On function fields with free absolute Galois groups

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Abstract. We prove that certain fields have the property that their absolute Galois groups are free as profinite groups: the function field of a real curve with no real points; the maximal abelian extension of a 2-variable Laurent series field over a separably closed field; and the maximal abelian extension of the function field of a curve over a finite field. These results are related to generalizations of Shafarevich's conjecture. Related results about quasi-free groups are also shown, in particular that the commutator subgroup of a quasi-free group is quasi-free.

§1: Introduction.

This paper shows that the absolute Galois groups of certain fields are free as profinite groups. Although these fields arise in geometric contexts, our results are related to Shafarevich's conjecture on absolute Galois groups, which he posed in the context of number theory. In its original form, that conjecture states that the absolute Galois group of $\mathbb{Q}^{\rm ab}$ is free, where $\mathbb{Q}^{\rm ab}$ is the maximal abelian extension of $\mathbb{Q}$. Since $\mathbb{Q}^{\rm ab}$ is also the maximal cyclotomic extension of $\mathbb{Q}$, Shafarevich's conjecture has been generalized to assert that for any global field K, the absolute Galois group of the maximal cyclotomic extension of K is free (see e.g. [Ha2]). That conjecture remains open in the number field case (where it would be sufficient to prove it for $\mathbb{Q}$, by [FJ], Prop. 17.6.2); but it was proven in the function field case in [Ha1] and [Po1] (cf. also [HJ4]). In that case, the conjecture is equivalent to saying that for any curve over $\bar{\mathbb{F}}_p$, the absolute Galois group of the function field is free. In [Ha1] and [Po1] even more was shown: that freeness holds for any curve over *any* algebraically closed field.

The result in [Ha1] and [Po1] suggests asking what happens over fields K that are "almost" algebraically closed; i.e. such that $[\bar{K} : K]$ is finite, where $\bar{K}$ is the algebraic closure. By the theorem of Artin-Schreier, these are precisely the real closed fields (e.g. $\mathbb{R}$); and in Theorem 4.2 we show that the function field of a curve X over a real closed field R has free absolute Galois group if and only if X has no R-points.

Another natural generalization of Shafarevich's conjecture is to assert that for any global field K, the absolute Galois group of the maximal abelian extension of K is free. In Theorem 4.1 we show that this holds in the function field case. The proof relies on the corresponding result in [Ha1] and [Po1] about the maximal cyclotomic extension. The number field case of this conjecture, too, remains open.

While Shafarevich's conjecture and its generalizations above concern one-dimensional function fields, it is also possible to consider higher dimensional versions. There, for

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cohomological reasons one must rule out the case of finite coefficient fields; and here we consider function fields over a separably closed field. In the local case of a smooth surface over a separably closed field k, Theorem 4.6 shows that the absolute Galois group of the maximal abelian extension of $k((x,y))$ is free. The first example of the global case would be to ask whether the same holds for $k(x,y)$ for k separably closed. This remains open.

The key approach here is to use that a profinite group is free if and only if it is projective and quasi-free ([HS], Theorem 2.1); see Section 2 below for definitions. In the current paper it is shown that the absolute Galois group of the function field of any curve over a real closed field is quasi-free, as is the maximal abelian extension of $k((x,y))$ for k *any* field (using Theorem 2.4). Projectivity is classical in the situation of Theorem 4.2, and it follows from [COP] (as generalized in Theorem 4.4 below) in the situation of Theorem 4.6; so in each case freeness then results. This same approach could also be used to provide another proof of the freeness result of [Ha1] and [Po1] referred to above, which we use here to obtain the freeness of the maximal abelian extension of the function field of a curve over a finite field (Theorem 4.1).

One could also use the above approach in considering the two-dimensional global version of Shafarevich's conjecture. Namely, let K^{ab} be the maximal abelian extension of $K = k(x,y)$, for k algebraically closed. Then the absolute Galois group $G_{K^{\mathrm{ab}}}$ is free if and only if it is projective and quasi-free. Here $G_{K^{\mathrm{ab}}}$ is the commutator subgroup of G_K; and projectivity would follow from knowing for every ℓ that a Sylow ℓ-subgroup of this commutator is a free pro-ℓ group. This condition would imply a conjecture of Bogomolov [Bo], asserting that the commutator subgroup of every Sylow ℓ-subgroup of G_K is a free pro-ℓ group. That conjecture is open, as is the quasi-freeness of $G_{K^{\mathrm{ab}}}$, even in the case of $K = \mathbb{C}(x,y)$. The above considerations suggest a stronger conjecture that if k is an algebraically closed field (or even a field containing all roots of unity) and K is a function field over k, then $G_{K^{\mathrm{ab}}}$ is free. Such a conjecture has been proposed by F. Pop.

After providing background material and definitions concerning profinite groups, Section 2 of this paper proves a key result on quasi-free groups (Theorem 2.4): that the commutator subgroup of a quasi-free group is quasi-free, of the same rank. Section 3, which discusses aspects of field arithmetic, reinterprets the results of the previous section in terms of absolute Galois groups of abelian extensions, and shows (in Theorem 3.4) that the absolute Galois group of the function field of a curve over a large field is quasi-free. The results on free absolute Galois groups are shown in Section 4, using results from the previous sections together with other results.

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Section 2. Profinite groups.

Let Π be a profinite group (i.e. an inverse limit of finite groups). An *embedding problem* $\mathcal{E}$ for Π is a pair of epimorphisms $(\alpha : \Pi \to G,\ f : \Gamma \to G)$ of profinite groups; it is *non-trivial* if ker$f$ is non-trivial and it is *finite* if Γ is finite. (Here and below, homomorphisms are required to be continuous.) A *weak solution* to $\mathcal{E} = (\alpha,\ f)$ consists of a homomorphism $\lambda : \Pi \to \Gamma$ such that $f \circ \lambda = \alpha$. A solution is called *proper* if it is surjective. A finite embedding (α, f) in which we also have a splitting $s : G \to \Gamma$ of f is called a *finite split embedding problem*. Every finite split embedding problem has a weak solution given by $s \circ \alpha$. A profinite group Π is *projective* if every finite embedding problem for Π has a weak solution. Being projective is equivalent to having cohomological dimension at most 1 ([Gru], Theorem 4; or [Se], I, §3.4 Prop. 16 and §5.9 Prop. 45).

A subset S of a profinite group Π *converges to* 1 if $S \cap (\Pi - N)$ is finite for every open normal subgroup N of Π. Similarly, a map $\phi : S \to G$ to a profinite group G *converges to* 1 if $S \cap \phi^{-1}(G - N)$ is finite for every open normal subgroup N of G. The *rank* of Π is the smallest cardinal number $m = d(\Pi)$ such that Π has a set of (topological) generators of cardinality m that converges to 1. In fact, if the rank of Π is infinite, any two such generating sets have the same cardinality ([FJ], Prop. 17.1.2). Note that if the rank m of Π is infinite, then there are at most m (continuous) homomorphisms from Π to any finite group, since the kernel must be open. Thus a finite embedding problem for Π can have at most m (weak or proper) solutions, if $m = d(\Pi)$ is infinite.

A profinite group Π is *free* on a generating set S that converges to 1 if every map $S \to G$ to a profinite group G that converges to 1 uniquely extends to a group homomorphism $\Pi \to G$. For every cardinal m there is a free profinite group of rank m ([FJ], §17.4), denoted $\hat{F}_m$; this is unique up to isomorphism. A profinite group Π is *ω-free* if every finite embedding problem for Π has a proper solution. Every free profinite group is ω-free. And by a theorem of Iwasawa ([Iw], p.567), a profinite group of countable rank is free if and only if it is ω-free. But this equivalence fails for uncountably generated profinite groups ([Ja1], Example 3.1). Instead, there is the following result of Melnikov and Chatzidakis ([Ja1], Lemma 2.1): if m is an infinite cardinal, then a profinite group Π is free of rank m if and only if every non-trivial finite embedding problem for Π has exactly m proper solutions.

Following [HS] and [RSZ], we say that a profinite group Π is *quasi-free* if there is a cardinal number m such that every non-trivial finite split embedding problem for Π has exactly m proper solutions; to indicate the cardinal, we may say that Π is *m-quasi-free*. It is easy to see that m is necessarily infinite. Also, m is necessarily equal to the rank of Π [RSZ]; so being m-quasi-free is equivalent to being quasi-free of rank m. As a variant on the result of Melnikov and Chatzidakis, if m is an infinite cardinal, then a profinite group Π is free of rank m if and only if it is projective and m-quasi-free ([HS], Theorem 2.1, or [FJ], Lemma 25.1.8). If Π has countable rank, freeness is also equivalent to the condition that Π is projective and every finite split embedding problem for Π has a proper solution ([HS], Corollary 2.8), in analogy with Iwasawa's theorem.

The main goal of this section is to show (in Theorem 2.4 below) that if Π is a quasi-free profinite group, then its commutator subgroup Π' is also quasi-free, of the same rank. We begin with some lemmas.

Lemma 2.1. *Let Π be a profinite group, let Π_1 be a closed subgroup that contains the commutator subgroup of Π, and let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be a finite embedding problem for Π with proper solutions β_1, β_2 having kernels M_1, M_2. Assume that $\Pi_1 \cap M_1 = \Pi_1 \cap M_2$ and that $Z \cap N = 1$, where Z is the center of Γ and $N = \ker f$. Then $M_1 = M_2$.*

Proof. Let $\Pi_0 = \Pi_1(M_1 \cap M_2) \subset \Pi$. We first claim that $M_1 \cap M_2 = \Pi_0 \cap M_2$. Here the containment $M_1 \cap M_2 \subset \Pi_0 \cap M_2$ is clear. For the other containment, let $m_2 \in \Pi_0 \cap M_2$. Since $m_2 \in \Pi_0$, we may write $m_2 = p_1 m$ for some $p_1 \in \Pi_1$ and $m \in M_1 \cap M_2 \subset M_2$. Here $p_1 = m_2 m^{-1} \in M_2$. Therefore $p_1 \in \Pi_1 \cap M_2 = \Pi_1 \cap M_1 \subset M_1$. Thus $p_1 \in M_1 \cap M_2$, and so $m_2 = p_1 m \in M_1 \cap M_2$. This proves the claim.

Define $\tilde{\beta} : \Pi \to \Gamma \times_G \Gamma$ by $\tilde{\beta}(x) = (\beta_1(x), \beta_2(x))$, and let $\tilde{\Gamma} = \tilde{\beta}(\Pi)$. So $\ker \tilde{\beta} = M_1 \cap M_2 = \Pi_0 \cap M_2$, by the previous paragraph. We next claim that $\tilde{\beta}(\Pi_0) \cap \tilde{\beta}(M_2) = 1$. To see this, let $p \in \tilde{\beta}(\Pi_0) \cap \tilde{\beta}(M_2)$. So $p = \tilde{\beta}(p_0) = \tilde{\beta}(m_2)$ for some $p_0 \in \Pi_0$ and $m_2 \in M_2$. Hence $\tilde{\beta}(p_0 m_2^{-1}) = 1$; i.e., $p_0 m_2^{-1} \in \ker(\tilde{\beta}) \subset M_2$. Since $m_2 \in M_2$, so is p_0; thus $p_0 \in \Pi_0 \cap M_2 = \ker, \tilde{\beta}$. So $p = \tilde{\beta}(p_0) = 1$, thereby showing the claim.

By assumption, the commutator subgroup Π' of Π is contained in Π_1 and hence in Π_0; so the commutator subgroup $\tilde{\Gamma}' = \tilde{\beta}(\Pi')$ of $\tilde{\Gamma}$ is contained in $\tilde{\beta}(\Pi_0)$. Now $\tilde{\beta}(M_2) \subset \tilde{\beta}(\Pi) = \tilde{\Gamma}$, so the commutator group $[\tilde{\Gamma}, \tilde{\beta}(M_2)]$ is contained in $[\tilde{\Gamma}, \tilde{\Gamma}] = \tilde{\Gamma}' \subset \tilde{\beta}(\Pi_0)$. Meanwhile, $\tilde{\beta}(M_2)$ is a normal subgroup of $\tilde{\Gamma} = \tilde{\beta}(\Pi)$ since M_2 is normal in Π. So $[\tilde{\Gamma}, \tilde{\beta}(M_2)]$ is contained in $\tilde{\beta}(M_2)$ and hence in $\tilde{\beta}(\Pi_0) \cap \tilde{\beta}(M_2) = 1$ by the previous claim. Thus $\tilde{\beta}(M_2)$ is contained in $\tilde{Z}$, the center of $\tilde{\Gamma}$. But the projection $\Gamma \times_G \Gamma \to \Gamma$ onto the first coordinate maps $\tilde{\beta}(M_2)$ onto $\beta_1(M_2)$ and maps $\tilde{Z}$ into Z. So the containment $\tilde{\beta}(M_2) \subset \tilde{Z}$ implies that $\beta_1(M_2) \subset Z$. Now $f\beta_1(M_2) = \alpha(M_2) = f\beta_2(M_2) = 1$, so $\beta_1(M_2) \subset \ker f = N$. Hence $\beta_1(M_2) \subset Z \cap N = 1$. Thus $M_2 \subset \ker \beta_1 = M_1$. Similarly $M_1 \subset M_2$. □

Lemma 2.2. *Let Π be a quasi-free profinite group of rank m, and let Π' be its commutator subgroup. Let p be a prime number. Then there exist m open normal subgroups of Π' having index p.*

Proof. Define a 2×2-matrix A_p over $\mathbb{F}_p$ as follows: If $p = 2$ the rows are $(0\ 1)$ and $(1\ 1)$; and if $p \neq 2$ the rows are $(0\ a)$ and $(1\ 0)$, where $a \in \mathbb{F}_p^*$ is not a square. Let r be the order of A_p in $\mathrm{GL}(2, p)$. So the cyclic group C_r acts on the two-dimensional $\mathbb{F}_p$-vector space $\mathbb{F}_p^2$ via left multiplication of the matrix A_p on column vectors. This action is irreducible over $\mathbb{F}_p$, since the minimal polynomial $f_p(x) \in \mathbb{F}_p[x]$ of A_p is irreducible. (Namely, $f_2(x) = x^2 - x - 1$ and $f_p(x) = x^2 - a$ for p odd.)

Let $\Gamma = C_p^2 \rtimes C_r$, where the conjugation action of C_r on the group $C_p^2 \approx \mathbb{F}_p^2$ is via A_p as above. Since this action is irreducible, C_p^2 has no non-trivial proper subgroup that is normal in Γ. Note that if Z is the center of Γ, then C_p^2 is not contained in Z; and hence $Z \cap C_p^2 = 1$, being a normal subgroup of Γ contained in C_p^2. Also, the commutator subgroup Γ' of Γ is non-trivial since Γ is non-abelian, and it is contained in C_p^2 since Γ/C_p^2 is abelian. Again by irreducibility, $\Gamma' = C_p^2$. So the maximal abelian quotient of Γ is C_r.

Since Π is quasi-free of rank m, Π has m distinct open normal subgroups with quotient group isomorphic to C_r. Picking one of them, say Λ, and an epimorphism $\alpha : \Pi \to C_r$ with kernel Λ, we obtain a finite split embedding problem for Π given by this map and the split exact sequence $1 \to C_p^2 \to \Gamma \to C_r \to 1$. Again, since Π is quasi-free of rank m, there are m distinct proper solutions $\beta : \Pi \to \Gamma$ to this embedding problem. Taking kernels, we have that there are m distinct normal subgroups $M \subset \Pi$ such that $\Pi/M \approx \Gamma$ and $M \subset \Lambda$ (using that there are only finitely many epimorphisms to a given finite group with a given kernel). For each such M, the subgroup of Π generated by Π' and M is the minimal normal subgroup of Π that contains M and has abelian quotient; i.e. $\Pi' M = \Lambda = \ker \alpha$ (by the maximality assertion in the previous paragraph). Thus $\Pi'/(\Pi' \cap M) \approx \Lambda/M = \ker((\Pi/M) \to (\Pi/\Lambda)) \approx \ker(\Gamma \to C_r) = C_p^2$.

So for any two such normal subgroups M_1, M_2 (among the m given by solutions to the embedding problem), we may apply Lemma 2.1; and conclude that if $\Pi' \cap M_1 = \Pi' \cap M_2$ then $M_1 = M_2$. That is, the m solutions to the embedding problem induce m distinct normal subgroups of Π' having quotient C_p^2. Taking the inverse images of $1 \times C_p$ and of $C_p \times 1$ under such quotient maps, we obtain m normal subgroups of Π' having quotient group C_p, possibly with repetitions. If there are precisely $m' \le m$ distinct normal subgroups of Π' with quotient C_p arising this way, then the number of normal subgroups of Π' with quotient C_p^2 obtained by taking intersections is also m'. But there are (at least) m of them, as noted above. So in fact $m' = m$. $\square$

Remark. a) Lemma 2.2 is a weak form of Theorem 2.4 below, in the case of a split embedding problem corresponding to a short exact sequence $1 \to C_p \to C_p \to 1 \to 1$.

b) In Lemma 2.2, Π' has at most m open normal subgroups (and hence *exactly* m of index p), because the cardinality of these is equal to the rank of Π' ([FJ], Prop. 17.1.2) and because the rank of the closed subgroup $\Pi' \subset \Pi$ is at most that of Π ([FJ], Cor. 17.1.5).

Let $\iota : \Pi_0 \to \Pi_1$ be a homomorphism of profinite groups (e.g. an inclusion), let $f : \Gamma \to G$ be an epimorphism of finite groups, and let $\alpha_i : \Pi_i \to G$ be an epimorphism for $i = 0, 1$. Thus $\mathcal{E}_i = (\alpha_i : \Pi_i \to G, f : \Gamma \to G)$ is a finite embedding problem for Π_i. We say that $\mathcal{E}_1$ *induces* $\mathcal{E}_0$ if $\alpha_0 = \alpha_1 \circ \iota$. If $\mathcal{E}_1$ induces $\mathcal{E}_0$ and if $\beta_i : \Pi_i \to \Gamma$ is a weak solution to $\mathcal{E}_i$ for $i = 0, 1$, we say that β_1 *induces* β_0 if $\beta_0 = \beta_1 \circ \iota$. Note that if β_1 is a proper solution to $\mathcal{E}_1$, then the induced solution β_0 to $\mathcal{E}_0$ need not be proper.

Lemma 2.3. *Let Π be a profinite group that is quasi-free of rank m, let Π' be its commutator subgroup, and let $\mathcal{E} = (\alpha : \Pi' \to G, f : \Gamma \to G)$ be a non-trivial finite split embedding problem for Π'. Then there is an open normal subgroup $\Pi_1 \subset \Pi$ containing Π' together with an embedding problem $\mathcal{E}_1 = (\alpha_1 : \Pi_1 \to G, f : \Gamma \to G)$ that induces $\mathcal{E}$, such that Π_1 has a set of m open subgroups each of which is the kernel of a proper solution to $\mathcal{E}_1$ that induces a proper solution to $\mathcal{E}$.*

Proof. Let $\tilde{\Lambda} = \ker\alpha \subset \Pi'$ and let $N = \ker f \neq 1$. By [FJ], Lemma 1.2.5(c), there is an open normal subgroup $\Pi_0 \subset \Pi$ containing Π', together with an embedding problem $\mathcal{E}_0 = (\alpha_0 : \Pi_0 \to G, f : \Gamma \to G)$ that induces $\mathcal{E}$. Let $\Lambda_0 = \ker\alpha_0 \subset \Pi_0$. So $\tilde{\Lambda} = \Pi' \cap \Lambda_0$.

Since Π is quasi-free of rank m, there are m open normal subgroups $\Phi \subset \Pi$ such that $\Pi/\Phi \approx C_2$ (arising from the embedding problem for Π corresponding to the exact sequence $1 \to C_2 \to C_2 \to 1 \to 1$). Thus in particular we may choose such a Φ that does not contain Λ_0 (since only finitely many subgroups of Π contain the finite index subgroup Λ_0). Note that Φ and Λ_0 generate Π, since $(\Pi : \Phi) = 2$. Let $\Pi_1 = \Pi_0 \cap \Phi \subset \Pi$ and let $\Lambda_1 = \Lambda_0 \cap \Phi = \Lambda_0 \cap \Pi_1 \subset \Pi$. Thus Π_1 is an open normal subgroup of Π_0, and Λ_1 is an open normal subgroup of the groups Π_1, Λ_0 and Π_0. Here $(\Lambda_0 : \Lambda_1) = (\Pi_0 : \Pi_1) = 2$, since $(\Pi : \Phi) = 2$ and since Φ does not contain Λ_0. So Π_0 is generated by Λ_0 and Π_1. Hence the natural map $\Pi_1/\Lambda_1 \hookrightarrow \Pi_0/\Lambda_0 = G$ (through which the isomorphism $G = \Pi'/\tilde{\Lambda} \xrightarrow{\sim} \Pi_0/\Lambda_0$ factors) is an isomorphism, and we have isomorphisms $\Pi_0/\Lambda_1 \xrightarrow{\sim} \Pi_0/\Lambda_0 \times \Pi_0/\Pi_1 \xrightarrow{\sim} G \times C_2$. So the restriction $\alpha_1 : \Pi_1 \to G$ of $\alpha_0 : \Pi_0 \to G$ is surjective with kernel Λ_1, and α_1 in turn restricts to $\alpha : \Pi' \to G$, whose kernel $\tilde{\Lambda}$ is contained in Λ_1. Thus the embedding problem $\mathcal{E}_1 = (\alpha_1 : \Pi_1 \to G, f : \Gamma \to G)$ induces $\mathcal{E}$. So the natural map $G = \Pi'/\tilde{\Lambda} \to \Pi_1/\Lambda_1$ is an isomorphism; and hence $\Pi_1 = \Pi'\Lambda_1$. Moreover α_1 lifts to a surjection $\hat{\alpha}_1 : \Pi_0 \to G \times C_2$ having kernel Λ_1, corresponding to the above isomorphism $\Pi_0/\Lambda_1 \xrightarrow{\sim} G \times C_2$.

Every open subgroup of a quasi-free group is also quasi-free of the same rank [RSZ]. So Π_0 is quasi-free of rank m. Let $\hat{\Gamma}$ be the semi-direct product of $N \times N$ with the group $G \times C_2$, where G acts on each factor N as it does in Γ, and where C_2 acts by interchanging the two copies of N. Also, let $\hat{f}_1 : \hat{\Gamma} \to G \times C_2$ be the canonical surjection, and consider the finite split embedding problem $\hat{\mathcal{E}}_1 = (\hat{\alpha}_1 : \Pi_0 \to G \times C_2,\ \hat{f}_1 : \hat{\Gamma} \to G \times C_2)$ for Π_0. Since Π_0 is quasi-free of rank m, this embedding problem has m proper solutions.

Consider any proper solution to $\hat{\mathcal{E}}_1$, say $\hat{\beta}_1 : \Pi_0 \to \hat{\Gamma}$. So $M := \ker\hat{\beta}_1$ is normal in Λ_1 and in Π_0, with quotient groups $\Lambda_1/M \approx N \times N$ and $\Pi_0/M \approx \hat{\Gamma}$. Here $H := \Lambda_0/M = (N \times N) \rtimes C_2$, with $C_2 = \Lambda_0/\Lambda_1$ interchanging the two factors of $N \times N = \Lambda_1/M$. Let M_1 be the inverse image of $1 \times N$ under the quotient map $\Lambda_1 \to N \times N$; thus $\Pi_1/M_1 \approx \Gamma$. Also, M is the largest normal subgroup of Π_0 that is contained in M_1 (since any such subgroup would also have to be contained in the inverse image of $N \times 1$); so M is determined by M_1 and thus distinct choices of M lead to distinct choices of M_1. Thus there are m distinct choices for M_1, arising from the m choices for M. Each such choice for M_1 is the kernel of a proper solution β_1 to the embedding problem $\mathcal{E}_1$, inducing a weak solution $\beta := \beta_1|_{\Pi'} : \Pi' \to \Gamma$ to $\mathcal{E}$ with kernel $\Pi' \cap M_1$. It remains to show that β is surjective.

If $n \in N$ and ι is the involution in C_2, the commutator $[(n,1),\iota] \in [N \times N, C_2] \subset H = (N \times N) \rtimes C_2$ is equal to $(n, n^{-1}) \in N \times N$. Thus $N \times N = \Lambda_1/M$ is generated by $1 \times N = M_1/M$ and the commutator subgroup H' of $H = \Lambda_0/M$ (where $H' \subset N \times N$ because $H/(N \times N)$ is abelian). So Λ_1 is generated by M_1 and Λ_0', the commutator subgroup of Λ_0. Since $\Lambda_0' \subset \Pi'$, we have that $\Pi' M_1 = \langle \Pi', \Lambda_0', M_1 \rangle = \Pi'\Lambda_1 = \Pi_1$. Hence the natural inclusion $\beta(\Pi') \approx \Pi'/\ker\beta = \Pi'/(\Pi' \cap M_1) \hookrightarrow \Pi_1/M_1 = \Gamma$ is an isomorphism. So $\beta : \Pi' \to \Gamma$ is surjective, as desired. $\square$

Theorem 2.4. *Let m be an infinite cardinal. If Π is a quasi-free profinite group of rank m, then so is its commutator subgroup Π'.*

Proof. Let $\mathcal{E} = (\alpha : \Pi' \to G, f : \Gamma \to G)$ be any non-trivial finite split embedding problem for Π'. Let $N = \ker f \neq 1$ and let Z be the center of Γ. We wish to show that $\mathcal{E}$ has exactly m distinct proper solutions.

By Remark (b) after Lemma 2.2 above, Π' has at most m open normal subgroups. So for any finite group H, $\mathrm{Hom}(\Pi', H)$ has cardinality at most m (using that each open normal subgroup of Π' is the kernel of at most finitely many homomorphisms $\Pi' \to H$). Taking $H = \Gamma$, we have that $\mathcal{E}$ has at most m distinct proper solutions; and it suffices to show that there are at least that many.

By Lemma 2.3, there is an open normal subgroup $\bar{\Pi}$ of Π containing Π', together with an embedding problem $\bar{\mathcal{E}} = (\bar{\alpha} : \bar{\Pi} \to G, f : \Gamma \to G)$ that induces $\mathcal{E}$, such that $\bar{\Pi}$ has m open subgroups $\bar{M}$ each of which is the kernel of a proper solution $\bar{\beta}$ to $\bar{\mathcal{E}}$ that induces a proper solution β to $\mathcal{E}$, say with kernel M. Since $\bar{\mathcal{E}}$ induces $\mathcal{E}$, we have $M = \Pi' \cap \bar{M}$. Also, Π' contains the commutator subgroup of $\bar{\Pi}$. By Lemma 2.1 (with $\bar{\Pi}$, Π', $\bar{\mathcal{E}}$ here playing the roles of Π, Π_1, $\mathcal{E}$ there), if $Z \cap N = 1$ then distinct choices of $\bar{\beta}$ that have distinct kernels $\bar{M}$ yield distinct open subgroups $M \subset \Pi'$ and hence distinct proper solutions β to $\mathcal{E}$. So in this case we are done; and we are therefore reduced to the case that $Z \cap N \neq 1$.

We may thus assume that there is a cyclic subgroup C of prime order p in $Z \cap N$. By Lemma 2.2, Π' has m distinct open normal subgroups of index p; and so $\mathrm{Hom}(\Pi', C)$ has cardinality m. Let $\beta : \Pi' \to \Gamma$ be the proper solution to $\mathcal{E}$ given by *some* choice of α_1 and M_1 in the previous paragraph. Since C is central in Γ, for each $\varepsilon \in \mathrm{Hom}(\Pi', C)$ we obtain a homomorphism $\beta \cdot \varepsilon : \Pi' \to \Gamma$ given by $(\beta \cdot \varepsilon)(a) = \beta(a)\varepsilon(a)$ for $a \in \Pi'$. The composition of $\beta \cdot \varepsilon$ with the quotient map $\Gamma \to G$ is the surjection $\alpha : \Pi' \to G$, since this is true for β and since $C \subset N = \ker(\Gamma \to G)$. Moreover, the compositions of β and of $\beta \cdot \varepsilon$ with $\Gamma \to \Gamma/C$ also agree, and the former is surjective; so Γ is generated by C and the image of $\beta \cdot \varepsilon$. In particular, $\beta \cdot \varepsilon$ is surjective if and only if its image contains C. Also, distinct choices of ε yield distinct homomorphisms $\beta \cdot \varepsilon$. So it suffices to show that the image of $\beta \cdot \varepsilon$ contains C for m choices of $\varepsilon \in \mathrm{Hom}(\Pi', C)$.

Let $\Delta \subset \Pi'$ be the inverse image of C under β. Since $\beta : \Pi' \to \Gamma$ is surjective, the image of $\beta|_\Delta$ is C. Also, for each $\varepsilon \in \mathrm{Hom}(\Pi', C)$, the image of the restriction $(\beta \cdot \varepsilon)|_\Delta$ is either C or 1. Let S be the set of $\varepsilon \in \mathrm{Hom}(\Pi', C)$ such that this image is C; thus $\mathrm{card}\, S \leq m$. For any $\varepsilon \in S$, the map $\beta \cdot \varepsilon$ is surjective, since its image contains C; and so it suffices to show that the cardinality of S is m. If the complement of S in $\mathrm{Hom}(\Pi', C)$ has cardinality less than m, then the cardinality of S is m, and we are done. On the other hand, if the cardinality of the complement of S is m, then fix some ε_0 in this complement. For any *other* ε in the complement of S, consider the map $u_\varepsilon := \varepsilon_0^{-1} \cdot \varepsilon : \Pi' \to C$ sending $a \in \Pi'$ to $\varepsilon_0(a)^{-1}\varepsilon(a)$. The restriction of this map to Δ is trivial, since $(\beta \cdot \varepsilon)(a) = (\beta \cdot \varepsilon_0)(a) = 1$ for $a \in \Delta$. So $(\beta \cdot u_\varepsilon)|_\Delta = \beta|_\Delta$, whose image is C; and hence $u_\varepsilon \in S$. Since distinct ε's in the complement of S induce distinct u_ε's in S, it follows that the cardinality of S is m. $\square$

Theorem 2.4 has the following analog for free profinite groups:

Proposition 2.5. *Let Π be a free profinite group of infinite rank m, and let Π_1 be a closed subgroup of Π that contains the commutator subgroup Π' of Π. Then Π_1 is also free profinite of rank m.*

Proof. If Π_1 has finite index in Π, then it is an open subgroup, and [FJ], Proposition 17.6.2, says that it is free of rank m. On the other hand, suppose that Π_1 has infinite index in Π. Then Π_1 is normal in Π with (infinite) abelian quotient Π/Π_1, because Π_1 contains Π'. So the desired conclusion follows from [FJ], Corollary 25.4.8. □

Remark. a) In Proposition 2.5, if we instead allow m to be a finite cardinal greater than 1, then the profinite group Π_1 is still free. Namely, if Π_1 has index i in Π, then Π_1 is free of rank $1 + i(m-1)$ if i is finite ([FJ], Proposition 17.6.2), and of rank m if i is infinite ([FJ], Corollary 25.4.8).

b) In the case that $\Pi_1 = \Pi'$, Proposition 2.5 can also be deduced from Theorem 2.4 above. Namely, since Π is free profinite of rank m, and since m is infinite, we have that Π is quasi-free of rank m. So by Theorem 2.4, Π' is also quasi-free of rank m. Since Π' is a closed subgroup of the free profinite group Π, it follows from [FJ], Corollary 22.4.6, that Π' is projective. Since Π' is projective and quasi-free of rank m, it is free of rank m by [HS], Theorem 2.1.

The following result is another variant of Theorem 2.4, considering just the existence of finite quotients rather than embedding problems.

Proposition 2.6. *Let Π be a profinite group with the property that every finite group is a quotient of Π by an open normal subgroup. Then the commutator subgroup Π' of Π also has this property.*

Proof. We proceed as at the end of the proof of Lemma 2.3. Let N be any finite group, and let $H = (N \times N) \rtimes C_2$, where C_2 acts by interchanging the two copies of N. By hypothesis, Π has a closed normal subgroup M such that $\Pi/M = H$. Let $p : \Pi \to H$ be the canonical surjection, let $\Pi_1 = p^{-1}(N \times N)$, and let $M_1 = p^{-1}(1 \times N)$. Thus $M \subset M_1 \subset \Pi_1 \subset \Pi$, and $\Pi' \subset \Pi_1$ since Π/Π_1 is abelian. As in the proof of Lemma 2.3, M is the largest normal subgroup of Π contained in M_1, and $N \times N = \Pi_1/M$ is generated by $1 \times N = M_1/M$ and the commutator subgroup H' of $H = \Pi/M$. So Π_1 is generated by M_1 and Π'. Hence the natural inclusion $\Pi'/(\Pi' \cap M_1) \hookrightarrow \Pi_1/M_1 \approx N$ is an isomorphism. Thus $\Pi' \cap M_1$ is a closed normal subgroup of Π' with quotient group isomorphic to N. □

Section 3. Field arithmetic.

Let K be a field, with separable closure K^{s}. The *absolute Galois group* of K is the profinite group $G_K := \mathrm{Gal}(K^{\mathrm{s}}/K)$. An *embedding problem* for K is an embedding problem $\mathcal{E} = (\alpha : G_K \to G,\ f : \Gamma \to G)$ for G_K. Here the epimorphism α corresponds to a G-Galois field extension L of K together with a K-inclusion $i : L \hookrightarrow K^{\mathrm{s}}$. A proper solution to $\mathcal{E}$

corresponds to a Γ-Galois field extension M of K that contains L, together with a K-inclusion $j : M \hookrightarrow K^{\mathrm{s}}$ that extends i, where the restriction map $\mathrm{Gal}(M/K) \twoheadrightarrow \mathrm{Gal}(L/K)$ corresponds to f. Thus $\mathcal{E}$ has a proper solution if and only if the given G-Galois field extension of K can be embedded into a Γ-Galois field extension (hence the terminology). Note that if G and Γ are finite, then there are only finitely many K-inclusions i and j as above, for given field extensions L and M. Also, if $m := \mathrm{card}\, K$ is infinite, then K has at most m field extensions of finite degree; and so a finite embedding problem for K can have at most m (weak or proper) solutions.

In the above situation, suppose that K is a function field over a subfield F (i.e. separable and of finite transcendence degree over F, with F algebraically closed in K), and let β be a proper solution to $\mathcal{E} = (\alpha : G_K \to G,\, f : \Gamma \to G)$ corresponding to a pair (M, j) extending (L, i). We say that the proper solution β is *regular* (with respect to F) if the algebraic closures of F in L and in M are the same (regarding $L \subset M$).

The Galois cohomology of a field K is the same as the group cohomology of G_K, and so K and G_K have the same cohomological dimension. We say that K is *free* [resp. *quasi-free*, *ω-free*, *projective*] if G_K is. So K is projective if and only if it has cohomological dimension ≤ 1. Also, if K is quasi-free of rank m_0, then $\mathrm{card}\, K \ge m_0$. We say that a profinite group G *is a Galois group over* K if there is a Galois field extension L of K with Galois group isomorphic to G; this is equivalent to saying that G_K has a closed normal subgroup N such that G_K/N is isomorphic to G.

For any field K, let K^{ab} denote its maximal abelian extension (in a given separable closure). By considering the absolute Galois group $\Pi = G_K$ and its commutator $\Pi' = G_{K^{\mathrm{ab}}}$, we may restate Proposition 2.6, Theorem 2.4 and Proposition 2.5 in field-theoretic terms as follows:

Proposition 3.1. *Let K be a field.*

a) If K has the inverse Galois property (i.e. every finite group is a Galois group over K), then the same holds for K^{ab}.

b) Let m be an infinite cardinal. If K is quasi-free of rank m, then so is K^{ab}.

c) Let m be an infinite cardinal. Let K_1 be an abelian extension of K. If the absolute Galois group of K is free of rank m, then the same holds for K_1.

Recall that a field K is called *large* [Po2] (or *ample*; see [FJ], Remark 16.12.3) if every smooth K-curve (i.e. 1-dimensional K-scheme of finite type) with a K-point has infinitely many K-points. Examples of large fields include fraction fields of henselian (e.g. complete) discrete valuation rings; real closed fields (e.g. $\mathbb{R}$); the field of totally real (or totally p-adic) algebraic numbers; algebraically closed fields; more generally pseudo-algebraically closed fields (PAC fields: fields K such that smooth geometrically integral K-variety has a K-rational point); and algebraic extensions of large fields [Po2]. The property of being large is equivalent to the property that for every smooth integral K-variety X, if X has a K-point then $X(K)$ is Zariski dense (using that the union of smooth K-curves containing a given smooth K-point on an integral K-variety X is Zariski dense in X). It is also equivalent to the condition that K is existentially closed in $K((t))$; i.e. every K-variety with a $K((t))$ point has a K-point. See [Po2], Proposition 1.1.

A key property of large fields (first shown by F. Pop) is the following: Let F be a large field and let K be the function field of a smooth projective F-curve. Then every finite split embedding problem for K has a proper regular solution. Versions of this result have appeared in [Po1], [Po2], [HJ2], and [HJ3] (see also [Ha3], §5.1, for a further discussion). Hence large Hilbertian fields K have the property that every finite split embedding problem has a proper solution. If in addition G_K is projective then G_K is ω-free; and if also K is countable then Iwasawa's theorem ([Iw], p.567) applies and so G_K is free of countable rank ([Po2], Theorem 2.1).

Remark 3.2. In the case that m is countable (which is the case that we will use in Theorem 4.1), Proposition 2.5 and hence also Proposition 3.1(c) follow from ideas related to the above. Namely, if Π is free profinite of countably infinite rank, then Π is isomorphic to the absolute Galois group of any countable Hilbertian PAC field of characteristic 0 ([FV], Theorem A). Any algebraic extension of a PAC field is PAC ([FJ], Corollary 11.2.5), and any abelian extension of a Hilbertian field is Hilbertian ([FJ], Theorem 16.11.3). So K_1 is also a countable Hilbertian PAC field, and hence its absolute Galois group is also free of countable rank.

As in [HS], call a field K *very large* if every smooth K-curve with a K-point has exactly m K-points, where m is the cardinality of K. This is equivalent to the property that for every smooth integral K-variety X, if X has a K-point then every non-empty open subset of X contains exactly m K-points (using the same reasoning as for the corresponding characterization of large).

Observe that every large field is infinite, as is every very large field (e.g. by considering the curve $\mathbb{P}^1_K$). Hence every very large field is large. Also, if K is an infinite field of cardinality m, then every K-variety (of finite type) has at most m K-points.

The proof of the following proposition is due to F. Pop (not previously published).

Proposition 3.3. (Pop) *Let K be a large field of cardinality m. Then K is very large.*

Proof. Let X be a smooth K-curve with a K-point P, where K is large. We wish to show that the cardinality of $X(K)$ is equal to m. Since X is a K-variety, $X(K)$ has cardinality at most $m = \operatorname{card} K$. So it suffices to prove the reverse inequality; and for this we may assume that X is connected. Possibly after deleting finitely many points (other than P) from X, we may embed X in $\mathbb{A}^2_K$. After replacing X by its image in $\mathbb{A}^2_k$, and making a change of variables in the plane, we may assume that X is a smooth plane curve containing the origin, defined by a polynomial f such that $\partial f/\partial y$ does not vanish at the origin. We claim that for each $a \in K$ we may choose a pair of K-points $(x_1, y_1), (x_2, y_2) \in X(K)$ such that $x_2 \neq 0$ and $x_1/x_2 = a$. If this is shown, we obtain an injection $i : K \hookrightarrow X(K) \times X(K)$; and this then implies that the cardinality of $X(K)$ is at least m, as desired.

So it suffices to prove the claim. Let $a \in K$. Consider affine 4-space $\mathbb{A}^4_K$ with coordinates X_1, Y_1, X_2, Y_2, and the subvariety $V_a \subset \mathbb{A}^4_k$ defined by:

$$f(X_1, Y_1) = 0, \ f(X_2, Y_2) = 0, \ X_1 - aX_2 = 0.$$

Here $V_a \cong (X \times_K X) \cap H_a \subset \mathbb{A}^2_K \times_K \mathbb{A}^2_K = \mathbb{A}^4_K$, where H_a is the affine hyperplane $X_1 - aX_2 = 0$ in $\mathbb{A}^4_K$. The partial derivatives of the above three polynomials with respect to Y_1, Y_2, X_1 respectively are non-zero at the origin in $\mathbb{A}^4$, since $\partial f/\partial y$ is non-zero at the origin in $\mathbb{A}^2$. So in a neighborhood of the origin in $\mathbb{A}^4$, V_a is a K-curve having the origin as a smooth K-point.

Let X_a be the unique irreducible component of V_a containing the origin. The smooth locus X_a° of X_a contains the origin and is a geometrically irreducible K-curve, since it is irreducible and has a smooth K-point. Here $X_a^\circ(K)$ is infinite because K is large and $X_a^\circ(K)$ is non-empty. But there are only finitely many points $(x_1, y_1, x_2, y_2) \in X_a^\circ(K)$ with $x_2 = 0$, since at such points $x_1 = 0$ and there are then only finitely many possible values of y_1, y_2. So there exists $(x_1, y_1, x_2, y_2) \in X_a^\circ(K)$ with $x_2 \neq 0$, and hence there exist $(x_1, y_1), (x_2, y_2) \in X(K)$ with $x_2 \neq 0$ and $a = x_1/x_2$. This proves the claim. □

Theorem 3.4. *The function field K of a smooth projective curve over a large field F is quasi-free, of rank equal to the cardinality of F.*

Proof. In [HS], Theorem 4.3, it was shown that if F is a very large field of cardinality m, and K is the function field of a smooth projective F-curve, then every non-trivial finite split embedding problem $\mathcal{E}$ for K has m proper *regular* solutions. Hence the set of *all* proper solutions to $\mathcal{E}$ also has cardinality m. That is, (the absolute Galois group of) K is m-quasi-free, or equivalently quasi-free of rank m (by [RSZ]; see the discussion before Lemma 2.1 above). The result now follows from Proposition 3.3 above. □

Remark 3.5. (a) As the proof of Theorem 3.4 shows, under the hypotheses of the theorem, every non-trivial finite split embedding problem for K has exactly m proper *regular* solutions, where $m = \operatorname{card} F$. So this theorem strengthens Pop's result ([Po1], [Po2]) that if K is the function field of a smooth projective curve over a large field, then every finite split embedding problem for K has at least one proper regular solution.

(b) The property of being large (or PAC) can be regarded as complementary to the property of being Hilbertian (see [La], [FJ]). Namely, consider a Galois branched cover $\phi : Y \to X = \mathbb{A}^1_K$. If K is Hilbertian, then there are infinitely many K-points of X that remain prime in Y. Meanwhile, to say that K is PAC or large is to say that there are infinitely many K-points of X that are totally split in Y (in the latter case, assuming there is one such point). Moreover, as for large fields, this property for curves implies a corresponding property in higher dimensions. (Note also these properties are analogous to the two extremes in the Tchebotarev Density Theorem.)

(c) Remark (b) suggests introducing a notion of *very Hilbertian*; i.e. that for $Y \to X$ as in (b), the cardinality of the set of K-points of X that remain prime in Y is equal to the cardinality of K. And in fact, the strategy of the proof of Theorem 3.3 also shows that every Hilbertian field is very Hilbertian. Namely, if ϕ is generically given by a polynomial $f(x,y) \in K[x,y]$, consider for each $a \in K$ the variety $V_a \subset \mathbb{A}^4_K$ as in the proof of Theorem 3.3. Then there exist $(x_1, x_2) \in \mathbb{A}^2(K)$ which remains prime in V_a, such that $x_2 \neq 0$; i.e. such that $f(x_1, Y)$ and $f(x_2, Y)$ are irreducible in $K[Y]$, with $x_1/x_2 = a$. The property of being very Hilbertian then follows.

(d) If K is a Hilbertian large field, then every finite split embedding problem over K has a proper solution ([Po2], Main Theorem B), since every finite split embedding problem over the function field of the K-line has a proper regular solution. In fact, each such non-trivial embedding problem $\mathcal{E} = (\alpha : \Pi \to G,\ f : \Gamma \to G)$ has infinitely many solutions. (For example, for each $n > 0$ there is a proper solution to $\mathcal{E}_n = (\alpha : \Pi \to G,\ f_n : \Gamma^n_G \to G)$, where Γ^n_G is the n^{th} fibre power of Γ over G. Taking projections $\Gamma^n_G \to \Gamma$ yields n distinct proper solutions to $\mathcal{E}$.) Since the properties of large and Hilbertian imply the properties of being very large and very Hilbertian, this suggests that a large Hilbertian field K is quasi-free (and of rank equal to the cardinality of K). Surprisingly, this is false, by an example of Jarden. Namely, according to Examples 3.1 and 3.2 of [Ja1], there is a profinite group G of uncountable rank that is projective and ω-free but not free, and which is the absolute Galois group of a Hilbertian PAC (and hence large) field K. Since $G = G_K$ is projective but not free, it cannot be quasi-free.

(e) By another example (also due to Jarden), it is also possible for the absolute Galois group of a large Hilbertian field K to be quasi-free, yet have rank strictly smaller than the cardinality of K. Namely, by [FJ], Theorem 23.1.1, there is an uncountable PAC field K whose absolute Galois group G_K is free of countable rank. So K is large, and G_K is quasi-free of countable rank (and in particular ω-free). Also K is Hilbertian by a theorem of Roquette ([FJ], Theorem 27.3.3), because it is ω-free and PAC. So K is as claimed. Combining this example with Remark (c) above exposes a subtle point: for such a Hilbertian field K and any finite Galois extension L of $K(x)$, there will be $\operatorname{card} K$ elements of K for which the specialization of L is irreducible; but the corresponding Galois field extensions of K are not linearly disjoint (and *up to isomorphism* there are fewer than $\operatorname{card} K$ of them).

Section 4. Main results.

This section contains the main results of this paper, viz. the freeness of the absolute Galois groups of the function field of a real curve without real points, of the maximal abelian extension of $\mathbb{C}((x,y))$, and of the maximal abelian extension of the function field of a curve over a finite field. Each of these is stated in somewhat stronger form below.

Theorem 4.1. ("Geometric Shafarevich Conjecture") *Let p be a prime and let k be a subfield of $\bar{\mathbb{F}}_p$ (e.g. a finite field). Let F be a one-variable function field over k, and let F^{ab} be its maximal abelian extension. Then the absolute Galois group of F^{ab} is free of countable rank.*

Proof. Let $\tilde{F}$ be the compositum of F and $\bar{\mathbb{F}}_p$ in an algebraic closure of F. Then $\tilde{F}$ is the function field of a smooth projective curve over $\bar{\mathbb{F}}_p$. Moreover we have containments $F \subset \tilde{F} \subset F^{\mathrm{ab}} \subset \tilde{F}^{\mathrm{ab}}$; i.e. F^{ab} is abelian over $\tilde{F}$. By [Ha1] or [Po1], the absolute Galois group of $\tilde{F}$ is a free profinite group of countably infinite rank. So the same holds for F^{ab}, by Proposition 3.1(c) (or Remark 3.2). □

Recall that a field K with algebraic closure $\bar{K}$ is *formally real* if -1 is not a sum of squares in K; and K is *real closed* if it is a maximal element in the set of formally real subfields of $\bar{K}$. If K is real closed then $K[\sqrt{-1}]$ is algebraically closed; and so the absolute Galois group of a real closed field is cyclic of order 2. According to [CT], p.360, and [Ja2], a field is large if its absolute Galois group is a pro-p group for some prime p; in particular, real closed fields are large. (More generally, according to [Po2], pp. 18-19, "pseudo-real closed" fields are large because they satisfy a universal local-global principle.)

Theorem 4.2. *Let X be a smooth projective curve over a real closed field R (e.g. $R = \mathbb{R}$), and let K be the function field of X. Then the absolute Galois group of K is free if and only if X has no R-points; and if it is free, its rank is equal to the cardinality of R.*

Proof. As noted above, every real closed field is large. So Theorem 3.4 says that K is quasi-free of rank equal to $m := \operatorname{card} R$. Thus K is free (necessarily of rank m) if and only if it is projective, by [HS], Theorem 2.1.

In general, the function field of an integral variety of dimension d over a real closed field R with no R-points has cohomological dimension d ([CP], Proposition 1.2.1). So in our situation, if $X(R)$ is empty then K has cohomological dimension 1, which implies that it is projective (as noted at the beginning of Section 2 above).

Conversely, if X has an R-point, then it is classical that K is not projective. Namely, let $C = R[\sqrt{-1}]$. If $K = R(X)$ is projective, then the $\mathbb{Z}/2$-Galois extension $C(X)/R(X)$ can be embedded in a $\mathbb{Z}/4$-Galois field extension $L/R(X)$ (since the kernel of $\mathbb{Z}/4 \to \mathbb{Z}/2$ is Frattini). Since R is large, $X(R)$ is infinite; so some $P \in X(R)$ is unramified in this extension. But a decomposition group over P would then surject onto $Z/2$ and thus be $\mathbb{Z}/4$, which is impossible since $\mathbb{Z}/4$ is not a Galois group over R. (Alternatively, one can argue that an R-point on X yields an involution in the absolute Galois group G_K of K; so the cohomological dimension of G_K is infinite and thus G_K is not projective.) □

Remarks 4.3. a) As an example of the theorem, the fraction field of $\mathbb{R}[x, y]/(x^2 + y^2 + 1)$ has free absolute Galois group, of rank equal to the cardinality of $\mathbb{R}$.

b) The proof of [CP], Proposition 1.2.1, is due to Ax and relies on a result of Serre. But in the proof above, only the dimension 1 case of [CP], Proposition 1.2.1, is needed; and that case is more classical, essentially going back to Witt [Wi].

c) In the context of Theorem 4.2, one can give a more explicit description of the absolute Galois group G_K of K in the case that X has R-points. Namely, G_K is a free product $A * B$, where A is a free profinite group of rank $m = \operatorname{card}(R)$, and B is a free product of groups of order 2 indexed by a profinite (i.e. compact, Hausdorff and totally disconnected) topological space of cardinality m. (See [Za] for the definition of a free product in this sense.) This assertion was proven for the projective R-line in [HJ1]. As M. Jarden has observed to the author, if X is *any* smooth projective R-curve, then its function field $K(X)$ can be viewed as a finite extension of $R(x)$, and hence its absolute Galois group is an open subgroup of that of $R(x)$. By a profinite version of the Kurosh subgroup theorem [Za], an open subgroup of $A * B$ is a profinite group of the same form, except that the index space of the second factor could have cardinality less than m. (Indeed, by Theorem 4.2, if X has no R-points, then G_K is free, and there are no factors of order 2.) But if the smooth curve X has an R-point, then it has m such points. Each of them contains an involution in its decomposition group, and for distinct points these involutions are non-conjugate. So in this situation, the second factor of the free product has a (possibly different) index space of cardinality m, and G_K has the general form asserted above.

Finally, we turn to consideration of the absolute Galois group of the maximal abelian extension K of $k((x,y))$, where k is a separably closed field of arbitrary characteristic. As in the previous theorem, we prove that this is free by using that it is projective and quasi-free. As noted before, projectivity is equivalent to the condition of having cohomological dimension at most 1; and to show that latter condition, we use a result proven by J.-L. Colliot-Thélène, M. Ojanguren and R. Parimala ([COP], Theorem 2.3, which was numbered Theorem 2.2 in their preprint). Their result, though, assumed characteristic zero. Following a sketch provided by Parimala, we generalize their result and proof to the characteristic p case (Theorem 4.4), for use in Theorem 4.6 below.

Before stating Theorem 4.4, we recall some notions related to Brauer groups that are used in the proof (see [COP], §§1,2, and [Gr2], II]). Following [COP], we denote the *cohomological Brauer group* $H^2_{\text{ét}}(X, \mathbb{G}_m)$ of a scheme X by $\mathrm{Br}(X)$; and denote by $\mathrm{Br}_{\mathrm{Az}}(X)$ the *Azumaya Brauer group* of X, which is a torsion group classifying equivalence classes of Azumaya algebras over X. If $X = \operatorname{Spec} R$, we also write $\mathrm{Br}(R) = \mathrm{Br}(X)$ and $\mathrm{Br}_{\mathrm{Az}}(R) = \mathrm{Br}_{\mathrm{Az}}(X)$. For a field K, $\mathrm{Br}(K) = \mathrm{Br}_{\mathrm{Az}}(K)$, classifying equivalence classes of central simple algebras over K, with the trivial class consisting of those that are split (i.e. isomorphic to some $M_n(K)$). More generally, there is a natural inclusion $\mathrm{Br}_{\mathrm{Az}}(X) \subset \mathrm{Br}(X)$, which is an isomorphism if X is Noetherian of dimension ≤ 1, or is Noetherian and regular of dimension 2 ([Gr2], II, Cor. 2.2).

Let ${}_n\mathrm{Br}(K)$ denote the n-torsion subgroup $H^2_{\text{ét}}(K, \mu_n)$ of $\mathrm{Br}(K)$ for $n > 0$, and let $\mathrm{Br}(K)(\ell)$ denote the ℓ-primary part of $\mathrm{Br}(K)$ for ℓ a prime. If v is a discrete valuation on a field K with valuation ring R, we say that an element $\alpha \in \mathrm{Br}(K)$ (or a central simple algebra D that it represents) is *unramified at* v if α is in the image of the natural map $\mathrm{Br}(R) \to \mathrm{Br}(K)$. If X is an integral scheme with function field K, and x is a regular codimension 1 point of X corresponding to v, we also say that α (or D) is *unramified at* x. In this context, if $n > 0$ is invertible on X, then a class α in ${}_n\mathrm{Br}(K)$ is unramified at x if and only if α is in the kernel of the natural residue map $\partial_x : {}_n\mathrm{Br}(K) \to H^1_{\text{ét}}(\kappa(x), \mathbb{Z}/n)$, where $\kappa(x)$ is the residue field at x ([COP], §2). Moreover a given $\alpha \in {}_n\mathrm{Br}(K)$ is unramified at all but finitely many height one primes x; and the *ramification divisor* of α on X is the sum of the closures of the ramified codimension 1 points of X. If X is a regular connected

Noetherian scheme of dimension ≤ 2 with function field K, then $\mathrm{Br}(X)$ consists of the classes in $\mathrm{Br}(K)$ whose ramification divisor on X is empty ([Gr2], II, Prop. 2.3, using $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{Az}}(X)$ by [Gr2], II, Cor. 2.2).

If X is an excellent integral scheme with function field K, then the residue map can also be interpreted in terms of tame symbols. Namely, if there is a primitive n^{th} root of unity on X, where n is invertible on X, then this root of unity yields an isomorphism $\mathbb{Z}/n \xrightarrow{\sim} \mu_n$ over X, and hence identifications ${}_n\mathrm{Br}(X) \xrightarrow{\sim} H^2_{\text{ét}}(K, \mu_n^{\otimes 2})$ and $H^1_{\text{ét}}(\kappa(x), \mathbb{Z}/n) \xrightarrow{\sim} H^1_{\text{ét}}(\kappa(x), \mu_n) = \kappa(x)^\times/(\kappa(x)^\times)^n$. With respect to these identifications, the residue map ∂_x becomes identified with the tame symbol map $\delta_x : H^2_{\text{ét}}(K, \mu_n^{\otimes 2}) \to \kappa(x)^\times/(\kappa(x)^\times)^n$ sending $(a,b)_n$ to $(-1)^{v_x(a)v_y(b)}\overline{(a^{v_x(b)}/b^{v_x(a)})}$, where $(a,b)_n \in H^2_{\text{ét}}(K, \mu_n^{\otimes 2})$ is the cup product of $a, b \in K^\times$. (See [COP], §2, following [Ka], §1.)

We now have the following generalization of [COP], Theorem 2.3/2.2, for use in Theorem 4.6:

Theorem 4.4. *Let A be an excellent henselian two-dimensional local domain, with fraction field K and separably closed residue field k, of equal characteristic $p \geq 0$. Then the maximal abelian extension K^{ab} of K has cohomological dimension at most 1, as does the maximal pro-prime-to-p abelian extension K' of K.*

In the proof of this theorem, we will rely on the following technical lemma, which was shown in Section 2 of [COP] and used in proving [COP], Theorem 2.3/2.2. (The lemma was shown in the proof of [COP], Theorem 2.1, though it was not stated as a separate result there.)

Lemma 4.5. ([COP], §2) *Let A, K, k, p be as in Theorem 4.4. Let L/K be a finite Galois extension, let B be the integral closure of A in L, and let $X \to \operatorname{Spec} A$ and $Y \to \operatorname{Spec} B$ be regular models such that $Y \to \operatorname{Spec} B \to \operatorname{Spec} A$ factors through $X \to \operatorname{Spec} A$. Let $\xi \in {}_n\mathrm{Br}(K)$, where $p \not| n$, such that the ramification divisor of ξ on X is a normal crossing divisor $C + E$, where C and E are regular curves. Let $y \in Y$ be a point of codimension 1 lying over a point $x \in X$ of codimension 2. Then the induced element $\xi_L \in {}_n\mathrm{Br}(L)$ is unramified at y provided either*
(i) $x \notin C \cap E$, or
(ii) $(\pi, \delta)_n \in H^2_{\text{ét}}(K, \mu_n^{\otimes 2}) \approx {}_n\mathrm{Br}(K)$ induces an element of ${}_n\mathrm{Br}(L)$ that is unramified at y, where π, δ locally define the ideals of C, E on X at $x \in C \cap E$.

Proof of Theorem 4.4. The case $p = 0$ was shown in [COP], Theorem 2.3/2.2. For $p > 0$, we modify that proof:

A field of characteristic $p \neq 0$ has p-cohomological dimension at most 1 ([Se], II 2.2 Proposition 3). So it suffices to show that $\mathrm{cd}_\ell \leq 1$ for all $\ell \neq p$. Regard $K \subset K' \subset K^{\mathrm{ab}} \subset \bar{K}$, where $\bar{K}$ is a separable closure of K. Since the extension K^{ab}/K' is algebraic, $\mathrm{cd}_\ell(K^{\mathrm{ab}}) \leq \mathrm{cd}_\ell(K')$ ([Se], II 4.1 Proposition 10). So it suffices to consider just the case of K'. By [Se], II 2.3 Proposition 4 and II 1.2 Proposition 1, $\mathrm{cd}_\ell(K') \leq 1$ for $\ell \neq p$ if and only if every finite separable extension F/K' satisfies $\mathrm{Br}(F)(\ell) = 0$. So it suffices to show that for every finite separable extension F/K' (contained in $\bar{K}$), every central simple F-algebra of exponent prime to p is split.

Case I: F is Galois over K. An F-algebra as above is induced via base change from a central simple algebra D over L, of exponent n prime to p, where L is a finite Galois field extension of K. Let $d = [L : K]$ and write $d = d'p^m$ with d' prime to p and $m \geq 0$. Thus p does not divide $N := nd'$. The henselian ring A (and hence K, L, F) contains all prime-to-p roots of unity since the residue field k is separably closed of characteristic p.

Let B be the integral closure of A in the field L. Since L is finite over K and A is an excellent domain, B is finite over A ([Gr1], 2^e partie, 7.8.3(vi)); so B is also an excellent domain ([Gr1], 2^e partie, 7.8.6(i)). Being finite over a two-dimensional henselian local domain, B is a two-dimensional semi-local domain which is henselian ([Gr1], 4^e partie, 18.5.10) and hence local ([Gr1], 4^e partie, 18.5.9(i)). Its residue field $\tilde{k}$ is algebraic over the separably closed field k; thus $\tilde{k}$ is purely inseparable over k and hence is itself separably closed. There are finitely many points of codimension 1 on $\operatorname{Spec} B$ at which the (class of the) algebra D ramifies. We may chose a Weil divisor Δ on $\operatorname{Spec} B$ that contains these points and the closed point of B, and which is invariant under $G := \operatorname{Gal}(L/K)$.

Observe that there is a projective birational morphism $\pi : X \to \operatorname{Spec} B$ such that X is connected and regular with function field L and the reduced inverse image of Δ is a G-invariant normal crossing divisor on X of the form $C + E$, where C and E are each regular. Namely, since $\operatorname{Spec} B$ is a normal surface, after finitely many blow-ups we obtain a regular surface; after finitely many more, the reduced inverse image of Δ is a curve Δ' with only ordinary double points; and after blowing up those double points, the proper transform C of Δ' is a regular curve, with the remaining components of the inverse image meeting them normally and forming a disjoint union E of projective lines (viz. the exceptional divisors of the last blow-ups). The composition of these blow ups is then the desired $\pi : X \to \operatorname{Spec} B$, since at each step the surface and the set of blown-up points is G-invariant.

Let S be a finite, G-invariant set of closed points of X that contains all the points of $C \cap E$ and at least one point on each irreducible component of $C + E$. Since $X \to \operatorname{Spec} B$ is projective and S is finite, there is an affine open subset $U = \operatorname{Spec} A \subset X$ that contains S. Let A_S be the semi-localization of A at the primes corresponding to the points of S. Being semi-local and regular, A_S is a unique factorization domain. So there is a non-zero element $g \in A_S$ whose divisor on $\operatorname{Spec} A_S$ is the restriction of $C + E$. Viewing $g \in L^\times$, the divisor of g on X is of the form $C + E + J$, where J is a divisor that does not contain any point of S, and in particular has no component in common with $C + E$. Note that the norm $f = N_{L/K}(g) \in K^\times$ has divisor given by $\operatorname{div}_X(f) = d(C + E) + \sum_{\sigma \in G} \sigma J$.

Let $M = L(f^{1/N}) \subset \bar{K}$, and let D_M be the extension of D to M. Since L contains a primitive N^{th} root of unity, M is a cyclic extension of L. Also $M \subset F$, since $L \subset F$ (by definition of L) and since $h := f^{1/N} \in K' \subset F$. So in order to show that the given central simple algebra is split over F it suffices to show that D_M is split over M.

Let B_1 be the integral closure of B in M and let X_1 be the normalization of the fibre product $X \times_B B_1$. Applying resolution of singularities to X_1, we obtain a projective birational morphism $Y \to X_1$ whose composition with $X_1 \to \operatorname{Spec} B_1$ gives a projective birational morphism $Y \to \operatorname{Spec} B_1$, and whose composition with $X_1 \to X$ gives a morphism $q : Y \to X$ compatible with $\operatorname{Spec} B_1 \to \operatorname{Spec} B$. Since Y is a Noetherian regular connected surface with function field M, the class of D_M in $\operatorname{Br}(M)$ lies in $\operatorname{Br}(Y)$ provided that the ramification locus of D on Y is empty. But $\operatorname{Br}(Y) = 0$ by Corollary 1.10(b) of [COP] (Corollary 1.9(b) in the preprint). So to show that D_M is split it suffices to show that D_M is unramified at every codimension 1 point y on Y.

So let y be a codimension 1 point of Y, and let $x = q(y)$. We may assume x lies on the locus of $C + E$, since otherwise D is unramified at x and so D_M is unramified at y. Now a point x on the locus of $C + E$ may have codimension 1 or 2. If such an x has codimension 1, then $nd' \cdot \operatorname{div}_Y(h) = N \cdot \operatorname{div}_Y(h) = \operatorname{div}_Y(f) = d'p^m \cdot q^{-1}(C + E) + q^{-1}(\sum_{\sigma \in G} \sigma J)$ because $h^N = f \in M^\times$. Since n is prime to p, it follows that n divides the ramification index e of y over x. But the residue $\partial_x([D]) \in H^1(\kappa(x), \mathbb{Z}/n)$, and $\partial_y([D_M]) = e\partial_x([D])$; so $\partial_y([D_M]) = 0$ and thus D_M is unramified at y.

So it remains to consider the case that $x = q(y)$ is a codimension 2 point of X. Choose an identification $\mu_n \approx \mathbb{Z}/n$ on X. By Lemma 4.5 (with L, M, B, B_1 playing the roles of K, L, A, B there), it suffices to show that if x is a point on $C \cap E$, with a regular system of parameters $\pi, \delta \in \mathcal{O}_{X,x}$ respectively defining C and E locally, then $(\pi, \delta)_n$ induces an element of $\operatorname{Br}(M)$ that is unramified at y. For this, it suffices to show that $(\pi, \delta)_n$ induces the trivial element of $\operatorname{Br}(M_y)$, where M_y is the fraction field of the henselization $\mathcal{O}^h_{Y,y}$. This in turn is equivalent to showing that $p^m \cdot (\pi, \delta)_n = 0$ in $\operatorname{Br}(M_y)$, because $(\pi, \delta)_n$ is n-torsion and n is relatively prime to p. Since units in the multiplicative group of $\mathcal{O}^h_{X,x}$ are divisible by integers that are prime to p, and since $f = u\pi^d\delta^d \in \mathcal{O}_{X,x}$ for some $u \in \mathcal{O}^*_{X,x}$, we have $h^{nd'} = f = v^{nd'}\pi^{d'p^m}\delta^{d'p^m} \in \mathcal{O}^h_{Y,y}$ for some $v \in (\mathcal{O}^h_{X,x})^\times$. The residue field of $\mathcal{O}^h_{Y,y}$ contains the separably closed field $\tilde{k}$, and so the group of roots of unity in $\mathcal{O}^h_{Y,y}$ is d'-divisible. Thus $(\pi\delta)^{p^m} = \rho^n$ for some $\rho \in M_y$. So in ${}_n\operatorname{Br}(M_y) = H^2_{\text{ét}}(M_y, \mu_n) \approx H^2_{\text{ét}}(M_y, \mu_n^{\otimes 2})$, we obtain as desired $p^m \cdot (\pi, \delta)_n = p^m \cdot (\pi, \pi^{-1})_n + (\pi, \rho^n)_n = 0 + 0 = 0$.

Case II: General case. Let $M \subset \bar{K}$ be the Galois closure of F over K; this is finite over F. By Case I, $\operatorname{Br}(M)(\ell) = 0$; so $[M : F] \operatorname{Br}(F)(\ell) = 0$. Choosing an isomorphism of $\mathbb{Z}[1/\ell]/\mathbb{Z}$ with the ℓ-power roots of unity of F, the Merkurjev-Suslin theorem [MS] gives an isomorphism $K_2(E) \otimes (\mathbb{Z}[1/\ell]/\mathbb{Z}) \approx \operatorname{Br}(E)(\ell)$; so $\operatorname{Br}(E)(\ell)$ is ℓ-divisible. But being an ℓ-group, $\operatorname{Br}(E)(\ell)$ is also r-divisible for every integer r that is prime to ℓ. So $\operatorname{Br}(E)(\ell)$ is divisible, and hence is trivial, being $[M : F]$-torsion. $\square$

Remark. The above proof breaks down in the unequal characteristic case, where $\operatorname{char} K = 0$ and $\operatorname{char} k = p \neq 0$, because of the need in that case to show that $\operatorname{cd}_p \leq 1$.

Using the above result, we obtain:

Theorem 4.6. *Let k be a field and let K^{ab} be the maximal abelian extension of $K = k((x, y))$, with absolute Galois group $G_{K^{\text{ab}}}$.*

a) Then $G_{K^{\text{ab}}}$ is quasi-free of rank equal to the cardinality of K^{ab} ($= \operatorname{card} K$).

b) If k is separably closed, then the absolute Galois group of K^{ab} is a free profinite group of rank equal to the cardinality of K^{ab}.

Proof. a) According to [HS], Theorem 5.1, the absolute Galois group of K is quasi-free of rank equal to $\operatorname{card} K$ (even without any assumptions on k). By Proposition 3.1(b), it follows that the absolute Galois group of K^{ab} is also quasi-free of rank $\operatorname{card} K$. But K and K^{ab} have the same cardinality; so the assertion follows.

b) By [HS], Theorem 2.1, a profinite group is free of infinite rank m if and only if it is projective and is quasi-free of that rank. As noted before, $G_{K^{\mathrm{ab}}}$ is projective if and only if K^{ab} has cohomological dimension 1; and that latter property holds by Theorem 4.4. So the assertion follows from part (a). □

Remark. (a) The above proof of Theorem 4.6(b) relies on 4.6(a), hence on Proposition 3.1 and thus Theorem 2.4. But if one is willing to omit 4.6(a), one can prove 4.6(b) using just a weak form of Theorem 2.4 in which one adds the hypothesis that the commutator subgroup Π' is projective. This weak form of 2.4 can be shown using a proof that is somewhat shorter than the proof of the full theorem, by using projectivity to reduce to the case that the kernel of a given split embedding problem is a minimal normal subgroup. In fact, this was an earlier strategy of the author, before obtaining a proof of the full Theorem 2.4 and hence Theorem 4.6(a); and this approach has now been carried out in detail by M. Jarden, in correspondence with the author about this paper.

(b) As M. Jarden pointed out to the author, a weaker version of Theorem 4.6(b) — that K^{ab} is ω-free — can be proven still more briefly without relying on Theorems 4.6(a) or 2.4, by proceeding as follows: By a theorem of Weissauer ([FJ], Theorem 15.4.6), $K = k((x,y))$ is Hilbertian, being the fraction field of the two dimensional Krull domain $k[[x,y]]$. So its maximal abelian extension K^{ab} is also Hilbertian, by a theorem of Kuyk ([FJ], Theorem 16.11.3). Thus every finite split embedding problem for K^{ab} with an abelian kernel has a proper solution, by a theorem of Ikeda ([FJ], Proposition 16.4.5). Since $G_{K^{\mathrm{ab}}}$ is projective by Theorem 4.4 above (using that k is separably closed), every finite embedding problem for K^{ab} is dominated by a finite split embedding problem; and so solving any finite embedding problem for K^{ab} can be reduced to solving a finite sequence of finite split embedding problems each of which has a minimal normal subgroup as its kernel. So it suffices to show that such embedding problems have proper solutions. If the kernel of such an embedding problem is abelian, then we are done by the theorem of Ikeda cited above. Otherwise, the kernel of the embedding problem is a product of finitely many isomorphic non-abelian finite simple groups ([As], Chap. 3, 8.3, 8.2). This embedding problem for K^{ab} is induced by a finite split embedding problem for some finite extension K_1 of K that is contained in K^{ab}. But K is quasi-free by [HS], Theorem 5.1; and hence so is K_1, by [RSZ]. So there is a proper solution to the embedding problem for K_1; and this induces a proper solution to the embedding problem over K^{ab} because of linear disjointness, since K^{ab} is abelian over K_1 whereas the kernel of the embedding problem has no non-trivial abelian quotients. So K^{ab} is ω-free.

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