Embedding Problems and Adding Branch Points

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Abstract: If $Y \to X$ is a G-Galois branched cover of curves over an algebraically closed field k, and if G is a quotient of a finite group Γ, then $Y \to X$ is dominated by a Γ-Galois branched cover $Z \to X$. This is classical in characteristic 0, and was proven in characteristic p by the author [Ha6] and F. Pop [Po1] in conjunction with the proof of the geometric case of the Shafarevich Conjecture on free absolute Galois groups. The resulting cover $Z \to X$, though, may acquire additional branch points. The present paper shows how many new branch points are needed, and shows that there is some control on the positions of these branch points and on the inertia groups of $Z \to X$.

Section 1. Introduction and survey of results.

This paper concerns an aspect of the fine structure of the fundamental group of an affine curve U over an algebraically closed field k of characteristic p. In [Ra], [Ha3], it was shown which finite groups G are quotients of $\pi_1(U)$ — namely, according to Abhyankar's Conjecture, the set of such G depends only on the pair (g, n), where g is the genus of the smooth compactification X of U and n is the number of points in $X - U$. But the structure of the profinite group $\pi_1(U)$ remains a mystery, even in the case of the affine line. Moreover, the group $\pi_1(U)$ (unlike the set $\pi_A(U)$ of its finite quotients) does not depend just on (g, n) (cf. [Ha4, §1], [Ta, Thm. 3.5]), though it is unclear how it varies in moduli. In the current paper we study the structure of $\pi_1(U)$ by investigating how the finite quotients of this group fit together, and how $\pi_1(U)$ grows as additional points are deleted.

A preliminary result in this direction appeared in [Ha6], [Po1], in connection with proving the geometric case of the Shafarevich Conjecture. Namely, it was shown there that the absolute Galois group G_K of the function field K of U is a free profinite group (of rank equal to the cardinality of k). This was done by showing that every finite embedding problem for K has a proper solution, i.e. that if $\Gamma \twoheadrightarrow G$ is a surjection of finite groups, then every unramified G-Galois cover $V \to U$ of affine curves is dominated by a Γ-Galois *branched*

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cover $W \to U$. In fact, the proof (which used patching techniques in formal or rigid geometry) showed a bit more — viz. it bounded the number of branch points of $W \to U$. That bound was not sharp, however, and here we obtain the sharp bound (Theorem 5.4 below).

More precisely, let $G = \Gamma/N$, let $p(N)$ be the subgroup of N generated by its p-subgroups (so that $\bar{N} = N/p(N)$ is the maximal prime-to-p quotient of N), and let r be the *rank* of $\bar{N}$ (i.e. the minimal number of elements in any generating set). In [Ha6, Theorem 3.5] it was shown that $W \to U$ as above can be chosen with at most $r+1$ branch points; and it was asked if it is always possible to choose $W \to U$ with at most r branch points. (It is not in general possible with only $r-1$ branch points even in characteristic 0, as topological considerations show; and that implies the same for characteristic p.) In [Po1], it was shown that this is always possible in the case that $r = 0$, thus answering [Ha6, Question 3.7]. So if N above is a *quasi-p group* (i.e. is generated by its p-subgroups, or equivalently if $\bar{N} = 1$) then the cover $W \to U$ can be chosen to be unramified.

Here we show that for arbitrary r (not just $r = 0$), the dominating cover $W \to U$ can be chosen with at most r branch points (where as above, $r = \mathrm{rk}(\bar{N})$). In fact, we show a bit more. Namely, for finite group Γ and normal subgroup N of Γ, we will define the *relative rank* of N in Γ, denoted $\mathrm{rk}_\Gamma(N)$. This will be a non-negative integer that is $\le \mathrm{rk}(N)$ (but is often strictly less). What we will show is that in the above situation, the cover $W \to U$ can be chosen with at most $\mathrm{rk}_{\bar{\Gamma}}(\bar{N})$ branch points, where $\bar{\Gamma} = \Gamma/p(N)$. By using the $r = 0$ case, the proof of this result is reduced to the case that N is of order prime to p; and there we use methods of patching and lifting. In addition, we show that there is often control over the positions of the new branch points, and over the inertia groups of the resulting cover (cf. Props. 3.3, 3.5, 4.1, 5.1).

The results in this paper can be phrased in the language of embedding problems. This and other group-theoretic notions (along with some notions about covers) are discussed in Section 2. Then, in Section 3, we use formal patching to prove the above result in a key special case (when N has order prime to p, and one of the branch points of $Y \to X$ is tame, where $Y \to X$ is the smooth compactification of $V \to U$). In Section 4, we use a lifting result of Garuti [Ga, Theorem 2] to prove the above result in the case that $\mathrm{rk}_\Gamma(N) \le 1$, again assuming that N has order prime to p. Section 5 combines the two special cases, and applies Pop's result [Po1] in the case $r = 0$, to prove the full result (Theorem 5.4).

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Section 2. Notions concerning covers and groups.

In this paper we work over a fixed algebraically closed field k of characteristic $p \geq 0$, and consider covers of k-curves. A *cover* will be a finite generically separable morphism $Y \to X$ of k-schemes, where X is connected. If G is a finite group, then a G-*Galois* cover consists of a cover $Y \to X$ together with a homomorphism $\rho : G \to \mathrm{Aut}(Y/X)$, such that G acts simply transitively on any generic geometric fibre of the cover, via ρ. (The top space Y is not required to be connected. For example, the trivial G-Galois cover of X is a disjoint union of copies of X indexed by the elements of G, on which G acts by the regular representation.)

Since k is algebraically closed of characteristic p, there is a primitive mth root of unity $\zeta_m \in k$ for each positive integer m not divisible by p. Here we may choose the elements ζ_m to be compatible; i.e. such that $\zeta_{mm'}^{m'} = \zeta_m$ for all m, m'. From now on, these will be fixed. For any G-Galois cover $\psi : Y \to X$ of smooth connected k-curves and any tame ramification point η lying over a branch point $\xi \in X$, the corresponding extension of complete local rings is given by $y^m = x$, for some choice of local parameters x, y. The inertia group is generated by $c : y \mapsto \zeta_m y$, and the element $c \in G$ (which is independent of the choice of local parameters) is called the *canonical generator* of the inertia group at η. (Here and just below, we follow the terminology of [St] and [HS, §§2,3].) If all the branch points of $Y \to X$ are tame, and if the branch points are given with an ordering, say $\xi_1, \dots, \xi_r$, then we say that the cover has *description* $(c_1, \dots, c_r)$, where c_j is a canonical generator of inertia at a point over ξ_j, and where each c_j is determined up to (individual) conjugacy. In the case that $k = \mathbf{C}$ and $\zeta_m = e^{2\pi i/m}$, the fundamental group of $U = X - \{\xi_1, \dots, \xi_r\}$ has presentation

$$\pi_1(U) = \langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_r \mid \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} c_j = 1 \rangle, \qquad (*)$$

where g is the genus of X. Here the G-Galois cover corresponding to a surjection $\phi : \pi_1(U) \to G$ has description $(\phi(c_1), \dots, \phi(c_r))$. If p does not divide the order of G, then this also holds for an arbitrary algebraically closed field k of characteristic $p \geq 0$, via the same presentation $(*)$ of the maximal prime-to-p quotient $\pi_1(U)^{p'}$ of $\pi_1(U)$ [Gr, XIII, Cor. 2.12]. (This presentation arises via a specialization morphism between k and $\mathbf{C}$, which should be chosen so that the given roots of unity $\zeta_m \in k$ correspond to $e^{2\pi i/m} \in \mathbf{C}$. Cf. [Gr, XIII] and [GM, Thm. 4.3.2, Lemma 4.1.3].)

Consider more generally a G-Galois cover $Y \to X$ of a semi-stable k-curve X (i.e. X is connected and its only singularities are nodes). Let $\tilde{Y} \to \tilde{X}$ be the pullback of $Y \to X$ over the normalization $\tilde{X}$ of X. We say that $Y \to X$ is *admissible* if for each singular point $\eta \in Y$, the canonical generators at the two points $\eta_1, \eta_2 \in \tilde{Y}$ over η are inverses in G. A *thickening* of $Y \to X$ is a G-Galois cover of normal $k[[t]]$-curves $Y^* \to X^*$ whose closed fibre is $Y \to X$, and whose completion along the smooth locus of X is a trivial deformation of its closed fibre. If $Y \to X$ is an admissible cover, then such a $Y^* \to X^*$ is called an *admissible thickening* of $Y \to X$ if at the complete local ring at every singular point of Y the cover is given by the extension $k[[t, x_1, x_2]]/(x_1x_2 - t^m) \hookrightarrow k[[t, y_1, y_2]]/(y_1y_2 - t)$ for some m prime to p, where $x_i \mapsto y_i^m$ under the inclusion, and where an associated canonical generator of inertia acts by $x \mapsto \zeta_m x$, $y \mapsto \zeta_m^{-1} y$. Observe that in this situation, the singular points of the closed fibre X are isolated branch points of $Y^* \to X^*$ (and this does not contradict Purity of Branch Locus since X^* is not regular and Y^* is not flat over X^*). Since these points are branch points of the irreducible components of $Y \to X$, the process of constructing an admissible thickening can be regarded as a way of patching together these components in such a way that some of the branch points "cancel" on the general fibre (and cf. [HS, Thm. 7]). This observation will be key to the results of §3 below, and thus to the paper's main theorem, by yielding a cover with fewer branch points than would otherwise be expected.

The remainder of this section is devoted to discussing some group-theoretic notions that will be used in this paper.

If Γ is any finite group, then (following [FJ]), we define its *rank* to be the smallest non-negative integer $r = \mathrm{rk}(\Gamma)$ such that Γ has a generating set of r elements. (In the literature, this integer is also sometimes denoted by $d(\Gamma)$.) More generally, let E be a subgroup of a finite group Γ. A subset $S \subset E$ will be called a *relative generating set* for E in Γ if for every subset $T \subset \Gamma$ such that $E \cup T$ generates Γ, the subset $S \cup T$ also generates Γ. We define the *relative rank* of E in Γ to be the smallest non-negative integer $s = \mathrm{rk}_\Gamma(E)$ such that there is a relative generating set for E in Γ consisting of s elements. Thus every generating set for E is a relative generating set, and so $0 \le \mathrm{rk}_\Gamma(E) \le \mathrm{rk}(E)$. Also, $\mathrm{rk}_\Gamma(E) = \mathrm{rk}(E)$ if $E = 1$ or $E = \Gamma$, while $\mathrm{rk}_\Gamma(E) = 0$ if and only if E is contained in the Frattini subgroup $\Phi(\Gamma)$ of Γ.

A related notion is the following: Let G be a subgroup of a group Γ. A subset $T \subset \Gamma$ is a *supplementary generating set* for Γ *with respect to* G if $T \cup G$ generates Γ. Suppose that Γ is a finite group that is generated by two subgroups E, G. We then define the *relative rank* of $E \subset \Gamma$ *with* *respect to* G to be the smallest non-negative integer $t = \mathrm{rk}_\Gamma(E, G)$ such that there is a supplementary generating set T for Γ with respect to G, with T of cardinality t and $T \subset E$. Note that every relative generating set for E in Γ is a supplementary generating set for Γ with respect to G. So $0 \le \mathrm{rk}_\Gamma(E, G) \le \mathrm{rk}_\Gamma(E) \le \mathrm{rk}(E)$.

If p is a prime number, then the *quasi-p part* of a finite group Γ is the subgroup $p(\Gamma) \subset \Gamma$ that is generated by the p-subgroups of Γ (or equivalently, by the Sylow p-subgroups of Γ). Thus $p(\Gamma)$ is a characteristic subgroup of Γ, and in particular is normal. A group Γ is defined to be a *quasi-p* group if $\Gamma = p(\Gamma)$. Thus for any finite group Γ, the subgroup $p(\Gamma)$ is a quasi-p group and $\Gamma/p(\Gamma)$ is the (unique) maximal quotient of Γ whose order is not divisible by p. (In the other case, viz. $p = 0$, we set $p(\Gamma) = 1$.)

If Π, Γ, H are groups (not necessarily finite), then an *embedding problem* for Π consists of a pair of surjective group homomorphisms $\mathcal{E} = (\alpha : \Pi \to H, f : \Gamma \to H)$. A *weak solution* to the embedding problem consists of a group homomorphism $\beta : \Pi \to \Gamma$ such that $f\beta = \alpha$. If moreover β is surjective, then it is referred to as a *proper solution* to the embedding problem. An embedding problem is *finite* if Γ is finite. The motivation for the notion of embedding problems comes from Galois theory: If $K \subset L$ is a Galois field extension with group H, and if Π is the absolute Galois group G_K of K, then Galois theory yields a corresponding surjection $\alpha : G_K \twoheadrightarrow H$. Let $f : \Gamma \twoheadrightarrow H$ be a surjective homomorphism of finite (or profinite) groups. Then a proper [resp. weak] solution to the embedding problem (α, f) corresponds to a Γ-Galois field extension of K [resp. to a Γ-Galois K-algebra] containing the H-Galois extension L, such that the actions of Γ and H are compatible with the surjection $\Gamma \twoheadrightarrow H$. That is, the H-Galois extension L is embedded in a Γ-Galois extension via a solution to the embedding problem.

Observe that if $\phi : \Pi' \twoheadrightarrow \Pi$ is a surjective homomorphism of groups, then every embedding problem for Π induces an embedding problem for Π'. Namely, if $\mathcal{E} = (\alpha : \Pi \to H, f : \Gamma \to H)$ is an embedding problem for Π, then there is an *induced embedding problem* $\mathcal{E}' = (\alpha' : \Pi' \to H, f : \Gamma \to H)$ for Π', where $\alpha' = \alpha\phi$. Moreover, a weak or proper solution to the given embedding problem induces such a solution to the new problem. On the other hand, not every solution to the new problem need come from a solution to the original problem. These observations will be useful later, when considering the fundamental groups $\Pi = \pi_1(U)$ and $\Pi' = \pi_1(U')$ of two affine curves $U' \subset U$. In that context, solutions to embedding problems for Π correspond to certain unramified covers of U, whereas solutions to embedding problems for Π' are required merely to be unramified over U'.

As above, let $\mathcal{E} = (\alpha : \Pi \to H, f : \Gamma \to H)$ be an embedding problem for Π. If the exact sequence $1 \to N \to \Gamma \to H \to 1$ is split, where $N = \ker f$, then we say that $\mathcal{E}$ is a *split* embedding problem. A split embedding problem $\mathcal{E} = (\alpha, f)$ always has a weak solution, viz. $s\alpha : \Pi \to \Gamma$, where s is a section of $\Gamma \to H$. Often, finding proper solutions to embedding problems can be reduced to doing so for split embedding problems — e.g. see [FJ, §20.4], [Ha3, proofs of Thm. 5.4, Prop. 6.2], [Ha6, proof of Prop. 3.3], and [Po2, §1B(2)]. For the sake of completeness, we conclude this section with a precise statement of this reduction, in a form that can be cited later (in sections 3 and 5 below).

Proposition 2.1. *Let $\mathcal{E} = (\alpha : \Pi \to H, f : \Gamma \to H)$ be an embedding problem, and let $N = \ker f$. Suppose that $\mathcal{E}$ has a weak solution $\alpha_0 : \Pi \to \Gamma$, and let $H_0 \subset \Gamma$ be the image of α_0. Consider the semi-direct product $\Gamma_0 = N \rtimes H_0$, with respect to the conjugation action of H_0 on $N \triangleleft \Gamma$, and let $f_0 : \Gamma_0 \twoheadrightarrow H_0$ be the natural quotient map. If the split embedding problem $\mathcal{E}_0 = (\alpha_0 : \Pi \to H_0, f_0 : \Gamma_0 \to H_0)$ has a proper solution, then so does $\mathcal{E}$.*

Proof. Since α_0 is a weak solution to $\mathcal{E}$, we have $f\alpha_0 = \alpha$; or equivalently $\bar{\mu}\alpha_0 = \alpha$, where $\bar{\mu} : H_0 \to H$ is the restriction of $f : \Gamma \twoheadrightarrow H$ to H_0. Since f has kernel N, and since its restriction $f|_{H_0} = \bar{\mu}$ is surjective onto H (because $\bar{\mu}\alpha_0 = \alpha$ is), it follows that Γ is generated by N and H_0. Let $\mu : \Gamma_0 \to \Gamma$ be the homomorphism defined by taking the identity inclusion on each factor of $\Gamma_0 = N \rtimes H_0$. (This is a homomorphism since the conjugation action of H_0 on N in Γ_0 is the same as the conjugation action of H_0 on N in Γ.) Then μ is a surjection since N and H_0 generate Γ, and it is straightforward to check that $f\mu = \bar{\mu}f_0$. We thus obtain the following commutative diagram (where as above $\bar{\mu}\alpha_0 = \alpha : \Pi \to H$):

$$\begin{tikzcd}
 & & & \Pi \arrow[d, "\alpha_0"'] & \\
1 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}"'] & \Gamma_0 \arrow[r, "f_0"] \arrow[d, "\mu"'] & H_0 \arrow[r] \arrow[d, "\bar{\mu}"'] & 1 \\
1 \arrow[r] & N \arrow[r] & \Gamma \arrow[r, "f"] & H \arrow[r] & 1
\end{tikzcd}$$

So any proper solution $\beta_0 : \Pi \twoheadrightarrow \Gamma_0$ to the split embedding problem $\mathcal{E}_0 = (\alpha_0 : \Pi \to H_0, f_0 : \Gamma_0 \to H_0)$ yields a proper solution $\beta : \Pi \twoheadrightarrow \Gamma$ to the original embedding problem $\mathcal{E}$, viz. $\beta = \mu\beta_0$. $\square$

In particular, the reduction in the above proposition can be always accomplished in the case that the group Π is *projective* (which by definition [FJ, §20.4] means that every finite embedding problem for Π has a weak solution). Indeed, in that situation, the given embedding problem $\mathcal{E}$ has a weak solution α_0, and so the above hypotheses are satisfied.

Section 3. Results via patching.

This section uses formal patching methods in order to prove the main result of the paper in a special case. Namely, we consider a finite group Γ and a quotient $G = \Gamma/N$, together with a G-Galois étale cover of smooth affine k-curves $V \to U$ (where, as always, k is algebraically closed of characteristic $p \geq 0$). We consider the smooth completions X, Y of U, V, and assume that $Y \to X$ is tamely ramified at some branch point ξ. We will also assume that p does not divide the order of N. In this situation, we will show that there is a Γ-Galois cover $W \to U$ dominating $V \to U$, having at most $\mathrm{rk}_\Gamma(N)$ branch points, and with specified inertia groups over those points (Prop. 3.5). This solves a certain embedding problem (Cor. 3.6). Moreover we will obtain greater control on the number of branch points of the constructed cover and on the inertia groups over $X - U$ in the case that the embedding problem is split (under an additional assumption on normalizers). Cf. Prop. 3.3 and Cor. 3.4. A more general and more precise version of these results appears first, in Prop. 3.1 an Cor. 3.2).

Patching methods, in formal or rigid geometry, have previously been used to prove a number of results concerning fundamental groups of varieties, especially for curves in characteristic p — e.g. [Ha1], [Ha2], [Ra,§§3-5], [St], [Sa], [Ha6], [Po1], [HS]; see also [Ha5, §2]. The basic idea is to build a simpler, but possibly degenerate, cover with similar properties, and then to deform it to a family of covers whose generic member is as desired. In order to reduce the number of branch points of the cover we construct here, and thus achieve the desired sharp bound on that number, we will use a construction involving *admissible* covers; cf. §2 above and the remark following Proposition 3.3 below.

Below we preserve the terminology of Section 2, and begin with an assertion concerning the problem of modifying a cover so as to expand its Galois group. (Cf. also [Ha2, Theorem 2] for a related result.) Note that here, and in the next few results, it suffices to use the value $\mathrm{rk}_\Gamma(E, G)$, rather than having to use the possibly larger value $\mathrm{rk}_\Gamma(E)$.

Proposition 3.1. *Let Γ be a finite group generated by two subgroups G, E, where p does not divide $|E|$, and let $r \geq \mathrm{rk}_\Gamma(E, G)$. Let $V \to U$*

be a G-Galois étale cover of smooth connected affine k-curves with smooth completion $Y \to X$. Suppose that $Y \to X$ is tamely ramified over some point ξ of $B = X - U$, and that some inertia group over ξ normalizes E. Then there is a smooth connected Γ-Galois cover $W \to U$ having at most r branch points.

Moreover, if $\{e_1, \ldots, e_r\} \subset E$ is a supplementary generating set for Γ with respect to G, then the above cover can be chosen so that:

(i) The H-Galois cover $W/N \to U$ agrees with $V/(N \cap G) \to U$, where N is the normal closure of E in Γ and where $H = \Gamma/N = G/(N \cap G)$.

(ii) There are inertia groups of $W \to U$ over the branch points $\xi_1, \ldots, \xi_r$ having canonical generators $e_1, \ldots, e_r$, respectively.

(iii) Each inertia group of $Y \to X$ over any point $\chi \in B - \{\xi\}$ is also an inertia group of $Z \to X$ over χ, where Z is the smooth completion of W.

Proof. Let $R = k[[t]]$, let $\tilde{X} = X \times_k R$, and let X^* be the blow-up of $\tilde{X}$ at the closed point of $\bar{\xi} = \xi \times_k R$. Thus X^* is a regular two-dimensional scheme that is projective as an R-curve. Its closed fibre X_0 is connected and consists of two irreducible components: a proper transform that is isomorphic to X, and an exceptional divisor that is isomorphic to $\mathbf{P}^1_k$. These two components meet at the point on the proper transform corresponding to ξ on X, and to the point $s = 0$ on the projective s-line $\mathbf{P}^1_k$. (Here we take $s = t/x$, where x is a local parameter for X at ξ. Thus the locus of $(s = \infty)$ is the proper transform of $\bar{\xi}$.)

Let $\{e_1, \ldots, e_r\} \subset E$ be a supplementary generating set for Γ with respect to G, and let $\sigma_1, \ldots, \sigma_r$ be distinct points of $\mathbf{P}^1_k$ other than $s = 0, \infty$. By hypothesis we may choose a point $\eta \in Y$ over $\xi \in X$ for which the inertia group $I \subset G$ normalizes E. Let $g \in G$ be the canonical generator of the inertia group I. Thus the subgroup $E_0 \subset \Gamma$ generated by E and g is of the form $E_0 = E \rtimes I$, and hence its order is not divisible by p. Let $E_1 \subset E_0$ be the subgroup generated by $e_1, \ldots, e_r, g$, and let $h = (e_1 \cdots e_r)^{-1} g$. Thus p also does not divide the order of E_1, and $g^{-1} e_1 \cdots e_r h = 1$. As discussed in §2 above (and cf. [Gr, XIII, Cor. 2.12]), there exists a smooth connected E_1-Galois cover $M \to \mathbf{P}^1_k$ branched at $0, \sigma_1, \ldots, \sigma_r, \infty$ with description $(g^{-1}, e_1, \ldots, e_r, h)$. Let $\mu \in M$ be a point over 0 at which g^{-1} is a canonical generator of inertia. Consider the induced (disconnected) Γ-Galois covers $\mathrm{Ind}_G^\Gamma Y \to X$ and $\mathrm{Ind}_{E_1}^\Gamma M \to \mathbf{P}^1_k$, consisting of disjoint unions of copies of $Y \to X$ and $M \to \mathbf{P}^1_k$, respectively, indexed by the cosets of G and of E_1 in Γ. We may identify Y and M with the identity components of the respective induced covers. Identifying the two points $\gamma(\eta) \in \mathrm{Ind}_G^\Gamma Y$ and $\gamma(\mu) \in \mathrm{Ind}_{E_1}^\Gamma M$ for each $\gamma \in \Gamma$, we obtain a Γ-Galois cover Z_0 of the reducible curve X_0. Moreover Z_0 is admissible over X_0 by construction, and is connected since G and E_1 generate Γ (and cf. also [HS, §4, Prop. 2]). By [HS,§2, Cor. to Thm. 2], there is a Γ-Galois cover $Z^* \to X^*$ which is an admissible thickening of $Z_0 \to X_0$ (viz., in the terminology of [HS], the unique solution to the corresponding relative thickening problem).

Let $Z^\circ \to X^\circ$ be the fibre of $Z^* \to X^*$ over the generic point of $\operatorname{Spec} k[[t]]$. Since X^* is the blow-up of $\tilde{X} = X \times_k k[[t]]$ at the closed point of $\bar{\xi}$, there are isomorphisms of K-curves $X^\circ \approx X \times_k K \approx X^* \times_R K$, where $K = k((t))$. Since $Z^* \to X^*$ is a thickening of $Z_0 \to X_0$, the cover in particular restricts to a trivial deformation of the restriction of $\operatorname{Ind}_G^\Gamma Y \to X$ to $X - \{\xi\}$. Hence $Z^\circ \to X^\circ$ is branched at the points of $(B - \{\xi\}) \times_k K$, with the same inertia groups as the corresponding points of $B - \{\xi\}$ for $Y \to X$; and it is branched at no other point of X° except for those whose closure in $\tilde{X}$ passes through the point $(\xi, (t = 0))$. Among points of the latter type, $Z^\circ \to X^\circ$ is branched precisely at $r + 1$ points $\sigma_1^\circ, \dots, \sigma_r^\circ, \infty^\circ$ whose closures $\sigma_1^*, \dots, \sigma_r^*, \infty^*$ in X^* pass through the points $\sigma_1, \dots, \sigma_r, \infty$. (Note that the singular point of X_0 is an isolated point of the branch locus of $Z^* \to X^*$, as discussed in §2; so it does not contribute to the branch locus of $Z^\circ \to X^\circ$.) Over the point σ_i°, the inertia groups of $Z^\circ \to X^\circ$ are the same as those of $Z^* \to X^*$ over σ_i^*, and one of them has canonical generator e_i. Here the closure of ∞° in $\tilde{X}$ is $\bar{\xi} = \xi \times_k R$. So under the isomorphism $X \times_k K \approx X^\circ$, the branch locus consists of the r points σ_i° and the points of $B \times_k K$ (with $\xi \times_k K$ corresponding to the point ∞° in X°). Also, in the special case $E = 1$, the cover $Z^\circ \to X^\circ$ is just the base change of $Y \to X$ from k to K. Since the above construction commutes with taking quotients, we deduce for arbitrary E that the cover $Z^\circ/N \to X^\circ$ is the base change of $Y/(N \cap G) \to X$ from k to K.

Thus $Z^\circ \to X^\circ \approx X \times_k K$ is a smooth connected Γ-Galois cover whose restriction $W^\circ \to U^\circ := U \times_k K$ has the desired properties for W, but over K instead of over k. Being of finite type, this cover descends to a smooth connected Γ-Galois cover $Z_A \to X_A := X \times_k A$ over some finitely generated k-algebra $A \subset K$, whose restriction to $U_A := U \times_k A$ has the corresponding properties over A. Here $\operatorname{Spec} A$ is an absolutely irreducible variety, since $A \subset K$ and k is algebraically closed. By [Ha2, Prop. 5] (or [FJ, Props. 8.8, 9.29]) we conclude that the specialization $Z_\nu \to X$ of $Z_A \to X_A$ at a k-point $\nu \in \operatorname{Spec} A$ restricts to a G-Galois cover $W := Z_\nu \times_X U \to U$ having the desired properties. $\square$

Using the notion of embedding problems (cf. §2), we may rephrase Proposition 3.1 in more group-theoretic terms. In particular, we have the following corollary. In this connection, we recall that an inclusion $U' \hookrightarrow U$ of affine curves induces a surjection $\pi_1(U') \twoheadrightarrow \pi_1(U)$.

Corollary 3.2. *Let X be a smooth connected projective k-curve, let $U \subset X$ be a dense affine open subset, and let $\xi \in X - U$. Let $\Pi = \pi_1(U)$ and let Π^* be the quotient of Π corresponding to covers whose smooth completions are tamely ramified over ξ. Let $\mathcal{E} = (\alpha : \Pi^* \to H, f : \Gamma \to H)$ be a finite embedding problem for Π^*, and let β be a weak solution to $\mathcal{E}$. Suppose that Γ is generated by $G, E \subset \Gamma$, where E is a subgroup of $\ker(f)$ with p not dividing $|E|$. Suppose also that the normalizer of E in Γ contains $\beta(I)$, where $I \subset \Pi^*$ is an inertia group over ξ. Let $r \geq \mathrm{rk}_\Gamma(E, G)$. Then there is an open subset $U' \subset U$ such that $U - U'$ has cardinality r and the induced embedding problem $\mathcal{E}'$ for $\Pi' = \pi_1(U')$ has a proper solution.*

Proof. Let N be the normal closure of E in Γ. Since $E \subset \ker(f)$, it follows that $N \subset \ker(f)$, and H is a quotient of the group $H_1 := \Gamma/N = G/(N \cap G)$. Let $f_1 : \Gamma \twoheadrightarrow H_1$ and $f_0 : G \twoheadrightarrow H_1$ be the natural quotient maps, and let $\alpha_1 = f_0\beta : \Pi^* \twoheadrightarrow H_1$. Replacing $\mathcal{E} = (\alpha : \Pi^* \to H, f : \Gamma \to H)$ by the embedding problem $\mathcal{E}_1 = (\alpha_1 : \Pi^* \to H_1, f_1 : \Gamma \to H_1)$, we may assume that $H = H_1$, that $f|_G = f_0$, and that $\alpha = f_0\beta$.

Now β is a proper solution to the embedding problem $\mathcal{E}_0 = (\alpha : \Pi^* \to H, f_0 : G \to H)$, where $f_0 = f|_G : G \to H$. Under the Galois correspondence, the surjection $\beta : \Pi^* \twoheadrightarrow G$ corresponds to a connected étale G-Galois cover $V \to U$ whose smooth completion $Y \to X$ is tamely ramified over ξ, and such that some inertia group over ξ normalizes E. By Proposition 3.1, there is a smooth connected Γ-Galois cover $W \to U$ having at most r branch points, such that there is an isomorphism of H-Galois covers $W/N \approx V/(N \cap G)$ of U. (Here, as above, N is the normal closure of E in Γ, and $H = \Gamma/N = G/(N \cap G)$.) So over the complement $U' \subset U$ of the r-point branch locus of $W \to U$, we obtain a Γ-Galois étale cover $W' \to U'$ corresponding to a proper solution to the embedding problem $\mathcal{E}'$. □

Remarks. (a) The above corollary does not rely on the full statement of Proposition 3.1, since neither (ii) nor (iii) there are used. But if Π' is replaced by a suitably refined quotient ${\Pi'}^*$ (containing additional information about inertia groups), then a corresponding result can be proven, with the aid of (ii) and (iii) of 3.1, about embedding problems for ${\Pi'}^*$; and this would correspond to the full content of 3.1.

(b) In the other direction, it would be desirable to state a version of Corollary 3.2 just for Π, rather than for Π^*, and without assumptions on normalizers. Correspondingly, it would be desirable to state a version of

Proposition 3.1 without the assumptions on tameness or normalizers. (The proof of 3.1 at least shows that it is possible to weaken the assumption that an inertia group I over ξ normalizes E, by instead assuming that E, I generate a prime-to-p subgroup of Γ.)

(c) The proofs of the above results, and those that follow in this section, require the base field k to be algebraically (or at least separably) closed, because of the use of [Gr, XIII, Cor. 2.12] in the proof of Proposition 3.1. In particular, the condition that k be *large* (cf. [Po2]) does not suffice, at least for the proofs here. See also Remark (b) at the end of Section 4 below.

In particular, in the split embedding problem situation, the above results give rise to the following proposition and corollary:

Proposition 3.3. *Let Γ be a finite group of the form $N \rtimes G$, where p does not divide the order of N, and let $\{n_1, \ldots, n_r\} \subset N$ be a supplementary generating set for Γ with respect to G. Let $V \to U$ be a G-Galois étale cover of smooth connected affine k-curves whose smooth completion $Y \to X$ is tamely ramified over some point ξ of $B = X - U$. Then there is a smooth connected Γ-Galois cover $W \to U$ branched only at r points $\xi_1, \ldots, \xi_r$, with smooth completion $Z \to X$, such that: $W/N \approx V$ as G-Galois covers of U; the element n_i is the canonical generator of an inertia group of $W \to U$ over ξ_i; $Z \to X$ is tamely ramified over ξ; and each inertia group of $Y \to X$ over any point $\chi \in B - \{\xi\}$ is also an inertia group of $Z \to X$ over χ.*

Proof. Since N is normal in Γ, any inertia group of $Y \to X$ over ξ must normalize N. So Proposition 3.1 applies, with $E = N$, and with the H of Proposition 3.1 being the same group as G here. This yields the result (with tameness over ξ following since $Z/N = Y$ and p does not divide the order of N). $\square$

Remark. In the special case that the cover $Y \to X$ has trivial inertia groups over ξ (so that the given tamely ramified point is not actually a true branch point), the assertion of Proposition 3.3 is closely related to [Ha6, Theorem 3.5] (by taking the point ξ_0 of [Ha6, Theorem 3.5] to be ξ above), and the proofs are also related. But in the general case, the assertion of [Ha6, Theorem 3.5] is weaker than Proposition 3.3 above, since it requires an extra branch point (beyond the r points in Prop. 3.3). The difference is that in the result above, admissible covers can be used to avoid adding the extra branch point, provided that we have a tameness assumption. (The result in [Ha6] also uses a weaker notion of rank.)

Corollary 3.4. *Let X be a smooth connected projective k-curve, let $U \subset X$ be a dense affine open subset, and let $\xi \in X - U$. Let $\Pi = \pi_1(U)$ and let*

Π^* be the quotient of Π corresponding to covers whose smooth completions are tamely ramified over ξ. Consider a finite split embedding problem $\mathcal{E} = (\alpha : \Pi^* \to G, f : \Gamma \to G)$ for Π^*, such that p does not divide the order of $N = \ker(f)$. Let $r \geq \mathrm{rk}_\Gamma(N, \iota(G))$, where $\iota : G \to \Gamma$ is a section of f. Then there is an open subset $U' \subset U$ such that $U - U'$ has cardinality r and the induced embedding problem $\mathcal{E}'$ for $\Pi' = \pi_1(U')$ has a proper solution.

Proof. Identifying G with its image under ι, we may identify Γ with the semidirect product $N \rtimes G$. By the assumption on rank, there is a supplementary generating set $\{n_1, \dots, n_r\} \subset N$ for Γ with respect to G. Also, the homomorphism $\alpha : \Pi^* \twoheadrightarrow G$ corresponds to a G-Galois connected étale cover of affine k-curves whose smooth completion $Y \to X$ is tamely ramified over ξ. So the hypotheses of Proposition 3.3 are satisfied, yielding a Γ-Galois cover $W \to U$ that is étale over some $U' \subset U$ with $U - U'$ of cardinality r. This cover corresponds to a homomorphism $\Pi' \twoheadrightarrow \Gamma$ that is a solution to the induced embedding problem $\mathcal{E}'$. $\square$

Remarks. (a) The proper solution to $\mathcal{E}'$ in 3.4 is automatically a proper solution to the induced embedding problem for Π'^*, the quotient of Π' corresponding to covers of U' whose smooth completions are tamely ramified over ξ. As in 3.3, this is because p does not divide the order of N.

(b) Corollary 3.4 can also be deduced directly from Corollary 3.2, by taking $H = G$, $N = E$, and $\beta = \iota\alpha$.

As discussed in Section 2, results about split embedding problems for a group Π can sometimes be extended to results about arbitrary embedding problems for Π, e.g. in situations in which the group Π is projective. By [Se2, Proposition 1], the fundamental group of an affine k-curve has cohomological dimension ≤ 1; and hence it is a projective group [Se1, I.5.9, Prop. 45]. Using this projectivity, we obtain the following variant of Proposition 3.3 that applies even in the non-split case. It does, however, provide a bit less control on the number of punctures needed (and cf. Remark (c) after the proof of Corollary 3.6 below).

Proposition 3.5. *Let Γ be a finite group, let N be a normal subgroup of order prime to p, and let $G = \Gamma/N$. Let $\{n_1, \dots, n_r\} \subset N$ be a relative generating set for N in Γ. Let $V \to U$ be a G-Galois étale cover of smooth connected affine k-curves whose smooth completion $Y \to X$ is tamely ramified over some point ξ of $B = X - U$. Then there is a smooth connected Γ-Galois cover $W \to U$ branched only at r points $\xi_1, \dots, \xi_r$, such that $W/N \approx V$ as G-Galois covers of U; n_i is the canonical generator of an inertia group over ξ_i; and the smooth completion of $W \to U$ is tamely ramified over ξ.*

Proof. The fundamental group $\Pi := \pi_1(U)$ is a projective group, since $\mathrm{cd}(\Pi) \le 1$ [Se2, Prop. 1]). So the surjective homomorphism $\Pi \twoheadrightarrow G$ corresponding to $V \to U$ lifts to a homomorphism $\Pi \to \Gamma$, say with image G_0. Let $V_0 \to U$ be the G_0-Galois cover corresponding to this lift. Thus we have an unramified $N \cap G_0$-Galois cover $V_0 \to V$. Let Y_0 be the smooth completion of V_0. Then p does not divide the degree of $Y_0 \to Y$, since that degree divides $|N|$. Since $Y \to X$ is tame over ξ, it follows that so is $Y_0 \to X$. Moreover any inertia group of $Y_0 \to X$ over ξ must normalize N, since N is normal in Γ.

Since $G_0 \twoheadrightarrow G = \Gamma/N$, the group Γ is generated by N and G_0. Hence Γ is generated by $n_1, \dots, n_r$ and G_0; i.e. $\{n_1, \dots, n_r\} \subset N$ is a supplementary generating set for Γ with respect to G_0. So by Proposition 3.1, there is a Γ-Galois cover $W \to U$ having at most r branch points $\xi_1, \dots, \xi_r$ such that the G-Galois cover $W/N \to U$ agrees with $V_0/(N \cap G_0) \to U$, and such that n_i is the canonical generator of an inertia group of $W \to U$ over ξ_i. Since $V_0/(N \cap G_0)$ is isomorphic to V as a G-Galois cover of U, and since p does not divide the order of N, it follows that $W \to U$ is as desired. □

Corollary 3.6. *The assertion of Corollary 3.4 carries over to finite embedding problems that are not necessarily split, provided that one instead takes* $r \ge \mathrm{rk}_\Gamma(N)$.

Proof. Since $r \ge \mathrm{rk}_\Gamma(N)$, there is a relative generating set $\{n_1, \dots, n_r\} \subset N$ for N in Γ. The proof then proceeds parallel to that of Corollary 3.4, but using Proposition 3.5 instead of Proposition 3.3. □

Remarks. (a) Remark (a) after Corollary 3.4 carries over as well to Corollary 3.6.

(b) Corollary 3.6 can also be proven by applying Proposition 2.1 to Corollary 3.4. This uses that Π is projective; that $\mathrm{rk}_\Gamma(N) \ge \mathrm{rk}_\Gamma(N, G_0)$ (where G_0 is as in the proof of 3.5); and that the G_0-cover $Y_0 \to X$ is tamely ramified over ξ (as in the proof of 3.5).

(c) As mentioned above, 3.3 and 3.4 apply only to split embedding problems, whereas 3.5 and 3.6 apply more generally to embedding problems that need not be split. But in the process of reducing to the split case, we obtain weaker conclusions in 3.5 and 3.6 than in 3.3 and 3.4 (though under more general hypotheses). Specifically, different notions of generators and rank are used in the two pairs of results, and the notion of rank in 3.5 and 3.6 will typically be larger (when both make sense). The need for these variant notions here is due to the fact that one does not in advance know the group $G_0 \subset \Gamma$ that arises in the proof of 3.5 (and indirectly, in 3.6), in the process of reducing to the split case. Another way in which the generalized hypothesis leads to a weaker conclusion here is that one no longer has the same control on the inertia groups over $B - \{\xi\}$ in the situation of 3.5 that one had in 3.3. This is because the map $G_0 \to G$ in 3.5 need not be an isomorphism, and because one does not know *a priori* which choice of $G_0 \subset \Gamma$ over G will be needed in the construction.

(d) In the results in this section of the paper, it would be desirable to prove that the positions of the new r branch points can be specified in advance. In a related situation, such a result with control on the additional branch locus appears in Section 4 below. But there, unfortunately, connectivity cannot always be guaranteed.

Section 4. Results via lifting.

In this section another special case of the main theorem in proven, by means of lifting to characteristic 0. As before in Proposition 3.5, we have a finite group Γ and a quotient $G = \Gamma/N$, and a G-Galois étale cover of smooth affine k-curves $V \to U$. And as before, the problem is to show that there is a Γ-Galois cover $W \to U$ dominating $V \to U$, having at most $\mathrm{rk}_\Gamma(N)$ branch points with specified inertia there, under the assumption that N has order prime to p. But unlike the situation of the previous section, we need not make any tameness assumption here on the smooth completion of $V \to U$. What is shown here (Prop. 4.1) is that if the relative rank $\mathrm{rk}_\Gamma(N)$ is at most 1 (or if $p = 0$), then such a connected W exists, and moreover that the position of the extra branch point can be given in advance. (On the other hand if $\mathrm{rk}_\Gamma(N) > 1$ and $p > 0$, then we still obtain a W with specified branch locus and inertia, but conceivably it might not be connected.) Thus if $\mathrm{rk}_\Gamma(N) \leq 1$ or $p = 0$ then we can obtain a proper solution to the corresponding embedding problem (Cor. 4.2).

The method of lifting and specializing to characteristic 0, in order to study fundamental groups in characteristic p, was used by Grothendieck (cf. [Gr], [GM]) in the situation of the *tame* fundamental group — with the strongest conclusions obtained on the maximal prime-to-p quotient of π_1. The idea is to work with a mixed characteristic complete discrete valuation ring R, whose residue field is the given algebraically closed field k of characteristic p. By using the knowledge of π_1 in characteristic 0, one can construct a cover over the general fibre; close this up over R; and then specialize to the closed fibre to obtain a cover over k. The main difficulty in extending this method to more general covers is that the restriction to the closed fibre may be inseparable over the generic point or it may have wild ramification there. Nevertheless, in [Ra, §6], Raynaud was able to use this method, in conjunction with a careful analysis of ramification along the closed fibre of a semistable model, in order to construct covers of the affine line in characteristic p in a key case (and thereby complete the proof of Abhyankar's Conjecture for $\mathbf{A}^1$).

In addition to the problem of specializing characteristic 0 covers to characteristic p, there is also the problem of lifting a given characteristic p cover to characteristic 0. Again, this was done by Grothendieck in the tame case ([Gr], [GM]). In the wild case, this is not in general possible, since some characteristic p curves violate the Hurwitz bound on the number of automorphisms that a curve of genus g can have. But M. Garuti has proven a modified lifting result, which will be sufficient for our purposes. Namely, he has shown [Ga, Thm. 2] that if we are given a G-Galois cover $Y \to X$ over k, and if a lift X^* of X to R as above is given, then (possibly after enlarging R) there is a normal G-Galois cover $Y^* \to X^*$ over R whose closed fiber $Y_k^* \to X$ is closely related to $Y \to X$. Specifically, Y_k^* is an irreducible curve whose only singularities are cusps over wildly ramified branch points of $Y \to X$, and Y is the normalization of Y_k^*.

Using Garuti's result to lift, followed by a construction in characteristic 0 and then descent to characteristic p, we obtain the following version (Proposition 4.1) of the main theorem of the paper. Note that in the proof, after constructing a Γ-Galois cover W^* in characteristic 0, we do not in general know that its closed fibre W_k^* is irreducible. So instead we will choose a suitable irreducible component W of W_k^*, which will be Γ'-Galois for some $\Gamma' \subset \Gamma$. But if $\mathrm{rk}_\Gamma(N) \leq 1$ then W_k^* will in fact be irreducible, and so we will have $\Gamma' = \Gamma$ in this special case.

We state the result in a slightly more general form, in which we specify in advance the extra inertial elements $n_1, \ldots, n_r$, but do not require them to constitute a relative generating set for N. In this generality we still obtain a Γ-Galois cover with the desired properties except for connectivity (and so the Galois group of a connected component will be a subgroup of Γ). But when the n_i form a relative generating set, and $r \leq 1$ or $p = 0$, then we do obtain connectivity (cf. part (c) below).

Proposition 4.1. *Let Γ be a finite group, let N be a normal subgroup of order not divisible by p, and let $G = \Gamma/N$. Let $S = \{n_1, \ldots, n_r\}$ be a finite subset of N, with $r \geq 0$. Let $V \to U$ be a G-Galois étale cover of smooth connected affine k-curves, and let $\xi_1, \ldots, \xi_r \in U$ be distinct points.*

a) Then there is a subgroup $\Gamma' \subset \Gamma$ and a smooth connected Γ'-Galois cover $W \to U$ branched only at $\xi_1, \ldots, \xi_r$, such that $\Gamma = N\Gamma'$ and $n_1 \in \Gamma'$ (if $r \geq 1$); $W/(N \cap \Gamma')$ is isomorphic to V as a G-Galois cover of U; and the *canonical generator of each inertia group over ξ_i is conjugate to n_i in Γ, with n_i being equal to the canonical generator of some inertia group over ξ_i if $r = 1$ or $p = 0$.*

b) If $r \le 1$ or $p = 0$ then we may also require that $S \subset \Gamma'$.

c) If S is a relative generating set for N in Γ, and if either $r \le 1$ or $p = 0$, then we may take $\Gamma' = \Gamma$.

Proof. Let $Y \to X$ be the smooth completion of $V \to U$. Let $B = X - U$, which is a non-empty finite set of $|B|$ points. Let $B' = \{\xi_1, \dots, \xi_r\}$, which is a subset of U; and let $U' = U - B'$. There are two cases to consider:

Case A: $p = 0$.

(a), (b): Let g be the genus of X and let $n = |B|$. Thus $n \ge 1$. By [Gr, XIII, Cor. 2.12], the fundamental group $\pi_1(U)$ is generated by elements $a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n$ subject to the presentation $(*)$ of §2 above. Similarly, $\pi_1(U')$ is generated by elements $\tilde{a}_1, \dots, \tilde{a}_g, \tilde{b}_1, \dots, \tilde{b}_g, \tilde{c}_1, \dots, \tilde{c}_{n+r}$ subject to the analogous presentation $(*)'$. The natural map $\pi_1(U') \twoheadrightarrow \pi_1(U)$ takes $\tilde{a}_i \mapsto a_i$, $\tilde{b}_i \mapsto b_i$, $\tilde{c}_j \mapsto c_j$ for $1 \le j \le n$, and $\tilde{c}_j \mapsto 1$ for $n < j \le n + r$.

The G-Galois étale cover $V \to U$ corresponds to a surjective group homomorphism $\phi : \pi_1(U) \to G$. Let $\alpha_i = \phi(a_i)$ and $\beta_i = \phi(b_i)$ for $1 \le i \le g$, and let $\gamma_j = \phi(c_j)$ for $1 < j \le n$. For each such i and j choose elements $\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_j \in \Gamma$ over $\alpha_i, \beta_i, \gamma_j \in G$. Also, let $\tilde{\gamma}_{n+j} = n_j \in N$ for $1 \le j \le r$. Then there is a unique homomorphism $\tilde{\phi} : \pi_1(U') \to \Gamma$ given by $\tilde{\phi}(\tilde{a}_i) = \tilde{\alpha}_i$, $\tilde{\phi}(\tilde{b}_i) = \tilde{\beta}_i$, and $\tilde{\phi}(\tilde{c}_j) = \tilde{\gamma}_j$ for $1 < j \le n + r$. (The image of $\tilde{c}_1$ is uniquely determined by the single relation $(*)'$ for $\pi_1(U')$ and by the requirement that $\tilde{\phi}$ be a homomorphism.) Let Γ' be the image of $\tilde{\phi}$. Thus $S \subset \Gamma'$. Also, $\tilde{\phi}$ lifts the surjection ϕ, and so $N\Gamma' = \Gamma$. Thus $\tilde{\phi}$ corresponds to a connected Γ'-Galois étale cover $W \to U$ that dominates $V \to U$. The smooth completion $Z \to X$ of $W \to U$ has description $(\tilde{\phi}(\tilde{c}_1), \dots, \tilde{\phi}(\tilde{c}_{n+r}))$ (cf. §2 above). In particular, $n_j = \tilde{\phi}(\tilde{c}_{n+j}) \in N$ is a canonical generator of inertia above ξ_j for $1 \le j \le r$. This cover is thus as desired, for parts (a) and (b).

(c): By part (b), Γ' contains S. But S is assumed to be a relative generating set for N. Since $N\Gamma' = \Gamma$ it follows that $\Gamma' = \Gamma$.

Case B: $p > 0$.

(a): Let R be a complete discrete valuation ring of mixed characteristic and residue field k (e.g. the Witt ring over k). By [Gr, III, Cor. 7.4], there is a smooth complete R-curve X^* whose closed fibre is X. By [Ga, Thm. 2], it follows that after replacing R by a finite extension (and X^* by its pullback to that extension), there is an irreducible G-Galois cover $Y^* \to X^*$ of proper R-curves with Y^* normal, such that Y is the normalization of the closed fibre

Y^*_k of Y^*, and such that Y^*_k is everywhere unibranched, with the morphism $Y \to Y^*_k$ being an isomorphism away from the wildly ramified points of Y. (Here the residue field of the enlarged R is still k, since k is algebraically closed.)

Suppose that $R \subset T$ is a finite extension of complete discrete valuation rings, and let $Y^*_T \to X^*_T$ be the normalized pullback of $Y^* \to X^*$ from R to T. Thus Y^*_T is the normalization of $Y^* \times_{X^*} X^*_T$. Now the closed fibre of $Y^* \to X^*$ is generically étale, but the closed fibre of $X^*_T \to X^*$ is totally ramified. So the irreducible schemes Y^* and X^*_T dominate no common non-trivial covers of X^*. But Y^* and X^*_T are normal; so the intersection of their function fields is that of X^*. Also, since Y^* is Galois over X^*, the corresponding extension of function fields is also Galois. So by [FJ, p. 110], the function fields of Y and X^*_T are linearly disjoint over that of X^*. Thus $Y^* \times_{X^*} X^*_T$ is irreducible and hence so is Y^*_T . The conclusion is that the generic fibre Y° of Y^* is geometrically irreducible.

Since X^* is regular and Y^* is normal, Purity of Branch Locus applies to $Y^* \to X^*$ [Na, 41.1]; and since the closed fibre is generically étale, it follows that the branch locus B^* defines a cover of $\operatorname{Spec} R$. Also, since $X^* \to \operatorname{Spec} R$ is smooth, it follows from [Gr, III, Thm. 3.1] that there are R-points ξ^*_i of X^* that lift the k-points ξ_i of $X = X^*_k$. The ξ^*_i have pairwise disjoint support, since the points ξ_i are distinct and since R is a complete local ring. Let B'^* be the union of the loci of the ξ^*_i, and write $U^* = X^* - B^*$, $U'^* = U^* - B'^*$, $V^* = Y^* \times_{X^*} U^*$, and $V'^* = V^* \times_{U^*} U'^*$.

Let K be the fraction field of R, and let $\bar{K}$ be the algebraic closure of K. Let $\bar{X}^\circ = X^* \times_R \bar{K}$; $\bar{B}^\circ = B^* \times_R \bar{K}$; $\bar{U}^\circ = \bar{X}^\circ - \bar{B}^\circ$; and $\bar{V}^\circ = V^* \times_{U^*} \bar{U}^\circ$. Similarly, write $\bar{B}'^\circ = B'^* \times_R \bar{K}$; $\bar{U}'^\circ = \bar{U}^\circ - \bar{B}'^\circ$; and $\bar{V}'^\circ = \bar{V}^\circ \times_{\bar{U}^\circ} \bar{U}'^\circ$. Thus $\bar{V}^\circ \to \bar{U}^\circ$ and $\bar{V}'^\circ \to \bar{U}'^\circ$ are G-Galois étale covers which are connected (since Y° is geometrically irreducible). Also, $\bar{B}'^\circ$ consists of r distinct $\bar{K}$-points $\bar{\xi}^\circ_1, \ldots, \bar{\xi}^\circ_r$ specializing respectively to the k-points $\xi_1, \ldots, \xi_r$ — viz. $\bar{\xi}^\circ_i = \xi^*_i \times_R \bar{K}$. By Hensel's Lemma, the compatible system of roots of unity $\{\zeta_n\}$ in k lifts uniquely to such a system in K, and so in $\bar{K}$. With respect to this system, we may consider the canonical generators of inertia of covers defined over K or $\bar{K}$.

By Case A over the characteristic 0 field $\bar{K}$, there is a subgroup $\Gamma_0 \subset \Gamma$ containing $n_1, \ldots, n_r$ such that $N\Gamma_0 = \Gamma$, together with a smooth connected Γ_0-Galois cover $\bar{W}^\circ \to \bar{U}^\circ$ of $\bar{K}$-curves that dominates $\bar{V}^\circ \to \bar{U}^\circ$ and whose canonical generators of inertia at points $\bar{\omega}^\circ_i$ over $\bar{\xi}^\circ_i$ are equal to n_i. Let $N_0 = N \cap \Gamma_0$, so that $\Gamma_0/N_0 \approx G$. For some subfield $\tilde{K} \subset \bar{K}$ that is finite over K, this cover descends to a connected Γ_0-Galois cover of $\tilde{K}$-curves with the corresponding properties (and so in particular it dominates the induced G-Galois cover $\tilde{V}^\circ \to \tilde{U}^\circ$ of $\tilde{K}$-curves). Replacing K by $\tilde{K}$, and R by its integral closure in $\tilde{K}$, we may assume that there is a smooth connected Γ_0-Galois cover $W^\circ \to U^\circ$ that dominates $V^\circ \to U^\circ$ and is étale away from the general fibre B'° of B'^*; and that there is a K-point ω_i° on W° over ξ_i° at which the canonical generator of inertia is n_i.

Let W^* be the normalization of U^* in W°, and let $W'^* = W^* \times_{U^*} U'^*$. Thus $W^* \to U^*$ is Γ_0-Galois and $W^*/N_0 \approx V^*$ as G-Galois covers of U^*, and similarly for $W'^* \to U'^*$. Moreover $W'^* \to U'^*$ is étale except possibly on the closed fibre. Since p does not divide the order of $N_0 \subset N$, any ramification of $W'^* \to V'^*$ along the closed fibre is tame. So applying Abhyankar's Lemma, and after replacing K by a finite separable extension (and R by its normalization in this extension), we may assume that $W'^* \to V'^*$ is étale along the general point of the closed fibre. Since $V'^* \to U'^*$ is étale, it follows that the Γ_0-Galois cover $W'^* \to U'^*$ is étale along the general point of the closed fibre, as well as away from the closed fibre. But U'^* is regular and W'^* is normal. So Purity of Branch Locus implies that $W'^* \to U'^*$ is étale. Hence the closed fibre $W_k^* \to U_k^*$ of $W^* \to U^*$ is étale over U'.

Let ω_i^* be the closure of ω_i° in W^* and let ω_i be the closed point of ω_i^*. Thus ω_i^* is an R-point of W^* over ξ_i^*, and ω_i is a k-point of W_k^* over ξ.

Claim: The k-curve W_k^* is smooth at the point ω_i, and the inertia group I_i there is cyclic with canonical generator n_i.

To prove the claim, we first apply [GM, Corollary 2.3.6] over the complete local ring of X^* at $\xi_i \in X \subset X^*$. In our situation, that result says that the restriction of the tame I_i-Galois cover $\operatorname{Spec} \hat{\mathcal{O}}_{W^*,\omega_i} \to \operatorname{Spec} \hat{\mathcal{O}}_{U^*,\xi_i}$ over its closed fibre is also tame (in the sense of [GM, Def. 2.2.2]). In particular, this restriction $\operatorname{Spec} \hat{\mathcal{O}}_{W_k^*,\omega_i} \to \operatorname{Spec} \hat{\mathcal{O}}_{U,\xi_i}$ is normal, and the inertia group I_i is a cyclic group. Thus the k-curve W_k^* is regular at ω_i, and hence smooth there (since k is perfect). Moreover I_i is abelian, and $n_i \in I_i$ (since n_i is in the inertia group of W° at ω_i°), so $\langle n_i \rangle$ is a normal subgroup of I_i. The intermediate cover $\operatorname{Spec} \hat{\mathcal{O}}_{W^*,\omega_i}/\langle n_i \rangle \to \operatorname{Spec} \hat{\mathcal{O}}_{U^*,\xi_i}$ is then a normal $I_i/\langle n_i \rangle$-Galois cover which is unramified except at the closed point (where it is totally ramified). By Purity of Branch Locus it follows that this intermediate cover is trivial, and so $I_i = \langle n_i \rangle$. That is, the inertia group of $W^* \to U^*$ at ω_i is generated by n_i. (This conclusion can also be reached by reasoning as in the proof of [Fu, Theorem 3.3, Case 1], instead of using [GM].) So $\hat{\mathcal{O}}_{W^*,\omega_i}$ is of the form $\hat{\mathcal{O}}_{U,\xi_i}[f_i^{1/m_i}]$, where f_i is a local uniformizer along ξ_i^*, where m_i is the order of n_i. Since n_i is the canonical generator of W° at ω_i°, it follows that the generator $n_i \in I_i$ acts on the overring by $f_i^{1/m_i} \mapsto \zeta_{m_i} f_i^{1/m_i}$.

Restricting to the closed fibre, we find that n_i is the canonical generator of $W_k^* \to U$ at ω_i. This proves the claim.

Now $W_k^* \to U$ is Galois, étale over $U' = U - \{\xi_1, \dots, \xi_n\}$, and smooth at the point ω_i over ξ_i. So W_k^* is smooth. Hence there is a unique irreducible component W of W_k^* that contains ω_1. Let $\Gamma' \subset \Gamma_0$ be the decomposition group of the generic point of W. Thus $W \to U$ is a Γ'-Galois smooth connected cover that is branched only at $\{\xi_1, \dots, \xi_n\}$. Also, by the claim, the canonical generator of inertia at any point of W_k^* over ξ_i is conjugate to n_i in Γ_0; and in particular this is true for the points of W over ξ_i. Moreover the inertia group at $\omega_1 \in W$ has canonical generator n_1, and so we have that $n_1 \in \Gamma'$. And since V is irreducible, the composition $W \hookrightarrow W_k^* \to W_k^*/N_0 \approx V$ (which is a morphism of covers of U) is surjective on points. Thus $W/(N_0 \cap \Gamma') \approx V$ as G-Galois covers of U. This implies that $N_0\Gamma'/N_0 \approx \Gamma'/(N_0 \cap \Gamma') \approx G \approx \Gamma_0/N_0$; but $N_0, \Gamma' \subset \Gamma_0$. So $N_0\Gamma' = \Gamma_0$ and thus $N\Gamma' = N\Gamma_0 = \Gamma$. Thus W has the desired properties.

(b): This is automatic: If $r = 0$ then S is empty; and if $r = 1$ then $S = \{n_1\}$, and $n_1 \in \Gamma'$ by (a).

(c): The same proof works as in Case A(c). □

Reinterpreting the above result in terms of embedding problems, we obtain:

Corollary 4.2. *Let X be a smooth connected projective k-curve, let $U \subset X$ be a dense affine open subset, and let $\Pi = \pi_1(U)$. Let $\mathcal{E} = (\alpha : \Pi \to G,$ $f : \Gamma \to G)$ be a finite embedding problem for* Π, *such that p does not divide the order of $N = \ker(f)$. Let $U' \subset U$ such that $U - U'$ has cardinality $r \geq \mathrm{rk}_\Gamma(N)$. If $p = 0$ or $r \leq 1$, then the induced embedding problem $\mathcal{E}'$ for $\Pi' = \pi_1(U')$ has a proper solution.*

Proof. The surjection $\alpha : \Pi \twoheadrightarrow G$ corresponds to a connected G-Galois étale cover $V \to U$. By Proposition 4.1, there is a smooth connected Γ-Galois cover $W \to U$ branched only at the r points of $B' = U - U'$, and which dominates $V \to U$. Its restriction $W' \to U'$ is étale, and dominates the restriction $V' \to U'$. Hence it corresponds to a proper solution to the induced embedding problem $\mathcal{E}'$ for $\Pi' = \pi_1(U')$. □

Remarks. (a) It would be desirable to extend Proposition 4.1(c) to the case that $r > 1$ with $p > 0$. While the main theorem of the paper (Theorem 5.4) does give a connected Γ-Galois cover $W \to U$ that dominates a given G-Galois étale cover $V \to U$ and has r additional branch points (even if

$r > 1$), it does not allow control over the positions of those branch points (except for the first).

The difficulty in extending the proof of 4.1(c) to $r > 1$, $p > 0$ is this: In the proof of Case B of 4.1, if S is a relative generating set for N in Γ, then the cover $W^* \to U^*$ is irreducible and Γ-Galois (using (c) in the characteristic 0 version); but it is unclear whether its closed fibre W_k^* is also irreducible, if $r > 1$. It would suffice to show that $Z_k^* \to X$ is unibranched at its wildly ramified points, where Z_k^* is the closed fibre of the normalization Z^* of X^* in W^*. One approach to this would be to use Purity of Branch Locus over Y^*; but for this, one wants Y^* to be regular. This raises the question of whether Garuti's result [Ga, Thm. 2] can be strengthened to show that Y^* can always be chosen to be regular.

(b) The proofs of the above results require the base field k to be algebraically closed, because of the use of [Ga, Thm. 2] in Proposition 4.1.

Section 5. The main result.

This section contains the main result of this paper, that a given Γ/N-Galois étale cover $V \to U$ of an affine k-curve is dominated by a Γ-Galois cover of U that is branched at $\mathrm{rk}_{\bar\Gamma}(\bar N)$ points of U, where $\bar\Gamma, \bar N$ are the reductions of Γ, N modulo $p(N)$. In the case that the order of N is prime to p (so that $\bar\Gamma = \Gamma$ and $\bar N = N$), we prove this essentially by combining the special cases in which it has already been shown: the case that the smooth completion of $V \to U$ has a tamely ramified branch point (cf. §3), and the case where $\mathrm{rk}_\Gamma(N) \le 1$ (cf. §4). This is done in Proposition 5.1. Afterwards, this result is combined with a result of F. Pop (cf. Thm. 5.2 and Cor. 5.3 below) to prove the general case (Theorem 5.4). This is then interpreted in terms of embedding problems (Corollary 5.5).

Proposition 5.1. *Let Γ be a finite group, let N be a normal subgroup whose order is not divisible by p, and let $G = \Gamma/N$. Let $\{n_1, \ldots, n_r\} \subset N$ be a relative generating set for N in Γ, with $r \ge 0$. Let $V \to U$ be a G-Galois étale cover of smooth connected affine k-curves. Then there is a smooth connected Γ-Galois cover $W \to U$ branched only at r distinct points $\xi_1, \ldots, \xi_r \in U$, such that W/N is isomorphic to V as a G-Galois cover of U, and n_i is a canonical generator of inertia over ξ_i for $1 < i \le r$. Moreover we may specify the position of ξ_1 in advance, if $r > 0$.*

Proof. If $r \le 1$ then this assertion is contained in the statement of Proposition 4.1. So we may assume $r > 1$.

Let $\xi_1 \in U$ be any point. We may apply Proposition 4.1 with $S = \{n_1\}$. Doing so yields a subgroup $\tilde G \subset \Gamma$ and a smooth connected $\tilde G$-Galois cover

$\tilde{V} \to U$ branched only at ξ_1, such that $\Gamma = N\tilde{G}$ and $n_1 \in \tilde{G}$; $\tilde{V}/(N \cap \tilde{G})$ is isomorphic to V as a G-Galois cover of U; and n_1 is the canonical generator of an inertia group over ξ_1.

Let X be the smooth completion of U, let $U' = U - \{\xi_1\}$, let $\tilde{V}'$ be the restriction of $\tilde{V}$ to U', and let $\tilde{Y}$ be the normalization of X in $\tilde{V}'$ (or equivalently, in $\tilde{V}$). Thus $\tilde{Y} \to X$ is a $\tilde{G}$-Galois cover that is tamely ramified over the point $\xi_1 \in B' := X - U'$. Moreover n_1 is the canonical generator of some inertia group over ξ_1, and this inertia group normalizes N since N is normal. In addition, $\Gamma = N\tilde{G}$, and so the relative generating set $\{n_1, \dots, n_r\}$ for N is in particular a supplementary generating set for Γ with respect to $\tilde{G}$. Since $n_1 \in \tilde{G}$, it follows that $\{n_2, \dots, n_r\}$ is a supplementary generating set for Γ with respect to $\tilde{G}$.

So we may apply Proposition 3.1 (with $\tilde{G}, N, r-1, \tilde{V}' \to U', n_2, \dots, n_r$ playing the roles of $G, E, r, V \to U, e_1, \dots, e_r$ there). As a result, we obtain a smooth connected Γ-Galois cover $W' \to U'$ having at most $r-1$ branch points $\xi_2, \dots, \xi_r \in U'$, satisfying the three conditions (i)-(iii) there. That is, the G-Galois cover $W'/N \to U'$ agrees with $\tilde{V}'/(N \cap \tilde{G}) \to U'$; there are inertia groups of $W' \to U'$ over the branch points $\xi_2, \dots, \xi_r$ having canonical generators $n_2, \dots, n_r$ respectively; and the inertia groups of $\tilde{Y} \to X$, over each point of $B' - \{\xi_1\} = X - U$, are also inertia groups of $Z \to X$ over that point, where Z is the smooth completion of W'. Let W be the normalization of U in W'. Thus $W \to U$ is a smooth connected Γ-Galois cover branched only at $\xi_1, \dots, \xi_r$, with n_i a canonical generator of inertia over ξ_i for each $i > 1$. As a G-Galois cover of U, the intermediate cover W/N is isomorphic to $\tilde{V}/(N \cap \tilde{G})$ and hence to V. □

Remarks. (a) Unlike Proposition 4.1, the above result yields a connected Γ-Galois cover over the given G-Galois cover even if there are two or more elements in the relative generating set, in characteristic $p > 0$. But on the other hand, in 5.1 we lose control of the positions of the branch points other than ξ_1, and of the inertia group over ξ_1. Similarly, unlike the results of §3, the above result does not require that the smooth completion $Y \to X$ of $V \to U$ have a tamely ramified branch point. But on the other hand, in 5.1 we have less control over inertia groups, and do not have a version that is analogous to 3.1 using the smaller number $\mathrm{rk}(E, G)$ of new branch points.

(b) In the proof of the above result, it is tempting to try to invoke Proposition 4.1 repeatedly on successive n_i's, rather than to use Proposition 3.1 (and thus to control the positions of all of the branch points, though losing control over the inertia groups). The strategy would be to take the subgroup $\tilde{G} \subset \Gamma$ containing n_1 and surjecting onto G, given by the use of

Prop. 4.1 as above; to take the minimal $\tilde{G}$-invariant subgroup $N_1 \subset N$ that contains n_2; and then to form the semi-direct product $\Gamma' = N_1 \rtimes \tilde{G}$ with respect to the conjugation action of $\tilde{G}$ on N_1 in Γ. Applying 4.1 again would yield a subgroup $\tilde{G}' \subset \Gamma'$ that contains $(n_2, 1)$ and surjects onto $\tilde{G}$ under the second projection; and then one could take the image of $\tilde{G}'$ in Γ under the multiplication homomorphism $\mu : \Gamma' \to \Gamma$ given by $(n, g) \mapsto ng$. This image $\mu(\tilde{G}') \subset \Gamma$ would then contain n_2 and surject onto G, and one might hope to repeat the process. But unfortunately, $\mu(\tilde{G}')$ need no longer contain n_1. For example, suppose that we are given $p > 3$; $N = S_3$; $G = \langle g \rangle$, cyclic of order p; and $\Gamma = N \times G$; with $n_1 = (12)$ and $n_2 = (13)$ being (relative) generators of N. Then we could have $\tilde{G} = \langle (12), g \rangle \subset \Gamma$ (viewing N, G as subgroups of Γ); $N_1 = N$; $\Gamma' = N \rtimes \tilde{G}$; $\tilde{G}' = \langle ((13), 1), ((12), (12)), (1, g) \rangle \subset N \rtimes \tilde{G} = \Gamma'$; and $\mu(\tilde{G}') = \langle (13), g \rangle \subset \Gamma$, which is strictly smaller than $\Gamma = N \times G$ and does not contain the first generator $n_1 = (12)$ of N.

Finally, we combine the above proposition with a result of F. Pop, to obtain our main theorem (Theorem 5.4 below). In this theorem, unlike the previous results in this paper, we permit the order of the kernel N to be divisible by p. The following is a rephrasing of Pop's result [Po1, Theorem B]:

Theorem 5.2. [Po1, Thm. B] *Let $\Gamma = Q \rtimes G$ where Q is a quasi-p group; let $Y \to X$ be a smooth connected G-Galois cover of k-curves; and let $\xi \in X$ (possibly a branch point of $Y \to X$). Then there is a smooth connected Γ-Galois cover $Z \to X$ dominating $Y \to X$, such that $Z \to Y$ is branched only at points of Y over ξ, and the inertia groups of $Z \to Y$ over those points are the Sylow p-subgroups of Q.*

Although this result is stated just for the split case, it implies a result in the more general case of group extensions by a quasi-p kernel. Namely, as the following corollary states, if U is a smooth connected affine k-curve and $\mathcal{E} = (\alpha, f)$ is a finite embedding problem for $\pi_1(U)$ with $\ker(f)$ a quasi-p group, then $\mathcal{E}$ has a proper solution.

Corollary 5.3. *Let Q be a normal quasi-p subgroup of a finite group Γ, and let $G = \Gamma/Q$. Let $V \to U$ be a connected G-Galois étale cover of smooth affine k-curves. Then there is a connected Γ-Galois étale cover $W \to U$ dominating $V \to U$.*

Proof. By [Se2, Prop. 1], the group $\Pi = \pi_1(U)$ has cohomological dimension 1 and hence is a projective group. The G-Galois étale cover $V \to U$ corresponds to a surjection $\alpha : \Pi \to G$, and there is a natural quotient map $f : \Gamma \to G$. Thus $\mathcal{E} = (f : \Gamma \to G, \alpha : \Pi \to G)$ is an embedding problem for the projective group Π, and so it has a weak solution $\alpha_0 : \Pi \to \Gamma$, say with image G_0. Let $\Gamma_0 = Q \rtimes G_0$, where the semi-direct product is taken with respect to the conjugation action of G_0 on $Q \lhd \Gamma$, and let $f_0 : \Gamma_0 \twoheadrightarrow G_0$ be the natural quotient map.

The surjection $\alpha_0 : \Pi \to G_0$ corresponds to a connected G_0-Galois étale cover $V_0 \to U$, say with smooth completion $Y_0 \to X$. Theorem 5.2 now applies to the group Γ_0, the G_0-Galois cover $Y_0 \to X$, and a point $\xi \in B := X - U$. So there is a smooth connected Γ_0-Galois cover $Z_0 \to X$ dominating $Y_0 \to X$ such that $Z_0 \to Y_0$ is branched only at points over ξ. Thus $Z_0 \to X$ is étale over U, and corresponds to a surjection $\beta_0 : \Pi \to \Gamma_0$.

Now $f\beta_0 = \alpha_0$ since $Z_0 \to X$ dominates $Y_0 \to X$. So β_0 is a proper solution to the split embedding problem $\mathcal{E}_0 = (\alpha_0 : \Pi \to G_0, f_0 : \Gamma_0 \to G_0)$. Thus by Proposition 2.1, the original embedding problem $\mathcal{E} = (f : \Gamma \to G, \alpha : \Pi \to G)$ has a proper solution $\beta : \Pi \to \Gamma$. Here $f\beta = \alpha$. The map β corresponds to a connected Γ-Galois étale cover $W \to U$, and this cover dominates $V \to U$ because $f\beta = \alpha$. $\square$

Combining the above with Proposition 5.1 yields the main theorem of the paper:

Theorem 5.4. *Let N be a normal subgroup of a finite group Γ, and let $G = \Gamma/N$. Let $\bar{N} = N/p(N)$ and $\bar{\Gamma} = \Gamma/p(N)$, and let $r = \mathrm{rk}_{\bar{\Gamma}}(\bar{N})$. Let $V \to U$ be a G-Galois étale cover of smooth connected affine k-curves. Then there is a smooth connected Γ-Galois cover $W \to U$ branched only at r distinct points $\xi_1, \dots, \xi_r \in U$, such that W/N is isomorphic to V as a G-Galois cover of U. Moreover we may specify the position of ξ_1 in advance, if $r > 0$.*

Proof. First observe here that $p(N)$ is a characteristic subgroup of N, and so is a normal subgroup of Γ. Hence $\bar{\Gamma} = \Gamma/p(N)$ is well defined.

Pick $\xi_1 \in U$. Since $r = \mathrm{rk}_{\bar{\Gamma}}(\bar{N})$, there is a relative generating set $\{\bar{n}_1, \dots, \bar{n}_r\} \subset \bar{N}$ for $\bar{N}$ in $\bar{\Gamma}$. Also, $\bar{\Gamma}/\bar{N} = (\Gamma/p(N))/(N/p(N)) \approx \Gamma/N = G$. So by Proposition 5.1, there are distinct points $\xi_2, \dots, \xi_r \in U - \{\xi_1\}$ together with a smooth connected $\bar{\Gamma}$-Galois cover $\bar{W} \to U$ branched only at $S = \{\xi_1, \dots, \xi_r\}$ such that $\bar{W}/\bar{N}$ is isomorphic to V as a G-Galois cover of U. Let $U' = U - S$ and let $\bar{W}'$ be the inverse image of U' in $\bar{W}$. We may now apply Corollary 5.3 with $Q = p(N)$, and with $\bar{W}' \to U'$ playing the role of $V \to U$ there. As a result, we obtain a connected Γ-Galois étale cover $W' \to U'$ dominating $\bar{W}' \to U'$. The normalization $W \to U$ of U in $W' \to U'$ is then as desired. $\square$

Remark. In the situation of Theorem 5.4 above, if the short exact sequence

$1 \to p(N) \to \Gamma \to \bar{\Gamma} \to 1$ is split, then in the proof we can apply Pop's Theorem (5.2 above) rather than Corollary 5.3. Doing so gives more information about inertia. In particular, suppose we are given an integer $r \geq \mathrm{rk}_{\bar{\Gamma}}(\bar{N})$ and a relative generating set $\{n_1, \ldots, n_r\} \subset N$ for N in Γ. Then in the split situation we may choose the Γ-Galois cover $W \to U$ so that the branching over $\xi_2, \ldots, \xi_r$ is tame, and so that n_i is the canonical generator of inertia at some point over ξ_i for each $i > 1$.

Reinterpreting the above result in terms of embedding problems, we immediately obtain:

Corollary 5.5. *Let U be a smooth connected affine k-curve, let $\Pi = \pi_1(U)$, and let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be a finite embedding problem for Π. Let $N = \ker(f)$ and let $r = \mathrm{rk}_{\bar{\Gamma}}(\bar{N})$. Then for some set $S \subset U$ of cardinality r, the induced embedding problem $\mathcal{E}'$ for $\Pi' = \pi_1(U - S)$ has a proper solution. Moreover, if $r > 0$, then one of the points of S can be chosen in advance.*

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