LOCAL-GLOBAL PRINCIPLES FOR GALOIS COHOMOLOGY

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Abstract. This paper proves local-global principles for Galois cohomology groups over function fields F of curves that are defined over a complete discretely valued field. We show in particular that such principles hold for $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$, for all $n > 1$. This is motivated by work of Kato and others, where such principles were shown in related cases for $n = 3$. Using our results in combination with cohomological invariants, we obtain local-global principles for torsors and related algebraic structures over F. Our arguments rely on ideas from patching as well as the Bloch-Kato conjecture.

1. Introduction

In this paper we present local-global principles for Galois cohomology, which may be viewed as higher-dimensional generalizations of classical local-global principles for the Brauer group. These results then lead to local-global principles for other algebraic structures as well, via cohomological invariants.

Recall that if F is a global field, the theorem of Albert-Brauer-Hasse-Noether says a central simple F-algebra is isomorphic to a matrix algebra if and only if this is true over each completion F_v of F. Equivalently, the natural group homomorphism

$$\mathrm{Br}(F) \to \prod_{v \in \Omega_F} \mathrm{Br}(F_v)$$

is injective, where Ω_F is the set of places of F.

Kato suggested a higher dimensional generalization of this in [Kat86], drawing on the observation that the above result provides a local-global principle for the m-torsion part of the Brauer group $\mathrm{Br}(F)[m] = H^2(F, \mathbb{Z}/m\mathbb{Z}(1))$. (Here $\mathbb{Z}/m\mathbb{Z}(n)$ denotes $\mu_m^{\otimes n}$, for m not divisible by $\mathrm{char}(F)$.) He proposed that the natural domain for higher-dimensional versions of local-global principles should be $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$, for $n > 1$. Cohomological invariants (such as the Rost invariant) often take values in $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$ for some $n > 1$; and thus such local-global principles for cohomology could be used to obtain local-global principles for other algebraic objects.

In Theorem 0.8(1) of [Kat86], Kato proved such a principle with $n = 3$ for the function field F of a smooth proper surface X over a finite field, both with respect to the discrete

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valuations on F that arise from codimension one points on X, and alternatively with respect to the set of closed points of X (in the latter case using the fraction fields of the complete local rings at the points). He also proved a related result [Kat86, Theorem 0.8(3)] for arithmetic surfaces, i.e. for curves over rings of integers of number fields. The corresponding assertions for $n > 3$ are vacuous in his situation, for cohomological dimension reasons; and the analogs for $n = 2$ do not hold there (e.g. if the unramified Brauer group of the surface is non-trivial).

Unlike the classical case of dimension one, in dimension two it is also meaningful to consider local-global principles for fields that are not global, e.g. $k((x, y))$ or $k((t))(x)$. In [COP02, Theorem 3.8], the authors start with an irreducible surface over a finite field of characteristic not dividing m; and they take the fraction field F of the henselization of the local ring at a closed point. In that situation, they prove a local-global principle for $H^3(F, \mathbb{Z}/m\mathbb{Z}(2))$ with respect to the discrete valuations on F. Also, while not explicitly said in [Kat86], it is possible to use Theorem 5.2 of that paper to obtain a local-global principle for function fields F of curves over a non-archimedean local field, with respect to $H^3(F, \mathbb{Z}/m\mathbb{Z}(2))$. This was relied on in [CPS12, Theorem 5.4] and [Hu12] (cf. also [PS98, pp. 139 and 148]).

1.1. **Results**

In this manuscript, we show that when F is the function field of a curve over an *arbitrary* complete discretely valued field K, local-global principles hold for the cohomology groups $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$ for *all* $n > 1$.

In particular we obtain the following local-global principle with respect to points on the closed fiber X of a model $\widehat{X}$ of F over the valuation ring of K (where k is the residue field):

Theorem (3.2.3)**.** *Let $n > 1$ and let A be one of the following algebraic groups over F:*

(i) $\mathbb{Z}/m\mathbb{Z}(n-1)$, where m is not divisible by the characteristic of k, or

(ii) $\mathbb{G}_{\mathrm{m}}$, if $\mathrm{char}(k) = 0$ and K contains a primitive m-th root of unity for all $m \geqslant 1$.

Then the natural map

$$H^n(F, A) \to \prod_{P \in X} H^n(F_P, A)$$

is injective, where P ranges through all the points of the closed fiber X.

Here F_P denotes the fraction field of the complete local ring of $\widehat{X}$ at P.

We also obtain a local-global principle with respect to discrete valuations if K is equicharacteristic:

Theorem (3.3.6)**.** *Suppose that K is an equicharacteristic complete discretely valued field of characteristic not dividing m, and that $\widehat{X}$ is a regular projective T-curve with function field F. Let $n > 1$. Then the natural map*

$$H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1)) \to \prod_{v \in \Omega_{\widehat{X}}} H^n(F_v, \mathbb{Z}/m\mathbb{Z}(n-1))$$

is injective.

Here $\Omega_{\widehat{X}}$ is the set of discrete valuations on F that arise from codimension one points on $\widehat{X}$. Also, in the above results and henceforth, the cohomology that is used is Galois

cohomology, where $H^n(F, A) = H^n(\mathrm{Gal}(F), A(F^{\mathrm{sep}}))$ for A a smooth commutative group scheme over F and $n \geqslant 0$, with $H^0(F, A) = A(F)$. (For non-commutative group schemes, we similarly have H^0 and H^1.)

These results also yield new local-global principles for torsors under linear algebraic groups by the use of cohomological invariants such as the Rost invariant ([GMS03, p. 129]), following a strategy used in [CPS12] and [Hu12]. We list some of these applications of our local-global principles in Section 4. Note that although we also obtained certain local-global principles for torsors for linear algebraic groups in [HHK11], the results presented here use a different set of hypotheses on the group. In particular, here we do not require that the group G be rational, unlike in [HHK11].

1.2. **Methods and structure of the manuscript**

Our approach to obtaining these local-global principles uses the framework of patching over fields, as in [HH10], [HHK09], and [HHK11]. The innovation is that these principles derive from long exact Mayer-Vietoris type sequences with respect to the "patches" that arise in this framework. These sequences are analogous to those in [HHK11] for linear algebraic groups that were not necessarily commutative (but where only H^0 and H^1 were considered for that reason).

In Section 2, we derive Mayer-Vietoris sequences and local-global principles in an abstract context of a field together with a finite collection of overfields (Section 2.5). This allows us to isolate the necessary combinatorial, group-theoretic, and cohomological properties of our fields and Galois modules. The combinatorial data of the collection of fields we use is encoded in the notion of a Γ-field; see Section 2.1. The key group-theoretic property of our Galois modules is "separable factorization", introduced in Section 2.2. The cohomological properties we require are formulated in the concept of global domination of Galois cohomology (Sections 2.3 and 2.4). An essential ingredient in our arguments is the Bloch-Kato conjecture.

In Section 3, we apply our results to the situation of a function field over a complete discretely valued field. In Section 3.1 we obtain a local-global principle with respect to "patches." This is used in Section 3.2 to obtain a local-global principle with respect to points on the closed fiber of a regular model. Finally, in Section 3.3, we obtain our local-global principle with respect to discrete valuations with the help of a result of Panin [Pan03] for local rings in the context of Bloch-Ogus theory. This step is related to ideas used in [Kat86].

In Section 4, we combine our local-global principles with cohomological invariants taking values in $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$, to obtain our applications to other algebraic structures.

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2. Patching and local-global principles for cohomology

This section considers patching and local-global principles for cohomology in an abstract algebraic setting, in which we are given a field and a finite collection of overfields indexed

by a graph. The results here will afterwards be applied to a geometric setting in Section 3, where we will consider curves over a complete discretely valued field.

In the situation here, we will obtain a new long exact sequence for Galois cohomology with respect to the given field and its overfields, which in a key special case can be interpreted as a Mayer-Vietoris sequence. In [HHK11, Theorem 2.4], we obtained such a sequence for linear algebraic groups that need not be commutative. Due to the lack of commutativity, the assertion there was just for H^0 and H^1; and that result was then used in [HHK11] to obtain local-global principles for torsors in a more geometric context. In the present paper, we consider commutative linear algebraic groups, and so higher cohomology groups H^n are defined. It is for these that we prove our long exact sequence, which we then use to obtain a local-global principle for Galois cohomology in the key case of $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$ with $n > 1$. This is carried out in Sections 2.5 and 3.1. (Note that the six-term cohomology sequence in [HHK11, Theorem 2.4] is used in our arguments here, in the proofs of Theorems 2.1.5 and 2.2.4.)

2.1. Γ-Fields and patching

Our local-global principles will be obtained by an approach that *formally* emulates the notion of a cover of a topological space by a collection of open sets, in the special case that there are no nontrivial triple overlaps. In this case, one may ask to what extent one may derive global information from local information with respect to the sets in the open cover. We encode this setup combinatorically in the form of a graph whose vertices correspond to the connected open sets in the cover and whose edges correspond to the connected components of the overlaps (though we do not introduce an associated topological space or Grothendieck topology).

In our setting the global space will correspond to a field F whose arithmetic we would like to understand, and the open sets and overlaps correspond to field extensions of F. This setup is formalized in the definitions below, which draw on terminology in [HH10] and [HHK11].

2.1.1. *Graphs and* Γ*-fields*

By a *graph* Γ, we will always mean a finite multigraph, with a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$; i.e. we will permit more than one edge to connect a pair of vertices. But we will not permit loops at a vertex: the two endpoints of an edge are required to be distinct vertices.

By an *orientation* on Γ we will mean a choice of labeling of the vertices of each edge $e \in \mathcal{E}$, with one chosen to be called the *left vertex* $l(e)$ and the other the *right vertex* $r(e)$ of e. This choice can depend on the edge (i.e. a vertex v can be the right vertex for one edge at v, and the left vertex for another edge at v).

Definition 2.1.1. Let Γ be a graph. A Γ*-field* $F_\bullet$ consists of the following data:

(1) For each $v \in \mathcal{V}$, a field F_v,
(2) For each $e \in \mathcal{E}$, a field F_e,
(3) An injection $\iota_v^e : F_v \to F_e$ whenever v is a vertex of the edge e.

Often we will regard ι_v^e as an inclusion, and not write it explicitly in the notation if the meaning is clear.

A Γ-field $F_\bullet$ can also be interpreted as an inverse system of fields. Namely, the index set of the inverse system is the disjoint union $\mathcal{V} \sqcup \mathcal{E}$; and the maps consist of inclusions of fields $\iota_v^e : F_v \hookrightarrow F_e$ as above.

Conversely, consider any finite inverse system of fields whose index set can be partitioned into two subsets $\mathcal{V} \sqcup \mathcal{E}$, such that for each $e \in \mathcal{E}$ there are exactly two elements in $v, v' \in \mathcal{V}$ having maps $F_v \hookrightarrow F_e$ and $F_{v'} \hookrightarrow F_e$ in the inverse system; and such that there are no other maps in the inverse system. Then such an inverse system of fields, called a *factorization inverse system* in [HHK11, Section 2], gives rise to a graph Γ and a Γ-field $F_\bullet$ as above.

Given a Γ-field $F_\bullet$, we may consider the inverse limit F_Γ of the fields in $F_\bullet$, with respect to the associated inverse system, in the category of rings. Equivalently,

$$F_\Gamma = \left\{ a_\bullet \in \prod_{v \in \mathcal{V}} F_v \;\middle|\; \iota_v^e a_v = \iota_w^e a_w \text{ for each } e \text{ incident to } v \text{ and } w \right\}.$$

We may also regard F_Γ as a subring of $\prod_{e \in \mathcal{E}} F_e$, by sending an element $a_\bullet = (a_v)_{v \in \mathcal{V}}$ to $(a_e)_{e \in \mathcal{E}}$, where $a_e = \iota_v^e a_v = \iota_w^e a_w$ if e is incident to v and w.

Note that if $F_\bullet$ is a Γ-field, then we may regard each field F_v, F_e naturally as an F_Γ-algebra in such a way that all the inclusions ι_v^e are F_Γ-algebra homomorphisms.

Lemma 2.1.2. *If $F_\bullet$ is a Γ-field, then F_Γ is a field if and only if Γ is connected.*

Proof. If Γ is disconnected, there are elements $a_\bullet$ of the inverse limit F_Γ such that $a_\xi = 0$ for all $\xi \in \mathcal{V} \sqcup \mathcal{E}$ that lie on one connected component of Γ, but $a_\xi = 1$ for all ξ on another component. Hence F_Γ has zero-divisors and is not a field. Conversely, if F_Γ is not a field, then there is a zero-divisor $a_\bullet$. The set of vertices and edges ξ such that $a_\xi = 0$ forms an open subset of Γ, since $\iota_v^e a_v = a_e = \iota_w^e a_w$ whenever v, w are the vertices of an edge e. This open subset is neither empty nor all of Γ, since $a_\bullet$ is a zero divisor. Hence Γ is disconnected. □

Notation 2.1.3. We will say for short that $F_\bullet$ is a *Γ/F-field* if Γ is a connected graph, F is a field, and $F_\bullet$ is a Γ-field with $F_\Gamma = F$.

2.1.2. *Patching problems*

Given a Γ/F-field $F_\bullet$, and a finite dimensional vector space V over F, we obtain an inverse system $V_{F_\xi} = V \otimes_F F_\xi$ of finite dimensional vector spaces over the fields F_ξ (for $\xi \in \mathcal{V} \sqcup \mathcal{E}$). Conversely, given such an inverse system, we can ask whether it is induced by an F-vector space V. More precisely, let $\mathrm{Vect}(F)$ be the category of finite dimensional F-vector spaces; define a *vector space patching problem* $V_\bullet$ over $F_\bullet$ to be an inverse system of finite dimensional F_ξ-vector spaces; and let $\mathcal{PP}(F_\bullet)$ be the category of vector space patching problems over $F_\bullet$. There is then a base change functor $\mathrm{Vect}(F) \to \mathcal{PP}(F_\bullet)$. If it is an equivalence of categories, we say that *patching holds for finite dimensional vector spaces* over the Γ/F-field $F_\bullet$.

We may consider analogous notions for other objects over F. In particular let A be a group scheme over F (which we always assume to be of finite type). Let $\mathrm{Tors}(A)$ denote the category of A-torsors over F; the objects in this category are classified by the elements in the Galois cohomology group $H^1(F, A)$.

An object $\mathcal{T}$ in $\mathrm{Tors}(A)$ induces an *A-torsor patching problem* $\mathcal{T}_\bullet$ over $F_\bullet$, i.e. an inverse system consisting of A_{F_ξ}-torsors $\mathcal{T}_\xi$ for each $\xi \in \mathcal{V} \sqcup \mathcal{E}$, together with isomorphisms ϕ_v^e :

$(\mathcal{T}_v)_{F_e} \to \mathcal{T}_e$ for v a vertex of an edge e. These patching problems form a category $\mathcal{PP}(F_\bullet, A)$, whose morphisms correspond to collections of morphisms of torsors which commute with the maps ϕ_v^e. (Once we choose an orientation on the graph Γ, an A-torsor patching problem can also be viewed as collection of A-torsors $\mathcal{T}_v$ for $v \in \mathcal{V}$, together with a choice of isomorphism $(\mathcal{T}_{l(e)})_{F_e} \to (\mathcal{T}_{r(e)})_{F_e}$ for every edge $e \in \mathcal{E}$.) As before, we obtain a base change functor $\mathrm{Tors}(A) \to \mathcal{PP}(F_\bullet, A)$; and we say that *patching holds for A-torsors* over the Γ/F-field $F_\bullet$ if this is an equivalence of categories. For short we say that *patching holds for torsors over $F_\bullet$* if it holds for all linear algebraic groups A over F. (Our convention is that a *linear algebraic group* over F is a smooth closed subgroups $A \subseteq \mathrm{GL}_{n,F}$ for some n.)

2.1.3. *Local-global principles and simultaneous factorization*

Local-global principles are complementary to patching. Given a Γ/F-field $F_\bullet$, and a group scheme A over F, we say that *A-torsors over F satisfy a local-global principle* over $F_\bullet$ if an A-torsor $\mathcal{T}$ is trivial if and only if each induced F_v-torsor $\mathcal{T}_v := \mathcal{T} \times_F F_v$ is trivial. In [HHK11], criteria were given for patching and for local-global principles in terms of factorization. Before recalling them, we introduce some terminology and notation.

If $F_\bullet$ is a Γ/F-field, and if Γ is given an orientation, then there are induced maps $\pi_l, \pi_r : \prod_{v\in\mathcal{V}} F_v \to \prod_{e\in\mathcal{E}} F_e$ defined by $(\pi_l(a))_e = a_{l(e)}$ and $(\pi_r(a))_e = a_{r(e)}$ for $a = (a_v)_{v\in\mathcal{V}} \in \prod_{v\in\mathcal{V}} F_v$. Similarly, if A is a group scheme over F, there are induced maps $\pi_l, \pi_r : \prod_{v\in\mathcal{V}} A(F_v) \to \prod_{e\in\mathcal{E}} A(F_e)$ given by the same expressions, for $a = (a_v)_{v\in\mathcal{V}} \in \prod_{v\in\mathcal{V}} A(F_v)$. We say that a group scheme A over F *satisfies simultaneous factorization* over a Γ/F-field $F_\bullet$ (or for short, is *factorizable* over $F_\bullet$) if the map of pointed sets $\pi_l \cdot \pi_r^{-1} : \prod_{v\in\mathcal{V}} A(F_v) \to \prod_{e\in\mathcal{E}} A(F_e)$, defined by $a \mapsto \pi_l(a)\pi_r(a)^{-1}$, is surjective. In other words, if we are given a collection of elements $a_e \in A(F_e)$ for all $e \in \mathcal{E}$, then there exist elements $a_v \in A(F_v)$ for all $v \in \mathcal{V}$ such that $a_e = a_{l(e)}a_{r(e)}^{-1}$ for all e, with respect to the inclusions $F_{l(e)}, F_{r(e)} \hookrightarrow F_e$. Note that this factorization condition does not depend on the choice of orientation, since if we reverse the orientation on an edge e then we may consider the element $a' \in \prod_{e\in\mathcal{E}} A(F_e)$ such that $a'_e = a_e^{-1}$ and where the other entries of a' are the same as for a.

2.1.4. *Relations between patching, local-global principles and factorization*

The following two results are essentially in [HHK11].

Theorem 2.1.4. *Let Γ be a connected graph, F a field, and $F_\bullet$ a Γ/F-field. Then the following conditions are equivalent:*

(i) GL_n *is factorizable over $F_\bullet$ for all $n \geqslant 1$.*
(ii) Patching holds for finite dimensional vector spaces over $F_\bullet$.
(iii) Patching holds for torsors over $F_\bullet$.

Proof. It was shown in [HHK11, Proposition 2.2] that (i) is equivalent to (ii); and in [HHK11, Theorem 2.3] it was shown that (ii) implies (iii). It remains to show that (iii) implies (i).

Fix an orientation for Γ and let $g = (g_e)_{e\in\mathcal{E}} \in \mathrm{GL}_n(\prod_{e\in\mathcal{E}} F_e)$. We wish to show that there exists $h \in \mathrm{GL}_n(\prod_{v\in\mathcal{V}} F_v)$ such that $g = \pi_l(h)\pi_r(h)^{-1}$.

Consider the patching problem for GL_n-torsors over $F_\bullet$ that is given by trivial torsors $\mathcal{T}_\xi := \mathrm{GL}_{n,F_\xi}$ over F_ξ for each $\xi \in \mathcal{V} \sqcup \mathcal{E}$, together with transition functions $\mathrm{GL}_{n,F_e} =$

$(\mathcal{T}_{r(e)})_{F_e} \to (\mathcal{T}_{l(e)})_{F_e} = \mathrm{GL}_{n,F_e}$ given by multiplication by $g_e \in \mathrm{GL}_n(F_e)$, for each $e \in \mathcal{E}$. By hypothesis (iii), there is a GL_n-torsor $\mathcal{T}$ over F that induces this patching problem. But $\mathcal{T}$ is trivial, since $H^1(F, \mathrm{GL}_n) = 0$ by Hilbert's Theorem 90 ([KMRT98, Theorem 29.2]); i.e. there is an isomorphism $\mathrm{GL}_{n,F} \to \mathcal{T}$. The induced isomorphisms $\mathrm{GL}_{n,F_v} \to \mathcal{T}_v = \mathrm{GL}_{n,F_v}$ are given by multiplication by elements $h_v \in \mathrm{GL}_n(F_v)$. Since $\mathcal{T}$ induces the given patching problem, we have $h_{l(e)} = g_e h_{r(e)} \in \mathrm{GL}_n(F_e)$ for every $e \in \mathcal{E}$. Therefore $g = \pi_l(h)\pi_r(h)^{-1}$, with $h = (h_v)_{v\in\mathcal{V}} \in \mathrm{GL}_n(\prod_{e\in\mathcal{E}} F_e)$, as desired. □

Theorem 2.1.5. *Let Γ be a connected graph, F a field, and $F_\bullet$ a Γ/F-field. Assume that patching holds for finite dimensional vector spaces over $F_\bullet$. Then a linear algebraic group A over F is factorizable over $F_\bullet$ if and only if A-torsors over F satisfy a local-global principle over $F_\bullet$.*

Proof. This assertion is contained in the exactness of the sequence given in [HHK11, Theorem 2.4]; i.e. $\pi_l \cdot \pi_r^{-1}$ is surjective if and only if $H^1(F, A) \to \prod_{v\in\mathcal{V}} H^1(F_v, A)$ is injective. □

Note that the hypothesis of Theorem 2.1.5 does not imply that the equivalent conditions in the conclusion of that theorem necessarily hold. (In particular, in Example 4.4 of [HHK09] there is a non-trivial obstruction to a local-global principle, by Corollaries 5.6 and 5.5 of [HHK11]). Thus patching need not imply factorization over $F_\bullet$ for all linear algebraic groups over F. But as shown in the next section (Corollary 2.2.5), patching does imply factorization for all linear algebraic groups if we are allowed to pass to the separable closure of F. This will be useful in obtaining local-global principles for higher cohomology.

2.2. **Separable factorization**

As asserted in Theorems 2.1.4 and 2.1.5, there are relationships between factorization conditions on the one hand, and patching and local-global properties on the other. Below, in Theorem 2.2.4 and Corollary 2.2.5, we prove related results of this type, concerning "separable factorization", which will be needed later in applying the results of Section 2.5. We also prove a result (Proposition 2.2.3) that will be used in obtaining our long exact sequence in Section 2.5, and hence our local-global principle there.

2.2.1. *Separably factorizable group schemes*

Let $F_\bullet$ be a Γ/F-field, and write $\mathrm{Gal}(F)$ for the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. Given an F-scheme A, we have morphisms $A(F^{\mathrm{sep}}) \to A(F_v \otimes_F F^{\mathrm{sep}})$ for each vertex $v \in \mathcal{V}$, and $A(F_v \otimes_F F^{\mathrm{sep}}) \to A(F_e \otimes_F F^{\mathrm{sep}})$ when $v \in \mathcal{V}$ is a vertex of Γ on the edge $e \in \mathcal{E}$. These are induced by the inclusions $F^{\mathrm{sep}} \to F_v \otimes_F F^{\mathrm{sep}}$ and $F_v \otimes_F F^{\mathrm{sep}} \to F_e \otimes_F F^{\mathrm{sep}}$.

If A is an F-scheme, and $L \subseteq L'$ are field extensions of F, then the natural map $A(L) \to A(L')$ is an inclusion. (This is immediate if A is affine, and then follows in general.) In particular, given a Γ/F-field $F_\bullet$ as above, the maps $A(F) \to A(F_v)$ and $A(F_v) \to A(F_e)$ are injective for v a vertex of an edge e in Γ.

If we choose an orientation for the graph Γ, then as in Section 2.1.3 we may define maps $\pi_l, \pi_r : \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}}) \to \prod_{e\in\mathcal{E}} A(F_e \otimes_F F^{\mathrm{sep}})$ by $(\pi_l(a))_e = a_{l(e)}$ and $(\pi_r(a))_e = a_{r(e)}$.

Lemma 2.2.1. *Consider an affine F-scheme A, a graph Γ with a choice of orientation, and a Γ/F-field $F_\bullet$. Then*

$$A(F^{sep}) \longrightarrow \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}}) \underset{\pi_r}{\overset{\pi_l}{\rightrightarrows}} \prod_{e\in\mathcal{E}} A(F_e \otimes_F F^{\mathrm{sep}})$$

is an equalizer diagram of sets.

Proof. The hypothesis that F equals F_Γ is equivalent to having an exact sequence of F-vector spaces $0 \to F \to \prod_{v\in\mathcal{V}} F_v \to \prod_{e\in\mathcal{E}} F_e$, given by $\pi_l \cdot \pi_r^{-1}$ on the right. Since F^{sep} is a flat F-module, we have an exact sequence $0 \to F^{\mathrm{sep}} \to \prod_{v\in\mathcal{V}} F_v \otimes F^{\mathrm{sep}} \to \prod_{e\in\mathcal{E}} F_e \otimes F^{\mathrm{sep}}$. This in turn tells us that in the category of rings,

$$F^{\mathrm{sep}} = \varprojlim_{\xi\in\mathcal{V}\sqcup\mathcal{E}} (F_\xi \otimes F^{\mathrm{sep}}).$$

Write $A = \mathrm{Spec}(R)$. By the inverse limit property above, it follows that a homomorphism $R \to F^{\mathrm{sep}}$ is equivalent to a homomorphism $\phi : R \to \prod_{v\in\mathcal{V}} F_v \otimes F^{\mathrm{sep}}$ such that $\pi_l\phi = \pi_r\phi$, where $\pi_l, \pi_r : \prod_{v\in\mathcal{V}} F_v \otimes F^{\mathrm{sep}} \to \prod_{e\in\mathcal{E}} F_e \otimes F^{\mathrm{sep}}$ are the two projections. This gives the desired equalizer diagram. □

This lemma, and the notion of factorizability in the previous section, motivate the following definition.

Definition 2.2.2. Let $F_\bullet$ be a Γ/F-field, and suppose that A is a group scheme over F. We say that A is *separably factorizable* (over $F_\bullet$) if the pointed set map $\pi_l \cdot \pi_r^{-1} : \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}}) \to \prod_{e\in\mathcal{E}} A(F_e \otimes_F F^{\mathrm{sep}})$ is surjective for some (hence every) orientation on Γ.

Lemma 2.2.1 and Definition 2.2.2 then yield:

Proposition 2.2.3. *Let $F_\bullet$ be a Γ/F-field, and let A be a group scheme over F. Choose any orientation on Γ, and take the associated maps π_l, π_r. Then A is separably factorizable if and only if*

$$0 \longrightarrow A(F^{\mathrm{sep}}) \longrightarrow \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}}) \xrightarrow{\pi_l\cdot\pi_r^{-1}} \prod_{e\in\mathcal{E}} A(F_e \otimes_F F^{\mathrm{sep}}) \longrightarrow 0$$

is an exact sequence of pointed $\mathrm{Gal}(F)$-sets (and in fact an exact sequence of Galois modules in the case that A is commutative).

2.2.2. *Patching and separable factorization*

The following theorem and its corollary complement Theorems 2.1.4 and 2.1.5.

Theorem 2.2.4. *Let Γ be a connected graph, F a field, and $F_\bullet$ a Γ/F-field. Then the following conditions are equivalent:*

(i) GL_n *is factorizable over $F_\bullet$, for all $n \geqslant 1$.*
(ii) GL_n *is separably factorizable over $F_\bullet$, for all $n \geqslant 1$.*
(iii) Every linear algebraic group over F is separably factorizable over $F_\bullet$.

Proof. We begin by showing that (i) implies (iii). Fix an orientation for Γ. Let A be a linear algebraic group over F, and suppose we are given $g \in \prod_{e\in\mathcal{E}} A(F_e \otimes_F F^{\mathrm{sep}})$. We wish to show that there exists $h \in \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}})$ such that $g = \pi_l(h)\pi_r(h)^{-1}$.

Since $\mathcal{E}$ is finite, there is a finite separable field extension L/F such that g is the image of $g' \in \prod_{e\in\mathcal{E}} A(F_e \otimes_F L)$. Let $A' = R_{L/F}A_L$, the Weil restriction of $A_L = A \times_F L$ from L to F (see [BLR90], Section 7.6); this is a linear algebraic group over F. We may then view $g' \in \prod_{e\in\mathcal{E}} A'(F_e)$.

Since GL_n is factorizable over $F_\bullet$ by condition (i), Theorem 2.1.4 implies that patching holds for finite-dimensional vector spaces over $F_\bullet$. Thus [HHK11, Theorem 2.4] applies, giving us a six-term cohomology sequence for A'; and we may consider the image of g under the coboundary map $\prod_{e\in\mathcal{E}} A'(F_e) \to H^1(F, A')$. This image defines an A'-torsor $\mathcal{T}'$ over F (viz. the solution to the patching problem that consists of trivial torsors over each F_v and for which the transition functions are given by g'). But $H^1(F, A')$ may be identified with $H^1(L, A)$ by Shapiro's Lemma ([Ser97], Corollary to Proposition I.2.5.10), since $A'(F^{\mathrm{sep}})$ is the Galois module induced from $A(F^{\mathrm{sep}})$ via the inclusion $\mathrm{Gal}(L) \to \mathrm{Gal}(F)$. So $\mathcal{T}'$ corresponds to an A-torsor $\mathcal{T}$ over L. There is then a finite separable field extension E/L over which $\mathcal{T}$ becomes trivial. After replacing L by E, we may assume that $\mathcal{T}$ and hence $\mathcal{T}'$ is trivial. Hence by the exactness of the six-term sequence in [HHK11, Theorem 2.4], g' is the image of an element $h' \in \prod_{v\in\mathcal{V}} A'(F_v) = \prod_{v\in\mathcal{V}} A(F_v \otimes_F L)$ under $\pi_l \cdot \pi_r^{-1}$. The image $h \in \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}})$ of h' is then as desired, proving that condition (iii) holds.

Condition (iii) trivially implies condition (ii). It remains to show that condition (ii) implies condition (i).

If condition (ii) holds, then Proposition 2.2.3 yields a short exact sequence of pointed $\mathrm{Gal}(F)$-sets

$$0 \longrightarrow \mathrm{GL}_n(F^{\mathrm{sep}}) \longrightarrow \prod_{v\in\mathcal{V}} \mathrm{GL}_n(F_v \otimes_F F^{\mathrm{sep}}) \xrightarrow{\pi_l\cdot\pi_r^{-1}} \prod_{e\in\mathcal{E}} \mathrm{GL}_n(F_e \otimes_F F^{\mathrm{sep}}) \longrightarrow 0.$$

This in turn yields an exact sequence of pointed sets in Galois cohomology that begins

$$0 \to H^0(F, \mathrm{GL}_n) \to \prod_{v\in\mathcal{V}} H^0(F, (\mathrm{GL}_n)_v) \to \prod_{e\in\mathcal{E}} H^0(F, (\mathrm{GL}_n)_e) \to H^1(F, \mathrm{GL}_n).$$

But the last term vanishes by Hilbert's Theorem 90. The remaining short exact sequence is then equivalent to the condition that GL_n is factorizable over $F_\bullet$, i.e. condition (i). □

Corollary 2.2.5. *Let Γ be a connected graph, F a field, and $F_\bullet$ a Γ/F-field. Then patching holds for finite dimensional vector spaces over $F_\bullet$ if and only if every linear algebraic group over F is separably factorizable over $F_\bullet$.*

Proof. This is immediate from Theorem 2.1.4 and Theorem 2.2.4, which assert that these two conditions are each equivalent to GL_n being factorizable over $F_\bullet$ for all $n \geqslant 1$. □

2.3. **Globally dominated field extensions and cohomology**

To obtain our results, we will want to relate the cohomologies $H^n(F, A)$ and $H^n(F_\xi, A)$ for ξ a vertex or edge of Γ. One difficulty with this in general is the potential difference between the absolute Galois groups of F and F_ξ. To bridge this gap, we will use the Galois module

$A(F_\xi \otimes_F F^{\mathrm{sep}})$, which was studied in Section 2.2.1. Its cohomology $H^n(\mathrm{Gal}(F), A(F_\xi \otimes_F F^{\mathrm{sep}}))$ is meant to approximate the cohomology group $H^n(F_\xi, A)$.

Our strategy will be carried out using the notion of "global domination," which we introduce and study below. The condition that a Galois module has globally dominated cohomology will provide an important ingredient in demonstrating the existence of Mayer-Vietoris type sequences and local-global principles for its Galois cohomology groups. These applications are developed in Section 2.5.

2.3.1. *Globally dominated extensions*

Definition 2.3.1. Fix a field F. For any field extension L/F, with separable closure L^{sep}, let L^{gd} denote the compositum of L and F^{sep} taken within L^{sep}. If E/L is a separable algebraic field extension, we say that E/L is *globally dominated* (with respect to F) if E is contained in L^{gd}.

Thus a separable algebraic field extension E/L is globally dominated if and only if E is contained in some compositum $E'L \subseteq L^{\mathrm{sep}}$, where E'/F is a separable algebraic field extension. Also, the subfield $L^{\mathrm{gd}} \subseteq L^{\mathrm{sep}}$ can be characterized as the maximal globally dominated field extension of L. Since the extension F^{sep}/F is Galois with group $\mathrm{Gal}(F)$, it follows that the extension L^{gd}/L is Galois and that $\mathrm{Gal}^{\mathrm{gd}}(L) := \mathrm{Gal}(L^{\mathrm{gd}}/L)$ can be identified with a subgroup of $\mathrm{Gal}(F)$.

Lemma 2.3.2. *Let L/F be a field extension, and let A be a commutative group scheme defined over F. Then we may identify:*

$$H^n(\mathrm{Gal}(F), A(L \otimes_F F^{\mathrm{sep}})) = H^n(\mathrm{Gal}^{\mathrm{gd}}(L), A(L^{\mathrm{gd}})).$$

Proof. We may identify the group $H^n(\mathrm{Gal}(F), A(L \otimes F^{\mathrm{sep}}))$ as a limit of groups

$$H^n(\mathrm{Gal}(E/F), A(L \otimes E)),$$

and the group $H^n(\mathrm{Gal}^{\mathrm{gd}}(L), A(L^{\mathrm{gd}}))$ as a limit of groups

$$H^n(\mathrm{Gal}(LE/L), A(LE)),$$

where both limits are taken over finite Galois extensions E/F, and where LE is a compositum of L and E. Therefore the result will follow from a (compatible) set of isomorphisms

$$H^n(\mathrm{Gal}(E/F), A(L \otimes E)) \cong H^n(\mathrm{Gal}(LE/L), A(LE)).$$

Write $L \otimes E = \prod_{i=1}^m E_i$ for finite Galois field extensions E_i/L. We can also choose $LE = E_1$. We have $A(L \otimes E) = \prod_i A(E_i)$. Let $G = \mathrm{Gal}(E/F)$ and let G_1 be the stabilizer of E_1 (as a set) with respect to the action of G on $L \otimes_F E$. Then we may identify the G-modules $A(L \otimes E)$ and $\mathrm{Ind}_{G_1}^G A(E_1)$. We therefore have

$$H^n(G, A(L \otimes E)) \cong H^n(G, \mathrm{Ind}_{G_1}^G A(E_1)) \cong H^n(G_1, A(E_1)) = H^n(\mathrm{Gal}(LE/L), A(LE))$$

by Shapiro's Lemma ([Ser97], Corollary to Proposition I.2.5.10), as desired. □

2.3.2. *Globally dominated cohomology*

It remains to compare the cohomology with respect to the maximal globally dominated extension and the full Galois cohomology. For this we make the following

Definition 2.3.3. Let A be a commutative group scheme over F and L/F a field extension. We say that *the cohomology of A over L is globally dominated* (with respect to F) if $H^n(L^{\mathrm{gd}}, A) = 0$ for every $n > 0$.

Proposition 2.3.4. *Let A be a commutative group scheme over F and L/F a field extension. Suppose that the cohomology of A over L is globally dominated. Then we have isomorphisms:*

$$H^n(\mathrm{Gal}(F), A(L \otimes F^{\mathrm{sep}})) = H^n(\mathrm{Gal}^{\mathrm{gd}}(L), A(L^{\mathrm{gd}})) = H^n(\mathrm{Gal}(L), A(L^{\mathrm{sep}}))$$

for all $n \geqslant 0$.

Proof. The identification of the first and second groups was given in Lemma 2.3.2, and it remains to prove the isomorphism between the second and third groups. By the global domination hypothesis, $H^n(\mathrm{Gal}(L^{\mathrm{gd}}), A(L^{\mathrm{sep}})) = H^n(L^{\mathrm{gd}}, A) = 0$ for all $n > 0$. Hence from the Hochschild-Serre spectral sequence

$$H^p(\mathrm{Gal}^{\mathrm{gd}}(L), H^q(\mathrm{Gal}(L^{\mathrm{gd}}), A(L^{\mathrm{sep}}))) \implies H^{p+q}(\mathrm{Gal}(L), A(L^{\mathrm{sep}}))$$

for the tower of field extensions $L \subseteq L^{\mathrm{gd}} \subseteq L^{\mathrm{sep}}$ (viz. by [HS53, Theorem III.2]), the desired isomorphism follows. □

The notion of globally dominated cohomology can also be described just in terms of finite extensions of fields. First we prove a lemma.

Lemma 2.3.5. *Suppose that a field E_0 is a filtered direct limit of subfields E_i, each of which is an extension of a field E. Let A be a commutative group scheme over E, and let $\alpha \in H^n(E, A)$ for some $n \geqslant 0$. If the induced element $\alpha_{E_0} \in H^n(E_0, A)$ is trivial, then there is some i such that $\alpha_{E_i} \in H^n(E_i, A)$ is trivial.*

Proof. Since $\alpha_{E_0} \in H^n(E_0, A)$ is trivial, we may find some finite Galois extension L/E_0 such that α_{E_0} may be written as a cocycle in $Z^n(L/E_0, A(L))$ and such that it is the coboundary of a cochain in $C^{n-1}(L/E_0, A(L))$. Now the Galois extension L/E_0 is generated by finitely many elements of L, and the splitting cochain is defined by an additional finite collection of elements in $A(L)$, each of which is defined over some finitely generated extension of E (since A is of finite type over E). So we may find finitely many elements $a_1, \ldots, a_r \in E_0$ such that $\alpha_{E(a_1,\ldots,a_r)} = 0$. But since E_0 is the filtered direct limit of the fields E_i, there is an i such that $a_1, \ldots, a_r \in E_i$; and then $\alpha_{E_i} = 0$ as desired. □

Proposition 2.3.6. *Let A be a commutative group scheme over F and L/F a field extension. Then the cohomology of A over L is globally dominated if and only if for every finite globally dominated field extension L'/L, every $n > 0$, and every $\alpha \in H^n(L', A)$, there exists a finite globally dominated extension E/L' such that $\alpha_E = 0$.*

Proof. First suppose that the cohomology of A over L is globally dominated, and let $\alpha \in H^n(L', A)$ for some finite globally dominated field extension L'/L and some $n > 0$. Then

$\alpha_{L^{\mathrm{gd}}} = 0$ by hypothesis; and so by Lemma 2.3.5 there is some finite globally dominated extension E/L' such that $\alpha_E = 0$, as desired.

Conversely, suppose that the above condition on every $\alpha \in H^n(L', A)$ holds. Let $\alpha \in H^n(L^{\mathrm{gd}}, A)$. Then α is in the image of some element $\widetilde{\alpha} \in H^n(L', A)$ for some finite extension L'/L that is contained in L^{gd}. Now L' is globally dominated, so by hypothesis there exists a finite globally dominated field extension E/L' such that $\widetilde{\alpha}_E = 0$. Thus $\alpha = \widetilde{\alpha}_{L^{\mathrm{gd}}} = (\widetilde{\alpha}_E)_{L^{\mathrm{gd}}} = 0$. This shows that $H^n(L^{\mathrm{gd}}, A)$ is trivial, so the cohomology is globally dominated. □

2.4. **Criteria for global domination**

In the case of cyclic groups, the condition for cohomology to be globally dominated will be made more explicit here, using the Bloch-Kato conjecture to reduce to consideration of just the first cohomology group. That conjecture asserts the surjectivity of the norm residue homomorphism $K^{\mathrm{M}}_\bullet(F) \to H^\bullet(F, \mathbb{Z}/m\mathbb{Z}(\bullet))$ of graded rings. This surjectivity was proven for m prime in [Voe11] and [Wei09], with the general case then following via [GS06], Proposition 7.5.9. Since every element in the Milnor K-group $K^{\mathrm{M}}_n(F)$ is by definition a sum of n-fold products of elements of $K^{\mathrm{M}}_1(F)$, the following assertion is then immediate:

Proposition 2.4.1. *Let F be a field and let m be a positive integer not divisible by* $\mathrm{char}(F)$. *Then for every $n \geqslant 1$, every element of $H^n(F, \mathbb{Z}/m\mathbb{Z}(n))$ is a sum of n-fold cup products of elements of $H^1(F, \mathbb{Z}/m\mathbb{Z}(1))$.*

2.4.1. *Global domination for cyclic groups*

Proposition 2.4.2. *Let L/F be a field extension, and m an integer not divisible by the characteristic of F. Then the following are equivalent:*

(i) The cohomology of $\mathbb{Z}/m\mathbb{Z}$ over L is globally dominated.
(ii) For every finite globally dominated field extension L'/L and every positive integer r dividing m, every $\mathbb{Z}/r\mathbb{Z}$-Galois field extension of L' is globally dominated.
(iii) The multiplicative group $(L^{\mathrm{gd}})^\times$ is m-divisible; i.e. $((L^{\mathrm{gd}})^\times)^m = (L^{\mathrm{gd}})^\times$.

Proof. (i) ⇒ (ii): A $\mathbb{Z}/r\mathbb{Z}$-Galois field extension of L' corresponds to an element $\alpha \in H^1(L', \mathbb{Z}/r\mathbb{Z}(1))$. Let β be the image of α in $H^1(L^{\mathrm{gd}}, \mathbb{Z}/r\mathbb{Z}(1)) = H^1(L^{\mathrm{gd}}, \mathbb{Z}/r\mathbb{Z})$, where the equality holds because the field $L^{\mathrm{gd}} = LF^{\mathrm{sep}}$ contains a primitive m-th root of unity. It suffices to show that $\beta = 0$.

In the long exact cohomology sequence associated to the short exact sequence of constant groups $0 \to \mathbb{Z}/r\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/(m/r)\mathbb{Z} \to 0$, the map $H^0(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z}) \to H^0(L^{\mathrm{gd}}, \mathbb{Z}/(m/r)\mathbb{Z})$ is surjective, so the map $H^1(L^{\mathrm{gd}}, \mathbb{Z}/r\mathbb{Z}) \to H^1(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z})$ is injective. But the latter group is trivial, by hypothesis. Hence $\beta = 0$.

(ii) ⇒ (iii): Given $a \in (L^{\mathrm{gd}})^\times$, we wish to show that $a \in ((L^{\mathrm{gd}})^\times)^m$. Let ζ be a primitive m-th root of unity in $F^{\mathrm{sep}} \subseteq L^{\mathrm{gd}}$, and let $L' = L(\zeta, a) \subseteq L^{\mathrm{gd}}$. Thus L'/L is finite and separable. The field $E = L'(a^{1/m}) \subseteq L^{\mathrm{sep}}$ is Galois over L', with Galois group cyclic of order r for some r dividing m. Thus the extension E/L' is globally dominated, by (ii); i.e. $E \subseteq L'^{\mathrm{gd}} = L^{\mathrm{gd}}$. Hence $a \in ((L^{\mathrm{gd}})^\times)^m$.

(iii) ⇒ (i): By (iii), $H^1(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z}) = H^1(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z}(1)) = (L^{\mathrm{gd}})^\times/((L^{\mathrm{gd}})^\times)^m$ is trivial.

It remains to show the triviality of $H^n(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z}) = H^n(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z}(n))$ for $n > 1$. So let α lie in this cohomology group. By Proposition 2.4.1, we may write $\alpha = \sum_{i=1}^{s} \alpha_i$, with each α_i having the form

$$\alpha_i = \alpha_{i,1} \cup \cdots \cup \alpha_{i,n},$$

where $\alpha_{i,j} \in H^1(L^{\mathrm{gd}}, \mathbb{Z}/m\mathbb{Z})$. But this H^1 is trivial. Hence each α_i is trivial, and α is trivial, as desired. □

2.4.2. *Global domination for commutative group schemes*

Using the above result, the question of global domination for the cohomology of a finite commutative group scheme A can be reduced to the case of cyclic groups of prime order. We restrict to the case that the characteristic of F does not divide the order of A (equivalently, $A_{F^{\mathrm{sep}}}$ is a finite constant group scheme of order not divisible by $\mathrm{char}(F)$).

Corollary 2.4.3. *Let L/F be a field extension, and S a collection of prime numbers unequal to $\mathrm{char}(F)$. Suppose that the cohomology of the finite constant group scheme $\mathbb{Z}/\ell\mathbb{Z}$ over L is globally dominated for each $\ell \in S$. Then for every finite commutative group scheme A over F of order divisible only by primes in S, the cohomology of A over L is globally dominated.*

Proof. We wish to show that $H^n(L^{\mathrm{gd}}, A) = 0$ for $n > 0$. Since A is a finite étale group scheme defined over F, it becomes split (i.e. a finite constant group scheme) over F^{sep} and hence over $L^{\mathrm{gd}} = LF^{\mathrm{sep}}$. In particular, the base change of A to L^{gd} is a product of copies of cyclic groups $\mathbb{Z}/m\mathbb{Z}$, where each prime dividing m lies in S. Since cohomology commutes with taking products of coefficient groups, we are reduced to the case that $A \cong \mathbb{Z}/m\mathbb{Z}$ for m as above. The result now follows from condition (iii) of Proposition 2.4.2, since a group is m-divisible if it is ℓ-divisible for each prime factor ℓ of m. □

In characteristic zero, we also obtain a result in the case of group schemes that need not be finite. First we prove a lemma. If A is a group scheme over a field E and $m \geqslant 1$, let $A[m]$ denote the m-torsion subgroup of A, i.e. the kernel of the map $A \to A$ given by multiplication by m. Thus there is a natural map $H^n(E, A[m]) \to H^n(E, A)$.

Lemma 2.4.4. *Let A be a connected commutative group scheme over a field E of characteristic zero, and let $n \geqslant 1$. Then every element of $H^n(E, A)$ is in the image of $H^n(E, A[m]) \to H^n(E, A)$ for some $m \geqslant 1$.*

Proof. The group $H^n(E, A)$ is torsion by [Ser97, I.2.2 Cor. 3], and so for every $\alpha \in H^n(E, A)$ there exists $m \geqslant 1$ such that $m\alpha = 0$ (writing A additively). Since $\mathrm{char}(E) = 0$ and A is connected, there is a short exact sequence $0 \to A[m] \to A \to A \to 0$ of étale sheaves. This yields an exact sequence $H^n(E, A[m]) \to H^n(E, A) \to H^n(E, A)$ of groups, where the latter map is multiplication by m. Thus α is sent to zero under this map, and hence it lies in the image of $H^n(E, A[m])$. □

Proposition 2.4.5. *Assume that $\mathrm{char}(F) = 0$, and let L/F be a field extension. Suppose that the cohomology of the finite constant group scheme $\mathbb{Z}/\ell\mathbb{Z}$ over L is globally dominated for every prime ℓ. Then for every smooth commutative group scheme A over F, the cohomology of A over L is globally dominated.*

Proof. Let $\alpha \in H^n(L^{\mathrm{gd}}, A)$, for some $n > 0$. We wish to show that $\alpha = 0$.

The short exact sequence $0 \to A^0 \to A \to A/A^0 \to 0$ of étale sheaves yields an exact sequence $H^n(L^{\mathrm{gd}}, A^0) \to H^n(L^{\mathrm{gd}}, A) \to H^n(L^{\mathrm{gd}}, A/A^0)$ of groups. But Corollary 2.4.3 asserts that the cohomology of the finite commutative group scheme A/A^0 over L is globally dominated, since $\mathrm{char}(F) = 0$; i.e. $H^n(L^{\mathrm{gd}}, A/A^0) = 0$. So $\alpha \in H^n(L^{\mathrm{gd}}, A)$ is the image of some element $\alpha^0 \in H^n(L^{\mathrm{gd}}, A^0)$. By Lemma 2.4.4, α^0 lies in the image of $H^n(L^{\mathrm{gd}}, A^0[m])$ for some $m \geqslant 1$. Since $A^0[m]$ is a finite commutative group scheme over L, a second application of Corollary 2.4.3 yields that $H^n(L^{\mathrm{gd}}, A^0[m]) = 0$. So $\alpha^0 = 0$ and hence $\alpha = 0$. □

2.5. **Mayer-Vietoris and local-global principles**

We now use the previous results to obtain our long exact sequence, which in particular gives the abstract form of our Mayer-Vietoris sequence, and we then prove the abstract form of a local-global principle for Galois cohomology.

Theorem 2.5.1. *Given an oriented graph Γ, fix a Γ/F-field $F_\bullet$ and consider a separably factorizable smooth commutative group scheme A over F. Suppose that for every $\xi \in \mathcal{V} \sqcup \mathcal{E}$, the cohomology of A over F_ξ is globally dominated. Then we have a long exact sequence of Galois cohomology:*

$$0 \longrightarrow H^0(F, A) \longrightarrow \prod_{v\in\mathcal{V}} H^0(F_v, A) \longrightarrow \prod_{e\in\mathcal{E}} H^0(F_e, A)$$

$$H^1(F, A) \longrightarrow \prod_{v\in\mathcal{V}} H^1(F_v, A) \longrightarrow \prod_{e\in\mathcal{E}} H^1(F_e, A) \longrightarrow \cdots$$

Proof. By hypothesis, the cohomology of A over F_ξ is globally dominated. By Proposition 2.3.4, with $L = F_\xi$, we may identify $H^n(\mathrm{Gal}(F), A(F_\xi \otimes_F F^{\mathrm{sep}})) \cong H^n(F_\xi, A)$.

Since A is separably factorizable, by Proposition 2.2.3 we have a short exact sequence of $\mathrm{Gal}(F)$-modules

$$0 \longrightarrow A(F^{\mathrm{sep}}) \longrightarrow \prod_{v\in\mathcal{V}} A(F_v \otimes_F F^{\mathrm{sep}}) \xrightarrow{\pi_l \cdot \pi_r^{-1}} \prod_{e\in\mathcal{E}} A(F_e \otimes_F F^{\mathrm{sep}}) \longrightarrow 0.$$

This induces a long exact sequence in Galois cohomology over F. Applying the above identification to the terms of this sequence, we obtain the exact sequence asserted in the theorem. □

Corollary 2.5.2. *Given a separably factorizable smooth commutative group scheme A over F and a Γ/F-field $F_\bullet$, the long exact sequence in Theorem 2.5.1 holds in each of the following cases:*

- *(i) A is finite; and for every $\xi \in \mathcal{V} \sqcup \mathcal{E}$, and every prime ℓ dividing the order of A, the cohomology of $\mathbb{Z}/\ell\mathbb{Z}$ over F_ξ is globally dominated.*
- *(ii) F is a field of characteristic zero; and for every $\xi \in \mathcal{V} \sqcup \mathcal{E}$, and every prime number ℓ, the cohomology of $\mathbb{Z}/\ell\mathbb{Z}$ over F_ξ is globally dominated.*

Proof. By Theorem 2.5.1 it suffices to show that the cohomology of A over F_ξ is globally dominated. In these two cases, this condition is satisfied by Corollary 2.4.3 and Proposition 2.4.5 respectively. □

An important case is that of a graph Γ that is *bipartite*, i.e. for which there is a partition $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1$ such that for every edge $e \in \mathcal{E}$, one vertex is in $\mathcal{V}_0$ and the other is in $\mathcal{V}_1$. Given a bipartite graph Γ together with such a partition, we will choose the orientation on Γ given by taking $l(e)$ and $r(e)$ to be the vertices of $e \in \mathcal{E}$ lying in $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively.

Corollary 2.5.3 (Abstract Mayer-Vietoris)**.** *In the situation of Theorem 2.5.1, assume that the graph Γ is bipartite, with respect to a partition $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1$ of the set of vertices. Then the long exact cohomology sequence in Theorem 2.5.1 becomes the Mayer-Vietoris sequence*

$$\begin{array}{l} 0 \longrightarrow A(F) \xrightarrow{\Delta} \prod_{v\in\mathcal{V}_0} A(F_v) \times \prod_{v\in\mathcal{V}_1} A(F_v) \xrightarrow{-} \prod_{e\in\mathcal{E}} A(F_e) \\ \qquad\qquad \searrow \\ \qquad H^1(F,A) \xrightarrow{\Delta} \prod_{v\in\mathcal{V}_0} H^1(F_v,A) \times \prod_{v\in\mathcal{V}_1} H^1(F_v,A) \xrightarrow{-} \prod_{e\in\mathcal{E}} H^1(F_e,A) \\ \qquad\qquad \searrow \\ \qquad H^2(F,A) \xrightarrow{\Delta} \quad \cdots \end{array}$$

where the maps Δ and $-$ are induced by the diagonal inclusion and by subtraction, respectively.

Theorem 2.5.4 (Abstract Local-Global Principle)**.** *Fix a Γ/F-field $F_\bullet$, and fix a positive integer m not divisible by the characteristic of F. Suppose that the following conditions hold:*

(i) Γ is bipartite, with respect to a partition $\mathcal{V} = \mathcal{V}_0 \sqcup \mathcal{V}_1$ of the set of vertices;
(ii) for every $\xi \in \mathcal{V} \sqcup \mathcal{E}$, the cohomology of $\mathbb{Z}/m\mathbb{Z}$ over F_ξ is globally dominated;
(iii) given $v \in \mathcal{V}_0$, and elements $a_e \in F_e^\times$ for all $e \in \mathcal{E}$ that are incident to v, there exists $a \in F_v^\times$ such that $a_e/a \in (F_e^\times)^m$ for all e (where we identify F_v with its image $i_v^e(F_v) \subseteq F_e$).

Then for all $n > 0$, the natural local-global maps

$$\sigma_n : H^{n+1}(F, \mathbb{Z}/m\mathbb{Z}(n)) \to \prod_{v\in\mathcal{V}} H^{n+1}(F_v, \mathbb{Z}/m\mathbb{Z}(n))$$

are injective.

Proof. Given hypothesis (i), as above we choose the orientation on Γ such that $l(e) \in \mathcal{V}_0$ and $r(e) \in \mathcal{V}_1$ for all $e \in \mathcal{E}$. Consider the homomorphisms:

$$\rho^{i,j}, \rho_0^{i,j} : \prod_{v\in\mathcal{V}} H^i(F_v, \mathbb{Z}/m\mathbb{Z}(j)) \to \prod_{e\in\mathcal{E}} H^i(F_e, \mathbb{Z}/m\mathbb{Z}(j)),$$

where for $\alpha \in \prod_{v\in\mathcal{V}} H^i(F_v, \mathbb{Z}/m\mathbb{Z}(j))$ the e-th entries of $\rho^{i,j}(\alpha)$, $\rho_0^{i,j}(\alpha)$ are given by

$$\rho^{i,j}(\alpha)_e = (\alpha_{l(e))})_{F_e} - (\alpha_{r(e)})_{F_e}, \quad \rho_0^{i,j}(\alpha)_e = (\alpha_{l(e)})_{F_e}.$$

Using hypothesis (ii), Theorem 2.5.1 allows us to identify the kernel of σ_n with the cokernel of

$$\rho^{n,n} : \prod_{v\in\mathcal{V}} H^n(F_v, \mathbb{Z}/m\mathbb{Z}(n)) \to \prod_{e\in\mathcal{E}} H^n(F_e, \mathbb{Z}/m\mathbb{Z}(n)).$$

Thus it suffices to show that $\rho^{n,n}$ is surjective for $n \geqslant 1$. This in turn will follow from showing that $\rho_0^{n,n}$ is surjective, since the image of $\rho_0^{n,n}$ is contained in that of $\rho^{n,n}$ (using that Γ is bipartite, and setting $\alpha_v = 0$ for all $v \in \mathcal{V}_1$).

Writing $H_{\mathcal{V}}^n = \prod_{v\in\mathcal{V}} H^n(F_v, \mathbb{Z}/m\mathbb{Z}(n))$ and $H_{\mathcal{E}}^n = \prod_{e\in\mathcal{E}} H^n(F_e, \mathbb{Z}/m\mathbb{Z}(n))$, we note that $\rho_0^{\bullet,\bullet} : H_{\mathcal{V}}^\bullet \to H_{\mathcal{E}}^\bullet$ is a homomorphism of graded rings. By hypothesis (iii), $\rho_0^{1,1}$ is surjective, since $H^1(E, \mathbb{Z}/m\mathbb{Z}(1)) = E^\times/(E^\times)^m$ for any field E of characteristic not dividing m. By Proposition 2.4.1, every element in $H_{\mathcal{E}}^n$ is a sum of n-fold products of elements in $H_{\mathcal{E}}^1$, for $n \geqslant 1$. But since the map $\rho_0^{\bullet,\bullet}$ is a ring homomorphism, and $\rho_0^{1,1}$ is surjective, it follows that $\rho_0^{n,n}$ is surjective as well for all $n \geqslant 1$. □

3. Curves over complete discrete valuation rings

We now apply the previous general results to the more specific situation that we study in this paper: function fields F over a complete discretely valued field K. In Section 3.1 we will obtain a Mayer-Vietoris sequence and a local-global principle in the context of finitely many overfields F_ξ of F ("patches"). This can be compared with Theorem 3.5 of [HHK11]. We will afterwards use that to obtain local-global principles with respect to the points on the closed fiber of a model (in Section 3.2), and with respect to the discrete valuations on F or on a regular model of F (in Section 3.3). These will later be used in Section 4 to obtain applications to other algebraic structures.

3.0.1. *Notation*

We begin by fixing the standing notation for this section, which follows that of [HH10], [HHK09], and [HHK11]. Let T be a complete discrete valuation ring with fraction field K and residue field k and uniformizer t, and let $\widehat{X}$ be a projective, integral and normal T-curve. Let F be the function field of $\widehat{X}$. We let X be the closed fiber of $\widehat{X}$, and we choose a non-empty collection of closed points $\mathcal{P} \subset X$, containing all the points at which distinct irreducible components of X meet. Thus the open complement $X \smallsetminus \mathcal{P}$ is a disjoint union of finitely many irreducible affine k-curves U. Let $\mathcal{U}$ denote the collection of these open sets U.

For a point $P \in \mathcal{P}$, we let R_P be the local ring $\mathcal{O}_{\widehat{X},P}$ at P, and we let $\widehat{R}_P$ be the completion at its maximal ideal. Let F_P be the fraction field of $\widehat{R}_P$. For a component $U \in \mathcal{U}$, we let R_U be the subring of F consisting of rational functions on $\widehat{X}$ that are regular at the points of U, i.e.

$$R_U = \{f \in F | f \in \mathcal{O}_{\widehat{X},Q} \text{ for all } Q \in U\}.$$

We also let $\widehat{R}_U$ be the t-adic completion of R_U, and we let F_U be the field of fractions of $\widehat{R}_U$. Here $\widehat{R}_P$ and $\widehat{R}_U$ are Noetherian integrally closed domains (because $\widehat{X}$ is normal), and in particular Krull domains.

For a point $P \in \mathcal{P}$ and a component $U \in \mathcal{U}$, we say that P and U are *incident* if P is contained in the closure of U. Given $P \in \mathcal{P}$ and $U \in \mathcal{U}$ that are incident, the prime ideal sheaf $\mathcal{I}$ defining the reduced closure $\overline{U}^{\mathrm{red}}$ of U in $\widehat{X}$ induces a (not necessarily prime) ideal $\mathcal{I}_P$ in the complete local ring $\widehat{R}_P$. We call the height one prime ideals of $\widehat{R}_P$ containing $\mathcal{I}_P$ the *branches* on U at P. We let $\mathcal{B}$ denote the collection of branches on all points in $\mathcal{P}$ and all components in $\mathcal{U}$. For a branch $\wp$ on $P \in \mathcal{P}$ and $U \in \mathcal{U}$, the local ring of $\widehat{R}_P$ at $\wp$ is a discrete

valuation ring $R_\wp$. Let $\widehat{R}_\wp$ be its $\wp$-adic (or equivalently t-adic) completion, and let $F_\wp$ be the field of fractions of $\widehat{R}_\wp$. Note that this is a complete discretely valued field containing F_U and F_P (see [HH10], Section 6, and [HHK09], page 241).

Associated to the curve $\widehat{X}$ and our choice of points $\mathcal{P}$, we define a *reduction graph* $\Gamma = \Gamma_{\widehat{X},\mathcal{P}}$ whose vertex set is the disjoint union of the sets $\mathcal{P}$ and $\mathcal{U}$ and whose edge set is the set $\mathcal{B}$ of branches. The incidence relation on this (multi-)graph, which makes it bipartite, is defined by saying that an edge corresponding to a branch $\wp \in \mathcal{B}$ is incident to the vertices $P \in \mathcal{P}$ and $U \in \mathcal{U}$ if $\wp$ is a branch on U at P. We choose the orientation on Γ that is associated to the partition $\mathcal{P} \sqcup \mathcal{U}$ of the vertex set. We will consider the Γ-field $F_\bullet = F_\bullet^{\widehat{X},\mathcal{P}}$ defined by $F_\xi^{\widehat{X},\mathcal{P}} = F_\xi$ for $\xi \in \mathcal{P}, \mathcal{U}, \mathcal{B}$.

3.1. **Mayer-Vietoris and local-global principles with respect to patches**

Using the results of Section 2.5, we now obtain the desired Mayer-Vietoris sequence for the Γ-field $F_\bullet$ that is associated as above to the function field F and a choice of points $\mathcal{P}$ on the closed fiber of a normal model $\widehat{X}$ (see Theorem 3.1.3). In certain cases we show that this sequence splits into short exact sequences, possibly starting with the H^2 term (Corollaries 3.1.6 and 3.1.7). Related to this, we obtain a local-global principle for $H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1))$, in this patching context.

Theorem 3.1.1. *With F and $F_\bullet$ as above, $F_\bullet$ is a Γ/F-field, and patching holds for finite dimensional vector spaces over $F_\bullet$. Thus every linear algebraic group over F is separably factorizable over $F_\bullet$.*

Proof. According to [HHK11, Corollary 3.4], the fields F_ξ for $\xi \in \mathcal{P} \sqcup \mathcal{U} \sqcup \mathcal{B}$ form a factorization inverse system with inverse limit F. That is, $F_\bullet$ is a Γ/F-field. That result also asserts that patching holds for finite dimensional vector spaces over $F_\bullet$. The assertion about being separably factorizable then follows from Corollary 2.2.5. □

3.1.1. *Global domination and Mayer-Vietoris*

The following result relies on a form of the Weierstrass Preparation Theorem that was proven in [HHK12], and which extended related results in [HH10] and [HHK09]. Another result that is similarly related to Weierstrass Preparation appears at Lemma 3.1.4 below.

Theorem 3.1.2 (Global domination for patches)**.** *If $\xi \in \mathcal{U} \sqcup \mathcal{P} \sqcup \mathcal{B}$ and if m is a positive integer not divisible by* $\mathrm{char}(k)$*, then the cohomology of $\mathbb{Z}/m\mathbb{Z}$ over F_ξ is globally dominated.*

Proof. By Proposition 2.4.2, it suffices to show that $(F_\xi^{\mathrm{gd}})^\times = ((F_\xi^{\mathrm{gd}})^\times)^m$. So let $a \in (F_\xi^{\mathrm{gd}})^\times$. Thus $a \in F_\xi F' \subseteq F_\xi^{\mathrm{sep}}$ for some finite separable extension F'/F. Let $\widehat{X}' \to \widehat{X}$ be the normalization of $\widehat{X}$ in F', so that $\widehat{X}'$ is a normal projective T-curve with function field F'. Using [HH10, Lemma 6.2], we may identify $F_\xi \otimes_F F'$ with $\prod_{\xi'} F'_{\xi'}$, where ξ' ranges through the points, components or branches, respectively, lying above ξ on $\widehat{X}'$. We also see by this description that for each ξ', the field $F'_{\xi'}$ is the compositum of its subfields F_ξ and F'. Applying [HHK12, Theorems 3.3 and 3.7] to the curve $\widehat{X}'$ and the field $F'_{\xi'}$, and again

using [HH10, Lemma 6.2], it follows that there is an étale cover $\widehat{X}''$ of $\widehat{X}'$ such that $a = bc^m$ for some $b \in F'' \subseteq F^{\mathrm{sep}}$ and $c \in F''_{\xi''} = F_\xi F'' \subseteq F_\xi^{\mathrm{gd}}$; here F'' is the function field of $\widehat{X}''$ and ξ'' is any point, component or branch, respectively, on $\widehat{X}''$ that lies over ξ' on $\widehat{X}'$. Now $b \in (F^{\mathrm{sep}})^\times$, and $\mathrm{char}(F)$ does not divide m, so $b \in ((F^{\mathrm{sep}})^\times)^m$. Thus $a \in ((F_\xi^{\mathrm{gd}})^\times)^m$. □

Theorem 3.1.3 (Mayer-Vietoris for Curves)**.** *Let A be a commutative linear algebraic group over F. Assume that either*

(i) A is finite of order not divisible by the characteristic of k; or
(ii) $\mathrm{char}(k) = 0$.

Then we have a long exact Mayer-Vietoris sequence:

$$0 \longrightarrow A(F) \xrightarrow{\Delta} \prod_{P\in\mathcal{P}} A(F_P) \times \prod_{U\in\mathcal{U}} A(F_U) \xrightarrow{-} \prod_{\wp\in\mathcal{B}} A(F_\wp)$$

$$\longrightarrow H^1(F,A) \xrightarrow{\Delta} \prod_{P\in\mathcal{P}} H^1(F_P,A) \times \prod_{U\in\mathcal{U}} H^1(F_U,A) \xrightarrow{-} \prod_{\wp\in\mathcal{B}} H^1(F_\wp,A)$$

$$\longrightarrow H^2(F,A) \xrightarrow{\Delta} \cdots$$

Proof. Let Γ be the bipartite graph $\Gamma_{\widehat{X},\mathcal{P}}$ as above. By Theorem 3.1.1, A is separably factorizable over $F_\bullet$. Now for each prime ℓ unequal to the characteristic of k, and each $\xi \in \mathcal{U} \sqcup \mathcal{P} \sqcup \mathcal{B}$, the cohomology of $\mathbb{Z}/\ell\mathbb{Z}$ over F_ξ is globally dominated, by Theorem 3.1.2. The conclusion now follows from Corollaries 2.5.2 and 2.5.3. □

3.1.2. *Local-global principles with respect to patches*

Lemma 3.1.4. *Let m be a positive integer that is not divisible by* $\mathrm{char}(K)$*. Let P be a closed point of X, let $\wp_1, \ldots, \wp_s$ be the branches of X at P, and let $a_i \in F_{\wp_i}^\times$. Then there exists $a \in F_P^\times$ such that $a_i/a \in (F_{\wp_i}^\times)^m$ for every i.*

Proof. Since $F_{\wp_i}$ is the $\wp_i$-adic completion of F_P, the Approximation Theorem [Bou72, VI.7.3, Theorem 2] implies that the elements a_i can all be $\wp_i$-adically approximated arbitrarily well by an element $a \in F_P^\times$. The result now follows by applying the strong form of Hensel's Lemma (see [Bou72, III.4.5, Corollary 1 to Theorem 2]) to the polynomials $Y^m - a_i/a \in \widehat{R}_{\wp_i}[Y]$. □

Theorem 3.1.5 (Local-Global Principle)**.** *Let $\widehat{X}$ be a normal projective curve over a complete discrete valuation ring T with residue field k, let $\mathcal{P}$ be a non-empty finite subset of the closed fiber X that includes the points at which distinct irreducible components of X meet, and let $\mathcal{U}$ be the set of components of $X \smallsetminus \mathcal{P}$. Suppose that m is an integer not divisible by the characteristic of k. Then for each integer $n > 1$, the natural map*

$$H^n(F, \mathbb{Z}/m\mathbb{Z}(n-1)) \to \prod_{P\in\mathcal{P}} H^n(F_P, \mathbb{Z}/m\mathbb{Z}(n-1)) \times \prod_{U\in\mathcal{U}} H^n(F_U, \mathbb{Z}/m\mathbb{Z}(n-1))$$

is injective.

Proof. The graph $\Gamma_{\widehat{X},\mathcal{P}}$ is bipartite, with the set of vertices $\mathcal{V}$ partitioned as $\mathcal{V}_0 \sqcup \mathcal{V}_1$ with $\mathcal{V}_0 = \mathcal{P}$ and $\mathcal{V}_1 = \mathcal{U}$. So hypothesis (i) of Theorem 2.5.4 holds. Hypothesis (ii) of that theorem,

concerning global domination, also holds, by Theorem 3.1.2. Finally, hypothesis (iii), in this case concerning the lifting of elements of the fields $F_{\wp_i}^\times$ to an element of $F_P^\times$ modulo m-th powers, holds by Lemma 3.1.4. Thus Theorem 2.5.4 applies, and the conclusion follows. □

In some cases we can allow arbitrary Tate twists, and as a result the Mayer-Vietoris sequence splits into shorter exact sequences:

Corollary 3.1.6. *Let m be an integer not divisible by the characteristic of k, and suppose that the degree $[F(\mu_m) : F]$ is prime to m (e.g. if m is prime or F contains a primitive m-th root of unity). Let r be any integer. Then the Mayer-Vietoris sequence in Theorem 3.1.3 for $A = \mathbb{Z}/m\mathbb{Z}(r)$ splits into exact sequences*

$$0 \longrightarrow A(F) \longrightarrow \prod_{P\in\mathcal{P}} A(F_P) \times \prod_{U\in\mathcal{U}} A(F_U) \longrightarrow \prod_{\wp\in\mathcal{B}} A(F_\wp)$$

$$H^1(F, A) \longrightarrow \prod_{P\in\mathcal{P}} H^1(F_P, A) \times \prod_{U\in\mathcal{U}} H^1(F_U, A) \longrightarrow \prod_{\wp\in\mathcal{B}} H^1(F_\wp, A) \longrightarrow 0$$

and

$$0 \longrightarrow H^n(F, A) \longrightarrow \prod_{P\in\mathcal{P}} H^n(F_P, A) \times \prod_{U\in\mathcal{U}} H^n(F_U, A) \longrightarrow \prod_{\wp\in\mathcal{B}} H^n(F_\wp, A) \longrightarrow 0$$

for all $n > 1$.

Proof. If F contains a primitive m-th root of unity, then $A = \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/m\mathbb{Z}(n-1)$ over F and its extension fields, for all n. Hence in the Mayer-Vietoris sequence in Theorem 3.1.3(i), the maps $\iota_F : H^n(F, A) \to \prod_{P\in\mathcal{P}} H^n(F_P, A) \times \prod_{U\in\mathcal{U}} H^n(F_U, A)$ are injective for all $n > 1$, by Theorem 3.1.5. The result now follows in this case.

More generally, let $F' = F(\mu_m)$ and similarly for F_P and F_U. As above, $\iota_{F'}$ is injective. Using the naturality of ι_F with respect to F, we have $\ker(\iota_F) \subseteq \ker(\iota_{F'} \circ \mathrm{res}_{F'/F})$. Further, by the injectivity of $\iota_{F'}$, $\ker(\iota_{F'} \circ \mathrm{res}_{F'/F}) = \ker(\mathrm{res}_{F'/F}) \subseteq \ker(\mathrm{cor}_{F'/F} \circ \mathrm{res}_{F'/F})$. But $\mathrm{cor} \circ \mathrm{res} : H^n(F, A) \to H^n(F, A)$ is multiplication by $[F' : F]$ ([GS06], Proposition 3.3.7), which is injective since $|A| = m$ and $[F' : F]$ is prime to m. Thus these kernels are all trivial, and again the result follows. □

In Corollary 3.1.6, the initial six terms need *not* split into two three-term short exact sequences; i.e. the map on $H^1(F, A)$ need not be injective. In fact, for $A = \mathbb{Z}/m\mathbb{Z}$ with $m > 1$, a necessary and sufficient condition for splitting is that the reduction graph Γ is a tree ([HHK11], Corollaries 5.6 and 6.4). But in the next result, there is splitting at every level.

Corollary 3.1.7. *Suppose that* $\mathrm{char}(k) = 0$ *and that K contains a primitive m-th root of unity for all $m \geqslant 1$. Then the Mayer-Vietoris sequence in Theorem 3.1.3(ii) for $\mathbb{G}_\mathrm{m}$ splits into exact sequences*

$$0 \longrightarrow H^n(F, \mathbb{G}_\mathrm{m}) \longrightarrow \prod_{P\in\mathcal{P}} H^n(F_P, \mathbb{G}_m) \times \prod_{U\in\mathcal{U}} H^n(F_U, \mathbb{G}_\mathrm{m}) \longrightarrow \prod_{\wp\in\mathcal{B}} H^n(F_\wp, \mathbb{G}_\mathrm{m}) \longrightarrow 0$$

for all $n \geqslant 0$.

Proof. By Theorem 3.1.3(ii), it suffices to prove the injectivity of the maps $H^n(F, \mathbb{G}_{\mathrm{m}}) \to \prod_{P \in \mathcal{P}} H^n(F_P, \mathbb{G}_{\mathrm{m}}) \times \prod_{U \in \mathcal{U}} H^n(F_U, \mathbb{G}_{\mathrm{m}})$ for all $n \geqslant 1$. The case $n = 1$ follows from the vanishing of $H^1(F, \mathbb{G}_{\mathrm{m}})$ by Hilbert's Theorem 90. It remains to show injectivity for $n > 1$. Since K contains all roots of unity, for each m we may identify the Galois module $\mathbb{G}_{\mathrm{m}}[m] = \mu_m$ with $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}(n-1)$.

By Theorem 3.1.3(ii), the desired injectivity will follow from the surjectivity of the map

$$\prod_{P \in \mathcal{P}} H^{n-1}(F_P, \mathbb{G}_{\mathrm{m}}) \times \prod_{U \in \mathcal{U}} H^{n-1}(F_U, \mathbb{G}_{\mathrm{m}}) \to \prod_{\wp \in \mathcal{B}} H^{n-1}(F_\wp, \mathbb{G}_{\mathrm{m}}).$$

So let $\alpha \in \prod_{\wp \in \mathcal{B}} H^{n-1}(F_\wp, \mathbb{G}_{\mathrm{m}})$, and write $\alpha = (\alpha_\wp)_{\wp \in \mathcal{B}}$, with $\alpha_\wp \in H^{n-1}(F_\wp, \mathbb{G}_{\mathrm{m}})$. For each $\wp \in \mathcal{B}$, the element $\alpha_\wp$ is the image of some $\widetilde{\alpha}_\wp \in H^{n-1}(F_\wp, \mu_{m_\wp})$ for some $m_\wp \geqslant 1$, by Lemma 2.4.4. Since $\mathcal{B}$ is finite, we may let m be the least common multiple of the integers $m_\wp$. Thus α is the image of $\widetilde{\alpha} = (\widetilde{\alpha}_\wp) \in \prod_{\wp \in \mathcal{B}} H^{n-1}(F_\wp, \mu_m)$.

By Theorem 3.1.5 and Theorem 3.1.3(i), the map

$$\prod_{P \in \mathcal{P}} H^{n-1}(F_P, \mathbb{Z}/m\mathbb{Z}(n-1)) \times \prod_{U \in \mathcal{U}} H^{n-1}(F_U, \mathbb{Z}/m\mathbb{Z}(n-1)) \to \prod_{\wp \in \mathcal{B}} H^{n-1}(F_\wp, \mathbb{Z}/m\mathbb{Z}(n-1))$$

is surjective. So by the identification $\mu_m = \mathbb{Z}/m\mathbb{Z}(n-1)$, it follows that $\widetilde{\alpha}$ is the image of some element $\widetilde{\beta} \in \prod_{P \in \mathcal{P}} H^{n-1}(F_P, \mu_m) \times \prod_{U \in \mathcal{U}} H^{n-1}(F_U, \mu_m)$. Let β be the image of $\widetilde{\beta}$ in $\prod_{P \in \mathcal{P}} H^{n-1}(F_P, \mathbb{G}_{\mathrm{m}}) \times \prod_{U \in \mathcal{U}} H^{n-1}(F_U, \mathbb{G}_{\mathrm{m}})$. Since the diagram

$$\begin{array}{ccc} \widetilde{\beta} \in \prod_{P \in \mathcal{P}} H^{n-1}(F_P, \mu_m) \times \prod_{U \in \mathcal{U}} H^{n-1}(F_U, \mu_m) & \longrightarrow & \prod_{\wp \in \mathcal{B}} H^{n-1}(F_\wp, \mu_m) \ni \widetilde{\alpha} \\ \downarrow & & \downarrow \\ \beta \in \prod_{P \in \mathcal{P}} H^{n-1}(F_P, \mathbb{G}_{\mathrm{m}}) \times \prod_{U \in \mathcal{U}} H^{n-1}(F_U, \mathbb{G}_{\mathrm{m}}) & \longrightarrow & \prod_{\wp \in \mathcal{B}} H^{n-1}(F_\wp, \mathbb{G}_{\mathrm{m}}) \ni \alpha \end{array}$$

commutes, β maps to α, as desired. □

Note that Corollaries 3.1.6 and 3.1.7 also provide patching results for cohomology, in addition to local-global principles. Namely, for $n \neq 1$ in Corollary 3.1.6, or any n in Corollary 3.1.7, those assertions show the following. Given a collection of elements $\alpha_\xi \in H^n(F_\xi, A)$ for all $\xi \in \mathcal{P} \sqcup \mathcal{U}$ such that α_P, α_U induce the same element of $H^n(F_\wp, A)$ whenever $\wp$ is a branch on U at P, there exists a unique $\alpha \in H^n(F, A)$ that induces all the α_ξ. In the situation of Theorem 3.1.3, where splitting is not asserted, a weaker patching statement still follows: given elements α_ξ as above, there exists such an α, but it is not necessarily unique.

3.2. **Local-global principles with respect to points**

In this section we will investigate how to translate our results into local-global principles in terms of the points on the closed fiber X of $\widehat{X}$, rather than in terms of patches. Extending our earlier notation, if $P \in X$ is any point (not necessarily closed), we let F_P denote the fraction field of the complete local ring $\widehat{R}_P := \widehat{\mathcal{O}}_{\widehat{X},P}$. In particular, if η is the generic point of an irreducible component X_0 of the closed fiber X, then F_η is a complete discretely valued field, and it is the same as the η-adic completion of F.

3.2.1. *The field F_η^h*

In order to bridge the gap between the fields F_U and F_η, where η is the generic point of the irreducible component $X_0 \subseteq X$ containing U, we will consider a subfield F_η^h of F_η that has many of the same properties but is much smaller.

Namely, with notation as above, let R_η^h be the direct limit of the rings $\widehat{R}_V$, where V ranges over the non-empty open subsets of X_0 that do not meet any other irreducible component of X. Equivalently, we may fix one such non-empty open subset U, and consider the direct limit over the non-empty open subsets V of U. Here R_η^h is a subring of $\widehat{R}_\eta$; and we let F_η^h be its fraction field. Thus F_η^h is a subfield of F_η.

Lemma 3.2.1. *Let $X_0 \subseteq X$ be an irreducible component with generic point η, and let $U \subset X_0$ be a non-empty open subset meeting no other component. Then R_η^h is a Henselian discrete valuation ring with respect to the η-adic valuation, having residue field $k(U) = k(X_0)$. Its fraction field F_η^h is the filtered direct limit of the fields F_V, where V ranges over the non-empty open subsets of U.*

Proof. Each F_V is contained in F_η^h, and every element of F_η^h is of the form a/b with a, b in some common F_V. So F_η^h is the direct limit of the fields F_V.

Viewing η as a prime ideal of $\widehat{R}_V$, the fields F_V each have a discrete valuation with respect to η, and these are compatible. It follows that F_η^h is a discretely valued field with respect to the η-adic valuation. We wish to show that the valuation ring of F_η^h is R_η^h, with residue field $k(U)$. Note that the t-adic and η-adic metrics on $\widehat{R}_V$ are equivalent, since $\sqrt{(t)} = \eta$.

Since $\widehat{R}_V$ is contained in the η-adic valuation ring of F_V, it follows that R_η^h is contained in the valuation ring of F_η^h. To verify the reverse containment, consider a non-zero element $\alpha \in F_\eta^h$ with non-negative η-adic valuation. Thus $\alpha \in F_V^\times$ for some V; and so $\alpha = a/b$ with $a, b \in \widehat{R}_V$ non-zero and $v_\eta(a) \geqslant v_\eta(b)$. Since $\widehat{R}_V$ is a Krull domain, the element $b \in \widehat{R}_V$ has a well defined divisor, which is a finite linear combination of prime divisors; and other than the irreducible closed fiber V of $\mathrm{Spec}(\widehat{R}_V)$, each of them has a locus that meets this closed fiber at only finitely many points. After shrinking V by deleting these points, we may assume that b is invertible in $\widehat{R}_V[t^{-1}]$. But also $v_\eta(a/b) \geqslant 0$; and thus a/b has no poles on $\mathrm{Spec}(\widehat{R}_V)$. So the element $\alpha = a/b \in F_V$ actually lies in $\widehat{R}_V$, and hence in R_η^h as desired. Thus R_η^h is indeed the valuation ring of F_η^h. Since the valuations on the rings R_V are compatible and induce that of R_η^h, the maximal ideal ηR_η^h of R_η^h is the direct limit of the prime ideals $\eta\widehat{R}_V$ of the rings $\widehat{R}_V$. But $\widehat{R}_V/\eta\widehat{R}_V = k(V) = k(U)$ for all V. So the residue field of R_η^h is $k(U)$.

It remains to show that R_η^h is Henselian. Let S be a commutative étale algebra over R_η^h, together with a section $\sigma : \eta \to \mathrm{Spec}(S)$ of $\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R_\eta^h)$ over the point η. To show that R_η^h is Henselian, we will check that σ may be extended to a section over all of $\mathrm{Spec}(R_\eta^h)$. Now since S is a finitely generated R_η^h-algebra, it is induced by an étale $\widehat{R}_V$-algebra S_V for some V, together with a morphism $\pi_V : \mathrm{Spec}(S_V) \to \mathrm{Spec}(\widehat{R}_V)$ that induces π and has a section $\sigma_V^0 : \eta \to \mathrm{Spec}(S_V)$ over the generic point η of the closed fiber of $\mathrm{Spec}(\widehat{R}_V)$. Here σ_V^0 defines a rational section over V, and hence a section over a non-empty affine open subset of V. So after shrinking V, we may assume that σ_V^0 is induced by a section $\sigma_V : V \to \mathrm{Spec}(S_V)$.

But the ring $\widehat{R}_V$ is t-adically complete; so by a version of Hensel's Lemma (Lemma 4.5 of [HHK09]) the section σ_V over V extends to a section of π_V, over all of $\mathrm{Spec}(\widehat{R}_V)$. This in turn induces a section of π over $\mathrm{Spec}(R^h_\eta)$ that extends σ, thereby showing that R^h_η is Henselian. □

Proposition 3.2.2. *Let η be the generic point of an irreducible component X_0 of X, and let U be a non-empty affine open subset of X_0 that does not meet any other irreducible component of X. Let A be a smooth commutative group scheme over F. Suppose $\alpha \in H^n(F_U, A)$ satisfies $\alpha_{F_\eta} = 0$. Then there is a Zariski open neighborhood V of η in U such that $\alpha_{F_V} = 0$.*

Proof. The ring $\widehat{R}_U$ is excellent, by [Val76, Corollary 5] and regularity; hence so is its localization $(\widehat{R}_U)_\eta$ at η. The henselization R^h_U of $(\widehat{R}_U)_\eta$ contains $\widehat{R}_U$, and its completion is $\widehat{R}_\eta$; and it is minimal for these properties among henselian discrete valuation rings. So R^h_U is contained in R^h_η, and its fraction field F^h_U is contained in F^h_η. Let $c \in Z^n(F_U, A)$ represent the class α. Since $\alpha_{F_\eta} = 0$, there is a finite Galois extension L/F_η such that c_{F_η} is the coboundary of a cochain in $C^{n-1}(L/F_\eta, A(L))$. This can be expressed by finitely many polynomial equations. By excellence, Artin Approximation ([Art69, Theorem 1.10]) applies to R^h_U; and it follows that $c_{F^h_U}$ is the coboundary of an element of $C^{n-1}(F^h_U, A)$. Thus $\alpha_{F^h_U} = 0$ and hence $\alpha_{F^h_\eta} = 0$. The conclusion now follows from Lemma 2.3.5, since F^h_η is the filtered direct limit of the fields F_V, by the second part of Lemma 3.2.1. □

3.2.2. *Local-global principles with respect to points*

We now obtain a local-global principle in terms of points on the closed fiber X.

Theorem 3.2.3. *Let A be a commutative linear algebraic group over F and let $n > 1$. Assume that either*

(i) $A = \mathbb{Z}/m\mathbb{Z}(r)$, where m is an integer not divisible by $\mathrm{char}(k)$, and where either $r = n-1$ or else $[F(\mu_m) : F]$ is prime to m; or
(ii) $A = \mathbb{G}_\mathrm{m}$, $\mathrm{char}(k) = 0$, and K contains a primitive m-th root of unity for all $m \geqslant 1$.

Then the natural map

$$H^n(F, A) \to \prod_{P \in X} H^n(F_P, A)$$

is injective, where P ranges through all the points of the closed fiber.

Proof. Let $\alpha \in H^n(F, A)$ be an element of the above kernel. Consider the irreducible components X_i of X, and their generic points $\eta_i \in X_i \subseteq X$. Thus $\alpha_{F_{\eta_i}} = 0$ for each i (taking $P = \eta_i$). By Proposition 3.2.2, we may choose a non-empty Zariski affine open subset $U_i \subset X_i$, not meeting any other component of X, such that $\alpha_{F_{U_i}}$ is trivial. Let $\mathcal{U}$ be the collection of these sets U_i, and let $\mathcal{P}$ be the complement in X of the union of the sets U_i. Then α is in the kernel of the map on $H^n(F, A)$ in Theorem 3.1.5, Corollary 3.1.6, or Corollary 3.1.7 respectively. Since that map is injective, it follows that $\alpha = 0$. □

3.3. **Local-global principles with respect to discrete valuations**

Using the previous results, we now investigate how to translate our results into local-global principles involving discrete valuations on our field F, and in particular those valuations arising from codimension one points on our model $\widehat{X}$ of F. Our main result here is Theorem 3.3.6, which parallels Theorem 3.2.3(i), and asserts the vanishing of the obstruction $Ш^n(F, A)$ to such a local-global principle, for $n > 1$ and A an appropriate twist of $\mathbb{Z}/m\mathbb{Z}$.

In the case $n = 1$, a related result appeared at [HHK11, Corollary 8.11], but with different hypotheses and for different groups. In fact, for a constant finite group A, the obstruction $Ш^1(F, A)$ is non-trivial unless the reduction graph Γ of a regular model $\widehat{X}$ of F is a tree; see [HHK11], Proposition 8.4 and Corollary 6.5. (As in [HHK11], "discrete valuations" are required to have value group isomorphic to $\mathbb{Z}$, and in particular to be non-trivial.)

For the remainder of this section we make the *standing assumption that $\widehat{X}$ is regular.*

Lemma 3.3.1. *Let P be a point of X and let v be a discrete valuation on F_P. Then the restriction v_0 of v to F is a discrete valuation on F. Moreover if v is induced by a codimension one point of* $\mathrm{Spec}(\widehat{R}_P)$ *(or equivalently, a height one prime of $\widehat{R}_P$), then v_0 is induced by a codimension one point of $\widehat{X}$ whose closure contains P.*

Proof. The first assertion is given at [HHK11, Proposition 7.5]. For the second assertion, if v is induced by a height one prime of $\widehat{R}_P$, then $\widehat{R}_P$ is contained in the valuation ring of v. Hence so is the local ring R_P, which is then also contained in the valuation ring of v_0. Thus v_0 is induced by a codimension one point of $\mathrm{Spec}(R_P)$, and so by a codimension one point of $\widehat{X}$ whose closure contains P. □

Given a field E, let Ω_E denote the set of discrete valuations on E. For $v \in \Omega_E$, write E_v for the v-adic completion of E. If A is a commutative group scheme over E, let

$$Ш^n(E, A) = \ker\left(H^n(E, A) \to \prod_{v \in \Omega_E} H^n(E_v, A) \right).$$

Similarly, given a normal integral scheme Z with function field E, let $\Omega_Z \subseteq \Omega_E$ denote the subset consisting of the discrete valuations on E that correspond to codimension one points on Z. If A is as above, let

$$Ш^n_Z(E, A) = \ker\left(H^n(E, A) \to \prod_{v \in \Omega_Z} H^n(E_v, A) \right).$$

Here if $Z = \mathrm{Spec}(R)$, we also write $Ш^n_R(E, A)$ for $Ш^n_Z(E, A)$.

We will be especially interested in the case that $E = F$, the function field of a regular projective curve $\widehat{X}$ over our complete discrete valuation ring T; and where Z is either $\widehat{X}$ or $\mathrm{Spec}(\widehat{R}_P)$ for some closed point $P \in \widehat{X}$.

In the case that $Z = \widehat{X}$, with closed fiber X and function field F as before, there is a related group

$$Ш^n_{\widehat{X},X}(F, A) = \ker\left(H^n(F, A) \to \prod_{P \in X} H^n(F_P, A) \right).$$

Note that $Ш^n_{\widehat{X},X}(F,A)$ is contained in $Ш^n_{\widehat{X}}(F,A)$ by [HHK11, Proposition 7.4], which asserts that every field of the form F_v contains a field of the form F_P. In the above notation, Theorem 3.2.3 asserts that $Ш^n_{\widehat{X},X}(F,A)=0$ if condition (i) or (ii) of that result is satisfied.

3.3.1. *Relating local-global obstruction on a regular model to obstructions at closed points*

A key step in relating our patches to discrete valuations is the following result, which parallels Proposition 8.4 of [HHK11]. That result considered only the case $n=1$, but did not require the linear algebraic group to be commutative (since H^1 is defined even for non-commutative groups).

Here $X_{(0)}$ denotes the set of closed points of X, and $\prod'$ denotes the restricted product, i.e. the subgroup of the product consisting of elements in which all but finitely many entries are trivial.

Proposition 3.3.2. *Let A be a linear algebraic group over F.*

(a) The natural map $H^n(F,A)\to\prod_{P\in X}H^n(F_P,A)$ induces an exact sequence

$$0\to Ш^n_{\widehat{X},X}(F,A)\xrightarrow{\iota} Ш^n_{\widehat{X}}(F,A)\xrightarrow{\phi}\prod_{P\in X_{(0)}}{}' Ш^n_{\widehat{R}_P}(F_P,A).$$

(b) If A is finite and of order not divisible by $\mathrm{char}(k)$, or if $\mathrm{char}(k)=0$, the exact sequence extends to

$$0\to Ш^n_{\widehat{X},X}(F,A)\xrightarrow{\iota} Ш^n_{\widehat{X}}(F,A)\xrightarrow{\phi}\prod_{P\in X_{(0)}}{}' Ш^n_{\widehat{R}_P}(F_P,A)\to 0.$$

(c) If $n>1$ and A satisfies hypothesis (i) or (ii) of Theorem 3.2.3, then ϕ is an isomorphism.

Proof. It follows from Lemma 3.3.1 that for each $P\in X_{(0)}$, the image of $Ш^n_{\widehat{X}}(F,A)$ under $H^n(F,A)\to H^n(F_P,A)$ lies in $Ш^n_{\widehat{R}_P}(F_P,A)$. Thus we obtain a group homomorphism $Ш^n_{\widehat{X}}(F,A)\to\prod_{P\in X_{(0)}}Ш^n_{\widehat{R}_P}(F_P,A)$. To obtain the map ϕ, we wish to show that the image is contained in the restricted product. If $\alpha\in Ш^n_{\widehat{X}}(F,A)\subseteq H^n(F,A)$ and η is the generic point of an irreducible component of X, then the image of α in $H^n(F_\eta,A)$ is trivial, since η is a codimension one point on $\widehat{X}$. By Proposition 3.2.2, α has trivial image in $H^n(F_U,A)$ for some Zariski open neighborhood U of η, and hence in $H^n(F_P,A)$ for each $P\in U$. The union of these sets U, as η varies, contains all but finitely many closed points of X. So indeed the image of α lies in the restricted product.

The composition $\phi\iota$ is trivial by definition of $Ш^n_{\widehat{X},X}(F,A)$. To complete the proof of part (a), let $\alpha\in Ш^n_{\widehat{X}}(F,A)$ be any element in the kernel of this map. Then the image of α in $Ш^n_{\widehat{R}_P}(F_P,A)\subseteq H^n(F_P,A)$ is trivial for every closed point P on the closed fiber X. Meanwhile, for any non-closed point η of X (viz. the generic point of an irreducible component of X), the image of α in $H^n(F_\eta,A)$ is also trivial, by the definition of $Ш^n$, since η is a codimension one point of $\widehat{X}$. Hence α lies in $Ш^n_{\widehat{X},X}(F,A)\subseteq Ш^n_{\widehat{X}}(F,A)$, as required.

To prove part (b), i.e. that ϕ is surjective, take an element $(\alpha_P)_{P\in X_{(0)}}$ in the above restricted product. Thus $\alpha_P=0$ for all $P\in X_{(0)}$ outside of some finite set $\mathcal{P}$ that can be chosen to

include the points where distinct components of X meet. Since $\alpha_P \in Ш^n_{\hat{R}_P}(F_P, A)$, its image in $H^n(F_\wp, A)$ is trivial for every branch $\wp$ of X at P. Let $\mathcal{U}$ be the set of components of the complement of $\mathcal{P}$ in X, and let $\alpha_U = 0$ for each $U \in \mathcal{U}$. The hypotheses of Theorem 3.1.3 are satisfied in our situation, and the exact sequence there yields that the tuple $(\alpha_\xi)_{\xi \in \mathcal{P} \sqcup \mathcal{U}}$ is of the form $\Delta(\alpha)$ for some $\alpha \in H^n(F, A)$. The image of α under $H^n(F, A) \to \prod_{P \in X} H^n(F_P, A)$ is $(\alpha_P)_{P \in X_{(0)}} \in \prod'_{P \in X_{(0)}} Ш^n_{\hat{R}_P}(F_P, A)$. To complete the proof of (b), we show that $\alpha \in Ш^n_{\hat{X}}(F, A)$, i.e. $\alpha_v = 0$ for each $v \in \Omega_{\hat{X}}$. By [HHK11, Proposition 7.4], F_v contains F_P for some $P \in X$. If P is a closed point, then the discrete valuation on F_v restricts to a discrete valuation v_P on F_P (which in turn restricts to v on F). But $\alpha_P \in Ш^n_{\hat{R}_P}(F_P, A)$, so α becomes trivial over $(F_P)_{v_P}$ and hence over F_v. If instead P is a point of codimension one, i.e. the generic point η of some $U \in \mathcal{U}$, then $v = v_\eta$, and $\alpha_v = 0$ because $\alpha_U = 0$ and $F_U \subset F_\eta$.

Part (c) now follows from part (b) and Theorem 3.2.3, which says $Ш^n_{\hat{X},X}(F, A) = 0$. □

3.3.2. *Local-global principles at closed points*

We will use the following statement of Panin which asserts a particular case of the analog of the Gersten conjecture in the context of the theory of Bloch and Ogus. Here $\kappa(z)$ denotes the residue field at a point z, and $Z^{(i)}$ denotes the set of points of Z having codimension i. As usual, $\mathbb{Z}/m\mathbb{Z}(-r)$ denotes $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}(r), \mathbb{Z}/m\mathbb{Z})$ for $r > 0$, where m is not divisible by the characteristic of the field. Also, for m as above and for any $r \in \mathbb{Z}$, if A is an m-torsion group scheme then $A(r)$ denotes $A \otimes \mathbb{Z}/m\mathbb{Z}(r)$.

Theorem 3.3.3 ([Pan03, Theorem C])**.** *Suppose that R is an equicharacteristic regular local ring with fraction field F, and let $Z = \mathrm{Spec}(R)$. Then for any positive integer m that is not divisible by the characteristic, and any m-torsion finite étale commutative group scheme A over R, the Cousin complex*

$$0 \to H^n(Z, A) \to H^n(F, A) \to \bigoplus_{z \in Z^{(1)}} H^{n-1}(\kappa(z), A(-1)) \to \bigoplus_{z \in Z^{(2)}} H^{n-2}(\kappa(z), A(-2)) \to \cdots$$

of étale cohomology groups is exact.

Proposition 3.3.4. *Under the hypotheses of Theorem 3.3.3, assume that R is complete. Then $Ш^n(F, A) = Ш^n_Z(F, A) = 0$ for $n \geqslant 1$.*

Proof. Let d be the Krull dimension of R. The assertion is trivial if $d \leqslant 1$, so we may assume $d \geqslant 2$. Since $Ш^n(F, A) \subseteq Ш^n_Z(F, A)$, it suffices to show the vanishing of the latter group.

Let $\alpha \in Ш^n_Z(F, A) \subseteq H^n(F, A)$. Consider the exact sequence in Theorem 3.3.3. For each $z \in Z^{(1)}$, the ramification map $H^n(F, A) \to H^{n-1}(\kappa(z), A(-1))$ factors through the map to the completion $H^n(F_z, A)$. But the image of α in $H^n(F_z, A)$ vanishes, since $\alpha \in Ш^n_Z(F, A)$. Hence α maps to zero in $\bigoplus_{z \in Z^{(1)}} H^{n-1}(\kappa(z), A(-1))$, and thus it is induced by a class $\tilde{\alpha} \in H^n(Z, A)$.

Let k' be the residue field of R at its maximal ideal (corresponding to the closed point of Z). Let $\sigma_1, \ldots, \sigma_d$ be a regular system of parameters in R. Write $\sigma = \sigma_1$ and write $\widehat{R}_\sigma$ for the completion of the local ring of R at the prime ideal (σ). Thus $\widehat{R}_\sigma$ is a complete discrete valuation ring with uniformizer σ; let F_σ and $\kappa(\sigma)$ denote its fraction field and residue field, respectively. Here $\kappa(\sigma)$ is the fraction field of $\mathcal{O}_{\kappa(\sigma)} := R/(\sigma)$, an equicharacteristic regular

complete local ring of dimension $d-1$, such that the residues $\bar{\sigma}_2, \ldots, \bar{\sigma}_d$ of $\sigma_2, \ldots, \sigma_d$ form a regular system of parameters. By Theorem 3.3.3, the natural maps

$$H^n(\widehat{R}_\sigma, A) \to H^n(F_\sigma, A) \quad \text{and} \quad H^n(\mathcal{O}_{\kappa(\sigma)}, A) \to H^n(\kappa(\sigma), A)$$

are injections. The complete local rings R and $\mathcal{O}_{\kappa(\sigma)}$ each have residue field k', so by [Art62, Theorem III.4.9] we may identify

$$H^n(Z, A) = H^n(k', A) = H^n(\mathcal{O}_{\kappa(\sigma)}, A),$$

via restriction to the closed point. The natural map $H^n(Z, A) \to H^n(\mathcal{O}_{\kappa(\sigma)}, A)$ is thus an isomorphism. We have the following commutative diagram:

$$\begin{array}{ccc} H^n(\mathcal{O}_{\kappa(\sigma)}, A) & \hookrightarrow & H^n(\kappa(\sigma), A) \\ \wr\uparrow & & \uparrow \\ \tilde{\alpha} \in H^n(Z, A) & \longrightarrow & H^n(\widehat{R}_\sigma, A) \\ \downarrow & & \downarrow \\ \alpha \in H^n(F, A) & \longrightarrow & H^n(F_\sigma, A) \end{array}$$

Since σ defines a codimension one point of Z, the image of $\alpha \in Ш^n_Z(F, A) \subseteq H^n(F, A)$ in $H^n(F_\sigma, A)$ is trivial. Since $\tilde{\alpha}$ maps to α, a diagram chase then shows that $\tilde{\alpha}$ is trivial and hence so is α. □

In our situation, with $R = \widehat{R}_P$ arising from a regular model $\widehat{X}$, Proposition 3.3.4 asserts:

Corollary 3.3.5. *Suppose that K is an equicharacteristic complete discretely valued field of characteristic not dividing m, and that $\widehat{X}$ is regular. Then for every $P \in X$ and m-torsion finite étale commutative group scheme A over $\widehat{R}_P$, $Ш^n(F_P, A) = Ш^n_{\widehat{R}_P}(F_P, A) = 0$.*

3.3.3. *Local-global principles for function fields*

Finally, we obtain our local-global principles over our field F with respect to discrete valuations:

Theorem 3.3.6. *Suppose that K is an equicharacteristic complete discretely valued field of characteristic not dividing m, and that $\widehat{X}$ is regular. Let $n > 1$. Then*

$$Ш^n(F, \mathbb{Z}/m\mathbb{Z}(n-1)) = Ш^n_{\widehat{X}}(F, \mathbb{Z}/m\mathbb{Z}(n-1)) = 0.$$

If $[F(\mu_m) : F]$ is prime to m then also $Ш^n(F, \mathbb{Z}/m\mathbb{Z}(r)) = Ш^n_{\widehat{X}}(F, \mathbb{Z}/m\mathbb{Z}(r)) = 0$ for all r.

Proof. In each of the two cases considered, hypothesis (i) of Theorem 3.2.3 is satisfied. Since $n > 1$, Proposition 3.3.2(c) then applies. The theorem now follows by Corollary 3.3.5 and the containment $Ш^n(F, A) \subseteq Ш^n_{\widehat{X}}(F, A)$. □

Remark 3.3.7. The case $n = 2$, concerning Brauer groups, holds even without assuming equal characteristic ([CPS12, Theorem 4.3(ii)], [HHK11, Corollary 9.13]). It would be interesting to know if the same is true for $n > 2$, and also if Theorem 3.3.6 has an analog for $\mathbb{G}_\mathrm{m}$ in characteristic zero as in Theorem 3.2.3(ii). But carrying over the above proof would require versions of Panin's result [Pan03, Theorem C] in those situations.

4. Applications to torsors under noncommutative groups

As an application of our results, in this section we give local-global principles for G-torsors over F for certain connected noncommutative linear algebraic groups G, and for related structures. Our method is to use cohomological invariants in order to reduce to our local-global principles in Galois cohomology (viz. to Theorems 3.2.3(i) and 3.3.6).

We preserve the notation and terminology established at the beginning of Section 3. In particular, we write T for the valuation ring of K, and k for the residue field. We let $\widehat{X}$ be a normal, integral projective curve over T, with closed fiber X and function field F. As before, we write Ω_F for the set of discrete valuations on the field F, and write $\Omega_{\widehat{X}}$ for the subset of Ω_F consisting of those discrete valuations that arise from codimension one points on $\widehat{X}$.

4.1. **Relation to prior results**

The basic strategy used in this section to obtain local-global principles for torsors was previously used in [CPS12, Theorem 5.4], to obtain a local-global principle for G-torsors over the function field F of a smooth projective geometrically integral curve over a p-adic field K, where G is a linear algebraic F-group that is quasisplit, simply connected, and absolutely almost simple without an E_8 factor. There they used the local-global principle of Kato for H^3 together with the fact that the fields under their consideration were of cohomological dimension three. Our applications arise from our new local-global principles for higher cohomology groups, and hence do not require any assumptions on cohomological dimension.

Local-global principles for G-torsors were also obtained in [HHK11] (as well as in [HHK09], in the context of patches). But there the linear algebraic groups G were required to be rational varieties, whereas here there is no such hypothesis. On the other hand, here we will be looking at specific types of groups, such as E_8 and F_4. Another difference is that in [HHK11], in order to obtain local-global principles with respect to discrete valuations, we needed to make additional assumptions (e.g. that k is algebraically closed of characteristic zero, or that G is defined and reductive over $\widehat{X}$; see [HHK11, Corollary 8.11]). Here the only assumption needed for local-global principles with respect to discrete valuations is that K is equicharacteristic. (If we wish to consider only those discrete valuations that arise from a given model $\widehat{X}$ of F, then we also need to assume that $\widehat{X}$ is regular.) Thus even in the cases where the groups considered below are rational, the results here go beyond what was shown for those groups in [HHK11].

4.2. **Injectivity vs. triviality of the kernel**

The local-global principles for G-torsors will be phrased in terms of local-global maps on $H^1(F,G)$. Because of non-commutativity, $H^1(F,G)$ is just a pointed set, not a group. Thus there are two distinct questions that can be posed about a local-global map: whether the kernel is trivial, and whether the map is injective (the latter condition being stronger). And as in Section 3, there are actually several local-global maps: $H^1(F,G) \to \prod_{v\in\Omega_F} H^1(F_v,G)$,

$H^1(F,G) \to \prod_{v\in\Omega_{\widehat{X}}} H^1(F_v,G)$, and $H^1(F,G) \to \prod_{P\in X} H^1(F_P,G)$, with kernels $Ш(F,G)$, $Ш_{\widehat{X}}(F,G)$, and $Ш_{\widehat{X},X}(F,G)$ respectively. (As is common, here we write Ш for $Ш^1$.) Note that if $Ш_{\widehat{X}}(F,G) = 0$ for some model $\widehat{X}$ then $Ш(F,G) = 0$; and similarly for injectivity of the corresponding maps. So we will emphasize the cases of $Ш_{\widehat{X}}(F,G)$ and $Ш_{\widehat{X},X}(F,G)$.

4.3. **Local-global principles via cohomological invariants**

The approach that we take here for obtaining our applications is to use cohomological invariants of algebraic objects.

Recall that an *invariant* over F is a morphism of functors $a : S \to H$, where $S : (\text{Fields}/F) \to (\text{Pointed Sets})$ and $H : (\text{Fields}/F) \to (\text{Abelian Groups})$([GMS03], Part I, Sect. I.1). Most often, as in [GMS03], S will have the form S_G given by $S_G(E) = H^1(E,G)$ for some linear algebraic group G over F; this classifies G-torsors over E, and also often classifies other types of algebraic structures over F. In practice, $H(E)$ will usually take values in Galois cohomology groups of the form $H^n(E, \mathbb{Z}/m\mathbb{Z}(n-1))$.

The simplest situation is described in the following general result, where we retain the standing hypotheses stated at the beginning of Section 3, with $\widehat{X}$ a normal model of F.

Proposition 4.3.1. *Let $a : S \to H$ be a cohomological invariant over F, where $H(E) = H^n(E, \mathbb{Z}/m\mathbb{Z}(r))$ for some integers n, m, r with n, m positive, and where m is not divisible by $\mathrm{char}(k)$. Assume either that $r = n-1$, or else that the degree $[F(\mu_m) : F]$ is prime to m.*

(a) If $a(F) : S(F) \to H(F)$ has trivial kernel, then so does the local-global map $S(F) \to \prod_{P\in X} S(F_P)$. Moreover, if K is equicharacteristic and $\widehat{X}$ is regular, then the same holds for $S(F) \to \prod_{v\in\Omega_{\widehat{X}}} S(F_v)$.

(b) If $a(F) : S(F) \to H(F)$ is injective, then so is the local-global map $S(F) \to \prod_{P\in X} S(F_P)$. If in addition K is equicharacteristic and $\widehat{X}$ is regular, then $S(F) \to \prod_{v\in\Omega_{\widehat{X}}} S(F_v)$ is injective as well.

Proof. Consider the commutative diagrams

$$\begin{array}{ccc} S(F) & \xrightarrow{a(F)} & H(F) \\ \downarrow & & \downarrow \\ \prod_{P\in X} S(F_P) & \xrightarrow{\prod a(F_P)} & \prod_{P\in X} H(F_P) \end{array} \qquad \begin{array}{ccc} S(F) & \xrightarrow{a(F)} & H(F) \\ \downarrow & & \downarrow \\ \prod_{v\in\Omega_{\widehat{X}}} S(F_v) & \xrightarrow{\prod a(F_v)} & \prod_{v\in\Omega_{\widehat{X}}} H(F_v). \end{array}$$

The result follows by a diagram chase, using the fact that the right-hand vertical map in the first diagram is injective by Theorem 3.2.3(i), and that the corresponding map in the second diagram is injective in the case that K is equicharacteristic and $\widehat{X}$ is regular, by Theorem 3.3.6. □

Recall that a linear algebraic group G over F is *quasi-split* if it has a Borel subgroup defined over F. It is *split* if it has a Borel subgroup over F that has a composition series whose successive quotient groups are each isomorphic to $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{G}_{\mathrm{a}}$. If G is reductive, this is equivalent to G having a maximal torus that is split (i.e. a product $\mathbb{G}_{\mathrm{m}}^n$).

Corollary 4.3.2. *Let G be a simply connected linear algebraic group over F. Consider the Rost invariant $R_G : H^1(*, G) \to H^3(*, \mathbb{Z}/m\mathbb{Z}(2))$ of G, and assume that the characteristic of k does not divide m. In each of the following cases, $Ш_{\widehat{X},X}(F, G) = 0$. If K is equicharacteristic then $Ш(F, G) = 0$; and $Ш_{\widehat{X}}(F, G) = 0$ if in addition the given model $\widehat{X}$ is regular.*

- *G is a quasi-split group of type E_6 or E_7.*
- *G is an almost simple group that is quasi-split of absolute rank at most 5.*
- *G is an almost simple group that is quasi-split of type B_6 or D_6.*
- *G is an almost simple group that is split of type D_7.*

Proof. In each of these cases, the Rost invariant R_G has trivial kernel. This is by [Gar01, Main Theorem 0.1] in the first case, and by [Gar01, Theorem 0.5] in the other cases. So the assertion follows from Proposition 4.3.1(a). □

Corollary 4.3.3. *Let m be a square-free positive integer that is not divisible by the characteristic of k, and let A be a central simple F-algebra of degree m. Then the local-global map $H^1(F, \mathrm{SL}_1(A)) \to \prod_{P \in X} H^1(F_P, \mathrm{SL}_1(A))$ is injective. If in addition K is equicharacteristic and $\widehat{X}$ is regular, then the map $H^1(F, \mathrm{SL}_1(A)) \to \prod_{v \in \Omega_{\widehat{X}}} H^1(F_v, \mathrm{SL}_1(A))$ is injective.*

Proof. By [MS82, 12.2] (see also [Ser95, 7.2]), given a division algebra A of degree m, there is a cohomological invariant $a : H^1(*, \mathrm{SL}_1(A)) \to H^3(*, \mathbb{Z}/m\mathbb{Z}(2))$ that is injective if m is square-free. So the result follows from Proposition 4.3.1(b). □

In particular, $Ш_{\widehat{X},X}(F, \mathrm{SL}_1(A))$ and $Ш(F, \mathrm{SL}_1(A))$ respectively vanish in the above situations. Also, via the identification of $H^1(F, \mathrm{SL}_1(A))$ with $F^\times / \mathrm{Nrd}(A^\times)$, the above result gives a local-global principle for elements of $F^\times$ to be reduced norms from a (central) division algebra A; cf. also [Kat86, p. 146].

Other applications can be obtained by using a combination of cohomological invariants. This is done in the next results.

Proposition 4.3.4. *Let G be a simple linear algebraic group of type E_8 over F.*

(a) Assume $\mathrm{char}(K) = 0$. Then the group G is split over some odd degree extension of F if and only if G_{F_P} is split over some odd degree extension of F_P for every $P \in X$.

(b) Assume $\mathrm{char}(K) \neq 2, 3, 5$. Then the same holds for extensions of degree prime to five (rather than of odd degree) over F and each F_P.

(c) Assume in addition that K is equicharacteristic and $\widehat{X}$ is regular. Then the assertions in parts (a) and (b) hold with the fields F_P replaced by the fields F_v for all $v \in \Omega_{\widehat{X}}$.

Proof. For the forward implications, observe that if G is split over a finite extension E/F of degree $d = [E : F]$, and if F'/F is any field extension, then G also splits over the compositum $E' := EF'$ in an algebraic closure of F', and $[E' : F']$ divides d. Taking F' equal to F_P or F_v yields the forward implications. We now show the reverse implications.

Proof of (a) and the corresponding part of (c):

Let G_0 be a split simple algebraic group over F of type E_8. Then $H^1(F, G_0)$ classifies simple algebraic groups of type E_8 over F, since $G_0 = \mathrm{Aut}(G_0)$. Given a group G as in the proposition, let $[G]$ be the class of G in $H^1(F, G_0)$, and let $r_G := R_{G_0}([G])$ be the associated Rost invariant, say with order m. Thus $r_G \in H^3(F, \mathbb{Z}/m\mathbb{Z}(2))$.

For each $P \in X$, the group G becomes split over some extension E_P/F_P of odd degree d_P. Thus the Rost invariant of G over F_P maps to zero in $H^3(E_P, \mathbb{Z}/m\mathbb{Z}(2))$, and hence it is d_P-torsion in $H^3(F_P, \mathbb{Z}/m\mathbb{Z}(2))$ by a standard restriction-corestriction argument. Thus it is also d'_P-torsion, where d'_P is the greatest common divisor of d_P and m. Let d be the least common multiple of the odd integers d'_P, each of which divides m. Thus $dr_G \in H^3(F, \mathbb{Z}/m\mathbb{Z}(2))$ has trivial image in $H^3(F_P, \mathbb{Z}/m\mathbb{Z}(2))$ for all P. It follows from Theorem 3.2.3(i) that dr_G is trivial. Hence the order of the Rost invariant r_G over F is odd.

Let $H^1(*, G_0)_0 \subseteq H^1(*, G_0)$ be the subset consisting of classes α such that $R_{G_0}(\alpha)$ has odd order. By the above, this contains $[G]$. Now by [Sem09, Corollary 8.7], since $\mathrm{char}(F) = \mathrm{char}(K) = 0$, there is a cohomological invariant $u : H^1(*, G_0)_0 \to H^5(*, \mathbb{Z}/2\mathbb{Z})$ such that for any field extension E/F, the invariant $u([G_E])$ vanishes if and only if G splits over a field extension of E of odd degree.

By functoriality of u, the class $u([G])$ maps to $u([G_{F_P}])$ for every $P \in X$. But for every $P \in X$, G_{F_P} is split over an extension of odd degree and hence $u([G_{F_P}])$ is trivial in $H^5(F_P, \mathbb{Z}/2\mathbb{Z})$. By Theorem 3.2.3(i), it follows that $u([G])$ is trivial in $H^5(F, \mathbb{Z}/2\mathbb{Z})$. The conclusion of (a) now follows from the defining property of u.

The corresponding part of (c) is proved in exactly the same way, but with F_v replacing F_P and with Theorem 3.3.6 replacing Theorem 3.2.3(i).

Proof of (b) and the corresponding part of (c):

By the main theorem in [Che94], since $\mathrm{char}(F) \neq 2, 3, 5$, the Rost invariant of G over a field extension E/F has trivial image in $H^3(E, \mathbb{Z}/5\mathbb{Z}(2))$ if and only if G splits over some finite extension of E having degree prime to five. The desired assertion now follows from Proposition 4.3.1, taking $S(E)$ to be the subset of $H^1(E, G_0)$ that consists of elements that split over some field extension of E having degree prime to five, and with a being the restriction to this subset of the Rost invariant modulo 5. □

Proposition 4.3.5. *Assume that* $\mathrm{char}(K) \neq 2, 3$. *Then Albert algebras over F have each of the following properties if and only if the respective properties hold after base change to F_P for each $P \in X$.*

- *The algebra is reduced.*
- *The algebra is split.*
- *The automorphism group of the algebra is anisotropic.*
- *Two reduced algebras are isomorphic.*

The same holds for base change to F_v for each $v \in \Omega_{\widehat{X}}$, in the case that K is equicharacteristic and $\widehat{X}$ is regular.

Proof. Albert algebras are classified by $H^1(F, G)$, where G is a split simple linear algebraic group over F of type F_4. Moreover (see [Ser95, 9.2,9.3]) there are cohomological invariants

$$f_3 : H^1(F, G) \to H^3(F, \mathbb{Z}/2\mathbb{Z}),\ f_5 : H^1(F, G) \to H^5(F, \mathbb{Z}/2\mathbb{Z}),\ g_3 : H^1(F, G) \to H^3(F, \mathbb{Z}/3\mathbb{Z}),$$

where $H^3(F, \mathbb{Z}/3\mathbb{Z}) = H^3(F, \mathbb{Z}/3\mathbb{Z}(2))$. The properties of Albert algebras listed in the proposition are respectively equivalent to the following conditions involving these invariants (see [Ser95, 9.4]):

- The invariant g_3 vanishes on the algebra.

- The invariants f_3 and g_3 each vanish on the algebra.
- The invariants f_5 and g_3 are each non-vanishing on the algebra.
- The two reduced algebras have the same pair of invariants f_3, f_5.

By the injectivity of the local-global maps on $H^3(F, \mathbb{Z}/2\mathbb{Z})$, $H^5(F, \mathbb{Z}/2\mathbb{Z})$, and $H^3(F, \mathbb{Z}/3\mathbb{Z}(2))$ (viz. by Theorems 3.2.3(i) and 3.3.6 respectively), and by the functoriality of the invariants f_3, f_5, g_3, the assertion then follows. □

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