of [Art69]), the henselization $\mathcal{O}_{\tilde{X}_0, \tilde{P}}^h$ of $\hat{\mathcal{O}}_{\tilde{X}_0, \tilde{P}}$ is algebraically closed in the completion $\hat{\mathcal{O}}_{\tilde{X}_0, \tilde{P}}$. So the algebraic closure of $k(\tilde{X}_0) = k(X_0) = k(\eta)$ in $k(\tilde{\wp})$ is the fraction field of $\mathcal{O}_{\tilde{X}_0, \tilde{P}}^h$, which is separable over $k(\eta)$. Let $\alpha \in k(\tilde{\wp})$ be an element that is transcendental over $k(X_0)$; i.e., does not lie in the fraction field of $\mathcal{O}_{\tilde{X}_0, \tilde{P}}^h$.

First consider the case in which $\text{char } K = p$. That is, K is a complete discretely valued field of equal characteristic p , hence the form $k((t))$, with $F_\eta = k(\eta)((t))$ and $F_\wp = k(\tilde{\wp})((t))$. We will regard $k(\tilde{\wp})$ as contained in R_\wp , and hence in F_P and F_\wp ; and in particular we will regard α as an element of those fields. Let E_P be the degree p Galois field extension of F_P given by adjoining a root of $Y^p - Y - \alpha/t$, and let $E_\wp = E_P \otimes_{F_P} F_\wp$. We now apply Lemma 2.12, taking k_1 equal to the algebraic closure of $k(\eta)$ in $k(\tilde{\wp})$, taking $k_2 = k(\tilde{\wp})$, and taking $E = E_\wp$. The lemma then says that E_\wp has the asserted property, and this proves the result in the equal characteristic case.

Next, consider the case in which $\text{char } K = 0$. Let D be the local ring of \tilde{X}_0 at \tilde{P} , and let $\hat{D} = \hat{\mathcal{O}}_{\tilde{X}_0, \tilde{P}}$ be its completion at the point \tilde{P} . Let x be an element in the local ring of \mathcal{X} at η whose image \bar{x} in the residue field $k(\eta) = k(\tilde{X}_0)$ is a uniformizer of \tilde{X}_0 at \tilde{P} . Let $g \in R_\varphi \subset \hat{R}_P$ be a lift of $\alpha \in k(\varphi)$. Then $E_P = F_P[Y]/(Y^p - g^p - x)$ has the asserted property, by applying Lemma 2.14, where we take $R_1 = \hat{R}_\eta$, $R_2 = \hat{R}_\varphi$, and $E = E_\varphi = E_P \otimes_{F_P} F_\varphi$. \square

Geometrically, the above proposition asserts in particular that if $\text{char}(k) = p > 0$, then there is a degree p Galois branched cover of $\text{Spec}(\hat{R}_P) = \text{Spec}(\hat{\mathcal{O}}_{\mathcal{X}, P})$ that is not induced by any degree p branched cover of \mathcal{X} .

3 Absolute Galois groups

3.1 Injectivity of local-global maps on Galois groups

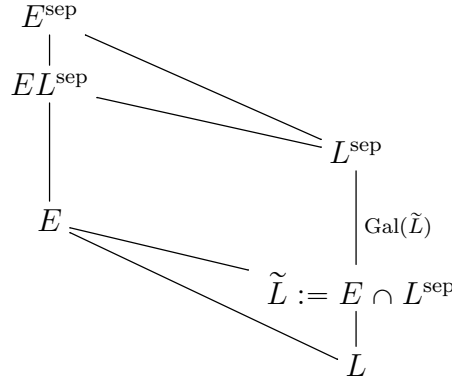
Given a finite group G , a field L , and a separable closure L^{sep} of L , homomorphisms $\text{Gal}(L) := \text{Gal}(L^{\text{sep}}/L) \rightarrow G$ (which we require to be continuous) are in bijection with pairs consisting of a G -Galois étale L -algebra E/L and an L -algebra homomorphism $E \rightarrow L^{\text{sep}}$. Here, an epimorphism $\phi : \text{Gal}(L) \rightarrow G$ corresponds to the G -Galois field extension $(L^{\text{sep}})^{\ker(\phi)}/L$ together with its inclusion into L^{sep} . A general homomorphism ϕ with image $H \subseteq G$ determines an H -Galois field extension E_0/L . The induced G -Galois étale L -algebra $E = \text{Ind}_H^G(E_0)$ is equipped with a projection map $\text{Ind}_H^G(E_0) \rightarrow E_0$, and consequently a map to L^{sep} given by $E \rightarrow E_0 = (L^{\text{sep}})^{\ker(\phi)} \subset L^{\text{sep}}$. (For further discussions, see [Gr71, Exp. V] and [KMRT98, Proposition 18.17].)

Consider an inclusion of fields $L \subseteq E$. If we pick a separable closure E^{sep} of E , then the separable closure of L in E^{sep} is a separable closure L^{sep} of L in the absolute sense. There is then an induced group homomorphism between the absolute Galois groups of E and L ; i.e., from $\text{Gal}(E) := \text{Gal}(E^{\text{sep}}/E)$ to $\text{Gal}(L) := \text{Gal}(L^{\text{sep}}/L)$. This is a special case of the fact that a morphism of pointed schemes $(V, v) \rightarrow (W, w)$ induces a homomorphism $\pi_1(V, v) \rightarrow \pi_1(W, w)$ between their étale fundamental groups. In our situation, there is the following result about the homomorphism $\text{Gal}(E) \rightarrow \text{Gal}(L)$:

Lemma 3.1. *In the above situation, the map $\text{Gal}(E) \rightarrow \text{Gal}(L)$ induced by the inclusion $L^{\text{sep}} \subseteq E^{\text{sep}}$ factors as $\text{Gal}(E) \twoheadrightarrow \text{Gal}(EL^{\text{sep}}/E) \hookrightarrow \text{Gal}(L)$, and its image is $\text{Gal}(E \cap L^{\text{sep}})$. Thus the map is injective if and only if $E^{\text{sep}} = EL^{\text{sep}}$, and it is surjective if and only if L is separably closed in E .*

This lemma is a special case of results on Galois categories in [Gr71, Exp. V.6], with the injectivity and surjectivity assertions respectively following from Corollaire 6.8 and Propo-

sition 6.9 there. The lemma also follows directly from this diagram of fields and inclusions:



Here the fields E and L^{sep} are linearly disjoint over \tilde{L} , because \tilde{L} is separably closed in E ; and so the natural map $\text{Gal}(EL^{\text{sep}}/E) \rightarrow \text{Gal}(\tilde{L})$ is an isomorphism.

In the above situation, if a different separable closure of L had been chosen, along with some embedding into E^{sep} , then the homomorphism $\text{Gal}(E) \rightarrow \text{Gal}(L)$ would be modified by conjugation, but the injectivity and surjectivity of $\text{Gal}(E) \rightarrow \text{Gal}(L)$ would not be affected.

We may apply the lemma in the situation of Notation 2.2, to the field extensions $F \rightarrow F_P$, $F \rightarrow F_U$, $F_P \rightarrow F_\varphi$, and $F_U \rightarrow F_\varphi$, where φ is a branch of X at P lying on U . We then obtain:

Theorem 3.2. *Let F be a semi-global field, and consider field extensions $F \subseteq F_P, F_U \subseteq F_\varphi$ as in Notation 2.2. Then the induced maps $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_P)$ and $\text{Gal}(F_U) \rightarrow \text{Gal}(F)$ between absolute Galois groups are injective. If the branch φ lies on U , then the induced maps $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_U)$ and $\text{Gal}(F_P) \rightarrow \text{Gal}(F)$ are injective if and only if the residue field k has characteristic zero. The above maps are never surjective.*

Proof. The last assertion is immediate from Lemma 3.1, since in each of the corresponding field extensions, the bottom field is not separably closed in the top field.

By Proposition 2.3, every finite separable extension of F_φ is the compositum of a finite separable extension of F_P with F_φ . Thus $F_\varphi^{\text{sep}} = F_\varphi F_P^{\text{sep}}$. So by Lemma 3.1, $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_P)$ is an injection. Similarly, $\text{Gal}(F_U) \rightarrow \text{Gal}(F)$ is injective, using Proposition 2.4. If $\text{char}(k) = 0$, then Proposition 2.5 implies that $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_U)$ is injective. If, in addition, P is a unbranched point of each component of X on which it lies, then Theorem 2.6 implies that $\text{Gal}(F_P) \rightarrow \text{Gal}(F)$ is injective.

If $\text{char}(k) = 0$ but we do not assume that each of these components is unbranched at P , then by [HHK15a, Proposition 6.2] there exists a finite Galois split cover $\mathcal{X}' \rightarrow \mathcal{X}$ such that for each closed point $P' \in \mathcal{X}'$ lying over $P \in \mathcal{X}$, each irreducible component of the closed fiber X' of \mathcal{X}' is unbranched at P' . (Recall from [HHK15a, Section 5] that a degree n morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is a *split cover* if $\mathcal{X}' \times_{\mathcal{X}} Q$ consists of n copies of Q for every point $Q \in \mathcal{X}$ other than the generic point of \mathcal{X} .) Every split cover is necessarily étale; and if we choose a point $P' \in \mathcal{X}'$ lying over P then the inclusion $F_P \hookrightarrow F'_{P'}$ is an isomorphism, where F' is the function field of \mathcal{X}' . Since $F' \subset F'_{P'}$ and since the extension F'/F is algebraic, we

obtain an inclusion of F' in the algebraic closure of F in F_P . So the map $\text{Gal}(F_P) \rightarrow \text{Gal}(F)$ factors through the inclusion $\text{Gal}(F') \hookrightarrow \text{Gal}(F)$. The map $\text{Gal}(F_P) \rightarrow \text{Gal}(F')$ is injective by Theorem 2.6 applied to the model \mathcal{X}' ; and so $\text{Gal}(F_P) \rightarrow \text{Gal}(F)$ is also injective.

For the converse, suppose that $\text{char } k = p > 0$. By Proposition 2.15(a), there is a degree p Galois field extension E_φ/F_φ that is not induced by any degree p separable field extension of the separable closure \tilde{F}_U of F_U in F_φ . If $E_\varphi \subseteq F_\varphi^{\text{sep}}$ is contained in $F_\varphi F_U^{\text{sep}} = F_\varphi \tilde{F}_U^{\text{sep}}$, then there is a finite Galois extension E/\tilde{F}_U , say with group G , such that $E_\varphi \subseteq F_\varphi E$. But E and F_φ are linearly disjoint over \tilde{F}_U , since \tilde{F}_U is separably closed in F_φ . So the compositum $F_\varphi E$ is a Galois field extension of F_φ having group G . Let $N = \text{Gal}(F_\varphi E/E_\varphi)$; this is a normal subgroup of index p . The fixed field E^N is then a degree p separable extension of \tilde{F}_U that induces E_φ . This is a contradiction, showing that actually E_φ is not contained in $F_\varphi F_U^{\text{sep}}$. Thus F_φ^{sep} is strictly larger than $F_\varphi F_U^{\text{sep}}$, and hence the map $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_U)$ is not injective. Similarly, using the extension E_P/F_P in Proposition 2.15(b), we deduce that the map $\text{Gal}(F_P) \rightarrow \text{Gal}(F)$ is not injective if $\text{char } k \neq 0$. \square

3.2 Van Kampen's theorem for diamonds

In this section, we prove an analog of van Kampen's theorem in our context. In the situation of Notation 2.8, the simplest case is the one in which $\mathcal{P}, \mathcal{U}, \mathcal{B}$ each contain just one element. That is, the closed fiber X of the normal model \mathcal{X} is irreducible; \mathcal{P} consists of a single unbranched closed point P of X ; the unique element $U \in \mathcal{U}$ is the complement of P in X ; and the element of \mathcal{B} is the unique branch φ of X at P . Thus $F \subset F_P, F_U \subset F_\varphi$, and we have a diamond of fields as at the beginning of Section 2.1. In our result, we express the absolute Galois group of our field F as the amalgamated product of the absolute Galois groups of F_P, F_U over that of F_φ . This parallels the usual form of van Kampen's theorem in topology, which concerns a space $S = S_1 \cup S_2$ with S, S_1, S_2 , and $S_0 := S_1 \cap S_2$ connected, and which expresses the fundamental group of S as the amalgamated product of the fundamental groups of S_1, S_2 over that of S_0 . There, one takes fundamental groups with respect to a common base point in S_0 ; here we will take absolute Galois groups with respect to a chosen separable closure of F_φ and corresponding separable closures of F, F_P, F_U . See Theorem 3.4. Afterwards, in Theorem 3.7 and Theorem 3.9, we prove analogous results that consider more general choices of $\mathcal{P}, \mathcal{U}, \mathcal{B}$.

For the proof of our analog of van Kampen's theorem, it will be convenient to use the language of torsors. Let L be a field with separable closure L^{sep} , and let G be a finite group. As discussed at the beginning of Section 3.1, the group homomorphisms $\phi : \text{Gal}(L) = \text{Gal}(L^{\text{sep}}/L) \rightarrow G$ are in natural bijection with isomorphism classes of pairs consisting of a G -Galois étale L -algebra E together with an L -algebra homomorphism $i : E \rightarrow L^{\text{sep}}$. On the geometric level, $\text{Spec}(E)$ is a G -torsor over L , corresponding to the cohomology class $\sigma \in H^1(L, G)$ of the cocycle $\phi \in \text{Hom}(\text{Gal}(L), G) = Z^1(L, G)$. (Here G is regarded as a constant finite group scheme over L , and so the action of $\text{Gal}(L)$ on G is trivial.) This torsor over L is *geometrically pointed*; i.e., it is equipped with a distinguished L^{sep} -point, corresponding to the L -algebra map $i : E \rightarrow L^{\text{sep}}$. A *morphism* between two geometrically

pointed G -torsors over L consists of a morphism of torsors that carries the first base point to the second. Thus the isomorphism classes of geometrically pointed G -torsors over L are in natural bijection with $\text{Hom}(\text{Gal}(L), G)$. Note that there is at most one morphism (necessarily, an isomorphism) between any two geometrically pointed G -torsors over L , since a morphism of torsors is determined by the image of a given geometric point. In particular, a geometrically pointed G -torsor has no non-trivial automorphisms.

Proposition 3.3. *Let F be a semi-global field, and let X be the closed fiber of a normal model \mathcal{X} . Assume that $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as in Notation 2.8 each consist of a single element, P, U, φ . Let $F^{\text{sep}}, F_P^{\text{sep}}, F_U^{\text{sep}}$ be the separable closures of F, F_P, F_U in a fixed separable closure F_φ^{sep} of F_φ . Let G be a finite group, and consider geometrically pointed G -torsors over F, F_P, F_U, F_φ with respect to the above separable closures. Then base change induces a bijection between*

- (i) *the set of isomorphism classes of geometrically pointed G -torsors over F ; and*
- (ii) *the set of pairs consisting of isomorphism classes of geometrically pointed G -torsors over F_P and over F_U that induce the same isomorphism class over F_φ .*

This proposition is the key step in proving our van Kampen theorem for diamonds:

Theorem 3.4. *Let F be a semi-global field, and assume that $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as in Notation 2.8 each contain just one element. Let $F^{\text{sep}}, F_P^{\text{sep}}, F_U^{\text{sep}}$ be the separable closures of F, F_P, F_U in a fixed separable closure F_φ^{sep} of F_φ . Then*

$$\text{Gal}(F) = \text{Gal}(F_P) *_{\text{Gal}(F_\varphi)} \text{Gal}(F_U),$$

the amalgamated product of $\text{Gal}(F_P)$ with $\text{Gal}(F_U)$ over $\text{Gal}(F_\varphi)$.

Proof. The theorem asserts that $\text{Gal}(F)$ is the direct limit of the directed system consisting of the other three groups, with corresponding diagram

$$\begin{array}{ccc} & \text{Gal}(F_\varphi) & \\ & \swarrow \quad \searrow & \\ \text{Gal}(F_P) & & \text{Gal}(F_U) \\ & \searrow \quad \swarrow & \\ & \text{Gal}(F) & \end{array} \quad . \tag{1}$$

That is, for every finite group G , the natural map of sets

$$\text{Hom}(\text{Gal}(F), G) \rightarrow \text{Hom}(\text{Gal}(F_P), G) \times_{\text{Hom}(\text{Gal}(F_\varphi), G)} \text{Hom}(\text{Gal}(F_U), G)$$

is a bijection. But as noted above, $\text{Hom}(\text{Gal}(F), G)$ is in natural bijection with the set of isomorphism classes of geometrically pointed G -torsors over F , and similarly for F_P, F_U, F_φ . So the assertion follows from Proposition 3.3. \square

Remark 3.5. (a) If one considers schemes rather than function fields, then the analog of Theorem 3.4 holds but not that of Theorem 3.2. More precisely, take a nonempty finite set of closed points of the generic fiber of \mathcal{X} ; let Σ be its closure in \mathcal{X} ; and let $\Sigma_P, \Sigma_U, \Sigma_\varphi$ be the pullbacks of Σ from \mathcal{X} to $\text{Spec}(\widehat{R}_P), \text{Spec}(\widehat{R}_U), \text{Spec}(\widehat{R}_\varphi)$ respectively. Replacing $\text{Gal}(F), \text{Gal}(F_P), \text{Gal}(F_U), \text{Gal}(F_\varphi)$ in Theorem 3.4 by the fundamental groups of $\mathcal{X} \setminus \Sigma, \text{Spec}(\widehat{R}_P) \setminus \Sigma_P, \text{Spec}(\widehat{R}_U) \setminus \Sigma_U, \text{Spec}(\widehat{R}_\varphi) \setminus \Sigma_\varphi$, the analog of Theorem 3.4 holds, by using formal patching (e.g. [Har03, Theorem 3.2.8]) instead of patching over fields. But the analog of Theorem 3.2 fails if we let $\mathcal{X} = \mathbb{P}_{k[[t]]}^1$, $U = \mathbb{A}_k^1$, and $\Sigma = (x = 0)$, with $\text{char}(k) = 0$. Namely, $y^n = x$ defines a branched cover of $\text{Spec}(\widehat{R}_U)$ unramified away from Σ_U , but it is not induced by a branched cover of $\mathcal{X} \setminus \Sigma$. Thus, as in Lemma 3.1, the homomorphism $\pi_1(\text{Spec}(\widehat{R}_U) \setminus \Sigma_U) \rightarrow \pi_1(\mathcal{X} \setminus \Sigma)$ is not injective. Similarly, if P is the point $x = 0$ on the closed fiber, the map $\pi_1(\text{Spec}(\widehat{R}_P) \setminus \Sigma_P) \rightarrow \pi_1(\mathcal{X} \setminus \Sigma)$ is not injective.

(b) The absolute Galois groups $\text{Gal}(F), \text{Gal}(F_P)$, and $\text{Gal}(F_U)$ that arise in Theorem 3.4 have interesting structures. In particular, every finite group is a quotient of each of these profinite groups. In the case of the group $\text{Gal}(F)$, this was shown for $F = K(x)$ in [Har87, Corollary 2.4]; for a more general semi-global field F , the assertion is a special case of [Pop93, Theorem 2.7]. For the groups $\text{Gal}(F_P)$ and $\text{Gal}(F_U)$, the assertion is given by [Lef99, Corollary 3.20]. On the other hand, none of these absolute Galois groups are free; this is because the fields F, F_P, F_U each have cohomological dimension greater than one, and so they are not even projective (see [FJ05, Proposition 11.6.6, Corollary 22.4.6]). The group $\text{Gal}(F_\varphi)$ is also not free, because F_φ is a complete discretely valued field. In general, not every finite group is a quotient of $\text{Gal}(F_\varphi)$. For example, if $K = \mathbb{C}((t))$ and $F = K(x)$, with P being the point $x = 0$ on the closed fiber $\mathbb{P}_{\mathbb{C}}^1$ of \mathbb{P}_K^1 , then F_φ is isomorphic to $\mathbb{C}((x))((t))$, and so the finite quotients of $\text{Gal}(F_\varphi)$ are just the metacyclic groups.

Proposition 3.3 follows from patching of G -torsors (or equivalently, of G -Galois étale algebras; see [HH10, Theorem 7.1]) combined with the fact that a geometrically pointed G -torsor has no non-trivial automorphisms. Below we prove a more general result, Proposition 3.6, which will be used in obtaining variants of Theorem 3.4 in the next section, and which concerns torsors that are equipped with a family of geometric points, rather than just one such point. More precisely, let L be a field and G a finite group. Let $\mathcal{S} = \{L_s\}_{s \in S}$ be a nonempty indexed set of separable closures L_s of L . Define an \mathcal{S} -multipointed G -torsor over L to be a pair $(Z, \{Q_s\}_{s \in S})$, where Z is a G -torsor over L , and $Q_s : \text{Spec}(L_s) \rightarrow Z$ is an L -morphism for each $s \in S$, i.e., an L_s -point of Z . A *morphism* of \mathcal{S} -multipointed G -torsors is a morphism of the underlying torsors that carries the chosen geometric points of the first torsor to the corresponding points of the second. Write $\mathcal{T}_G^{\mathcal{S}}(L)$ for the category of \mathcal{S} -multipointed G -torsors over L . Since the indexed set \mathcal{S} is nonempty, the objects in this category have no non-trivial automorphisms, and between any two objects there is at most one morphism (necessarily an isomorphism); in the terminology of [Stacks, Tag 02XZ], one says that $\mathcal{T}_G^{\mathcal{S}}(L)$ is a *setoid*.

Let F be a semi-global field, and let $\mathcal{P}, \mathcal{U}, \mathcal{B}$ be as in Notation 2.8. For each branch $\wp \in \mathcal{B}$, choose a separable closure F_\wp^{sep} of F_\wp , and let \mathcal{S}_\wp denote the singleton set consisting of F_\wp^{sep} . If a branch \wp lies on $U \in \mathcal{U}$ at $P \in \mathcal{P}$, then we let $F_U^{\text{sep}}(\wp)$ (respectively $F_P^{\text{sep}}(\wp)$) denote the separable closure of F_U (resp. F_P) in F_\wp^{sep} . Let \mathcal{S}_U (respectively \mathcal{S}_P) denote the indexed collection of fields of the form $F_U^{\text{sep}}(\wp)$ (resp. $F_P^{\text{sep}}(\wp)$), where \wp ranges over the branches on U (resp. at P). Finally, let $F^{\text{sep}}(\wp)$ denote the separable closure of F in F_\wp^{sep} and let \mathcal{S} denote the indexed collection of fields $\{F^{\text{sep}}(\wp)\}_{\wp \in \mathcal{B}}$. We define product categories

$$\mathcal{T}_G^{\mathcal{U}} = \prod_{U \in \mathcal{U}} \mathcal{T}_G^{\mathcal{S}_U}(F_U), \quad \mathcal{T}_G^{\mathcal{P}} = \prod_{P \in \mathcal{P}} \mathcal{T}_G^{\mathcal{S}_P}(F_P), \quad \mathcal{T}_G^{\mathcal{B}} = \prod_{\wp \in \mathcal{B}} \mathcal{T}_G^{\mathcal{S}_\wp}(F_\wp).$$

The natural inclusions of fields $F^{\text{sep}}(\wp) \subset F_U^{\text{sep}}(\wp), F_P^{\text{sep}}(\wp) \subset F_\wp^{\text{sep}}$ induce functors

$$\mathcal{T}_G^{\mathcal{S}}(F) \longrightarrow \mathcal{T}_G^{\mathcal{U}}, \mathcal{T}_G^{\mathcal{P}} \longrightarrow \mathcal{T}_G^{\mathcal{B}}.$$

Recall that if $\alpha_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$ are functors ($i = 1, 2$), the 2-fiber product category $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ (with respect to α_1, α_2) is the category whose objects are triples (V_1, V_2, ϕ) consisting of objects $V_i \in \mathcal{C}_i$ and an isomorphism $\phi : \alpha_1(V_1) \rightarrow \alpha_2(V_2)$ in \mathcal{C}_0 . A morphism from (V_1, V_2, ϕ) to (V'_1, V'_2, ϕ') is a pair of morphisms $f_i : V_i \rightarrow V'_i$ in \mathcal{C}_i ($i = 1, 2$) such that $\phi' \circ \alpha_1(f_1) = \alpha_2(f_2) \circ \phi$.

Proposition 3.6. *The above functors induce an equivalence of categories*

$$\mathcal{T}_G^{\mathcal{S}}(F) \rightarrow \mathcal{T}_G^{\mathcal{U}} \times_{\mathcal{T}_G^{\mathcal{B}}} \mathcal{T}_G^{\mathcal{P}},$$

where the right hand side is a 2-fiber product of categories. The isomorphism classes of objects in the category on the right hand side are in natural bijection with the (1-)fiber product of isomorphism classes of objects in the respective categories.

Note that Proposition 3.3 is a special case of the second assertion in this proposition.

Proof. The second assertion follows from the first since all the categories involved are setoids. For the first assertion there are two steps.

Step 1: Essential surjectivity.

An object of the right hand side of the map in the statement of the theorem corresponds to the following data: G -torsors $Z_P = \text{Spec}(E_P), Z_U = \text{Spec}(E_U), Z_\wp = \text{Spec}(E_\wp)$ for all $P \in \mathcal{P}, U \in \mathcal{U}, \wp \in \mathcal{B}$; together with associated points $\zeta_\wp \in Z_\wp(F_\wp^{\text{sep}})$ and $\zeta_\xi(\wp) \in Z_\xi(F_\xi^{\text{sep}}(\wp))$ for \wp a point at (or on) $\xi \in \mathcal{P} \cup \mathcal{U}$; such that for each pair ξ, \wp as above there exists an isomorphism (necessarily unique) of G -torsors $(Z_\xi)_{F_\wp} \rightarrow Z_\wp$ that takes $\zeta_\xi(\wp)$ to ζ_\wp . Thus we obtain a patching problem of G -torsors, or equivalently of G -Galois algebras, which has a solution that is unique up to isomorphism (by [HH10, Theorem 7.1]); viz., a G -torsor $Z = \text{Spec}(E)$ over F that induces each of the torsors Z_P, Z_U, Z_\wp compatibly. If \wp is a branch at P on U , the points $\zeta_{P,\wp}, \zeta_{U,\wp}, \zeta_\wp$ correspond to F -homomorphisms $E_P \rightarrow F_P^{\text{sep}}(\wp) \subset F_\wp^{\text{sep}}, E_U \rightarrow F_U^{\text{sep}}(\wp) \subset F_\wp^{\text{sep}}, E_\wp \rightarrow F_\wp^{\text{sep}}$ such that the first two are restrictions of the third. These homomorphisms thus restrict to a common F -homomorphism $i_\wp : E \rightarrow F_\wp^{\text{sep}}$. Since E is a

finite étale F -algebra, the image of i_φ lies in $F^{\text{sep}}(\varphi)$, the separable closure of F in F_φ^{sep} . It follows that the pair $(Z, \{i_\varphi\}_{\varphi \in \mathcal{B}})$ maps to the isomorphism class of our initially chosen object.

Step 2: Full faithfulness.

Since the categories in question are setoids, we need only check that objects that become isomorphic under our functor were isomorphic to start with.

Consider two objects $(Z, \{i_s\}_{s \in S}), (Z', \{i'_s\}_{s \in S})$ from the left hand side. Since they have isomorphic images, the induced objects on the right hand side are isomorphic as multipointed G -torsors; i.e., for each $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$ there is a (unique) torsor isomorphism $j_\xi : Z_{F_\xi} \rightarrow Z'_{F_\xi}$ that carry the base points of each Z_{F_ξ} to the base points of Z'_{F_ξ} . The j_ξ are compatible, by uniqueness. Hence they define an isomorphism of G -torsor patching problems; and by [HH10, Theorem 7.1], this isomorphism is induced by a unique G -torsor isomorphism $j : Z \rightarrow Z'$. Necessarily, j takes the base points of Z to those of Z' , since j_ξ takes the base points of Z_{F_ξ} to those of Z'_{F_ξ} , and since in each case the geometric points are in bijection with G . So $(Z, \{i_s\}_{s \in S}), (Z', \{i'_s\}_{s \in S})$ are isomorphic as multipointed G -torsors. \square

3.3 Van Kampen's theorem for general reduction graphs

In this section we prove variants on Theorem 3.4 in which the sets $\mathcal{P}, \mathcal{U}, \mathcal{B}$ in Notation 2.8 can consist of more than one element, and so the configuration of fields need not be a diamond. The simplest generalization would assert that $\text{Gal}(F)$ is the direct limit of the system of absolute Galois groups $\text{Gal}(F_\xi)$ for $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$, with respect to homomorphisms $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_P)$ and $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_U)$ whenever φ is a branch at P on U . Here the absolute Galois groups are taken with respect to a suitable choice of separable closures F^{sep} of F and F_ξ^{sep} of F_ξ for $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$. For $\text{Gal}(F)$ to be the direct limit, we would need homomorphisms $\text{Gal}(F_\xi) \rightarrow \text{Gal}(F)$ for all $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$ such that the compositions $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_P) \rightarrow \text{Gal}(F)$ and $\text{Gal}(F_\varphi) \rightarrow \text{Gal}(F_U) \rightarrow \text{Gal}(F)$ agree.

By the discussion at the beginning of Section 3, such group homomorphisms would be induced by a choice of separable closures $F^{\text{sep}}, F_\xi^{\text{sep}}, F_\varphi^{\text{sep}}$ for $\xi \in \mathcal{P} \cup \mathcal{U}, \varphi \in \mathcal{B}$, such that F^{sep} is the separable closure of F in F_ξ^{sep} for every $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$, and F_ξ^{sep} is the separable closure of F_ξ in F_φ^{sep} for every branch $\varphi \in \mathcal{B}$ on (or at) $\xi \in \mathcal{P} \cup \mathcal{U}$. We call this a *compatible system of separable closures*; and it then makes sense to ask whether the van Kampen assertion holds.

The next result provides a necessary and sufficient condition for such a generalized van Kampen theorem to hold, in terms of the reduction graph associated to the sets $\mathcal{P}, \mathcal{U}, \mathcal{B}$. (As in [HHK14, Section 2.1.1], the *reduction graph* associated to these sets is the connected bipartite graph whose vertices are the elements of $\mathcal{P} \cup \mathcal{U}$ and whose edges are the elements of \mathcal{B} , where an edge φ connects two vertices P, U if φ is a branch at P lying on U .)

Theorem 3.7. *Let \mathcal{X} be a normal model for a semi-global field F , and let $\mathcal{P}, \mathcal{U}, \mathcal{B}$ be as in Notation 2.8. Then the following conditions are equivalent:*

- (i) *The associated reduction graph is a tree.*
- (ii) *There is a compatible system of separable closures of the fields F and F_ξ for $\xi \in \mathcal{P}, \mathcal{U}, \mathcal{B}$.*

(iii) $\text{Gal}(F)$ is the direct limit of the groups $\text{Gal}(F_\xi)$ for $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$ with respect to some compatible system of separable closures.

Under these equivalent conditions, $\text{Gal}(F)$ is the direct limit of the groups $\text{Gal}(F_\xi)$ for $\xi \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{B}$ with respect to **any** given compatible system of separable closures.

Proof. Throughout this proof, the letter Γ will denote the associated reduction graph. To show that (i) implies (ii), we construct these separable closures inductively, using that the reduction graph Γ is a tree. Namely, consider a subtree \mathcal{T} of Γ , let $\xi_0 \in \mathcal{P} \cup \mathcal{U}$ be a terminal vertex of \mathcal{T} , and let $\varphi_0 \in \mathcal{B}$ be the edge connecting ξ_0 to the rest of \mathcal{T} , which we call \mathcal{T}' . Suppose that we have compatible separable closures F_ξ^{sep} associated to the vertices and edges ξ of \mathcal{T}' . Let $\xi_1 \in \mathcal{P} \cup \mathcal{U}$ be the other vertex of the edge φ_0 ; this is a vertex of \mathcal{T}' . Choose a separable closure $F_{\varphi_0}^{\text{sep}}$ of F_{ξ_0} that contains $F_{\xi_1}^{\text{sep}}$. (For example, take any separable closure of the field $(F_{\varphi_0} \otimes_{F_{\xi_1}} F_{\xi_1}^{\text{sep}})/\mathfrak{m}$, where \mathfrak{m} is a maximal ideal.) Then take $F_{\xi_0}^{\text{sep}}$ to be the separable closure of F_{ξ_0} in $F_{\varphi_0}^{\text{sep}}$. Since $F_{\xi_1}^{\text{sep}}$ is the separable closure of F_{ξ_1} in $F_{\varphi_0}^{\text{sep}}$, it follows that the separable closure of F in $F_{\varphi_0}^{\text{sep}}$ is the same as the separable closure of F in $F_{\xi_1}^{\text{sep}}$. But the latter is the same as the separable closure of F in F_φ^{sep} , where $\varphi \in \mathcal{B}$ is any branch at (or on) ξ_1 , and that field is F^{sep} . So this system of separable closures on the vertices and edges of \mathcal{T} is compatible, thus completing the induction, and showing that (i) implies (ii).

Next, we show that with respect to any given compatible system of separable closures F^{sep} and F_ξ^{sep} , $\text{Gal}(F)$ is the direct limit of the groups $\text{Gal}(F_\xi)$. This will show that (ii) implies both (iii) and the stronger condition in the last part of the assertion. For this, note that given this compatible system, the construction preceding Proposition 3.6 yields indexed sets $\mathcal{S}_P, \mathcal{S}_U, \mathcal{S}$ of separable closures of F_P, F_U, F respectively, for $P \in \mathcal{P}$ and $U \in \mathcal{U}$. For any given $\xi \in \mathcal{P} \cup \mathcal{U}$, the fields $F_\xi^{\text{sep}}(\varphi)$ in the indexed set \mathcal{S}_ξ are each just the field F_ξ^{sep} of the previous paragraph; and similarly all of the fields in the indexed set \mathcal{S} are just the above field F^{sep} . A geometrically pointed G -torsor over F (with respect to F^{sep}) is the same as an \mathcal{S} -pointed G -torsor $(Z, \{Q_s\}_{s \in \mathcal{S}})$ such that the morphisms $Q_s : \text{Spec}(F^{\text{sep}}(\varphi)) \rightarrow Z$ are all the same morphism $\text{Spec}(F^{\text{sep}}) \rightarrow Z$. The corresponding statements hold for each F_P, F_U, F_φ . Let $\tilde{\mathcal{T}}_G^{\mathcal{S}}(F)$ denote the full subcategory of $\mathcal{T}_G^{\mathcal{S}}(F)$ whose objects are the geometrically pointed G -torsors over F under the above identification. Similarly, we define full subcategories $\tilde{\mathcal{T}}_G^{\mathcal{U}}, \tilde{\mathcal{T}}_G^{\mathcal{P}},$ and $\tilde{\mathcal{T}}_G^{\mathcal{B}}$ of the product categories $\mathcal{T}_G^{\mathcal{U}}, \mathcal{T}_G^{\mathcal{P}},$ and $\mathcal{T}_G^{\mathcal{B}}$ defined in the discussion leading up to Proposition 3.6. Under the equivalence of categories given in Proposition 3.6, an object in $\mathcal{T}_G^{\mathcal{S}}(F)$ is sent to an object in $\tilde{\mathcal{T}}_G^{\mathcal{U}} \times_{\tilde{\mathcal{T}}_G^{\mathcal{B}}} \tilde{\mathcal{T}}_G^{\mathcal{P}}$ if and only if it is an object in $\tilde{\mathcal{T}}_G^{\mathcal{S}}(F)$, since the reduction graph is connected. Thus the equivalence of categories in Proposition 3.6 restricts to an equivalence $\tilde{\mathcal{T}}_G^{\mathcal{S}}(F) \rightarrow \tilde{\mathcal{T}}_G^{\mathcal{U}} \times_{\tilde{\mathcal{T}}_G^{\mathcal{B}}} \tilde{\mathcal{T}}_G^{\mathcal{P}}$. As in Proposition 3.6, isomorphism classes of objects in $\tilde{\mathcal{T}}_G^{\mathcal{U}} \times_{\tilde{\mathcal{T}}_G^{\mathcal{B}}} \tilde{\mathcal{T}}_G^{\mathcal{P}}$ are in natural bijection with the (1-)fiber product of isomorphism classes of objects in the respective categories. So the desired assertion now follows from the natural bijection between $\text{Hom}(\text{Gal}(F), G)$ and isomorphism classes of geometrically pointed G -torsors.

It remains to show that (iii) implies (i). Suppose to the contrary that the reduction graph Γ is not a tree. We claim that there is a non-trivial finite Galois field extension E/F with Galois group G that induces the trivial extension over each F_P and each F_U , for $P \in \mathcal{P}$

and $U \in \mathcal{U}$. Once this is shown, the corresponding non-trivial map $\text{Gal}(F) \rightarrow G$ induces the trivial maps $\text{Gal}(F_\xi) \rightarrow G$ for $\xi \in \mathcal{P} \cup \mathcal{U}$. But those trivial maps are also induced by the trivial map $\text{Gal}(F) \rightarrow G$. So $\text{Gal}(F)$ does not have the universal property for direct limits. This shows that it suffices to prove the claim.

To do this, first assume that the set \mathcal{P} contains all the points where the closed fiber X is not unibranched. By [HHK15a, Proposition 6.2], since Γ is not a tree (and thus has a non-trivial covering space), there exists a non-trivial finite connected split cover $\mathcal{Y} \rightarrow \mathcal{X}$. Let E/F be the corresponding finite separable field extension. By [HHK15a, Corollary 5.5], this split cover induces trivial extensions of each F_ξ , for $\xi \in \mathcal{P} \cup \mathcal{U}$; hence E/F satisfies the conditions of the claim.

If we do not make the above assumption on the set \mathcal{P} , then the proof of [HHK15a, Proposition 6.2] still shows that the finite connected covering spaces of Γ are in bijection with the split covers of \mathcal{X} that induce trivial extensions of each F_ξ . So again, since Γ has non-trivial covering spaces, there exists a non-trivial finite connected split cover $\mathcal{Y} \rightarrow \mathcal{X}$ that is trivial over each F_ξ ; and the corresponding field extension E/F again satisfies the conditions of the claim. \square

Thus, in the context of Notation 2.8, the absolute Galois group of F is the direct limit of the absolute Galois groups of the fields F_P, F_U, F_φ with respect to a compatible system of separable closures if the reduction graph is a tree, but not otherwise. We now consider the case where the reduction graph is not a tree, and state a van Kampen theorem in terms of groupoids rather than groups, to avoid this limitation. This assertion parallels a result in topology that generalizes the usual van Kampen theorem to the case in which the intersection is allowed to be disconnected, doing so in terms of groupoids (see [Br06], Section 6.7). Our approach here is also motivated by the use of fundamental groupoids of schemes in [Gr71, V.7]; there, as in the topological context, one uses a collection of base points, rather than just one point. In this way we can avoid the problem that in general there is no compatible system of separable closures (cf. Theorem 3.7).

Recall that a *groupoid* is a category in which every homomorphism is an isomorphism. Groups can be viewed as groupoids, by associating to each group G the groupoid BG consisting of one object, and with the morphisms corresponding to the elements of G . If L is a field, and $\mathcal{S} = \{L_s\}_{s \in S}$ is a nonempty indexed set of separable closures of L , then we may consider the *absolute Galois groupoid* $\pi_1(L, \mathcal{S})$ of L with respect to \mathcal{S} . Its objects are the elements of S , and its morphisms are isomorphisms between the corresponding separable closures of L . (Here $\pi_1(L, \mathcal{S})$ can be viewed as the fundamental groupoid $\pi_1(\text{Spec}(L), \mathcal{S})$ of $\text{Spec}(L)$ with respect to the indexed set of base points \mathcal{S} ; cf. [Gr71, V.7] and [Br06, Section 6.7].) This groupoid is small (i.e., is a small category) since S is a set.

If L^{sep} is a separable closure of a field L , and we write $\text{Gal}(L) = \text{Gal}(L^{\text{sep}}/L)$, then for each finite group G we have a natural bijection between the objects of $\text{Hom}(B\text{Gal}(L), BG)$ and the elements of $\text{Hom}(\text{Gal}(L), G)$. As discussed at the beginning of Section 3.2, this latter set is in natural bijection with the set of geometrically pointed G -torsors over L ; i.e., the set of isomorphism classes of pairs (Z, i) , where $Z = \text{Spec}(E)$ is a G -torsor over L and $i : E \rightarrow L^{\text{sep}}$ is an L -algebra map, and where i corresponds to a choice of a distinguished L^{sep} -point on

Z . This bijection can be extended to the context of groupoids, using multipointed torsors (with notation as in the discussion leading up to Proposition 3.6):

Lemma 3.8. *Let L be a field, $\mathcal{S} = \{L_s\}_{s \in S}$ an indexed set of separable closures of L , and G a finite group. Then the set $\text{Hom}(\pi_1(L, \mathcal{S}), BG)$ is in natural bijection with the set of isomorphism classes of \mathcal{S} -multipointed G -torsors $(Z, \{i_s\}_{s \in S})$ in $\mathcal{T}_G^{\mathcal{S}}(L)$.*

Proof. First, consider the isomorphism class of $(Z, \{i_s\}_{s \in S})$ as above, where $Z = \text{Spec}(E)$. For each $s \in S$, let $\zeta_s \in Z(L_s)$ be the L_s -point corresponding to $i_s : E \rightarrow L_s$. For $s \in S$, the restricted multipointed torsor $(Z, \{i_s\})$ defines an element $f_s \in \text{Hom}(\text{Gal}(L_s/L), G) = \text{Hom}(\pi_1(L, \{L_s\}), BG)$. Given $s, s' \in S$, each L -algebra isomorphism $\alpha : L_{s'} \rightarrow L_s$ induces a bijection $Z(\alpha) : Z(L_{s'}) \rightarrow Z(L_s)$; and there is a unique $g_\alpha \in G$ such that $Z(\alpha)(\zeta_{s'}) = \zeta_s \cdot g_\alpha$. Note that $g_\alpha = f_s(\alpha)$ if $s' = s$. It is then straightforward to check that there is a morphism $f \in \text{Hom}(\pi_1(L, \mathcal{S}), BG)$ given by $f(\alpha) = g_\alpha$ as above for all s, s' and $\alpha : L_{s'} \rightarrow L_s$; and that for $s \in S$, the restriction of f to $\text{Gal}(L_s/L)$ is f_s . This defines one direction of the bijection.

For the opposite direction, we begin by picking some $s_0 \in S$; and for every $s \in S$ we pick an L -algebra isomorphism $\alpha_s : L_{s_0} \rightarrow L_s$, with α_{s_0} being the identity automorphism of L_{s_0} . This induces a conjugation map $c_{\alpha_s} : \text{Gal}(L_{s_0}/L) \rightarrow \text{Gal}(L_s/L)$, sending σ to $\alpha_s \sigma \alpha_s^{-1}$. Say $f \in \text{Hom}(\pi_1(L, \mathcal{S}), BG)$. Then for every $s \in S$, f restricts to an element $f_s \in \text{Hom}(\pi_1(L, \{L_s\}), BG) = \text{Hom}(\text{Gal}(L_s/L), G)$, corresponding to the isomorphism class of a G -torsor $Z_s = \text{Spec}(E_s)$ over L together with an L -homomorphism $i_s : E_s \rightarrow L_s$; here i_s corresponds to an L_s -point ζ_s on Z_s . Write $Z = Z_{s_0}$, $E = E_{s_0}$, and $\zeta = \zeta_{s_0}$, and for each s let $Z(\alpha_s) : Z(L_{s_0}) \rightarrow Z(L_s)$ be the map induced by α_s . Since f is a morphism, the two maps $f_{s_0}, f_s c_{\alpha_s} \in \text{Hom}(\pi_1(L, \{L_{s_0}\}), BG) = \text{Hom}(\text{Gal}(L_{s_0}/L), G)$ differ by conjugation by $f(\alpha_s) \in G$. So there is a unique isomorphism $\alpha_{s*} : Z \rightarrow Z_s$ of G -torsors over L that carries $Z(\alpha_s)(\zeta) \in Z(L_s)$ to $\zeta_s \cdot f(\alpha_s) \in Z_s(L_s)$. We then obtain a multipointed torsor $(Z, \{i_s\}_{s \in S})$, where $i_s : E \rightarrow L_s$ is the homomorphism corresponding to the L_s -point $\alpha_{s*}^{-1}(\zeta_s) = Z(\alpha_s)(\zeta) \cdot f(\alpha_s)^{-1}$ on Z . In this way, for each f we obtain the isomorphism class of a multipointed torsor $(Z, \{i_s\}_{s \in S})$. It is straightforward to check that this association is independent of the choices of s_0 and α_s , and is inverse to the one in the previous paragraph. \square

Note that since there is at most one morphism between any two objects of $\mathcal{T}_G^{\mathcal{S}}(L)$, the above lemma yields an equivalence of categories $\text{Hom}(\pi_1(L, \mathcal{S}), BG) \cong \mathcal{T}_G^{\mathcal{S}}(L)$, if we regard the set $\text{Hom}(\pi_1(L, \mathcal{S}), BG)$ as a category with all arrows being identities.

In the context of the discussion leading up to Proposition 3.6, we may take the disjoint union groupoid $\coprod_{P \in \mathcal{P}} \pi_1(F_P, \mathcal{S}_P)$, whose objects and morphisms are the disjoint unions of the objects and morphisms of the groupoids $\pi_1(F_P, \mathcal{S}_P)$, for $P \in \mathcal{P}$. Similarly we may take $\coprod_{U \in \mathcal{U}} \pi_1(F_U, \mathcal{S}_U)$ and $\coprod_{\varphi \in \mathcal{B}} \pi_1(F_\varphi, \mathcal{S}_\varphi)$. We then have a commutative diagram of groupoids, which generalizes diagram (1) in Section 3.2, and in which the arrows induce bijections on

the (finite) sets of objects of the four categories:

$$\begin{array}{ccc}
 & \coprod \pi_1(F_\varphi, \mathcal{S}_\varphi) & \\
 \swarrow & & \searrow \\
 \coprod \pi_1(F_P, \mathcal{S}_P) & & \coprod \pi_1(F_U, \mathcal{S}_U) \\
 \searrow & & \swarrow \\
 & \pi_1(F, \mathcal{S}) &
 \end{array} \tag{2}$$

Here, the commutativity assertion is that the two vertical compositions give the same (not just equivalent) maps on objects, and on morphisms.

We now obtain a van Kampen-type theorem in terms of groupoids, which generalizes Theorem 3.4, and parallels the topological van Kampen result [Br06, 6.7.2] for groupoids:

Theorem 3.9. *The above diamond is a pushout diagram of small groupoids, in the sense that for every small groupoid \mathcal{G} , the natural map of sets*

$$\text{Hom}(\pi_1(F, \mathcal{S}), \mathcal{G}) \rightarrow \text{Hom}\left(\coprod \pi_1(F_P, \mathcal{S}_P), \mathcal{G}\right) \times_{\text{Hom}(\coprod \pi_1(F_\varphi, \mathcal{S}_\varphi), \mathcal{G})} \text{Hom}\left(\coprod \pi_1(F_U, \mathcal{S}_U), \mathcal{G}\right)$$

is a bijection. For any element of S , corresponding to a separable closure F^{sep} of F , the absolute Galois group $\text{Gal}(F^{\text{sep}}/F)$ of F is the automorphism group of that object in this groupoid.

Proof. The last assertion is immediate from the main assertion. The proof of the main assertion begins with several reduction steps.

First note that the category $\pi_1(F, \mathcal{S})$ is connected (viz. there is a morphism between each pair of objects in this category), because any two separable closures of a field L are L -isomorphic. So we may assume that \mathcal{G} is connected, by treating each connected component separately.

Second, we reduce to the case that \mathcal{G} has just one object, i.e., it is of the form BG for some group G . Pick an object t_0 in \mathcal{G} , and for every object t in \mathcal{G} pick an isomorphism $j_t : t_0 \rightarrow t$, with j_{t_0} being the identity on t_0 . Let $G = \text{Aut}(t_0)$, so that BG is the full subcategory of \mathcal{G} whose unique object is t_0 . Define a functor $J : \mathcal{G} \rightarrow BG$ by taking every object in \mathcal{G} to the object t_0 of BG , and taking every morphism $\alpha \in \text{Hom}(t, t')$ in \mathcal{G} to the morphism $j_{t'}^{-1} \alpha j_t \in \text{End}(t_0)$ in BG . Now given an element (ϕ_P, ϕ_U) in the right hand side of the above map of sets, by composing with J we obtain an element $(J\phi_P, J\phi_U)$ in

$$\text{Hom}\left(\coprod \pi_1(F_P, \mathcal{S}_P), BG\right) \times_{\text{Hom}(\coprod \pi_1(F_\varphi, \mathcal{S}_\varphi), BG)} \text{Hom}\left(\coprod \pi_1(F_U, \mathcal{S}_U), BG\right).$$

Once we prove the result for maps to groupoids that have just one object, we have that there is a unique $\tilde{\phi} \in \text{Hom}(\pi_1(F, \mathcal{S}), BG)$ that induces $(J\phi_P, J\phi_U)$. Define the functor $\phi : \pi_1(F, \mathcal{S}) \rightarrow \mathcal{G}$ by taking each object $\varphi \in \mathcal{B} = \text{Obj}(\pi_1(F, \mathcal{S}))$ to $\phi_P(\varphi) = \phi_U(\varphi)$, for φ a branch at P on U (so that φ is also an object in $\pi_1(F_P, \mathcal{S}_P)$ and in $\pi_1(F_U, \mathcal{S}_U)$, which allows us to apply ϕ_P and ϕ_U); and taking each morphism $\alpha : \varphi \rightarrow \varphi'$ (i.e., each F -algebra

isomorphism $\alpha : F^{\text{sep}}(\wp) \rightarrow F^{\text{sep}}(\wp')$ to $j_{\phi(\wp')} \tilde{\phi}(\alpha) j_{\phi(\wp)}^{-1} : \phi(\wp) \rightarrow \phi(\wp')$. Then ϕ is the unique element in $\text{Hom}(\pi_1(F, \mathcal{S}), \mathcal{G})$ that maps to $(\phi_{\mathcal{P}}, \phi_{\mathcal{U}})$. This establishes the desired bijection and completes this reduction step.

Third, since the set of objects in each of the groupoids in the diamond is finite, and since the automorphism group of each object is profinite, it suffices to prove the result in the case that $\mathcal{G} = BG$ for G a finite group. We now assume that we are in that case.

By Lemma 3.8, we may identify

$$\text{Hom}\left(\coprod \pi_1(F_P, \mathcal{S}_P), BG\right) = \prod \text{Hom}(\pi_1(F_P, \mathcal{S}_P), BG)$$

with the set of isomorphism classes of objects in $\mathcal{T}_G^{\mathcal{P}}$, and similarly for \mathcal{B}, \mathcal{U} ; and we may identify $\text{Hom}(\pi_1(F, \mathcal{S}), BG)$ with the set of isomorphism classes in $\mathcal{T}_G^{\mathcal{S}}(F)$. The result therefore follows from Proposition 3.6. \square

Following [Sti06], we can also describe the absolute Galois group $\text{Gal}(F^{\text{sep}}/F)$ of F more explicitly, by making a choice of maximal tree \mathcal{T} in the reduction graph Γ of $(\mathcal{X}, \mathcal{P})$. The vertices of \mathcal{T} are the same as those of Γ , and are indexed by $\mathcal{P} \cup \mathcal{U}$. For any two vertices v_1, v_2 of Γ , there is a unique minimal path in \mathcal{T} from v_1 to v_2 , and this provides an isomorphism between the fundamental groups $\pi_1(\Gamma, v_i)$ for $i = 1, 2$. These groups can also be identified with the fundamental group of Γ with respect to \mathcal{T} as a “base point”; or equivalently, the fundamental group of the graph Γ/\mathcal{T} obtained from Γ by contracting \mathcal{T} to a single vertex. The graph Γ/\mathcal{T} has just one vertex, and its edges are in bijection with the edges of Γ that do not lie in \mathcal{T} . This fundamental group is thus free of finite rank, with generators e_{\wp} indexed by those branches $\wp \in \mathcal{B}$ that correspond to the edges of Γ/\mathcal{T} . In this situation, Corollary 3.3 of [Sti06] gives:

Proposition 3.10. *Let F be a semi-global field, and let $\mathcal{P}, \mathcal{U}, \mathcal{B}$ be as in Notation 2.8. For each $\xi \in \mathcal{P} \cup \mathcal{U}$ and a branch $\wp \in \mathcal{B}$ at ξ , choose an inclusion $\bar{j}_{\xi, \wp} : F_{\xi}^{\text{sep}} \hookrightarrow F_{\wp}^{\text{sep}}$ extending the given inclusions $j_{\xi, \wp} : F_{\xi} \subset F_{\wp}$, and inducing homomorphisms $\alpha_{\wp, \xi} : \text{Gal}(F_{\wp}^{\text{sep}}/F_{\wp}) =: \text{Gal}(F_{\wp}) \rightarrow \text{Gal}(F_{\xi}^{\text{sep}}/F_{\xi}) =: \text{Gal}(F_{\xi})$. Choose a maximal tree \mathcal{T} in the reduction graph Γ ; thus the profinite completion $\hat{\pi}_1(\Gamma, \mathcal{T})$ is the free profinite group with generators e_{\wp} indexed by the edges of Γ that do not lie in \mathcal{T} . Let $e_{\wp} = 1 \in \hat{\pi}_1(\Gamma, \mathcal{T})$ for each $\wp \in \mathcal{B}$ that is an edge of \mathcal{T} . Then the absolute Galois group of F is isomorphic to the quotient of the free product $\ast_{\xi \in \mathcal{P} \cup \mathcal{U}} \text{Gal}(F_{\xi}) \ast \hat{\pi}_1(\Gamma, \mathcal{T})$ by the relations $\alpha_{\wp, U}(g) = e_{\wp} \alpha_{\wp, P}(g) e_{\wp}^{-1}$ for all triples P, U, \wp where $\wp \in \mathcal{B}$ is a branch at $P \in \mathcal{P}$ on $U \in \mathcal{U}$, and all $g \in \text{Gal}(F_{\wp})$.*

Note that the situation considered in [Sti06] involved a connected simplicial complex of dimension at most two with associated groups and group homomorphisms associated to boundary maps. This abstract situation is applied there to categories with descent data. But while some descent categories involve self-intersections (e.g. $U \times_X U \rightarrow X$ for the étale topology on X), patching provides a descent context without such self-intersections. As a result, the abstract framework described in [Sti06] simplifies in our situation, and it suffices to consider one-dimensional simplicial complexes, viz. graphs, as above.

Note also that if Γ is a tree, and if compatible separable closures $F^{\text{sep}}, F_{\xi}^{\text{sep}}, F_{\varnothing}^{\text{sep}}$ are chosen as in Theorem 3.7, then the above description of $\text{Gal}(F)$ simplifies to the description given there.

It would be interesting to show that Proposition 3.10 can be deduced from Theorem 3.9.

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