Correction and Addendum to “Embedding Problems with Local Conditions”

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This note is a correction and addendum to the author’s paper [Ha00], which concerned embedding problems with p-group kernel for affine varieties in characteristic p, and which showed that such embedding problems have proper solutions satisfying prescribed local conditions. The main results there relied on a general assertion about embedding problems with p-group kernel for profinite groups [Ha00, Theorem 2.3]. But the precise statement of that assertion needs to be corrected; and this is done here, in Theorem 1. As explained below, this correction to [Ha00, Theorem 2.3] does not affect the main results of [Ha00], nor the main results of [Ha03] (on strengthenings of Abhyankar’s Conjecture), where [Ha00, Theorem 2.3] was also cited. In particular, we show in Proposition 6 that every p-embedding problem for a Laurent series field $k((x))$ in characteristic p has a proper solution, thereby verifying [Ha03, Example 3.2(b)].

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Recall that if Π, Γ, G are profinite groups, then an *embedding problem* $\mathcal{E}$ *for* Π is a pair of surjective group homomorphisms $(\alpha : \Pi \to G, f : \Gamma \to G)$. A *weak solution* to $\mathcal{E}$ is a group homomorphism $\beta : \Pi \to \Gamma$ such that $f\beta = \alpha$; if β is surjective, it is a *proper solution* to $\mathcal{E}$. We call $\mathcal{E}$ *weakly* [resp. *properly*] *solvable* if it has a weak [resp. a proper] solution, and we call $\mathcal{E}$ a *p-embedding problem* if $\ker(\mathcal{E}) := \ker(f)$ is a p-group. Following standard terminology, we say here that $\mathcal{E}$ is *finite* if the group Γ is finite. (In [Ha00] and [Ha03], only a weaker condition was required in order for $\mathcal{E}$ to be called “finite”; but here we use the more usual definition. See also the parenthetical comment in the second paragraph following Theorem 1 below.)

Given a homomorphism $\phi_1 : \Pi_1 \to \Pi$ of profinite groups, we obtain an embedding problem $\phi_1^*(\mathcal{E}) := (\alpha\phi_1 : \Pi_1 \to G_1, f_1 : \Gamma_1 \to G_1)$ for Π_1, the *pullback* of $\mathcal{E}$ to Π_1; here $G_1 = \alpha\phi_1(\Pi_1) \subset G$, $\Gamma_1 = f^{-1}(G_1) \subset \Gamma$, and $f_1 = f|_{\Gamma_1}$. For any weak solution $\beta : \Pi \to \Gamma$, its *pullback* $\phi_1^*(\beta) := \beta\phi_1 : \Pi_1 \to \Gamma_1$ is a weak solution to $\phi_1^*(\mathcal{E})$. If $\phi = \{\phi_j\}_{j\in J}$ is a family of homomorphisms $\phi_j : \Pi_j \to \Pi$ of profinite groups, $\mathcal{E}$ is *weakly* [resp. *properly*] *ϕ-solvable* if for every collection $\{\beta_j\}_{j\in J}$ of weak solutions to the pulled back embedding problems $\phi_j^*(\mathcal{E})$, there is a weak [resp. proper] solution β to $\mathcal{E}$ and elements $n_j \in N = \ker(\mathcal{E})$ such that $\phi_j^*(\beta) = \mathrm{inn}(n_j) \circ \beta_j$ for all $j \in J$. (Here $\mathrm{inn}(n_j) \in \mathrm{Aut}(\Gamma)$ is left conjugation by n_j.) The family ϕ is *p-dominating* [resp. *strongly p-dominating*] if $\phi^* : H^1(\Pi, P) \to \prod_{j\in J} H^1(\Pi_j, P)$ is surjective [resp. surjective with infinite kernel] for every non-trivial finite elementary abelian p-group P on which Π acts continuously.

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Here, we also define a profinite group Π to be *strongly p-dominating* if the one-element family $\phi = \{1 \to \Pi\}$ is strongly p-dominating (in the above sense). This is equivalent to saying that $H^1(\Pi, P)$ is infinite for every non-trivial finite elementary abelian p-group P on which Π acts continuously. This condition implies that Π has infinite p-rank.

The following is a corrected form of [Ha00, Theorem 2.3]:

Theorem 1. *Let p be a prime number and let Π be a profinite group. Consider the following four conditions (i)-(iv):*

(i) Every finite p-embedding problem for Π is weakly solvable (i.e. $\mathrm{cd}_p(\Pi) \leq 1$).
(ii) Every finite p-embedding problem for Π is weakly ϕ-solvable, for every p-dominating family of homomorphisms $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$.
(iii) Every finite p-embedding problem for Π is properly solvable.
(iv) Every finite p-embedding problem for Π is properly ϕ-solvable, for every strongly p-dominating family of homomorphisms $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$.

a) Then we have the implications (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv).
b) Moreover if Π is strongly p-dominating then the four conditions are equivalent.

As originally stated in [Ha00], the theorem had asserted that if Π has infinite p-rank then the four conditions (i)-(iv) are equivalent. But that hypothesis is not sufficient to conclude that a strongly p-dominating family $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$ exists, which is what is needed to obtain the implication (iv) $\Rightarrow$ (iii). The new additional hypothesis in (b) is equivalent to the existence of such a family, thereby allowing (b) to follow from (a).

Part (a) above holds even without the assumption of Π having infinite p-rank, because in the proof of [Ha00, Theorem 2.3] that assumption was invoked only for (iv) $\Rightarrow$ (iii), and not in the proofs of the implications in (a) above. Note in particular that the implication (ii) $\Rightarrow$ (iv) is vacuously true if the infinite p-rank hypothesis does not hold for Π, since in that case there are *no* strongly p-dominating families of homomorphisms $\phi = \{\phi_j : \Pi_j \to \Pi\}_{j \in J}$. Namely, for $P = \mathbb{Z}/p\mathbb{Z}$ with trivial action, $H^1(\Pi, P)$ and hence $\ker \phi^*$ will be finite. (Note also that the proof of (ii) $\Rightarrow$ (iv) in [Ha00] used that $H^1(G, P)$ is finite if $P = (\mathbb{Z}/p\mathbb{Z})^m$ is the kernel of a finite p-embedding problem $\mathcal{E}$ as above. This holds under the above definition of $\mathcal{E}$ being finite, since then G is also finite; but it might not hold under the weaker definition given in [Ha00], where only $P = \ker(\mathcal{E})$ was required to be finite.)

Theorem 2.3 of [Ha00] was cited several times later in that paper, in order to solve geometric embedding problems: in Corollary 3.10, Corollary 4.2, and Corollary 5.5 of [Ha00]. In each of those cases, the only part of Theorem 2.3 that was used is (i) $\Rightarrow$ (iv). That implication was unaffected by the above correction to the statement of Theorem 2.3, and so those latter three results are also unaffected. (In fact, though, the profinite groups considered in those three results were all strongly p-dominating. This property was shown in Proposition 3.8, Proposition 4.1, and Proposition 5.3 of [Ha00], respectively, by taking $r = 0$. So part (b) above applies there too.)

Example 2. As a counterexample to the original form of [Ha00, Theorem 2.3] with $p > 2$, let F be a free pro-p-group of countably infinite p-rank, let ℓ be a prime dividing $p-1$, and let $\Pi = F \times \mathbb{Z}_\ell$, a profinite group of infinite p-rank. So Π is projective, i.e. $\mathrm{cd}(\Pi) = 1$, since Π is the product of a free pro-p group and a free pro-ℓ-group (and these Sylow subgroups respectively satisfy $\mathrm{cd}_p = 1$ and $\mathrm{cd}_\ell = 1$). Let $\Gamma = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/\ell\mathbb{Z}$, where $\mathbb{Z}/\ell\mathbb{Z}$ acts on $\mathbb{Z}/p\mathbb{Z}$ non-trivially, as a subgroup of $(\mathbb{Z}/p\mathbb{Z})^* \approx \mathbb{Z}/(p-1)\mathbb{Z}$. Consider the surjections $\alpha : \Pi \to \mathbb{Z}/\ell\mathbb{Z}$ and $f : \Gamma \to \mathbb{Z}/\ell\mathbb{Z}$. Then the p-embedding problem (α, f) has no proper solution, because Γ is not a quotient of Π (since $\mathbb{Z}_\ell$ acts trivially on F in Π). Hence (iii) fails for Π. But Π satisfies (i), since $\mathrm{cd}_p(\Pi) = 1$.

As mentioned above, [Ha00, Theorem 2.3] was cited in the proof of [Ha03, Lemma 3.1]. Because of the correction to the statement of [Ha00, Theorem 2.3] (in Theorem 1 above), the statement of [Ha03, Lemma 3.1] also needs to be adjusted, as follows:

Lemma 3. *Let R be an integral domain of characteristic p, let $X = \operatorname{Spec} R$, and suppose that $\pi_1(X)$ is strongly p-dominating. Let P be a non-trivial normal p-subgroup of a finite group Γ, and let $Y \to X$ be a connected Γ/P-Galois étale cover. Then there are infinitely many connected Γ-Galois étale covers $Z \to X$ that dominate $Y \to X$ and are linearly disjoint over Y.*

The original statement of [Ha03, Lemma 3.1] had assumed just that $R/\wp(R)$ is infinite (where $\wp(x) := x^p - x$), and not the stronger hypothesis that $\pi_1(X)$ is strongly p-dominating. But the proof of [Ha03, Lemma 3.1] had cited the implication (i) $\Rightarrow$ (iii) of [Ha00, Theorem 2.3]; and as in Theorem 1 above, this requires assuming the stronger hypothesis. With this change as above, the proof of the lemma goes through as before.

Note that the conclusion of Lemma 3 is equivalent to the (*a priori* weaker) assertion that every finite p-embedding problem for $\pi_1(X)$ has a proper solution. Namely, for any $n > 0$ let Γ^n_G be the nth fibre power of Γ over $G = \Gamma/P$, and consider the p-embedding problem $\mathcal{E}_n$ corresponding to the exact sequence $1 \to P^n \to \Gamma^n_G \to G \to 1$. Via projections $\Gamma^n_G \to \Gamma$, each proper solution to $\mathcal{E}_n$ induces n Γ-Galois étale covers $Z \to X$ that are linearly disjoint over Y; and this holds for every n, yielding infinitely many linearly disjoint choices.

Example 4. Example 2 above can be used to provide a counterexample to the original form of [Ha03, Lemma 3.1]. Namely, let k_0 be an algebraically closed field of characteristic $p \neq 2$, and let $k = k_0(t)$. So the absolute Galois group G_k of k is a free profinite group of infinite rank ([Ha95], [Po]). Hence there is a (continuous) surjection $G_k \to \Pi$, where Π is as in Example 2. Since Π is projective, the surjection $G_k \to \Pi$ has a section s. The fixed field of $s(\Pi) \subset G_k$ is a field K of characteristic p whose absolute Galois group is Π. (In fact, by [LvD, Prop. 4.8], there is even such a field K which is PAC.) Here $K/\wp(K)$ is infinite, because the free pro-p group F of countable rank is a quotient of Π. By Example 2, not every finite p-embedding problem for Π is properly solvable. So there is a finite group Γ with normal p-subgroup P such that some Γ/P-Galois field extension of K cannot be embedded in any Γ-Galois field extension of K. Taking $R = K$ gives a counterexample to the original form of [Ha03, Lemma 3.1].

Example 3.2 of [Ha03] had asserted that the conclusion of Lemma 3 above holds in characteristic p for irreducible affine varieties of dimension > 0 and for one-variable

Laurent series fields; and part (b) of that example was afterwards used in the proof of [Ha03, Proposition 3.4]. This example relied on the original form of [Ha03, Lemma 3.1]; but despite the stronger hypotheses of Lemma 3 above, the example remains correct:

Example 5. We verify Example 3.2 of [Ha03]:

(a) Let X be an irreducible affine variety of finite type over a field of characteristic p, other than a point. Then as asserted in Example 3.2(a) of [Ha03], the conclusion of [Ha03, Lemma 3.1] (or equivalently of Lemma 3 above) holds for X. This is because $\pi_1(X)$ is strongly p-dominating by [Ha00, Proposition 3.8].

(b) As asserted in Example 3.2(b) of [Ha03], the conclusion of [Ha03, Lemma 3.1] (i.e. of Lemma 3 above) holds in the case $R = K := k((x))$, with k any field of characteristic p. In the case that k is algebraically closed (this being the case that was used in [Ha03, Proposition 3.4]), this holds because $\pi_1(X) = G_K$ is strongly p-dominating by [Se, Proposition 5]. In the case of a more general field k, the assertion follows from the next proposition (cf. the comment just before Example 4 above), whose proof essentially applies the situation of Example (a) above to an associated "Katz-Gabber cover".

Proposition 6. *If k is an arbitrary field of characteristic p, then every finite p-embedding problem $\mathcal{E}$ for the absolute Galois group G_K of $K = k((x))$ has a proper solution.*

Proof. Write $\mathcal{E} = (\alpha : G_K \to G, f : \Gamma \to G)$, let $P = \ker f$, and let L be the G-Galois field extension of K corresponding to α. Identify K with the fraction field of the complete local ring of $\bar{X} := \mathbb{P}^1_k$ at the point at infinity. By [Ka, Theorem 1.4.1], there is a G-Galois branched cover $\bar{Y} \to \bar{X}$ that is étale away from the points $0, \infty$; is tamely ramified over the point 0; and whose pullback to $\operatorname{Spec} K$ is $\operatorname{Spec} L \to \operatorname{Spec} K$. Let $Y \to X := \mathbb{A}^1_k$ be the pullback of $\bar{Y} \to \bar{X}$ to the affine line. By [Ha00, Theorem 5.14], there is a connected P-Galois étale cover $Z \to Y$ such that $Z \to X$ is Γ-Galois. Let $\bar{Z}$ be the normalization of $\bar{Y}$ in Z, and let $\operatorname{Spec} M$ be the pullback of $\bar{Z}$ to $\operatorname{Spec} K$. So M is a Γ-Galois K-algebra, and it suffices to show that M is a field. For this, it suffices to show that $\bar{Z}$ has just one closed point over the point at infinity on $\bar{X}$.

Let $\bar{Y}_0 = \bar{Y}/I$, where $I \subset G$ is the inertia group of $\bar{Y} \to \bar{X}$ over ∞ (or equivalently, the inertia group of L/K). Then $\bar{Y}_0 \to \bar{X}$ is étale away from the point 0 and is at most tamely ramified at 0. So $\bar{Y}_0 \to \bar{X} = \mathbb{P}^1_k$ is unramified everywhere, and $\bar{Y}_0 = \bar{X}_\ell = \mathbb{P}^1_\ell$ for some finite Galois extension ℓ/k. Note that $\bar{Y} \to \bar{Y}_0$ is totally ramified over ∞, and that ℓ is the algebraic closure of k in the function field $k(\bar{Y})$. So $k(\bar{Y})$ is linearly disjoint over $k(\bar{X}_\ell)$ from the algebraic closure $\bar{k}$. Hence the Galois group $I = \operatorname{Gal}(\bar{Y}/\bar{X}_\ell)$ remains the same under base change from ℓ to $\bar{k}$. Thus I has a normal Sylow p-subgroup $\bar{P}$ with a cyclic quotient C; and the C-Galois cover $\bar{Y}_1 := \bar{Y}/\bar{P} \to \bar{X}_\ell = \mathbb{P}^1_\ell$, branched only over $0, \infty$, has genus 0. This cover is tamely ramified, and ℓ is algebraically closed in the function field $k(\bar{Y}_1)$. So this cover has no residue field extension over the ℓ-rational points $0, \infty$ of $\bar{X}_\ell = \mathbb{P}^1_\ell$; i.e. the points of $\bar{Y}_1$ over $0, \infty$ are also ℓ-points. Note that $\bar{Z} \to \bar{Y}_1$ is unramified away from ∞ because $\tilde{P} = \operatorname{Gal}(\bar{Z}/\bar{Y}_1)$ is a p-group and the ramification over 0 is tame.

Let D be the decomposition group of a closed point of $\bar{Z}$ over the unique closed point of

$\bar{Y}_1$ over ∞. To conclude the proof it suffices to show that $D = \tilde{P}$. If D is strictly contained in $\tilde{P}$, then it is contained in a proper normal subgroup N of $\tilde{P}$, since $\tilde{P}$ is a p-group. So $\bar{Z}_0 := \bar{Z}/N \to \bar{Y}_1$ is unramified everywhere, with trivial decomposition group over the point of $\bar{Y}_1$ at ∞, which is an ℓ-point. So $\bar{Z}_0$ has an ℓ-point, and thus ℓ is algebraically closed in the function field $k(\bar{Z}_0)$. But $\bar{Y}_1$ is an ℓ-curve of genus 0. So $\bar{Z}_0 \to \bar{Y}_1$ must be trivial, which contradicts the fact that N is a proper subgroup of $\tilde{P} = \mathrm{Gal}(\bar{Z}/\bar{Y}_1)$. □

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