

Patching over Fields

David Harbater*

Julia Hartmann†

Abstract

We develop a new form of patching that is both far-reaching and more elementary than the previous versions that have been used in inverse Galois theory for function fields of curves. A key point of our approach is to work with fields and vector spaces, rather than rings and modules. After presenting a self-contained development of this form of patching, we obtain applications to other structures such as Brauer groups and differential modules.

1 Introduction

This manuscript introduces a new form of *patching*, a method that has been used to prove results in Galois theory over function fields of curves (e.g. see the survey in [13]). Our approach here, which involves patching vector spaces given over a collection of fields, is more elementary than previous approaches, while facilitating various applications.

There are several forms of patching in the Galois theory literature, all drawing inspiration from “cut-and-paste” methods in topology and analysis, in which spaces are constructed on metric open sets and glued on overlaps. Underlying this classical approach are Riemann’s Existence Theorem (e.g. see [13], Theorem 2.1.1), Serre’s GAGA [29], and Cartan’s Lemma on factoring matrices [2]. In the case of formal patching (e.g. in [10], [17], [27]), one considers rings of formal power series, and “patches” them together using Grothendieck’s Existence Theorem on sheaves over formal schemes ([5], Corollary 5.1.6). In the context of rigid patching (e.g. in [22], [28], [26]), one relies on Tate’s rigid analytic spaces, where there is a form of “rigid GAGA” that takes the place of Grothendieck’s theorem. The variant known as algebraic patching (e.g. [8], [31], [7]) restricts attention to the line, and draws on ideas from the rigid approach (most notably, convergent power series rings). But that strategy avoids relying on more substantial geometric results, and instead works with normed rings and versions of Cartan’s Lemma.

The current approach differs from formal and rigid patching by focusing on vector spaces rather than modules; i.e. by working over (fraction) fields rather than rings. Doing so makes it possible for us to prove our patching results more directly, without the more substantial

*Supported in part by NSF Grant DMS-0500118.

†Supported by the German National Science Foundation (DFG).

foundations needed in the other approaches. As a result, our approach is conceptually simpler and should be more accessible to those not already familiar with patching methods. Moreover our method is not restricted to the case of the line. In addition to providing a framework in which one can prove the sort of results on inverse Galois theory that have been shown using previous methods (see Section 7.2 below), our approach also permits easy application of patching to other situations in which one works just with fields and not with rings. See Section 7.1 for an application to Brauer groups of fields, and Section 7.3 on an application to differential modules (which are in fact vector spaces). Further applications in these directions appear in [16] and [15].

A framework for stating patching results can be found in Section 2, followed by some preliminary results in Section 3. Our main patching result (Theorem 4.12) and a variant (Theorem 4.14) are shown in Section 4. Section 5 takes up related forms of patching, in which “more local” patches are used; the main result there is Theorem 5.9, along with a variant, Theorem 5.10. A further generalization to singular curves appears in Section 6. The versions in Section 5 and 6 are designed to allow the method of patching over fields to be used in a variety of future applications. (Those interested just in our main form of patching, Theorem 4.12, can skip Sections 5 and 6, as well as the second half of Section 2.) Finally, in Section 7, we show how our new version of patching can be used to prove both old and new results.

We thank Daniel Krashen for helpful discussions concerning the application to Brauer groups in Section 7.1, and both him and Moshe Jarden for further comments and suggestions. We also thank the Mathematical Sciences Research Institute for their hospitality during the writing of this paper.

2 The setup for patching over fields

The general framework for patching can be expressed in a categorical language that permits its use in various contexts. Here we provide such a framework for patching vector spaces over fields; later in Section 7, we show how our results can be extended and applied to patching other objects over fields. We begin with some notation.

If $\alpha_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$ are functors ($i = 1, 2$), then we may form the **2-fibre product** category $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_2$ (with respect to α_1, α_2), defined as follows: An object in the category consists of a pair (V_1, V_2) together with an isomorphism $\phi : \alpha_1(V_1) \xrightarrow{\simeq} \alpha_2(V_2)$ in \mathcal{C}_0 , where V_i is an object in \mathcal{C}_i ($i = 1, 2$). A morphism from $(V_1, V_2; \phi)$ to $(V'_1, V'_2; \phi')$ consists of morphisms $f_i : V_i \rightarrow V'_i$ in \mathcal{C}_i (for $i = 1, 2$) such that $\phi' \circ \alpha_1(f_1) = \alpha_2(f_2) \circ \phi$.

For any field F , we write $\text{Vect}(F)$ for the category of finite dimensional F -vector spaces. If F_1, F_2 are subfields of a field F_0 and we let $\mathcal{C}_i = \text{Vect}(F_i)$, then there are base change functors $\alpha_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$ given on objects by $\alpha_i(V_i) = V_i \otimes_{F_i} F_0$. So we can form the category $\mathcal{C} := \text{Vect}(F_1) \times_{\text{Vect}(F_0)} \text{Vect}(F_2)$ with respect to these functors (and in the sequel, the functors α_i will be understood, though suppressed in the 2-fibre product notation).

Given an object $(V_1, V_2; \phi)$ in the above category \mathcal{C} , let $V_0 = \alpha_2(V_2) = V_2 \otimes_{F_2} F_0$. Then V_0, V_1, V_2 are each vector spaces over $F := F_1 \cap F_2 \leq F_0$. Let $i_2 : V_2 \hookrightarrow V_0$ be the natural

inclusion, and let $i_1 : V_1 \hookrightarrow V_0$ be the composition of the natural inclusion $V_1 \hookrightarrow \alpha_1(V_1) = V_1 \otimes_{F_1} F_0$ with ϕ . With respect to the inclusions i_1, i_2 , we may form the vector space fibre product $V := V_1 \times_{V_0} V_2 = \{(v_1, v_2) \in V_1 \times V_2 \mid i_1(v_1) = i_2(v_2)\}$ over F ; we call V the **fibre product** of the object $(V_1, V_2; \phi)$. Note that if we identify V_1 (resp. V_2) with its image under i_1 (resp. i_2), then the fibre product V is just the *intersection* of V_1 and V_2 inside V_0 . Of course this identification depends on ϕ (since i_1 depends on ϕ).

The following is a special case of [9], Proposition 2.1.

Proposition 2.1. *Let $F_1, F_2 \leq F_0$ be fields, and let $F = F_1 \cap F_2$. Let*

$$\beta : \text{Vect}(F) \rightarrow \text{Vect}(F_1) \times_{\text{Vect}(F_0)} \text{Vect}(F_2)$$

be the natural map given by base change. Then the following two statements are equivalent:

- (1) *β is an equivalence of categories.*
- (2) *For every positive integer n and every matrix $A \in \text{GL}_n(F_0)$ there exist matrices $A_i \in \text{GL}_n(F_i)$ such that $A = A_1 A_2$.*

Moreover if these conditions hold, then the inverse of β (up to isomorphism) is given on objects by taking the fibre product.

Our main Theorems 4.12 and 5.9 assert that the base change functor $\beta : \text{Vect}(F) \rightarrow \text{Vect}(F_1) \times_{\text{Vect}(F_0)} \text{Vect}(F_2)$ is an equivalence of categories in situations where certain fields $F \leq F_1, F_2 \leq F_0$ arise geometrically. The above proposition reduces the proofs there to showing the following two statements in those contexts:

Factorization: For every $A \in \text{GL}_n(F_0)$ there exist $A_i \in \text{GL}_n(F_i)$ such that $A = A_1 A_2$.

Intersection: $F_1 \cap F_2 = F$.

In Sections 4 and 5, we prove each of these two conditions in turn, and as a result obtain the main theorems. Beforehand, in Section 3, we prove a factorization result that will be useful in proving both of the above two conditions. In later results (Theorems 5.10 and 6.4), we will consider a more general type of situation, and for this we introduce the following definitions (which give another perspective on the results of this paper, though they are not otherwise essential and may be skipped on a first reading).

Let $\mathcal{F} := \{F_i\}_{i \in I}$ be a finite inverse system of fields (not necessarily filtered), whose inverse limit is a field F . Let $\iota_{ij} : F_i \rightarrow F_j$ denote the inclusion map associated to $i, j \in I$ with $i \succ j$ in the partial ordering on the index set I . By a (vector space) **patching problem** for the system \mathcal{F} we will mean a system $\mathcal{V} := \{V_i\}_{i \in I}$ of finite dimensional F_i -vector spaces for $i \in I$, together with F_i -linear maps $\nu_{ij} : V_i \rightarrow V_j$ for all $i \succ j$ in I , such that for $i \succ j$ in I , the induced F_j -linear map $\nu_{ij} \otimes_{F_i} F_j : V_i \otimes_{F_i} F_j \rightarrow V_j$ is an isomorphism. Note that for any patching problem \mathcal{V} , the dimension $\dim_{F_i} V_i$ is independent of $i \in I$; and we call this the **dimension** of the patching problem, denoted $\dim \mathcal{V}$.

A **morphism of patching problems** $\{V_i\}_{i \in I} \rightarrow \{V'_i\}_{i \in I}$ for \mathcal{F} is a collection of F_i -linear maps $\phi_i : V_i \rightarrow V'_i$ (for $i \in I$) which are compatible with the maps $\nu_{ij} : V_i \rightarrow V_j$ and $\nu'_{ij} : V'_i \rightarrow V'_j$. The patching problems for \mathcal{F} thus form a category $\text{PP}(\mathcal{F})$. (One can also consider the analogous notion of *algebra patching problems*, in which each of the finite dimensional vector spaces is given the structure of an associative algebra over its base field. Similarly one can consider patching problems for (finite dimensional) commutative algebras, central simple algebras, etc. These also form categories.)

Every finite dimensional F -vector space V induces a patching problem $\beta(V)$ for \mathcal{F} , by taking $V_i = V \otimes_F F_i$ and taking $\nu_{ij} = \text{id}_V \otimes_F \iota_{ij}$. Here β defines a functor from the category $\text{Vect}(F)$ of finite dimensional F -vector spaces to the category $\text{PP}(\mathcal{F})$. If \mathcal{V} is a patching problem for \mathcal{F} , and $\beta(V)$ is isomorphic to \mathcal{V} , we say that V is **solution** to the patching problem \mathcal{V} . If $\beta : \text{Vect}(F) \rightarrow \text{PP}(\mathcal{F})$ is an equivalence of categories, then it defines a bijection on isomorphism classes of objects; i.e., every patching problem for \mathcal{F} has a unique solution up to isomorphism.

The situation described in Proposition 2.1 above can then be rephrased in terms of patching problems for the inverse system $\mathcal{F} := \{F_0, F_1, F_2\}$ with $0 \prec 1, 2$ in the partial ordering, and with corresponding inclusions $\iota_{i0} : F_i \rightarrow F_0$ for $i = 1, 2$. Namely, the proposition says that in this situation, the above functor β is an equivalence of categories if and only if the matrix factorization condition (2) of the proposition holds. As noted above, every patching problem $\{V_0, V_1, V_2\}$ for \mathcal{F} then has a unique solution V up to isomorphism; and by the last assertion in the proposition, V is given by the fibre product $V_1 \times_{V_0} V_2$, or equivalently by the inverse limit of the finite inverse system $\{V_i\}$. As also noted above, with respect to the inclusions of V_1, V_2 into V_0 , we may also regard this fibre product as the intersection $V_1 \cap V_2$ in V_0 . The above result then has the following corollary:

Corollary 2.2. *Let $F_1, F_2 \leq F_0$ be fields and write $F = F_1 \cap F_2$. Let $\mathcal{V} = \{V_i\}$ be a patching problem for $\mathcal{F} := \{F_i\}$, and let $V = V_1 \cap V_2$ inside V_0 . Then the patching problem \mathcal{V} has a solution if and only if $\dim_F V = \dim \mathcal{V}$; and in this case, V is a solution.*

Proof. If there is a solution V' to the patching problem, then $V' = V_1 \cap V_2$ inside V_0 by Proposition 2.1 as rephrased above in terms of patching problems, using the identification $V_1 \times_{V_0} V_2 = V_1 \cap V_2$. Here $\dim_F V = \dim_{F_i} V_i = \dim \mathcal{V}$ since $V \otimes_F F_i$ is F_i -isomorphic to V_i .

Conversely, if $\dim_F V = \dim \mathcal{V}$, then $\dim_{F_i}(V \otimes_F F_i) = \dim_F V = \dim_{F_i} V_i$; so the inclusion $V \otimes_F F_i \hookrightarrow V_i$ induced by the natural map $V \hookrightarrow V_i$ is an isomorphism of V_i -vector spaces for $i = 0, 1, 2$. Since V is a fibre product, the three maps $V \hookrightarrow V_i$ are compatible; and so V (together with these inclusions) is a solution to the patching problem. \square

Concerning the last assertion of Proposition 2.1, we have the following more general result:

Proposition 2.3. *Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a finite inverse system of fields whose inverse limit is a field F , and let $\mathcal{V} := \{V_i\}_{i \in I}$ be a patching problem for \mathcal{F} , with a solution V . Then V and the associated system of isomorphisms $V \otimes_F F_j \xrightarrow{\simeq} V_j$ (for $j \in I$) can be identified with the inverse limit $\varprojlim V_i$ (as F -vector spaces) along with the maps $(\varprojlim V_i) \otimes_F F_j \rightarrow V_j$.*

Proof. This is immediate from the F -vector space identity $V \otimes_F (\varprojlim F_i) = \varprojlim (V \otimes_F F_i)$. \square

A special case is that the index set I of the inverse system is of the form $\{0, 1, \dots, r\}$ with the partial ordering of I given by $i \succ 0$ for $i = 1, \dots, r$ and with no other order relations (as in Proposition 2.1, where $r = 2$). Then the above inverse limits can be interpreted as fibre products:

$$F = F_1 \times_{F_0} F_2 \times_{F_0} \cdots \times_{F_0} F_r, \quad V = V_1 \times_{V_0} V_2 \times_{V_0} \cdots \times_{V_0} V_r.$$

If we identify each field and each vector space with its image under the respective inclusion, we can also regard F as the intersection of F_1, \dots, F_r inside F_0 and similarly for V and the V_i , generalizing the context of Proposition 2.1 above. (This situation arises in Theorem 4.14 below.)

Another special case of the above set-up arises in the context of Theorem 6.4 below. Consider an index set I of the form $I_0 \cup I_1 \cup I_2$, where the partial ordering has the property that for every $i_0 \in I_0$ there are unique elements $i_1 \in I_1$ and $i_2 \in I_2$ such that $i_1 \succ i_0$ and $i_2 \succ i_0$; and where there are no other relations in I . For an inverse system of fields $\mathcal{F} = \{F_i\}$ indexed by I , giving a patching problem (up to isomorphism) for \mathcal{F} is equivalent to giving a finite dimensional F_i -vector space V_i for each $i \in I_1 \cup I_2$, and giving an F_{i_0} -vector space isomorphism $\mu_{i_1, i_2, i_0} : V_{i_1} \otimes_{F_{i_1}} F_{i_0} \xrightarrow{\simeq} V_{i_2} \otimes_{F_{i_2}} F_{i_0}$ for every choice of i_1, i_2, i_0 such that each $i_j \in I_j$ and $i_1, i_2 \succ i_0$. Namely, given a patching problem for \mathcal{F} , the collection of maps ν_{i_1, i_0} and ν_{i_2, i_0} (for $i_1, i_2 \succ i_0$) determines a collection of maps μ_{i_1, i_2, i_0} as above, given by $(\nu_{i_2, i_0} \otimes_{F_{i_2}} F_{i_0})^{-1} \circ (\nu_{i_1, i_0} \otimes_{F_{i_1}} F_{i_0})$. Conversely, given F_i -vector spaces V_i for $i \in I_1 \cup I_2$ and a collection of maps μ_{i_1, i_2, i_0} as above, for $i_0 \in I_0$ we may define $V_{i_0} := V_{i_2} \otimes_{F_{i_2}} F_{i_0}$ where i_2 is the unique index in I_2 with $i_2 \succ i_0$. We can then let ν_{i_2, i_0} be the natural inclusion $V_{i_2} \hookrightarrow V_{i_2} \otimes_{F_{i_2}} F_{i_0} = V_{i_0}$; and let ν_{i_1, i_0} be the composition of the natural inclusion $V_{i_1} \hookrightarrow V_{i_1} \otimes_{F_{i_1}} F_{i_0}$ with μ_{i_1, i_2, i_0} . In this way we obtain inverse transformations between families $\{\mu_{i_1, i_2, i_0}\}$ and families $\{\nu_{i_1, i_0}\}$, thereby establishing the asserted equivalence.

3 Preliminary results

3.1 Matrix factorization

Below we show two matrix factorization results that will be used in proving our main results, Theorems 4.12 and 5.9. We begin with a lemma that reduces the problem to factoring matrices that are close to the identity. This reduction parallels the strategy employed in [31], Section 11.3, and [6], Section 4.

Lemma 3.1. *Let \hat{R}_0 be a complete discrete valuation ring with uniformizer t , and let $\hat{R}_1, \hat{R}_2 \leq \hat{R}_0$ be t -adically complete subrings that contain t . Write F_0, F_1 for the fraction fields of \hat{R}_0, \hat{R}_1 , and assume that $\hat{R}_1/t\hat{R}_1$ is a domain whose fraction field equals $\hat{R}_0/t\hat{R}_0$.*

(a) *Then $R_0 := \hat{R}_0 \cap F_1 \subset F_0$ is t -adically dense in \hat{R}_0 .*

(b) Suppose that for each $A \in \mathrm{GL}_n(\hat{R}_0)$ satisfying $A \equiv I \pmod{t \mathrm{Mat}_n(\hat{R}_0)}$, there exist $A_1 \in \mathrm{GL}_n(F_1)$, $A_2 \in \mathrm{GL}_n(\hat{R}_2)$ such that $A = A_1 A_2$. Then the same conclusion holds for all matrices $A \in \mathrm{Mat}_n(\hat{R}_0)$ with non-zero determinant.

Proof. (a) To prove this, we will show by induction that for every $f \in \hat{R}_0$ and every $m \geq 0$, there is an element $f_m \in R_0$ such that $f - f_m \in t^m \hat{R}_0$. This is trivial for $m = 0$, taking $f_m = 0$. Suppose the assertion holds for $m - 1$, and write $f - f_{m-1} = t^{m-1} e$, with $e \in \hat{R}_0$. The reduction $\bar{e} \in \hat{R}_0/t\hat{R}_0$ modulo $t\hat{R}_0$ lies in the fraction field of $\hat{R}_1/t\hat{R}_1$, and so may be written as \bar{g}/\bar{h} , with $\bar{g}, \bar{h} \in \hat{R}_1/t\hat{R}_1$ and $\bar{h} \neq 0$. Pick $g, h \in \hat{R}_1$ that reduce to \bar{g}, \bar{h} modulo $t\hat{R}_1$. Since $\hat{R}_0/t\hat{R}_0$ is a field, \bar{h} is a unit there, and so h is a unit in the t -adically complete ring \hat{R}_0 (which is the valuation ring of F_0). Thus $g/h \in \hat{R}_0$, and $e - g/h \in t\hat{R}_0$. Taking $f_m = f_{m-1} + t^{m-1}g/h \in \hat{R}_0 \cap F_1 = R_0$, we have $f - f_m \in t^m \hat{R}_0$, proving part (a).

(b) Let $A \in \mathrm{Mat}_n(\hat{R}_0)$ be a matrix with non-zero determinant. So $A^{-1} \in t^{-r} \mathrm{Mat}_n(\hat{R}_0) \subset \mathrm{Mat}_n(F_0)$ for some $r \geq 0$. Since R_0 is t -adically dense in \hat{R}_0 by part (a), there is a $C_0 \in \mathrm{Mat}_n(R_0)$ that is congruent to $t^r A^{-1} \in \mathrm{Mat}_n(\hat{R}_0)$ modulo $t^{r+1} \mathrm{Mat}_n(\hat{R}_0)$. Let $C = t^{-r} C_0 \in t^{-r} \mathrm{Mat}_n(R_0) \subset \mathrm{Mat}_n(F_1)$. Then $C - A^{-1} \in t \mathrm{Mat}_n(\hat{R}_0)$ and so $CA - I \in t \mathrm{Mat}_n(\hat{R}_0)$. Hence $CA \in \mathrm{GL}_n(\hat{R}_0)$, and in particular, C has non-zero determinant; i.e. $C \in \mathrm{GL}_n(F_1)$. By hypothesis, there exist $A'_1 \in \mathrm{GL}_n(F_1)$, $A_2 \in \mathrm{GL}_n(\hat{R}_2)$ such that $CA = A'_1 A_2$ in $\mathrm{GL}_n(F_0)$. Let $A_1 = C^{-1} A'_1 \in \mathrm{GL}_n(F_1)$. Then $A = A_1 A_2$, as asserted. \square

Lemma 3.1 will be used in conjunction with the following proposition, which provides a condition under which the factorization hypothesis of the above lemma is satisfied.

Proposition 3.2. *Let T be a complete discrete valuation ring with uniformizer t , let \hat{R}_0 be a t -adically complete T -algebra which is a domain, and let $\hat{R}_1, \hat{R}_2 \leq \hat{R}_0$ be t -adically complete subrings containing T , with fraction fields F_i ($i = 0, 1, 2$). Assume that $M_1 \subset F_0$ is a t -adically complete (e.g. finitely generated) \hat{R}_1 -submodule of $\hat{R}_0 \cap F_1$ such that for every $a \in \hat{R}_0$, there exist $a_1 \in M_1$ and $a_2 \in \hat{R}_2$ for which $a \equiv a_1 + a_2 \pmod{t\hat{R}_0}$. Then every $A \in \mathrm{GL}_n(\hat{R}_0)$ with $A \equiv I \pmod{t \mathrm{Mat}_n(\hat{R}_0)}$ can be written as $A = A_1 A_2$ with $A_1 \in \mathrm{Mat}_n(M_1)$ and $A_2 \in \mathrm{GL}_n(\hat{R}_2)$. Necessarily, $A_1 \in \mathrm{GL}_n(F_1)$.*

Proof. The proof proceeds by constructing A_1 and A_2 , respectively, as the limits of a sequence of matrices B_i with coefficients in M_1 , and a sequence of matrices C_i with coefficients in \hat{R}_2 , such that

$$\begin{aligned} B_0 &= C_0 = I, \\ A &\equiv B_i C_i \pmod{t^{i+1} \mathrm{Mat}_n(\hat{R}_0)}, \\ B_i &\equiv B_{i-1} \pmod{t^i \mathrm{Mat}_n(M_1)}, \\ C_i &\equiv C_{i-1} \pmod{t^i \mathrm{Mat}_n(\hat{R}_2)}. \end{aligned}$$

By t -adic completeness, these limits exist and $A_2 \in \mathrm{GL}_n(\hat{R}_2)$ since $A_2 \equiv I \pmod{t\hat{R}_2}$. Also, $A_1 \in \mathrm{GL}_n(F_1)$ because $M_1 \subset F_1$ and since A, A_2 have non-zero determinant.

We now construct this sequence inductively. So suppose for some $n \geq 1$ and for all $i \leq n - 1$ that B_i, C_i have already been constructed, satisfying the above conditions; and we wish to construct B_n, C_n .

By the inductive hypothesis,

$$A - B_{n-1}C_{n-1} = t^n \tilde{A}_n$$

for some \tilde{A}_n with coefficients in \hat{R}_0 . By the hypothesis of the proposition (applied to the entries of \tilde{A}_n), there exist matrices $B'_n \in \text{Mat}_n(M_1)$ and $C'_n \in \text{Mat}_n(\hat{R}_2)$ so that

$$\tilde{A}_n \equiv B'_n + C'_n \pmod{t \text{Mat}_n(\hat{R}_0)},$$

and thus

$$t^n \tilde{A}_n \equiv t^n B'_n + t^n C'_n \pmod{t^{n+1} \text{Mat}_n(\hat{R}_0)}.$$

So if we define

$$\begin{aligned} B_n &= B_{n-1} + t^n B'_n \\ C_n &= C_{n-1} + t^n C'_n, \end{aligned}$$

then

$$\begin{aligned} A &= B_{n-1}C_{n-1} + t^n \tilde{A}_n \\ &\equiv B_{n-1}C_{n-1} + t^n B'_n + t^n C'_n \pmod{t^{n+1} \text{Mat}_n(\hat{R}_0)} \\ &\equiv (B_{n-1} + t^n B'_n)(C_{n-1} + t^n C'_n) \pmod{t^{n+1} \text{Mat}_n(\hat{R}_0)} \\ &\equiv B_n C_n \pmod{t^{n+1} \text{Mat}_n(\hat{R}_0)}, \end{aligned}$$

where the second to last congruence uses that

$$\begin{aligned} B_{n-1} &\equiv B_0 \equiv I \pmod{t \text{Mat}_n(M_1)} & \text{and} \\ C_{n-1} &\equiv C_0 \equiv I \pmod{t \text{Mat}_n(\hat{R}_2)}. \end{aligned}$$

This finishes the proof. □

In Proposition 4.5 and Lemma 5.3 it will be shown that the hypothesis of Proposition 3.2 (i.e. the sum decomposition with respect to some module M_1) holds in the situations of our main results.

3.2 An intersection lemma

Let T be a complete domain with $(t) \subset T$ prime (e.g. a complete discrete valuation ring T with uniformizer t). Let $M \subseteq M_1, M_2 \subseteq M_0$ be T -modules with $M \cap tM_i = tM$ and $M_i \cap tM_0 = tM_i$. Then $M/tM = M/(M \cap tM_i) \subseteq M_i/tM_i$ for $i = 0, 1, 2$, and similarly $M_i/tM_i \subseteq M_0/tM_0$ for $i = 1, 2$. Hence we can form the intersection $M_1/tM_1 \cap M_2/tM_2$ in M_0/tM_0 ; and this intersection contains M/tM . Under certain additional hypotheses, the next lemma asserts that if this containment is actually an equality then $M_1 \cap M_2 = M$.

Lemma 3.3. *Let T be a complete domain with $(t) \subset T$ prime, and let $M \subseteq M_1, M_2 \subseteq M_0$ be T -modules with no t -torsion such that M is t -adically complete, with $M \cap tM_i = tM$ and $M_i \cap tM_0 = tM_i$, and with $\bigcap_{j=1}^{\infty} t^j M_0 = (0)$. Assume that $M_1/tM_1 \cap M_2/tM_2 = M/tM$. Then $M_1 \cap M_2 = M$ (where the intersection is taken inside M_0).*

Proof. After replacing M_0 by its submodule $M_1 + M_2 \subseteq M_0$, we may assume that those two modules are equal. So by the intersection hypothesis, we have an exact sequence

$$0 \rightarrow M/tM \rightarrow M_1/tM_1 \times M_2/tM_2 \rightarrow M_0/tM_0 \rightarrow 0,$$

via the diagonal inclusion of M/tM and the subtraction map to M_0/tM_0 . Let $N = M_1 \cap M_2$ inside M_0 . So $M \subseteq N$, and $tN = tM_1 \cap tM_2$ because M_0 has no t -torsion. Tensoring the exact sequence $0 \rightarrow N \rightarrow M_1 \times M_2 \rightarrow M_0 \rightarrow 0$ over T with $T/(t)$ yields the exact sequence $N/tN \rightarrow M_1/tM_1 \times M_2/tM_2 \rightarrow M_0/tM_0 \rightarrow 0$. But the first map in this sequence is injective because $tN = tM_1 \cap tM_2$; so

$$0 \rightarrow N/tN \rightarrow M_1/tM_1 \times M_2/tM_2 \rightarrow M_0/tM_0 \rightarrow 0$$

is exact. Hence the natural map $M/tM \rightarrow N/tN$ is an isomorphism; thus $M \cap tN = tM$ and $N = M + tN$. For all $j \geq 0$, $M \cap t^{j+1}N = M \cap tN \cap t^{j+1}N = tM \cap t^{j+1}N = t(M \cap t^jN)$, where the last equality uses that N has no t -torsion. So by induction, for all $j \geq 0$ we have that $M \cap t^jN = t^jM$, and also that $N = M + t^jN$.

So for $n \in M_1 \cap M_2 = N$, there is a sequence of elements $m_j \in M$ with $n - m_j \in t^jN$. If $h > j$ then $m_h - m_j \in M \cap t^jN = t^jM$. Since M is t -adically complete, there exists an element $m \in M$ and a sequence $i_j \rightarrow \infty$ such that $m - m_j \in t^{i_j}M$ for all j . We may assume $i_j \leq j$ for all j . Thus $n - m = (n - m_j) - (m - m_j) \in t^{i_j}N \subseteq t^{i_j}M_0$ for all j . But $\bigcap_{j=1}^{\infty} t^{i_j}M_0 = \bigcap_{j=1}^{\infty} t^jM_0 = (0)$. So $n - m = 0$ and $n = m \in M$. \square

4 The global case

We now turn to proving our patching result in a global context, in which we consider a smooth projective curve \hat{X} over a complete discrete valuation ring T , and use patches that are obtained from subsets U_1, U_2 of the closed fibre X of \hat{X} . These subsets are permitted to be Zariski open subsets of X , but can also be more general. The strategy is to show that the factorization and intersection conditions of Section 2 hold, employing the results of Section 3.

4.1 Factorization

In order to apply the results from the last section to patching, we will need to show that the hypothesis of Proposition 3.2 is satisfied, i.e., that there is a certain additive decomposition.

As before, T is a complete discrete valuation ring with uniformizer t . Let \hat{X} be a projective T -curve with closed fibre X , and let $P \in X$ be a closed point at which \hat{X} is smooth. A **lift** of P to \hat{X} is an effective prime divisor \hat{P} on \hat{X} whose restriction to X is the divisor P .

Such a lift always exists. Specifically, given P , let $\bar{\pi}$ be a uniformizer of the local ring $\mathcal{O}_{X,P}$ and let $\pi \in \mathcal{O}_{\hat{X},P}$ be a lift of $\bar{\pi}$. Then the maximal ideal of $\mathcal{O}_{\hat{X},P}$ is generated by π and t , and we may take \hat{P} to be the connected component of the zero locus of π that contains P .

More generally, if $D = \sum_{i=1}^r a_i P_i$ is an effective divisor on X , and if \hat{P}_i is a lift of P_i to \hat{X} as above, we call $\hat{D} := \sum_{i=1}^r a_i \hat{P}_i$ a **lift** of D to \hat{X} .

The following two propositions are preliminary technical results, which can be avoided in the special case that $T = k[[t]]$ for some field k and $\hat{X} = X \times_k k[[t]]$. (Namely there, if we choose the lift $\hat{P} = P \times_k k[[t]]$, then the next two propositions hold easily by extending constants from k to $k[[t]]$. See also [14] for a discussion of this special case.)

As usual, for a Cartier divisor D on a scheme Z , we let $L(Z, D) = \Gamma(Z, \mathcal{O}_Z(D))$, the set of rational functions on Z whose pole divisor is at most D .

Proposition 4.1. *Let T be a complete discrete valuation ring with uniformizer t , and let \hat{X} be a smooth connected projective T -curve with closed fibre X of genus g . Let D be an effective divisor on X and let \hat{D} be a lift of D to \hat{X} . Then*

- (a) $L(\hat{X}, \hat{D})$ is a finitely generated T -module, and
- (b) if D has degree $> 2g - 2$, the sequence

$$0 \rightarrow tL(\hat{X}, \hat{D}) \rightarrow L(\hat{X}, \hat{D}) \rightarrow L(X, D) \rightarrow 0$$

is exact.

Proof. (a) Since \hat{X} is projective over T , the T -module $L(\hat{X}, \hat{D}) = \Gamma(\hat{X}, \mathcal{O}(\hat{D}))$ is finitely generated ([19], II, Theorem 5.19 and Remark 5.19.2); so the first part holds.

(b) Suppose that D is an effective divisor on X of degree $d > 2g - 2$, with a lift \hat{D} to \hat{X} . The general fibre X° of \hat{X} has genus equal to g because the arithmetic genus is constant for a flat family of curves, by [19], III, Corollary 9.10. Also, \hat{D} is flat over the discrete valuation ring T , since it is torsion-free because its support does not contain the closed fibre X . So by the same result in [19] on constancy of invariants in flat families, the degree d of the closed fibre D of \hat{D} is equal to the degree of the general fibre D° , viewed as a divisor on X° .

Applying the Riemann-Roch Theorem ([30], Chapter II, Theorem 3) to the curves X° and X , both $L(X^\circ, D^\circ)$ and $L(X, D)$ are vector spaces of dimension $r := d + 1 - g$ over the fraction field K of T and the residue field k of T , respectively. Since $L(\hat{X}, \hat{D})$ is a submodule of the function field F of \hat{X} , it is torsion-free. But T is a principal ideal domain and $L(\hat{X}, \hat{D}) \otimes_T K = L(X^\circ, D^\circ)$ is an r -dimensional K -vector space; so the finitely generated torsion-free T -module $L(\hat{X}, \hat{D})$ is free of rank r . Thus the injection $L(\hat{X}, \hat{D})/tL(\hat{X}, \hat{D}) \rightarrow L(X, D)$ induced by the map $L(\hat{X}, \hat{D}) \rightarrow L(X, D)$ is an isomorphism of k -vector spaces, which implies the result. \square

Remark 4.2. Alternatively, one could deduce this from Zariski's Theorem on Formal Functions ([19], III, Theorem 11.1 and Remark 11.1.2). But the proof given here is more elementary, and the above assertion will suffice for our purposes.

Before we proceed, we introduce some notation that will be frequently used in the sequel.

Notation 4.3. Let T be a complete discrete valuation ring with uniformizer t , and let \hat{X} be a smooth connected projective T -curve with closed fibre X and function field F . Let R_\emptyset denote the local ring of \hat{X} at the generic point of X . Given a subset U of X , we introduce the following objects:

- We set $R_U := \{f \in R_\emptyset \mid f \text{ is regular on } U\}$, and we let \hat{R}_U denote the t -adic completion of R_U .
- If $U \neq X$, then F_U denotes the fraction field of \hat{R}_U , and we set $\hat{U} := \text{Spec } \hat{R}_U$. If $U = X$, then $F_U := F$.

In particular, \hat{R}_\emptyset is the completion of the local ring of \hat{X} at the generic point of the closed fibre X ; this is a complete discrete valuation ring with uniformizer t , having as residue field the function field of X . Also, $F \leq F_U$ for all U , and $F_U \leq F_V$ if $V \subseteq U$. (As we will see in Corollary 4.8 below, for *any* $U \subseteq X$, the field F_U is the compositum of its subrings F and \hat{R}_U .)

The next result is an analog of Proposition 4.1 for subsets U of the closed fibre X . If D is an effective divisor on X that is supported on U , then we may regard a lift \hat{D} of D to \hat{X} as a divisor on the scheme \hat{U} ; and so as above we may consider $L(\hat{U}, \hat{D})$, the rational functions on \hat{U} with pole divisor at most \hat{D} . We similarly write $L(U, D) := \{f \in k(X) \mid ((f) + D)|_U \geq 0\}$.

Proposition 4.4. *Let T be a complete discrete valuation ring with uniformizer t , and let \hat{X} be a smooth connected projective T -curve with closed fibre X of genus g . Let U be a proper subset of X , let D be an effective divisor on U , and let \hat{D} be a lift of D to \hat{X} . Then*

- (a) $L(\hat{U}, \hat{D})$ is a finitely generated \hat{R}_U -module, and
(b) if D has degree greater than $2g - 2$, the sequence

$$0 \rightarrow tL(\hat{U}, \hat{D}) \rightarrow L(\hat{U}, \hat{D}) \rightarrow L(U, D) \rightarrow 0$$

is exact.

Proof. (a) We proceed by induction on $N = \deg(D)$. The result holds for $N = 0$ since then $D = 0$ and $L(\hat{U}, \hat{D}) = \hat{R}_U$. So take D of degree $N > 0$ and assume that the result holds for smaller degrees. Write $D = \sum_{i=1}^r a_i P_i$ for integers $a_i > 0$ and distinct closed points P_i on U ; and write $\hat{D} = \sum_{i=1}^r a_i \hat{P}_i$ for some lifts \hat{P}_i of the points P_i . Let $D' = D - P_1$ and $\hat{D}' = \hat{D} - \hat{P}_1$. Then D' is effective of degree less than N , and so $L(\hat{U}, \hat{D}')$ has a finite generating set S' over \hat{R}_U .

Pick a closed point $Q \in X \setminus U$ and a lift \hat{Q} of Q to \hat{X} . Let X° be the generic fibre of \hat{X} , and let P_1° and Q° be the generic points of \hat{P}_1 and \hat{Q} . The residue field K_1 of \hat{R}_U at P_1° is a finite extension of the fraction field K of T , and the local ring $\mathcal{O}_{\hat{U}, P_1^\circ}$ at P_1° is an equal characteristic discrete valuation ring with residue field K_1 (which can also be regarded as the constant subfield of $\mathcal{O}_{\hat{U}, P_1^\circ}$).

By the Strong Approximation Theorem ([3], Proposition 3.3.1), there is a rational function f on X° (or equivalently, on \hat{X}) that has a pole of order a_1 at P_1° and is regular on $X^\circ \setminus \{P_1^\circ, Q^\circ\}$. After multiplying f by an appropriate power of t , we may assume that f is a unit at the generic point of X . So $f \in L(\hat{U}, a_1 \hat{P}_1) \setminus L(\hat{U}, (a_1 - 1) \hat{P}_1)$.

For any $h \in L(\hat{U}, \hat{D})$, there is an element $\bar{c} \in K_1 \leq \mathcal{O}_{\hat{U}, P_1^\circ}$ such that $h - \bar{c}f \in \mathcal{O}_{\hat{U}, P_1^\circ}$ has a pole at P_1° of order at most $a_1 - 1$ (since the local ring at P_1° is an equal characteristic discrete valuation ring with constant field K_1 , and since f has a pole of order a_1 at P_1°). Now viewing K_1 as the residue field of $\mathcal{O}_{\hat{U}, P_1^\circ}$, lift $\bar{c} \in K_1$ to $c \in \hat{R}_U$. Then $h - cf \in L(\hat{U}, \hat{D}')$, and hence it is an \hat{R}_U -linear combination of the elements of S' . So $S := S' \cup \{f\}$ is a finite \hat{R}_U -generating set for $L(\hat{U}, \hat{D})$.

(b) The kernel of $L(\hat{U}, \hat{D}) \rightarrow L(U, D)$ is clearly $tL(\hat{U}, \hat{D})$. To show surjectivity of $L(\hat{U}, \hat{D}) \rightarrow L(U, D)$, let $\bar{b} \in L(U, D)$ and consider \bar{b} as a rational function on X . Let $\{P'_i \mid i = 1, \dots, m\}$ be the set of poles of \bar{b} that are not in U , of orders $n_i \in \mathbb{N}$. Then $\bar{b} \in L(X, \sum_{i=1}^m n_i P'_i + D)$. If D has degree greater than $2g - 2$, then so does $\sum_{i=1}^m n_i P'_i + D$. For each i choose a lift \hat{P}'_i of P'_i to \hat{X} . Proposition 4.1(b) then gives a preimage b of \bar{b} in $L(\hat{X}, \sum_{i=1}^m n_i \hat{P}'_i + \hat{D}) \subseteq L(\hat{U}, \hat{D})$, as desired. \square

Proposition 4.5. *Let T be a complete discrete valuation ring with uniformizer t . Let \hat{X} be a smooth connected projective T -curve with function field F and closed fibre X of genus g . Consider proper subsets $U_1, U_2 \subset X$, with $U_0 := U_1 \cap U_2$ empty. Let $\hat{R}_i := \hat{R}_{U_i}$, $F_i := F_{U_i}$. Then there exists a finite \hat{R}_1 -submodule M_1 of $\hat{R}_0 \cap F_1 \subseteq F_0$ with the following property: For every $a \in \hat{R}_0$ there exist $b \in M_1$ and $c \in \hat{R}_2$ so that $a \equiv b + c \pmod{t\hat{R}_0}$. More precisely, for any closed point $P \in U_1 \subset X$, for any lift \hat{P} of P to \hat{X} , and for any non-negative integer $N > 2g - 2$, the module M_1 can be chosen as $L(\hat{U}_1, N\hat{P})$.*

Proof. Let P and \hat{P} be as above, and let $M_1 = L(\hat{U}_1, N\hat{P})$ for some non-negative integer $N > 2g - 2$. Thus $M_1 \subseteq \hat{R}_0 \cap F_1$. By Proposition 4.4(a), the \hat{R}_1 -module M_1 is finitely generated.

Given $a \in \hat{R}_0$, its mod t reduction $\bar{a} \in \hat{R}_0/t\hat{R}_0$ may be viewed as a rational function on X . Consider the family of rational functions $\{f_Q\}_{Q \in X}$ on X given by $f_Q = \bar{a}$ for $Q \in U_1$ and $f_Q = 0$ for $Q \notin U_1$. Since $N > 2g - 2$, the Riemann-Roch Theorem ([30], Chapter II, Theorem 3) implies that $H^1(X, \mathcal{O}_X(NP)) = 0$. Hence by [30], Chapter II, Proposition 3, there is a rational function \bar{c} on X such that $f_Q - \bar{c}$ is regular at Q for all $Q \neq P$, and such that $f_P - \bar{c}$ has a pole at P of order at most N . In particular, \bar{c} is regular on U_2 . Thus $\bar{c} \in \hat{R}_2/t\hat{R}_2$, and \bar{c} is the reduction of some $c \in \hat{R}_2$. By the definition of \bar{c} , the rational function $\bar{b} := \bar{a} - \bar{c}$ on X is regular on U_1 except possibly at P , where it has a pole of order at most N ; i.e., $\bar{b} \in L(U_1, NP)$. Proposition 4.4(b) implies that \bar{b} is the image of an element $b \in L(\hat{U}_1, N\hat{P}) = M_1$, since $N > 2g - 2$; and then $a \equiv b + c \pmod{t\hat{R}_0}$. \square

The main result of this section is a factorization result for use in patching. As above, we use Notation 4.3.

Theorem 4.6. *Let T be a complete discrete valuation ring, and let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let U_1, U_2 be subsets of X and assume that $U_0 := U_1 \cap U_2$ is empty. Let $F_i := F_{U_i}$ and $\hat{R}_i = \hat{R}_{U_i}$ ($i = 0, 1, 2$). Then for every matrix $A \in \mathrm{GL}_n(F_0)$ there exist matrices $A_1 \in \mathrm{GL}_n(F_1)$ and $A_2 \in \mathrm{GL}_n(F_2)$ such that $A = A_1 A_2$.*

Proof. We may assume that U_1, U_2 are proper subsets of X ; otherwise the assertion is trivial. As observed at Notation 4.3, $\hat{R}_0 = \hat{R}_\emptyset$ is a complete discrete valuation ring whose residue field $\hat{R}_0/t\hat{R}_0$ is the function field of X (which is also the fraction field of $\hat{R}_1/t\hat{R}_1$). Moreover the uniformizer t of T is also a uniformizer for \hat{R}_0 . By Proposition 4.5, there exists a finite \hat{R}_1 -module $M_1 \subset \hat{R}_0 \cap F_1$ satisfying the hypothesis of Proposition 3.2. So by Proposition 3.2, for every $A \in \mathrm{GL}_n(\hat{R}_0)$ that is congruent to the identity modulo t , there exist $A_1 \in \mathrm{GL}_n(F_1)$ and $A_2 \in \mathrm{GL}_n(\hat{R}_2)$ such that $A = A_1 A_2$. By Lemma 3.1, the same conclusion then holds for *any* matrix $A \in \mathrm{Mat}_n(\hat{R}_0)$ having non-zero determinant. Finally, for any $A \in \mathrm{GL}_n(F_0)$, there is an $r \geq 0$ such that $t^r A \in \mathrm{Mat}_n(\hat{R}_0)$ with non-zero determinant. Since $t^r I \in \mathrm{GL}_n(F_1)$, the conclusion again follows. \square

The above proof actually shows a stronger result: Namely, every matrix $A \in \mathrm{GL}_n(F_0)$ may be factored as $A = A_1 A_2$, for some matrices $A_1 \in \mathrm{GL}_n(F_1)$ and $A_2 \in \mathrm{GL}_n(\hat{R}_2)$.

A generalization of Theorem 4.6 in which $U_1 \cap U_2$ can be non-empty appears in Theorem 4.10 below.

4.2 Intersection

We continue to use Notation 4.3.

Proposition 4.7 (Weierstrass Preparation). *Let T be a complete discrete valuation ring and let \hat{X} be a smooth connected projective T -curve with function field F and closed fibre X . Suppose that $U \subseteq X$. Then every element $f \in \hat{R}_U$ may be written as $f = bu$ with $b \in F$ and $u \in \hat{R}_U^\times$.*

Proof. If $U = X$ then $\hat{R}_U = T \subset F$, and the result is immediate. If $U = \emptyset$, then \hat{R}_U is a discrete valuation ring, and the uniformizer t of T is also a uniformizer of \hat{R}_U . In this case the result also follows easily, by taking b to be a power of t . So from now on we assume that $U \neq X, \emptyset$.

Let $U_1 := X \setminus U$. Thus $U_1 \cap U = \emptyset$ and $\hat{R}_\emptyset/t\hat{R}_\emptyset$ is the function field of X . Let $f \in \hat{R}_U$; we may assume $f \neq 0$ since otherwise the result is trivial. Since $t \in F$, after factoring out a power of t from f , we may assume that $f \notin t\hat{R}_U$. Let $\bar{f} \in \hat{R}_U/t\hat{R}_U \subset \hat{R}_\emptyset/t\hat{R}_\emptyset$ be the reduction of f modulo t . Here \bar{f} is a non-zero rational function on X whose pole divisor D is supported on U_1 . If $D = 0$, then \bar{f} is a nonzero constant function on X , and hence is a unit in $\hat{R}_U/t\hat{R}_U$, the ring of functions on U . In this case f is a unit in the t -adically complete ring \hat{R}_U , and we may take $u = f, b = 1$.

So now assume instead that D is a nonzero effective divisor, hence of degree at least 1. Let \hat{D} be a lift of D to \hat{X} , and pick a positive integer $N > 2g - 2$, where g is the genus of X . Thus $\bar{f} \in L(X, D) \subseteq L(X, ND)$. By Proposition 4.1(b), there exists some

$\hat{f} \in L(\hat{X}, N\hat{D}) \subset R_U \subseteq F \cap \hat{R}_U$ whose reduction mod $t\hat{R}_U$ is \bar{f} ; thus $f \equiv \hat{f} \pmod{t\hat{R}_U}$. Here $\hat{f} \notin t\hat{R}_U$ because $\bar{f} \neq 0$; so \hat{f} is invertible in the t -adically complete ring \hat{R}_\emptyset . Let $\tilde{f} = f/\hat{f} \in \hat{R}_\emptyset$. Hence $\tilde{f} \equiv 1 \pmod{t\hat{R}_\emptyset}$. Let P be a point of U_1 and let \hat{P} be a lift of P to \hat{X} . By Proposition 4.5 (with $U_2 = U$ and $U_0 = \emptyset$), Proposition 3.2 allows us to write $\tilde{f} = f_1 f_2$ with $f_1 \in L(\hat{U}_1, N\hat{P})$ and $f_2 \in \hat{R}_U^\times$. So $\hat{f} f_1 = f f_2^{-1} \in L \cap \hat{R}_U$, where $L := L(\hat{U}_1, N\hat{D} + N\hat{P})$, using that $\hat{f} \in L(\hat{X}, N\hat{D}) \subseteq L(\hat{U}_1, N\hat{D})$. By Proposition 4.4(b), $L/tL = L(U_1, ND + NP)$. Thus $L/tL \cap \hat{R}_U/t\hat{R}_U = L(X, ND + NP) = L(\hat{X}, N\hat{D} + N\hat{P})/tL(\hat{X}, N\hat{D} + N\hat{P})$ by Proposition 4.1(b). Applying Lemma 3.3 to the four T -modules $L(\hat{X}, N\hat{D} + N\hat{P}) \subseteq L, \hat{R}_U \subseteq \hat{R}_\emptyset$ yields that $\hat{f} f_1 \in L \cap \hat{R}_U = L(\hat{X}, N\hat{D} + N\hat{P}) \subset F$. Hence we may take $b = \hat{f} f_1 \in F$ and $u = f_2 \in \hat{R}_U^\times$. \square

Note that if $\hat{X} = \mathbb{P}_T^1$ and U consists of a single point, then this assertion is related to the classical form of the Weierstrass preparation theorem (e.g. see [4], p.8).

Corollary 4.8. *With notation as in Proposition 4.7, every element f in the fraction field of \hat{R}_U may be written as $f = bu$ with $b \in F$ and $u \in \hat{R}_U^\times$. Hence F_U is the compositum of \hat{R}_U and F .*

Here the first assertion is immediate from the above proposition, and the second assertion then follows from the definition of F_U in Notation 4.3, using $\hat{R}_X = T$.

We are now in a position to prove the intersection result needed for patching.

Theorem 4.9. *Let T be a complete discrete valuation ring, let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let U_1, U_2 be subsets of X , and write $U = U_1 \cup U_2$, $U_0 = U_1 \cap U_2$. Then $F_{U_1} \cap F_{U_2} = F_U$ inside F_{U_0} .*

Proof. Let $\hat{R}_i := \hat{R}_{U_i}$ and $F_i := F_{U_i}$, and let t be a uniformizer of T . We need only show that $F_1 \cap F_2 \subseteq F_U$, the reverse inclusion being trivial. Take an element $f \in F_1 \cap F_2$. By Corollary 4.8, $f = f_1 u_1 = f_2 u_2$ with $f_i \in F \leq F_U$ and $u_i \in \hat{R}_i^\times$. We wish to show that $f \in F$.

First, assume that $U \neq X$. Thus F_U is the fraction field of \hat{R}_U . Write $f_i = a_i/b_i$ with $a_i, b_i \in \hat{R}_U$. So $f = a_1 u_1 / b_1 = a_2 u_2 / b_2$. Hence $a_1 b_2 u_1 = a_2 b_1 u_2$, where the left side is in \hat{R}_1 and the right side is in \hat{R}_2 . Since $\hat{R}_1/t\hat{R}_1 \cap \hat{R}_2/t\hat{R}_2 = \hat{R}_U/t\hat{R}_U$, the hypotheses of Lemma 3.3 are seen to hold in this situation (with $M_i := \hat{R}_i$, $M := \hat{R}_U$); so $\hat{R}_1 \cap \hat{R}_2 = \hat{R}_U$ and $a_1 b_2 u_1 \in \hat{R}_U$. But then $f = a_1 u_1 / b_1 = a_1 b_2 u_1 / b_1 b_2$, where the numerator and denominator are both in \hat{R}_U ; i.e., $f \in F_U$.

Next suppose that $U = X$, so that $F_U = F$. We may assume that U_1, U_2 are proper subsets of X , since otherwise the assertion is trivial. So U_2 is not contained in U_1 . Pick a closed point $P \in U_2 \setminus U_1$ and a lift $\hat{P} \in \hat{U}_2 \subset \hat{X}$. By Propositions 4.1(b) and 4.4(b), the mod t reduction maps $L(\hat{X}, N\hat{P}) \rightarrow L(X, NP)$ and $L(\hat{U}_2, N\hat{P}) \rightarrow L(U_2, NP)$ are surjective for N sufficiently large. Then, by Lemma 3.3, $\hat{R}_1 \cap L(\hat{U}_2, N\hat{P}) = L(\hat{X}, N\hat{P})$ for $N \gg 0$, using in particular that the same statement is true modulo t . Let $R' = \bigcup_{N=0}^{\infty} L(\hat{X}, N\hat{P})$, the ring of regular functions on $\hat{X} \setminus \hat{P}$; and let $\hat{R}'_2 = \bigcup_{N=0}^{\infty} L(\hat{U}_2, N\hat{P})$, the ring of regular functions on $\hat{U}_2 \setminus \hat{P}$. The above intersection for $N \gg 0$ implies that $\hat{R}_1 \cap \hat{R}'_2 = R'$, and in particular

that $R' \subset \hat{R}_1$. Also, $\hat{R}_2 \subset \hat{R}'_2$, and F is the fraction field of R' . Proceeding as in the previous paragraph but with \hat{R}_U and \hat{R}_2 respectively replaced by R' and \hat{R}'_2 , we may write $f_i = a_i/b_i$ with $a_i, b_i \in R'$. Thus $a_1 b_2 u_1 = a_2 b_1 u_2 \in \hat{R}_1 \cap \hat{R}'_2 = R'$ and so $f = a_1 b_2 u_1 / b_1 b_2 \in F$. \square

Using Theorem 4.9, we next obtain a strengthening of the factorization result Theorem 4.6 that applies to more general pairs U_1, U_2 . This result, which may be regarded as a form of Cartan's Lemma ([2], Section 4, Théorème I), also generalizes Corollary 4.5 of [6] (which dealt just with the case that the U_i are Zariski open subsets of the line in order to make use of unique factorization of the corresponding rings).

Theorem 4.10. *Let T be a complete discrete valuation ring, let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let $U_1, U_2 \subseteq X$, let $U_0 = U_1 \cap U_2$, and let $F_i := F_{U_i}$ ($i = 0, 1, 2$) under Notation 4.3. Then for every matrix $A \in \mathrm{GL}_n(F_0)$ there exist matrices $A_i \in \mathrm{GL}_n(F_i)$ such that $A = A_1 A_2$.*

Proof. Let $U'_2 = U_2 \setminus U_0$, and write $F'_2 = F_{U'_2}$ and $F'_0 = F_\emptyset$. Any $A \in \mathrm{GL}_n(F_0)$ lies in $\mathrm{GL}_n(F'_0)$, and so by Theorem 4.6 we may write $A = A_1 A_2$ with $A_1 \in \mathrm{GL}_n(F_1) \leq \mathrm{GL}_n(F_0)$ and $A_2 \in \mathrm{GL}_n(F'_2)$. But also $A_2 = A_1^{-1} A \in \mathrm{GL}_n(F_0)$; and $F'_2 \cap F_0 = F_2$ by Theorem 4.9 since $U'_2 \cup U_0 = U_2$. So actually $A_2 \in \mathrm{GL}_n(F_2)$. \square

Remark 4.11. In Theorem 4.10, we cannot replace GL_n everywhere by Mat_n , as the following example shows. With notation as above, assume $U_0 \neq U_1, U_2$, and consider the matrix

$$A = \begin{pmatrix} 1 & a_1 \\ a_2 & a_1 a_2 \end{pmatrix} \in \mathrm{Mat}_n(F_0)$$

with $a_i \in F_i \setminus F_U$. If there is a factorization $A = A_1 A_2$ with $A_i \in \mathrm{Mat}_n(F_i)$, then either A_1 or A_2 has determinant 0, since $\det(A) = 0$. Without loss of generality, we may assume $\det(A_1) = 0$ (since otherwise we can interchange the roles of U_1, U_2 and consider the transpose of A). So there exist $r, s \in F_1$, not both zero, such that $r(A_1)_1 = s(A_1)_2$, where $(A_1)_i$ denotes the i th row of A_1 . Multiplying by A_2 on the right then gives the equality $r(A)_1 = s(A)_2$ for the rows of A ; in particular, $r = s a_2$. If $s \neq 0$, then $a_2 = r/s \in F_1$. By assumption, $a_2 \in F_2$, and thus $a_2 \in F_U$ by Theorem 4.9; a contradiction. Consequently, $s = 0$, and thus $r = s a_2 = 0$, contradicting the fact that r, s are not both zero. Hence no such factorization can exist.

4.3 Patching

We now turn to our global patching result for function fields. We consider an irreducible projective T -curve \hat{X} with closed fibre X . For any subset $U \subseteq X$ we write $V(U)$ for $\mathrm{Vect}(F_U)$, where F_U is as in Notation 4.3.

Theorem 4.12. *Let T be a complete discrete valuation ring and let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let U_1, U_2 be subsets of X . Then the base change functor*

$$V(U_1 \cup U_2) \rightarrow V(U_1) \times_{V(U_1 \cap U_2)} V(U_2)$$

is an equivalence of categories.

Proof. In view of Proposition 2.1, the result follows from the factorization result Theorem 4.10 and the intersection result Theorem 4.9. \square

By Proposition 2.1, the inverse of the above equivalence of categories (up to isomorphism) is given by taking the fibre product of vector spaces.

Remark 4.13. Theorem 4.12 can also be deduced just from Theorem 4.6 and Theorem 4.9, without using Theorem 4.10. Namely, the case that $U_0 = \emptyset$ follows with Theorem 4.6 replacing Theorem 4.10 in the above proof; and the general case then follows from that by setting $U'_2 = U_2 \setminus U_0$ and using the equivalences of categories

$$\begin{aligned} \mathbb{V}(U_1) \times_{\mathbb{V}(U_0)} \mathbb{V}(U_2) &= \mathbb{V}(U_1) \times_{\mathbb{V}(U_0)} (\mathbb{V}(U_0) \times_{\mathbb{V}(\emptyset)} \mathbb{V}(U'_2)) \\ &= \mathbb{V}(U_1) \times_{\mathbb{V}(\emptyset)} \mathbb{V}(U'_2) = \mathbb{V}(U_1 \cup U'_2) = \mathbb{V}(U_1 \cup U_2). \end{aligned}$$

Theorem 4.12 generalizes to a version that allows patching more than two vector spaces at the same time. This next result will become important in later applications, where sometimes U_0 is empty.

Theorem 4.14. *Let T be a complete discrete valuation ring and let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let U_1, \dots, U_r denote subsets of X , and assume that the pairwise intersections $U_i \cap U_j$ (for $i \neq j$) are all equal to a common subset $U_0 \subseteq X$. Let $U = \bigcup_{i=1}^r U_i$. Then the base change functor*

$$\mathbb{V}(U) \rightarrow \mathbb{V}(U_1) \times_{\mathbb{V}(U_0)} \cdots \times_{\mathbb{V}(U_0)} \mathbb{V}(U_r)$$

is an equivalence of categories.

Proof. We proceed by induction; the case $r = 1$ is trivial. Since

$$\left(\bigcup_{i=1}^{r-1} U_i \right) \cap U_r = \bigcup_{i=1}^{r-1} (U_i \cap U_r) = U_0,$$

Theorem 4.12 yields an equivalence of categories

$$\mathbb{V}\left(\bigcup_{i=1}^r U_i\right) = \mathbb{V}\left(\bigcup_{i=1}^{r-1} U_i\right) \times_{\mathbb{V}(U_0)} \mathbb{V}(U_r).$$

By the inductive hypothesis, the first factor on the right hand side is equivalent to the category $\mathbb{V}(U_1) \times_{\mathbb{V}(U_0)} \cdots \times_{\mathbb{V}(U_0)} \mathbb{V}(U_{r-1})$, proving the result. \square

Note that by Theorem 4.9 and induction, F_U is the intersection of the fields F_{U_1}, \dots, F_{U_r} inside F_{U_0} . So as with Theorem 4.12, the inverse to the equivalence of categories (up to isomorphism) in Theorem 4.14 is given by taking the fibre product of the given F_{U_i} -vector spaces ($i = 1, \dots, r$) over the given F_{U_0} -vector space; this is by Proposition 2.3.

5 The Complete Local Case

In this section, we will prove a different patching result, in which complete local rings are used at one or more points, and which is related to results in [11], Section 1. The proof here relies on the case dealt with in Section 4. Again, the ingredients we need are a factorization result and an intersection result. We use the following

Notation 5.1. Let \hat{R} be a 2-dimensional regular local domain with maximal ideal \mathfrak{m} and local parameters f, t , such that \hat{R} is t -adically complete. Let \hat{R}_1 be the \mathfrak{m} -adic completion of \hat{R} , let \hat{R}_2 be the t -adic completion of $\hat{R}[f^{-1}]$, and let \hat{R}_0 be the t -adic completion of $\hat{R}_1[f^{-1}]$. Also let $\bar{R} := \hat{R}/t\hat{R}$ and let $\bar{R}_i = \hat{R}_i/t\hat{R}_i$ for $i = 0, 1, 2$.

Lemma 5.2. *In the context of Notation 5.1, the following hold:*

- (a) \hat{R}_1 is the f -adic completion of \hat{R} , and $\hat{R} \leq \hat{R}_i \leq \hat{R}_0$ for $i = 1, 2$.
- (b) $t\hat{R}_i \cap \hat{R} = t\hat{R}$ for $i = 0, 1, 2$, and $t\hat{R}_0 \cap \hat{R}_i = t\hat{R}_i$ for $i = 1, 2$.
- (c) \hat{R}_2 and \hat{R}_0 are complete discrete valuation rings with uniformizer t ; and \bar{R} and \bar{R}_2 are discrete valuation rings with uniformizer \bar{f} , the mod t reduction of f .
- (d) \bar{R}_1 is the \bar{f} -adic completion of \bar{R} ; while \bar{R}_2 and \bar{R}_0 are respectively isomorphic to $\bar{R}[\bar{f}^{-1}]$ and $\bar{R}_1[\bar{f}^{-1}]$, the fraction fields of \bar{R} and \bar{R}_1 .
- (e) $\bar{R} \leq \bar{R}_i \leq \bar{R}_0$ for $i = 1, 2$, with $\bar{R}_1 \cap \bar{R}_2 = \bar{R}$ inside \bar{R}_0 .

Proof. (a) Since \hat{R} is t -adically complete, $\hat{R} = \varprojlim \hat{R}/t^j\hat{R}$. In \hat{R} , $(f^{2n}, t^{2n}) \subset \mathfrak{m}^{2n} \subset (f^n, t^n)$ for all $n \geq 0$. So the f -adic completion of \hat{R} is $\varprojlim \hat{R}/f^i\hat{R} = \varprojlim \varprojlim (\hat{R}/t^j\hat{R})/f^i(\hat{R}/t^j\hat{R}) = \varprojlim \varprojlim \hat{R}/(f^i, t^j) = \varprojlim \hat{R}/\mathfrak{m}^n = \hat{R}_1$. This proves the first part of (a).

According to [1], III, §3.2, Corollary to Proposition 5, given an ideal I in a commutative ring A , the intersection $\bigcap I^n$ is equal to (0) if no element of $1 + I$ is a zero-divisor. Hence the completion maps $\hat{R} \rightarrow \hat{R}_1$, $\hat{R}[f^{-1}] \rightarrow \hat{R}_2$, and $\hat{R}_1[f^{-1}] \rightarrow \hat{R}_0$ are injections. Thus so are $\hat{R} \rightarrow \hat{R}_2$ and $\hat{R}_1 \rightarrow \hat{R}_0$.

It remains to show that $\hat{R}_2 \rightarrow \hat{R}_0$ is injective. Since the image \bar{f} of f is in the maximal ideal of $\hat{R}/t^j\hat{R}$, no element of $1 + (\bar{f}) \subseteq \hat{R}/t^j\hat{R}$ is a zero-divisor. The above result in [1] then implies that $\bigcap_{n=1}^{\infty} (\bar{f}^n) = (0) \subset \hat{R}/t^j\hat{R}$ for $j \geq 1$. Hence $\bigcap_{n=1}^{\infty} (t^j, f^n) = (t^j) \subset \hat{R}$ for each j . Meanwhile, since \hat{R}_1 is the f -adic completion of \hat{R} (as shown above), $\hat{R} \cap f^n\hat{R}_1 = f^n\hat{R}$, and $t^j\hat{R}$ is f -adically dense in $t^j\hat{R}_1$. By this density, if $g \in t^j\hat{R}_1 \cap \hat{R}$, then $g = t^j r + f^n r_1$ for some $r \in \hat{R}$ and $r_1 \in \hat{R}_1$. But then $f^n r_1 \in \hat{R} \cap f^n\hat{R}_1 = f^n\hat{R}$; i.e. $r_1 \in \hat{R}$. This shows that g lies in the ideal $(t^j, f^n) \subset \hat{R}$. Since this holds for all n , and since $\bigcap_{n=1}^{\infty} (t^j, f^n) = t^j\hat{R}$, it follows that $g \in t^j\hat{R}$. That is, $t^j\hat{R}_1 \cap \hat{R} \subseteq t^j\hat{R}$. The reverse containment is trivial, and so $t^j\hat{R}_1 \cap \hat{R} = t^j\hat{R}$. Hence $t^j\hat{R}_1[f^{-1}] \cap \hat{R}[f^{-1}] = t^j\hat{R}[f^{-1}]$ for all $j \geq 1$; thus the map $\hat{R}_2 \rightarrow \hat{R}_0$ on completions is injective.

(b) It was shown in the proof of part (a) that $t\hat{R}_1 \cap \hat{R} = t\hat{R}$. So if $g \in t\hat{R}_1[f^{-1}] \cap \hat{R}$, then $f^n g \in t\hat{R}_1 \cap \hat{R} = t\hat{R}$ for some n , and hence $g \in t\hat{R}$ because f, t are a system of local

parameters. Passing to the t -adic completion preserves the t -adic metric, and so $t\hat{R}_0 \cap \hat{R} \subseteq t\hat{R}$. The reverse containment is trivial; hence $t\hat{R}_0 \cap \hat{R} = t\hat{R}$. Since $t\hat{R}_2 \subseteq t\hat{R}_0$, we then also have $t\hat{R}_2 \cap \hat{R} = t\hat{R}$.

Since the t -adic metric is preserved under t -adic completion, to prove that $t\hat{R}_0 \cap \hat{R}_1 = t\hat{R}_1$ it suffices to show that $t\hat{R}_1[f^{-1}] \cap \hat{R}_1 = t\hat{R}_1$. Say g is in the left hand side. Then for some $n \geq 1$, $f^n g \in t\hat{R}_1 \cap f^n \hat{R}_1 = t f^n \hat{R}_1$; i.e., $g \in t\hat{R}_1$. Thus $t\hat{R}_1[f^{-1}] \cap \hat{R}_1 \subseteq t\hat{R}_1$, and the reverse containment is trivial. Finally, the equality $t\hat{R}_0 \cap \hat{R}_2 = t\hat{R}_2$ follows from the assertion $t\hat{R}_1[f^{-1}] \cap \hat{R}[f^{-1}] = t\hat{R}[f^{-1}]$ shown in the proof of part (a).

(c) Since $f \in \mathfrak{m}$, the rings $\hat{R}[f^{-1}]$ and $\hat{R}_1[f^{-1}]$ are regular domains of dimension one; and their t -adic completions \hat{R}_2 and \hat{R}_0 are thus complete discrete valuation rings with uniformizer t .

Since f, t form a system of local parameters at the maximal ideals of the two-dimensional regular local domains \hat{R} and \hat{R}_1 , the reduction \bar{f} is a local parameter for the reductions \bar{R} and \bar{R}_1 , which are one-dimensional regular local domains. That is, \bar{R} and \bar{R}_1 are discrete valuation rings with uniformizer \bar{f} .

(d) The first assertion follows from the fact that \hat{R}_1 is the f -adic completion of \hat{R} (proven in part (a)). The second assertion follows from part (c) and the definitions of \hat{R}_2 and \hat{R}_0 .

(e) These assertions follow from the characterizations of $\bar{R}, \bar{R}_1, \bar{R}_2, \bar{R}_0$ in part (d). \square

5.1 Factorization

Lemma 5.3. *In the context of Notation 5.1, for every $a \in \hat{R}_0$ there exist $b \in \hat{R}_1$ and $c \in \hat{R}_2$ such that $a \equiv b + c \pmod{t\hat{R}_0}$.*

Proof. We may assume $a \neq 0$. Write $v_{\bar{f}}$ for the \bar{f} -adic valuation on \bar{R}_0 . Let \bar{a} be the image of a in $\bar{R}_0 = \hat{R}_0/t\hat{R}_0$. If $v_{\bar{f}}(\bar{a}) \geq 0$, then $\bar{a} \in \bar{R}_1$; and so there exists $b \in \hat{R}_1$ such that $a \equiv b \pmod{t\hat{R}_0}$. Taking $c = 0$ completes the proof in this case. Alternatively, if $v_{\bar{f}}(\bar{a}) = -r < 0$, then $f^r a$ has the property that its reduction modulo $t\hat{R}_0$ lies in $\bar{R}_1 \subset \bar{R}_0$, since the \bar{f} -adic valuation of this reduction is 0. Since \bar{R} is \bar{f} -adically dense in \bar{R}_1 by Lemma 5.2(d), there exists $\bar{d} \in \bar{R}$ such that $\bar{d} \equiv f^r \bar{a} \pmod{\bar{f}^r \bar{R}_1}$. Let $\bar{c} = \bar{f}^{-r} \bar{d} \in \bar{R}_2$. Then $\bar{f}^r(\bar{a} - \bar{c}) = \bar{f}^r \bar{a} - \bar{d} \in \bar{f}^r \bar{R}_1$, and so $\bar{a} - \bar{c}$ is equal to some element $\bar{b} \in \bar{R}_1$. Choosing $b \in \hat{R}_1$ lying over \bar{b} , and $c \in \hat{R}_2$ lying over \bar{c} , completes the proof. \square

Theorem 5.4. *In the context of Notation 5.1, let F_i be the fraction field of \hat{R}_i . Then for every $A \in \text{GL}_n(F_0)$ there exist $A_1 \in \text{GL}_n(F_1)$ and $A_2 \in \text{GL}_n(F_2)$ such that $A = A_1 A_2$.*

Proof. By Lemma 5.2(d), \bar{R}_0 is a field. By Lemma 5.3, the module $M_1 := \hat{R}_1 \subset \hat{R}_0$ satisfies the hypothesis of Proposition 3.2. So in the case of matrices $A \in \text{GL}_n(\hat{R}_0)$ that are congruent to the identity modulo $t\hat{R}_0$, the assertion follows from that proposition. The result for an arbitrary matrix $A \in \text{Mat}_n(\hat{R}_0)$ with non-zero determinant then follows from Lemma 3.1 (whose other hypotheses are satisfied, by parts (c) and (d) of Lemma 5.2). Finally, the general case of a matrix $A \in \text{GL}_n(F_0)$ then follows since $t^r A \in \text{Mat}(\hat{R}_0)$ with non-zero determinant for some $r \geq 0$, and since $t^r I \in \text{GL}_n(F_1)$. \square

5.2 Intersection

The proof of Weierstrass preparation in the local case does not entirely parallel the global case; instead, we require the following lemma.

Lemma 5.5. *In the context of Notation 5.1, every unit $a \in \hat{R}_0^\times$ may be written as $a = bc$ for some units $b \in \hat{R}_1^\times$ and $c \in \hat{R}_2^\times$.*

Proof. Since $a \in \hat{R}_0^\times$, $a \not\equiv 0 \pmod{t\hat{R}_0}$. So the reduction of a modulo $t\hat{R}_0$ is a non-zero element of $\bar{R}_0 = \bar{R}_1[\bar{f}^{-1}]$, and hence is of the form $\bar{f}^s \bar{u}$ for some integer s and some unit $\bar{u} \in \bar{R}_1$. Choose $u \in \hat{R}_1$ with reduction \bar{u} . Thus u is a unit in the t -adically complete ring \hat{R}_1 and f^s is a unit in \hat{R}_2 . Replacing a by $u^{-1}af^{-s}$, we may assume that $a \equiv 1 \pmod{t\hat{R}_0}$.

Since \hat{R}_1, \hat{R}_2 are t -adically complete, it now suffices to define sequences of units $b_m \in \hat{R}_1$, $c_m \in \hat{R}_2$ such that

$$b_{m+1} \equiv b_m \pmod{t^{m+1}\hat{R}_1}, \quad c_{m+1} \equiv c_m \pmod{t^{m+1}\hat{R}_2}, \quad a \equiv b_m c_m \pmod{t^{m+1}\hat{R}_0}$$

for all $m \geq 0$. This will be done inductively.

Take $b_0 = 1, c_0 = 1$. Suppose b_{m-1} and c_{m-1} have been defined, with $m \geq 1$. Thus $b_{m-1} \equiv 1 \pmod{t\hat{R}_1}$, $c_{m-1} \equiv 1 \pmod{t\hat{R}_2}$, and $d_m := ab_{m-1}^{-1} - c_{m-1}$ is divisible by t^m in \hat{R}_0 . So $d_m = \delta_m t^m$ for some $\delta_m \in \hat{R}_0$; denote its reduction modulo $t\hat{R}_0$ by $\bar{\delta}_m \in \bar{R}_0$. For some non-negative integer i we have $\bar{f}^i \bar{\delta}_m \in \bar{R}_1$. But \bar{R} is \bar{f} -adically dense in \bar{R}_1 ; so there exists $\bar{\varepsilon}_m \in \bar{R}$ such that $\bar{\varepsilon}_m \equiv \bar{f}^i \bar{\delta}_m \pmod{\bar{f}^i \bar{R}_1}$. So $\bar{b}'_m := \bar{\delta}_m - \bar{f}^{-i} \bar{\varepsilon}_m \in \bar{R}_1$ and $\bar{c}'_m := \bar{f}^{-i} \bar{\varepsilon}_m \in \bar{R}[\bar{f}^{-1}] = \bar{R}_2$. Choose elements $b'_m \in \hat{R}_1$ and $c'_m \in \hat{R}_2$ respectively lying over $\bar{b}'_m \in \bar{R}_1$ and $\bar{c}'_m \in \bar{R}_2$, and let $b_m = b_{m-1} + b'_m t^m \in \hat{R}_1$ and $c_m = c_{m-1} + c'_m t^m \in \hat{R}_2$. Thus $b_m \equiv b_{m-1} \pmod{t^m \hat{R}_1}$, $c_m \equiv c_{m-1} \pmod{t^m \hat{R}_2}$, and $ab_{m-1}^{-1} - c_{m-1} = d_m = \delta_m t^m \equiv b'_m t^m + c'_m t^m \pmod{t^{m+1}\hat{R}_0}$. So $a \equiv b_{m-1}c_{m-1} + b_{m-1}b'_m t^m + b_{m-1}c'_m t^m \equiv b_{m-1}c_m + b'_m t^m \equiv b_m c_m \pmod{t^{m+1}\hat{R}_0}$, using that $b_{m-1} \equiv 1 \pmod{t\hat{R}_1}$, $c_m \equiv 1 \pmod{t\hat{R}_2}$. \square

Proposition 5.6 (Local Weierstrass Preparation). *In the context of Notation 5.1, let F be the fraction field of \hat{R} . Then every element of \hat{R}_1 is the product of an element of F and a unit in \hat{R}_1 .*

Proof. We may assume $a \in \hat{R}_1$ is non-zero, and hence $a = t^s a'$ for some non-negative integer s and some $a' \in \hat{R}_1$ that is not divisible by t . Replacing a by a' , we may assume that $a \notin t\hat{R}_1$, and hence that a is a unit in the discrete valuation ring \hat{R}_0 . So by Lemma 5.5, $a = bc$ for some units $b \in \hat{R}_1^\times$ and $c \in \hat{R}_2^\times$, and then $c = ab^{-1} \in \hat{R}_1$. But $\hat{R}_1 \cap \hat{R}_2 = \hat{R}$ by Lemma 3.3, using in particular that $\bar{R}_1 \cap \bar{R}_2 = \bar{R}$. Hence $c \in \hat{R}_1 \cap \hat{R}_2 = \hat{R} \subset F$. \square

Theorem 5.7. *In the context of Notation 5.1, let F, F_1, F_2, F_0 be the fraction fields of $\hat{R}, \hat{R}_1, \hat{R}_2, \hat{R}_0$ respectively. Then $F_1 \cap F_2 = F$ in F_0 .*

Proof. Let $h \in F_1 \cap F_2$. Write $h = a/b$ with $a, b \in \hat{R}_1$. By Proposition 5.6, $b = uf$ for some unit $u \in \hat{R}_1$ and some non-zero $f \in F$. Thus $h = au^{-1}/f$; and replacing h by fh , we may assume $h = au^{-1} \in \hat{R}_1$. But \hat{R}_2 is a complete discrete valuation ring with uniformizer t (Lemma 5.2(c)); so after multiplying $h \in F_2$ by a non-negative power of t we may assume $h \in \hat{R}_2 \cap \hat{R}_1$. As noted in the above proof, Lemma 3.3 implies that $\hat{R}_1 \cap \hat{R}_2 = \hat{R} \subset F$ and hence $h \in F$. \square

5.3 Patching

We begin with a local patching result, using the above factorization and intersection results.

Theorem 5.8. *In the context of Notation 5.1, let F be the fraction field of \hat{R} and let F_i be the fraction field of \hat{R}_i for $i = 0, 1, 2$. Then the base change functor*

$$\mathrm{Vect}(F) \rightarrow \mathrm{Vect}(F_1) \times_{\mathrm{Vect}(F_0)} \mathrm{Vect}(F_2)$$

is an equivalence of categories.

Proof. This follows from Theorem 5.4 and Theorem 5.7, by Proposition 2.1. \square

Combining this with the global patching result Theorem 4.12, we obtain the following result on complete local/global patching:

Theorem 5.9. *Let T be a complete discrete valuation ring with uniformizer t , and let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let \hat{R}_Q be the completion of the local ring of \hat{X} at a closed point Q ; let \hat{R}_Q° be the t -adic completion of the localization of \hat{R}_Q at the height one prime $t\hat{R}_Q$; and let F_Q, F_Q° be the fraction fields of $\hat{R}_Q, \hat{R}_Q^\circ$. Let U be a subset of X that contains Q , let $U' = U \setminus \{Q\}$, and let F_U and $F_{U'}$ be as in Notation 4.3. Then the base change functor*

$$\mathrm{Vect}(F_U) \rightarrow \mathrm{Vect}(F_Q) \times_{\mathrm{Vect}(F_Q^\circ)} \mathrm{Vect}(F_{U'})$$

is an equivalence of categories.

Proof. As in Notation 4.3, we let \hat{R}_\emptyset be the t -adic completion of the local ring of \hat{X} at the generic point of X and let F_\emptyset be the fraction field of \hat{R}_\emptyset . Here \hat{R}_Q denotes the completion of the local ring of \hat{X} at Q with respect to its maximal ideal, whereas $\hat{R}_{\{Q\}}$ denotes the t -adic completion of this same local ring. Also, $F_Q, F_{\{Q\}}$ denote the fraction fields of $\hat{R}_Q, \hat{R}_{\{Q\}}$.

Since \hat{X} is a smooth projective T -curve, $\hat{R}_{\{Q\}}$ is a two-dimensional regular local domain, with maximal ideal \mathfrak{m}_Q , and with \mathfrak{m}_Q -adic completion \hat{R}_Q . Choose a lift $f \in \hat{R}_{\{Q\}}$ of a uniformizer \bar{f} of Q on the closed fibre X ; thus f, t form a system of local parameters for \hat{X} at Q . The localization $\hat{R}_{\{Q\}}[f^{-1}]_{(t)}$ contains the local ring R_\emptyset of \hat{X} at the generic point of X and is contained in the t -adic completion \hat{R}_\emptyset of R_\emptyset ; hence \hat{R}_\emptyset is the t -adic completion of $\hat{R}_{\{Q\}}[f^{-1}]_{(t)}$, or equivalently of $\hat{R}_{\{Q\}}[f^{-1}]$. Similarly, the localization $\hat{R}_Q[f^{-1}]_{(t)}$ contains the localization $(\hat{R}_Q)_{(t)}$, and is contained in the t -adic completion \hat{R}_Q° of that ring; hence \hat{R}_Q° is in fact the t -adic completion of $\hat{R}_Q[f^{-1}]_{(t)}$, or equivalently of $\hat{R}_Q[f^{-1}]$. Thus the four rings $\hat{R}_{\{Q\}}, \hat{R}_Q, \hat{R}_\emptyset, \hat{R}_Q^\circ$ satisfy the assumptions of Notation 5.1 for the rings $\hat{R}, \hat{R}_1, \hat{R}_2, \hat{R}_0$ there. So by Theorem 5.8, the base change functor

$$\mathrm{Vect}(F_{\{Q\}}) \rightarrow \mathrm{Vect}(F_Q) \times_{\mathrm{Vect}(F_Q^\circ)} \mathrm{Vect}(F_\emptyset)$$

is an equivalence of categories. By Theorem 4.12, the base change functor

$$\mathrm{Vect}(F_U) \rightarrow \mathrm{Vect}(F_{\{Q\}}) \times_{\mathrm{Vect}(F_\emptyset)} \mathrm{Vect}(F_{U'})$$

is also an equivalence. Hence the composition

$$\begin{aligned} \text{Vect}(F_U) &\rightarrow \text{Vect}(F_{\{Q\}}) \times_{\text{Vect}(F_\emptyset)} \text{Vect}(F_{U'}) \\ &\rightarrow \left(\text{Vect}(F_Q) \times_{\text{Vect}(F_Q^\circ)} \text{Vect}(F_\emptyset) \right) \times_{\text{Vect}(F_\emptyset)} \text{Vect}(F_{U'}) \\ &\rightarrow \text{Vect}(F_Q) \times_{\text{Vect}(F_Q^\circ)} \text{Vect}(F_{U'}), \end{aligned}$$

given by base change, is an equivalence of categories. \square

To illustrate the above result, let $T = k[[t]]$, let \hat{X} be the projective x -line over T , let Q be the point $x = t = 0$, let $U = \mathbb{P}_k^1$, and let $U' = U \setminus \{Q\}$. The ring $R_{U'}$ contains $k[[t]][x^{-1}]$, the ring of regular functions on an affine open subset \tilde{U}' of \mathbb{P}_T^1 ; and it is equal to the intersection of the localizations of $k[[t]][x^{-1}]$ at the maximal ideals corresponding to the points of U' (i.e. the closed points of the closed fibre of \tilde{U}'). So $R_{U'}$ is the localization $S^{-1}(k[[t]][x^{-1}])$, where S is the multiplicative set of elements that lie in none of these maximal ideals, or equivalently are units modulo t (and hence modulo t^n for all n). Thus the inclusion $k[[t]][x^{-1}] \hookrightarrow R_{U'}$ becomes an isomorphism modulo t^n for all n . Hence $k[[t]][x^{-1}]$ is t -adically dense in $R_{U'}$, and these two rings have the same t -adic completion; i.e. $k[x^{-1}][[t]] = \hat{R}_{U'}$. The local ring of \hat{X} at Q is $k[[t]][x]_{(x,t)}$, whose t -adic completion $\hat{R}_{\{Q\}}$ is $k[x]_{(x)}[[t]]$, this being the inverse limit of the mod t^n -reductions $k[t, x]_{(x)}/(t^n)$. The (x, t) -adic completion of this same local ring is $\hat{R}_Q = k[[x, t]]$; while $\hat{R}_\emptyset = k(x)[[t]]$ and $\hat{R}_Q^\circ = k((x))[[t]]$, these being the t -adic completions of $k[[t]][x]_{(x)}$ and $k[[t]][x^{-1}]$. The fields $F_{U'}$, $F_{\{Q\}}$, F_Q , F_\emptyset , and F_Q° are the respective fraction fields of the above complete rings. The function field F_U of \hat{X} is the fraction field $k((t))(x)$ of $k[[t]][x]$, the ring of functions on the dense open subset \mathbb{A}_T^1 .

The next result is a generalization of Theorem 5.9 that allows more patches.

Theorem 5.10. *Let T be a complete discrete valuation ring with uniformizer t , and let \hat{X} be a smooth connected projective T -curve with closed fibre X . Let Q_1, \dots, Q_r be distinct closed points on \hat{X} . For each $i = 1, \dots, r$, let \hat{R}_i be the complete local ring of \hat{X} at Q_i ; let \hat{R}_i° be the t -adic completion of the localization of \hat{R}_i at the height one prime $t\hat{R}_i$; and let F_i, F_i° be the fraction fields of $\hat{R}_i, \hat{R}_i^\circ$. Let U be a subset of X that contains $S = \{Q_1, \dots, Q_r\}$, let $U' = U \setminus S$, and let F_U and $F_{U'}$ be as in Notation 4.3. Then the base change functor*

$$\text{Vect}(F_U) \rightarrow \prod_{i=1}^r \text{Vect}(F_i) \times_{\prod_{i=1}^r \text{Vect}(F_i^\circ)} \text{Vect}(F_{U'})$$

is an equivalence of categories.

Proof. This follows by induction from Theorem 5.9, using the identification of

$$\prod_{i=1}^{j-1} \text{Vect}(F_i) \times_{\prod_{i=1}^{j-1} \text{Vect}(F_i^\circ)} \left(\text{Vect}(F_j) \times_{\text{Vect}(F_j^\circ)} \text{Vect}(F_{U \setminus \{Q_1, \dots, Q_j\}}) \right)$$

with

$$\prod_{i=1}^j \text{Vect}(F_i) \times_{\prod_{i=1}^j \text{Vect}(F_i^\circ)} \text{Vect}(F_{U \setminus \{Q_1, \dots, Q_j\}}).$$

□

In the terminology of Section 2, we may rephrase the above result in terms of patching problems. Namely, consider the partially ordered set $I = \{1, \dots, r, 1', \dots, r', U'\}$, where $i \succ i'$ for each i , and where $U' \succ i'$ for all i . Set $F_{i'} = F_i^\circ$ for each i , and consider the corresponding finite inverse system of fields $\mathcal{F} = \{F_i, F_{i'}, F_{U'}\}$ indexed by I . Then Theorem 5.10 asserts that the base change functor $\text{Vect}(F_U) \rightarrow \text{PP}(\mathcal{F})$ is an equivalence of categories.

Remark 5.11. The above theorem can be regarded as analogous to a special case of Theorem 4.14 — viz. where each of the sets U_i consists of a single point, except for one U_i which is disjoint from the others. Both results then make a patching assertion in the context of one arbitrary set and a finite collection of points not in that set. The main difference between the two results is that in the above special case of Theorem 4.14, the local patches correspond to the fraction fields of the t -adic completions of the local rings at the respective points Q_i ; whereas Theorem 5.10 uses the fraction fields of the \mathfrak{m}_{Q_i} -adic completions of the local rings at those points. In the special case of Theorem 4.14, the “overlap” fields associated to the pairwise intersections $U_i \cap U_j$ are each just the fraction field F_\emptyset of the t -adic completion of the local ring at the generic point of X ; whereas in Theorem 5.10, the “overlap” fields F_i° are different from each other (and are larger than F_\emptyset). So Theorem 5.10 can no longer be phrased as a fibre product over a single base as in Theorem 4.14.

Proposition 5.12. *In Theorems 5.8, 5.9, and 5.10, the inverse of the base change functor (up to isomorphism) is given by taking the inverse limit of the vector spaces on the patches. In Theorems 5.8 and 5.9, this inverse limit is given by taking the intersection of vector spaces.*

Proof. By Corollary 2.2, the assertion for Theorems 5.8 and 5.9 follows from verifying the intersection condition of Section 2 concerning fields (i.e. that $F_1 \cap F_2 = F$ in Theorem 5.8 and that $F_Q \cap F_{U'} = F_U$ in Theorem 5.9). That condition follows for Theorem 5.8 by Theorem 5.7; and for Theorem 5.9 by combining that in turn with Theorem 4.9.

To prove the result in the case of Theorem 5.10, we rephrase that theorem as asserting that $\text{Vect}(F_U) \rightarrow \text{PP}(\mathcal{F})$ is an equivalence of categories, where \mathcal{F} is as defined in the paragraph following the proof above. By Theorem 5.7, $F_Q \cap F_{U'} = F_U$ in the situation of Theorem 5.9. So by induction on r , it follows that F_U is the inverse limit of the fields in \mathcal{F} . Hence Proposition 2.3 asserts that if a patching problem $\mathcal{V} = \{V_i, V_{i'}, V_{U'}\}$ in \mathcal{F} is induced (up to isomorphism) by a finite dimensional F_U -vector space V , then V is isomorphic to the inverse limit of \mathcal{V} . Such a V exists since the functor in Theorem 5.10 is an equivalence of categories; hence the assertion follows. □

6 Allowing Singularities

In view of later applications, it is desirable to have a version of Theorem 5.10 that can be applied to a singular curve. Let T be a complete discrete valuation ring with uniformizer t .

In order to perform patching in the case of normal curves $\hat{X} \rightarrow T$ that are not smooth, we introduce some terminology that was used in a related context in [17], Section 1.

Let \hat{X} be a connected projective normal T -curve, with closed fibre X . Consider a non-empty finite set $S \subset X$ that contains every point where distinct irreducible components of X meet. Thus each connected component of $X \setminus S$ is contained in an irreducible component of X , and moreover is an affine open subset of that component, since each irreducible component of X contains at least one point of S (by connectivity of X and the fact that $X \neq \emptyset$). For any non-empty irreducible affine Zariski open subset $U \subseteq X \setminus S$, we consider as before the ring R_U of rational functions on \hat{X} that are regular at the points of U (and hence also at the generic point of the component of X containing U); and the fraction field F_U of the t -adic completion \hat{R}_U of R_U (which is a domain by the irreducibility of U). For each point $P \in S$, the complete local ring \hat{R}_P of \hat{X} at P is a domain, say with fraction field F_P . Each height one prime ideal \wp of \hat{R}_P that contains t determines a *branch* of X at P (i.e. an irreducible component of the pullback of X to $\text{Spec } \hat{R}_P$); and we let \hat{R}_\wp denote the complete local ring of \hat{R}_P at \wp , with fraction field F_\wp . Since $t \in \wp$, the contraction of $\wp \subset \hat{R}_P$ to the local ring $\mathcal{O}_{\hat{X},P}$ defines an irreducible component of $\text{Spec } \mathcal{O}_{X,P}$; hence an irreducible component of X containing P . This in turn is the closure of a unique connected component U of $X \setminus S$; and we say that \wp **lies on** U . (Note that several branches of X at P may lie on the same U , viz. if the closure of U is not unbranched at P .)

In this situation, we obtain a finite inverse system consisting of fields F_U, F_P, F_\wp . More precisely, let I_1 be the set of irreducible (or equivalently, connected) components U of $X \setminus S$; let $I_2 = S$; let I_0 be the set of branches \wp of X at points $P \in S$; and let $I = I_1 \cup I_2 \cup I_0$. Give I the structure of a partially ordered set by setting $U \succ \wp$ if \wp lies on U , and setting $P \succ \wp$ if \wp is a branch of X at P . This defines the asserted finite inverse system $\mathcal{F} = \mathcal{F}_{\hat{X},S} = \{F_i\}_{i \in I}$ consisting of the fields F_U, F_P, F_\wp under the natural inclusions $F_U \rightarrow F_\wp$ and $F_P \rightarrow F_\wp$.

Theorem 6.4 below, which states a patching result for singular curves, will be proven by relating a given singular curve to an auxiliary smooth curve. That theorem and the results preceding it will be in the following situation:

Hypothesis 6.1. We make the following assumptions:

- T is a complete discrete valuation ring with uniformizer t .
- $f : \hat{X} \rightarrow \hat{X}'$ is a finite morphism of connected projective normal T -curves with function fields F, F' , and closed fibres X, X' , such that \hat{X}' is smooth over T .
- $S' \neq \emptyset$ is a finite set of closed points of X' , say with complement $U' = X' - S'$, such that $S := f^{-1}(S') \subset X$ contains the points where distinct irreducible components of X meet.

Lemma 6.2. *Under Hypothesis 6.1 and the above notation, let $P' \in S'$ and let \wp' be the branch of X' at P' .*

(a) *The natural maps*

$$F \otimes_{F'} F_{U'} \rightarrow \prod F_U, \quad F \otimes_{F'} F_{P'} \rightarrow \prod F_P, \quad F \otimes_{F'} F_{\wp'} \rightarrow \prod F_\wp$$

are isomorphisms of F -algebras, where the products respectively range over the connected components U of $X \setminus S$, the points P of S lying over P' , and the branches \wp of X over \wp' .

- (b) The natural inclusions $F_{U'} \rightarrow F_{\wp'}$ and $F_{P'} \rightarrow F_{\wp'}$ are compatible with the natural inclusions $\prod F_U \rightarrow \prod F_{\wp}$ and $\prod F_P \rightarrow \prod F_{\wp}$, where U and P range as above and \wp ranges over the branches of X at points of S .

Proof. (a) Choose a Zariski affine open subset $\text{Spec } \tilde{R}'$ of \hat{X}' that meets X' in U' , and let $\text{Spec } \tilde{R}$ be its inverse image in \hat{X} . The fraction fields of \tilde{R}, \tilde{R}' are respectively the function fields F, F' of \hat{X}, \hat{X}' , since $\text{Spec } \tilde{R} \subset \hat{X}, \text{Spec } \tilde{R}' \subset \hat{X}'$ are Zariski open dense subsets.

At every closed point of U' , the rings \tilde{R}' and $R'_{U'}$ have the same localization and hence the same completion; so $R'_{U'}$ and its subring \tilde{R}' have the same t -adic completion $\hat{R}_{U'}$. By the hypothesis on S , the inverse image $f^{-1}(U') = X \setminus S \subset X$ is the disjoint union of its irreducible components U . Thus the corresponding ideals I_U of \tilde{R} are pairwise relatively prime; hence $t\tilde{R} = \prod I_U$ and more generally $t^n \tilde{R} = \prod I_U^n$ for $n \geq 1$. By the Chinese Remainder Theorem, $\tilde{R}/t^n \tilde{R}$ is isomorphic to the product $\prod_U \tilde{R}/I_U^n = \prod_U R_U/I_U^n R_U = \prod_U R_U/t^n R_U$; and taking inverse limits shows that the t -adic completion of \tilde{R} is $\prod \hat{R}_U$. By definition, the fraction fields of $\hat{R}_U, \hat{R}_{U'}$ are respectively $F_U, F_{U'}$, where as above U ranges over the connected components of $X \setminus S$.

The natural \tilde{R} -algebra homomorphism $\tilde{R} \otimes_{\tilde{R}'} \hat{R}_{U'} \rightarrow \prod \hat{R}_U$ is bijective, by [1], Theorem 3(ii) in §3.4 of Chapter III, and thus is an isomorphism. Hence so is $\tilde{R} \otimes_{\tilde{R}'} F_{U'} = \tilde{R} \otimes_{\tilde{R}'} \hat{R}_{U'} \otimes_{\hat{R}_{U'}} F_{U'} \rightarrow \prod \hat{R}_U \otimes_{\hat{R}_{U'}} F_{U'}$. Now \tilde{R} is finite over \tilde{R}' , and \hat{R}_U is finite over $\hat{R}_{U'}$; so $F = \tilde{R} \otimes_{\tilde{R}'} F'$ and $F_U = \hat{R}_U \otimes_{\hat{R}_{U'}} F_{U'}$. Hence $\prod F_U = \prod \hat{R}_U \otimes_{\hat{R}_{U'}} F_{U'}$ as F -algebras. Thus the natural map $F \otimes_{F'} F_{U'} = \tilde{R} \otimes_{\tilde{R}'} F' \otimes_{F'} F_{U'} = \tilde{R} \otimes_{\tilde{R}'} F_{U'} \rightarrow \prod F_U$ is an F -algebra isomorphism. This proves that the first map is an isomorphism. The proofs for the other two maps are similar.

- (b) This follows from the fact that each of these maps is given by base change. \square

In the situation of Hypothesis 6.1, consider the diagonal inclusion map $F \rightarrow \prod F_U \times \prod F_P$, where U ranges over the components of $X \setminus S$ and P ranges over S . For each such U consider the diagonal inclusion $\iota_U : F_U \rightarrow \prod F_{\wp}$, where \wp ranges over branches of X lying on U ; and for each P consider the diagonal inclusion $\iota_P : F_P \rightarrow \prod F_{\wp}$, where \wp ranges over branches of X at P . Consider the sum $\prod F_U \times \prod F_P \rightarrow \prod F_{\wp}$ of the maps ι_U and $-\iota_P$ on the respective components; here \wp ranges over all the branches at points of S .

Proposition 6.3. *Under Hypothesis 6.1 and the above notation, the sequence*

$$0 \rightarrow F \rightarrow \prod F_U \times \prod F_P \rightarrow \prod F_{\wp}$$

of F -vector spaces is exact. Equivalently, F is the inverse limit of the inverse system $\mathcal{F}_{\hat{X}, S}$ of F -algebras consisting of the fields F_U, F_P, F_{\wp} with the natural inclusions.

Proof. With respect to the inclusions $F_{U'} \rightarrow F_{\wp'}$ and $\iota_{P'} : F_{P'} \rightarrow F_{\wp'}$, we obtain an inverse system $\mathcal{F}' = \mathcal{F}'_{\hat{X}', S'}$ of fields $F_{U'}, F_{P'}, F_{\wp'}$, where P' ranges over S' and \wp' ranges over the corresponding branches. Applying Proposition 5.12 in the situation of Theorem 5.10, the function field F' of \hat{X}' is the inverse limit of the system \mathcal{F}' , viewing each of the fields $F_{U'}, F_{P'}, F_{\wp'}$ as a one-dimensional vector space over itself. Writing $F' \rightarrow F_{U'} \times \prod F_{P'}$ and $F_{U'} \rightarrow \prod F_{\wp'}$ for the diagonal inclusions, and writing $\prod F_{P'} \rightarrow \prod F_{\wp'}$ for the product of the maps $-\iota_{P'}$, the inverse limit assertion for \mathcal{F}' is equivalent to the exactness of the sequence of F' -vector spaces

$$0 \rightarrow F' \rightarrow F_{U'} \times \prod F_{P'} \rightarrow \prod F_{\wp'}.$$

The desired exactness now follows from tensoring this exact sequence over F' with F , and using Lemma 6.2. This exactness is then equivalent to the assertion that F is the inverse limit of the system $\mathcal{F}_{\hat{X}, S}$. \square

Note that in the above result, as in Proposition 5.12 in the situation of Theorem 5.10, we must phrase the assertion in terms of inverse limits rather than intersections, because the various fields in the inverse system are not all contained in some common overfield in the system.

Under Hypothesis 6.1 and the above notation, we define a **(field) patching problem** \mathcal{V} for (\hat{X}, S) to be a patching problem (in the sense of Section 2) for the inverse system $\mathcal{F} = \mathcal{F}_{\hat{X}, S}$. Because of the form of the index set of \mathcal{F} , and as noted in the last paragraph of Section 2, giving such a patching problem is equivalent to giving:

- (i) a finite dimensional F_U -vector space V_U for every irreducible component U of $X \setminus S$;
- (ii) a finite dimensional F_P -vector space V_P for every $P \in S$;
- (iii) an F_\wp -vector space isomorphism $\mu_{U, P, \wp} : V_U \otimes_{F_U} F_\wp \xrightarrow{\simeq} V_P \otimes_{F_P} F_\wp$ for each choice of U, P, \wp , where U is an irreducible component of $X \setminus S$; $P \in S$ is in the closure of U ; and \wp is a branch of X at P that lies on U .

We write $\text{PP}(\hat{X}, S)$ for the category $\text{PP}(\mathcal{F})$ of patching problems for (\hat{X}, S) (or equivalently, for \mathcal{F}). The function field F of \hat{X} is contained in each F_i for $i \in I$, and in fact F is the inverse limit of the F_i by Proposition 6.3.

By the above containments, every finite dimensional F -vector space V induces a patching problem $\beta_{\hat{X}, S}(V)$ for (\hat{X}, S) via base change, and $\beta_{\hat{X}, S}$ defines a functor from $\text{Vect}(F)$ to $\text{PP}(\hat{X}, S)$. There is also a functor $\iota_{\hat{X}, S}$ from $\text{PP}(\hat{X}, S)$ to $\text{Vect}(F)$ that assigns to each patching problem its inverse limit (which we view as the “intersection”, though as in the case of the fields there is in fact no common larger object within which to take an intersection).

The following result is similar to Theorem 1(a) of [17], §1, which considered a related notion of patching problems for rings and modules rather than for fields and vector spaces.

Theorem 6.4. *Under Hypothesis 6.1, the base change functor $\beta_{\hat{X}, S} : \text{Vect}(F) \rightarrow \text{PP}(\hat{X}, S)$ is an equivalence of categories, and $\iota_{\hat{X}, S} \beta_{\hat{X}, S}$ is isomorphic to the identity functor on $\text{Vect}(F)$.*

Proof. The equivalence of categories assertion is that $\beta_{\hat{X},S}$ is surjective on isomorphism classes; and that the natural maps $\text{Hom}_F(V_1, V_2) \rightarrow \text{Hom}_F(\beta_{\hat{X},S}(V_1), \beta_{\hat{X},S}(V_2))$ are bijective for V_1, V_2 in $\text{Vect}(F)$. We show that these hold, and that $\iota_{\hat{X},S}\beta_{\hat{X},S}$ is isomorphic to the identity functor, in steps (with the surjectivity requiring the bulk of the work).

Step 1: To show that $\beta_{\hat{X},S}$ is surjective on isomorphism classes.

As in the discussion before the theorem, a patching problem \mathcal{V} for (\hat{X}, S) corresponds to a collection of finite-dimensional F_U -vector spaces V_U and F_P -vector spaces V_P together with isomorphisms $\mu_{U,P,\varphi}$. Let $W_{U'} = \prod V_U$ (ranging over components U of $X \setminus S$); and for $P' \in S'$ let $W_{P'} = \prod V_P$, where P ranges over $S_{P'} := f^{-1}(P') \subseteq S$. Since the fields F_U and F_P are respectively finite over $F_{U'}$ and $F_{P'}$ (where $f(P) = P'$), $W_{U'}$ is a finite dimensional vector space over $F_{U'}$, and $W_{P'}$ is a finite dimensional vector space over $F_{P'}$.

For each $P' \in S'$ with associated branch φ' , we have identifications $\prod F_U \otimes_{F_{U'}} F_{\varphi'} = (F \otimes_{F'} F_{U'}) \otimes_{F_{U'}} F_{\varphi'} = F \otimes_{F'} F_{\varphi'} = \prod F_{\varphi}$ of $F_{\varphi'}$ -algebras, where the last product ranges over the branches φ lying over φ' . Using the algebra isomorphisms $F \otimes_{F'} F_{U'} \simeq \prod F_U$ and $F \otimes_{F'} F_{\varphi'} \simeq \prod F_{\varphi}$ from Lemma 6.2, we thus obtain identifications $W_{U'} \otimes_{F_{U'}} F_{\varphi'} = (\prod V_U) \otimes_{F_{U'}} F_{\varphi'} = (\prod V_U) \otimes_{\prod F_U} (\prod F_U \otimes_{F_{U'}} F_{\varphi'}) = (\prod V_U) \otimes_{\prod F_U} \prod F_{\varphi} = \prod_U (V_U \otimes_{F_U} \prod F_{\varphi}) = \prod_U \prod_{\varphi} V_U \otimes_{F_U} F_{\varphi}$ of $F_{\varphi'}$ -vector spaces, with the products ranging over components U over U' , and branches φ lying on U (and lying over φ'). Similarly, using the algebra isomorphisms $F \otimes_{F'} F_{P'} \simeq \prod F_P$ and $F \otimes_{F'} F_{\varphi'} \simeq \prod F_{\varphi}$ from Lemma 6.2, we obtain an identification $W_{P'} \otimes_{F_{P'}} F_{\varphi'} = \prod_P \prod_{\varphi} V_P \otimes_{F_P} F_{\varphi}$ of $F_{\varphi'}$ -vector spaces, where the products range over points $P \in S_{P'}$ and branches φ lying over φ' . Combining the above identifications with the product of the isomorphisms $\mu_{U,P,\varphi} : V_U \otimes_{F_U} F_{\varphi} \simeq V_P \otimes_{F_P} F_{\varphi}$ for $P \in S$ over $P' \in S'$, we obtain an isomorphism $\mu'_{U',P',\varphi'} : W_{U'} \otimes_{F_{U'}} F_{\varphi'} \simeq W_{P'} \otimes_{F_{P'}} F_{\varphi'}$ of $F_{\varphi'}$ -vector spaces.

As in the discussion before the theorem, the vector spaces $W_{U'}, W_{P'}$ together with the $F_{\varphi'}$ -isomorphisms $\mu'_{U',P',\varphi'}$ define a patching problem $\mathcal{W} =: f_*(\mathcal{V})$ for (\hat{X}', S') . By Theorem 5.10, there is a finite dimensional F' -vector space W which is a solution to the patching problem \mathcal{W} ; i.e., $W = \beta_{\hat{X}',S'}(W)$. In order to complete the proof of the surjectivity of $\beta_{\hat{X},S}$ on isomorphism classes of objects, it will suffice to give W the structure of an F -vector space and to show that with respect to this additional structure, $\beta_{\hat{X},S}(W)$ is isomorphic to the given patching problem \mathcal{V} for (\hat{X}, S) .

To do this, consider the ‘‘identity patching problem’’ $\beta_{\hat{X},S}(F)$ for (\hat{X}, S) , given by F_U , the F_P , and the identity maps on each F_{φ} . Let $f_*(F)$ denote F viewed as an F' -vector space; similarly let $f_*(F_U), f_*(F_P)$ denote F_U, F_P as vector spaces over $F_{U'}, F_{P'}$ respectively. The patching problem $\beta_{\hat{X}',S'}(f_*(F))$ for (\hat{X}', S') induced by $f_*(F)$ is thus given by $f_*(F_U)$, the $f_*(F_P)$, and the identity map on $F_{\varphi'}$. Let $\alpha_U : f_*(F_U) \rightarrow \text{End}_{F_{U'}}(W_U)$ and $\alpha_P : f_*(F_P) \rightarrow \text{End}_{F_{P'}}(W_P)$ (for $P \in S_{P'}$) be the maps corresponding to scalar multiplication by F_U and F_P on the factors V_U of $W_{U'}$ and the factors V_P of $W_{P'}$, respectively. These maps define a morphism in the category of patching problems $\bar{\alpha} : \beta_{\hat{X}',S'}(f_*(F)) \rightarrow \beta_{\hat{X}',S'}(\text{End}_{F'}(W))$ for (\hat{X}', S') . By the equivalence of categories assertion in Theorem 5.10 applied to the T -curve \hat{X}' and finite subset S' , the element $\bar{\alpha} \in \text{Hom}_{F'}(\beta_{\hat{X}',S'}(f_*(F)), \beta_{\hat{X}',S'}(\text{End}_{F'}(W)))$ is induced by a unique morphism $\alpha \in \text{Hom}_{F'}(f_*(F), \text{End}_{F'}(W))$ in the category of finite dimensional

F' -vector spaces. As a result, W is given the structure of a finite dimensional F -vector space, with α defining scalar multiplication. It is now straightforward to check that $\beta_{\hat{X},S}(W)$ is isomorphic to \mathcal{V} , showing the desired surjectivity on isomorphism classes.

Step 2: To show that $\iota_{\hat{X},S}\beta_{\hat{X},S}$ is isomorphic to the identity functor.

For any V in $\text{Vect}(F)$, the induced patching problem $\beta_{\hat{X},S}(V)$ corresponds to data $V_U, V_P, \mu_{U,P,\varphi}$. Tensoring the exact sequence in Proposition 6.3 over F with V gives an exact sequence

$$0 \rightarrow V \rightarrow \prod V_U \times \prod V_P \rightarrow \prod V_\varphi$$

of F -vector spaces. Here $V_\varphi := V_P \otimes_{F_P} F_\varphi$ for φ a branch of X at P ; $V_U \rightarrow V_\varphi$ is defined via $\mu_{U,P,\varphi}$; and $V_P \rightarrow V_\varphi$ is minus the natural inclusion. This shows that V is naturally isomorphic to $\iota_{\hat{X},S}(\beta_{\hat{X},S}(V))$; i.e., $\iota_{\hat{X},S}\beta_{\hat{X},S}$ is isomorphic to the identity functor on $\text{Vect}(F)$.

Step 3: To show that $\beta_{\hat{X},S}$ induces a bijection between maps between corresponding objects.

Consider V_1, V_2 in $\text{Vect}(F)$, with induced patching problems $\beta_{\hat{X},S}(V_1), \beta_{\hat{X},S}(V_2)$. Then $V_i \rightarrow V_{i,U} := V_i \otimes_F F_U$ and $V_i \rightarrow V_{i,P} := V_i \otimes_F F_P$ are inclusions for $i = 1, 2$, for all U and P ; and a set of compatible maps $V_{1,U} \rightarrow V_{2,U}$ and $V_{1,P} \rightarrow V_{2,P}$ determines a unique map $\iota_{\hat{X},S}(\beta_{\hat{X},S}(V_1)) \rightarrow \iota_{\hat{X},S}(\beta_{\hat{X},S}(V_2))$. So the natural map from $\text{Hom}_F(V_1, V_2)$ to $\text{Hom}_F(\beta_{\hat{X},S}(V_1), \beta_{\hat{X},S}(V_2))$ is bijective (and this concludes the proof that $\beta_{\hat{X},S}$ is an equivalence of categories). \square

Thus with \hat{X} and S as in the theorem, every patching problem for (\hat{X}, S) has a unique solution up to isomorphism, and this solution is given by the inverse limit of the fields defining the patching problem.

Remark 6.5. Given a non-empty finite subset $S' \subset X'$, the hypothesis on $S = f^{-1}(S')$ is satisfied if S contains every point at which X is not unbranched. In particular, it is satisfied if S contains all the points at which the reduced structure of X is not regular.

In order to apply Theorem 6.4 to T -curves \hat{X} that are not necessarily given in the context of Hypothesis 6.1, we prove the next result.

Proposition 6.6. *Let \hat{X} be a projective curve over a discrete valuation ring T , having closed fibre X . Let S be a finite set of closed points of \hat{X} . Then there is a finite T -morphism $f: \hat{X} \rightarrow \mathbb{P}_T^1$ such that $S \subseteq f^{-1}(\infty)$.*

Proof. Let t be a uniformizer of T . Since \hat{X} is projective, we may fix an embedding $\hat{X} \hookrightarrow \mathbb{P}_T^n$ for some $n \geq 1$. We proceed by induction on n . If $n = 1$ then $\hat{X} = \mathbb{P}_T^1$. Choosing a rational function f on \mathbb{P}_T^1 with poles at each point of $S \subset \hat{X} = \mathbb{P}_T^1$ yields the result in this case.

So assume $n > 1$ and that the result holds for $n - 1$. Pick a closed point $P \in \mathbb{P}_T^n \setminus \hat{X}$. Its residue field k' is a finite extension of the residue field k of T ; and P is defined over k' , say with homogeneous coordinates $(a_0 : \cdots : a_n)$ where each $a_i \in k'$. Possibly after permuting the coordinates, we may assume that $a_0 \neq 0$. For each $i \neq 0$, let $g_i(x) \in k[x]$ be the monic minimal polynomial of $a_i/a_0 \in k'$ over k . Since k is the residue field of T , we

may choose monic polynomials $G_i(x) \in T[x]$ whose reductions modulo $tT(x)$ are $g_i(x)$. Let d_i be the degree of G_i (or of g_i); let d be the least common multiple of d_1, \dots, d_n ; and let $H_i(x, y) = y^d G_i(x/y)^{d/d_i} \in T[x, y]$. Thus H_i is a homogeneous polynomial that has total degree d in x, y , and is congruent to x^d modulo $yT[x, y]$. Letting m_0, \dots, m_N be the distinct (unordered) monomials of degree d in x_0, \dots, x_n , we may write $H_i(x_i, x_0) = \sum_{j=0}^N c_{ij} m_j$ for some elements $c_{ij} \in T$. Note that the locus of $H_1(x_1, x_0) = \dots = H_n(x_n, x_0) = 0$ is a closed subset of \mathbb{P}_T^n that meets the closed fibre precisely at P , and so is disjoint from \hat{X} .

Define the morphism $p : \hat{X} \rightarrow \mathbb{P}_T^{n-1}$ by $p(x_0 : \dots : x_n) = (H_1(x_1, x_0) : \dots : H_n(x_n, x_0))$. Thus p is the composition of the d -uple embedding ι_d of $\hat{X} \subset \mathbb{P}_T^n$ into \mathbb{P}_T^N with the projection morphism $\pi : \mathbb{P}_T^N - L \rightarrow \mathbb{P}_T^{n-1}$ defined on the complement of the linear subspace $L \subset \mathbb{P}_T^N$ that is given by the linear forms $\sum_{j=0}^N c_{ij} m_j$ on \mathbb{P}_T^N for $i = 1, \dots, n$. Now the d -uple embedding ι_d is finite; and so is the restriction of the projection π to the closed subscheme $\iota_d(\hat{X}) \subset \mathbb{P}_T^N - L$, by Proposition 6 of Chapter II, Section 7 of [24]. So p is finite.

Let $\hat{X}_1 \subseteq \mathbb{P}_T^{n-1}$ be the image of p . By the inductive hypothesis there is a finite T -morphism $f_1 : \hat{X}_1 \rightarrow \mathbb{P}_T^1$ such that $\pi(S) \subseteq f_1^{-1}(\infty)$. So $f_1 \circ p : \hat{X} \rightarrow \mathbb{P}_T^1$ is a finite T -morphism such that $S \subseteq f^{-1}(\infty)$. \square

Using the above proposition, we may apply Theorem 6.4 to a given curve \hat{X} and a given finite set S after possibly enlarging S :

Corollary 6.7. *Let T be a complete discrete valuation ring; let \hat{X} be a connected projective normal T -curve with closed fibre X ; and let S be a finite set of closed points of X . Then there exists a finite subset $S_1 \subset X$ containing S such that Theorem 6.4 holds for \hat{X} , S_1 .*

Proof. After enlarging S , we may assume that S contains the (finitely many) closed points of X where distinct irreducible components of X meet. By Proposition 6.6, there is a finite morphism $f : \hat{X} \rightarrow \mathbb{P}_T^1$ such that $S \subseteq f^{-1}(\infty)$. So Hypothesis 6.1 is satisfied by the data T , $f : \hat{X} \rightarrow \mathbb{P}_T^1$, $S' = \{\infty\}$, $S_1 = f^{-1}(S')$. Thus Theorem 6.4 holds for \hat{X} , S_1 . \square

7 Applications

In this section, we give several short applications of the new version of patching.

7.1 Patching Algebras and Brauer Groups

Our patching results for vector spaces carry over to patching for algebras of various sorts, because patching was phrased as an equivalence of categories.

To be more precise, for a field F we will consider finite dimensional associative F -algebras, with or without a multiplicative identity. We will also consider additional structure that may be added to the algebra, e.g. commutativity, separability, and being Galois with (finite) group G . A finite commutative F -algebra is assumed to have an identity; and it is separable if and only if it is a product of finitely many separable field extensions of F . By a G -**Galois** F -algebra we will mean a commutative F -algebra E together with an F -algebra action of G

on E such that the ring of G -invariants of E is F , and such that the inertia group $I_{\mathfrak{m}} \leq G$ at each maximal ideal \mathfrak{m} of E is trivial. Such an extension is necessarily separable and the G -action is necessarily faithful. If E is a field, being a G -Galois F -algebra is equivalent to being a G -Galois field extension. We will also consider (finite dimensional) central simple algebras over F .

Theorem 7.1. *Under the hypotheses of the patching theorems of Sections 4, 5, and 6 (Theorems 4.12, 4.14, 5.9, 5.10, 6.4, patching holds with the category of finite dimensional vector spaces replaced by any of the following (all assumed finite dimensional over F):*

- (i) *associative F -algebras;*
- (ii) *associative F -algebras with identity;*
- (iii) *commutative F -algebras;*
- (iv) *separable commutative F -algebras;*
- (v) *G -Galois F -algebras;*
- (vi) *central simple F -algebras.*

Proof. We follow the strategy of [9], Prop. 2.8 (cf. also [13], 2.2.4).

The equivalence of categories in each of the patching results of Sections 4 and 5 is given by a base change functor β , which preserves tensor products. So β is an equivalence of tensor categories.

An associative F -algebra is an F -vector space A together with a vector space homomorphism $p : A \otimes_F A \rightarrow A$ that defines the product and satisfies an identity corresponding to the associative law. Since the base change patching functor β is an equivalence of tensor categories, the property of having such a homomorphism p is preserved; so (i) follows. Part (ii) is similar, since a multiplicative identity corresponds to an F -vector space homomorphism $i : F \rightarrow A$ satisfying the identity law.

Part (iii) follows from the fact that up to isomorphism, β has an inverse given by intersection (i.e. fibre product or inverse limit); see Propositions 2.1 and 2.3. So a commutative F -algebra induces commutative algebras on the patches and vice versa. Part (iv) holds because if F' is a field extension of F , then a finite F -algebra E is separable if and only if the F' -algebra $E \otimes_F F'$ is separable.

For part (v), the first condition (on G -invariants) follows using that the inverse to β is given by intersection, together with the fact that the intersection of the rings of G -invariants in fields $E_i := E \otimes_F F_i$ is the ring of G -invariants in the intersection of the E_i . The second condition, on inertia groups, holds because the residue fields of E are contained in those of E_i , with the G -actions on the latter being induced by those on the former.

For part (vi), we are reduced by (ii) to verifying that centrality and simplicity are preserved. If E is the center of an F -algebra A , then $E \otimes_F F'$ is the center of the F' -algebra $A' := A \otimes_F F'$. So centrality is preserved by β and its inverse. The same holds for simplicity (in the presence of centrality) by [25], Section 12.4, Lemma b. \square

On the other hand, Theorem 7.1 as phrased above does *not* apply to (finite dimensional central) division algebras over F . For example, in the context of global patching in Section 4, let $T = k[[t]]$ where $\text{char } k \neq 2$; $\hat{X} = \mathbb{P}_T^1$ (the projective x -line over T); $U_1 = \mathbb{A}_k^1 = \mathbb{P}_k^1 \setminus \{\infty\}$, $U_2 = \mathbb{P}_k^1 \setminus \{0\}$, and $U_0 = U_1 \cap U_2 = \mathbb{P}_k^1 \setminus \{0, \infty\}$. With notation as in Section 4, we consider the function field $F = k((t))(x)$ of \hat{X} , along with the fraction fields F_1, F_2, F_0 of the rings $k[x][[t]], k[x^{-1}][[t]], k[x, x^{-1}][[t]]$, respectively. Let D be the quaternion algebra over F generated by elements a, b satisfying $a^2 = b^2 = 1 - xt$, $ab = -ba$. Then $D \otimes_F F_1$ is split as an algebra over F_1 , i.e. is isomorphic to $\text{Mat}_2(F_1)$ (and not to a division algebra), because F_1 contains an element f such that $f^2 = 1 - xt$ (where f is given by the binomial power series expansion in t for $(1 - xt)^{1/2}$).

But the other direction of the above theorem does hold for division algebras: viz. if D_1, D_2, D_0 are division algebras over F_1, F_2, F_0 in the context of Theorem 4.12, then the resulting finite dimensional central simple F -algebra D (given by part (vi) of the above theorem) is in fact a division algebra. This is because D is contained in the division algebras D_i , hence it has no zero-divisors, and so is a division algebra (being finite dimensional over F).

Despite the failure of the above result for division algebras, below we state a patching result for Brauer groups. For any field F , let $\text{Br}(F)$ be the set of isomorphism classes of (finite dimensional central) division algebras over F . The elements of $\text{Br}(F)$ are in bijection with the set of **Brauer equivalence classes** $[A]$ of (finite dimensional) central simple F -algebras A . Namely, by Wedderburn's theorem, every central simple F -algebra A is isomorphic to a matrix ring $\text{Mat}_n(D)$ for some unique positive integer n and some F -division algebra D which is unique up to isomorphism; and two central simple algebras are called **Brauer equivalent** if the underlying division algebras are isomorphic. By identifying elements of $\text{Br}(F)$ with Brauer equivalence classes, $\text{Br}(F)$ becomes an abelian group under the multiplication law $[A][B] = [A \otimes_F B]$, called the **Brauer group** of F . (See also Chapter 4 of [20].)

If F' is an extension of a field F (not necessarily algebraic), and if A is a central simple F -algebra, then $A \otimes_F F'$ is a central simple F' -algebra ([25], 12.4, Proposition b(ii)). Moreover if A, B are Brauer equivalent over F , then $A \otimes_F F', B \otimes_F F'$ are Brauer equivalent over F' . So there is an induced homomorphism $\text{Br}(F) \rightarrow \text{Br}(F')$. In terms of this homomorphism, we can state the following patching result for Brauer groups, which says that giving a division algebra over a function field F is equivalent to giving compatible division algebras on the patches:

Theorem 7.2. *Under the hypotheses of Theorem 4.10, let $U = U_1 \cup U_2$ and form the fibre product of groups $\text{Br}(F_1) \times_{\text{Br}(F_0)} \text{Br}(F_2)$ with respect to the maps $\text{Br}(F_i) \rightarrow \text{Br}(F_0)$ induced by $F_i \hookrightarrow F_0$. Then the base change map $\beta : \text{Br}(F_U) \rightarrow \text{Br}(F_1) \times_{\text{Br}(F_0)} \text{Br}(F_2)$ is a group isomorphism.*

Proof. Base change defines a homomorphism β as above, and we wish to show that it is an isomorphism.

For surjectivity, consider an element in $\text{Br}(F_1) \times_{\text{Br}(F_0)} \text{Br}(F_2)$, represented by a triple (D_1, D_2, D_0) of division algebras over F_1, F_2, F_0 such that the natural maps $\text{Br}(F_i) \rightarrow \text{Br}(F_0)$ take the class of D_i to that of D_0 , for $i = 1, 2$. Since the dimension of a division algebra is

a square, there are positive integers n_0, n_1, n_2 such that the three integers $n_i^2 \dim_{F_i} D_i$ (for $i = 0, 1, 2$) are equal. Let $A_i = \text{Mat}_{n_i}(D_i)$ for $i = 0, 1, 2$. Then A_i is a central simple algebra in the class of D_i for $i = 0, 1, 2$; and $A_i \otimes_{F_i} F_0$ is F_0 -isomorphic to A_0 for $i = 1, 2$, compatibly with the inclusions $F_i \hookrightarrow F_0$ (because they lie in the same class and have the same dimension). So by part (vi) of Theorem 7.1, there is a (finite dimensional) central simple F_U -algebra A that induces A_0, A_1, A_2 compatibly with the above inclusions. The class of A is then an element of $\text{Br}(F_U)$ that maps under β to the given element of $\text{Br}(F_1) \times_{\text{Br}(F_0)} \text{Br}(F_2)$.

To show injectivity, consider an element in the kernel, represented by an F_U -division algebra D . Then $A_i := D \otimes_F F_i$ is split for $i = 0, 1, 2$; i.e. for each i there is an F_i -algebra isomorphism $\psi_i : \text{Mat}_n(F_i) \rightarrow A_i$, where $n^2 = \dim_F D$. For $i = 1, 2$ let $\psi_{i,0}$ be the induced isomorphism $\text{Mat}_n(F_0) \rightarrow A_0$ obtained by tensoring ψ_i over F_i with F_0 and identifying each $A_i \otimes_{F_i} F_0$ with A_0 . So $\psi_{2,0}^{-1} \circ \psi_{1,0}$ is an F_0 -algebra automorphism of $\text{Mat}_n(F_0)$, and hence is given by (right) conjugation by a matrix $C \in \text{GL}_n(F_0)$ (by [20], Corollary to Theorem 4.3.1). By Theorem 4.10, there are matrices $C_i \in \text{GL}_n(F_i)$ such that $C = C_1 C_2$. Let $\psi'_1 = \psi_1 \rho_{C_1^{-1}} : \text{Mat}_n(F_1) \xrightarrow{\simeq} A_1$ and $\psi'_2 = \psi_2 \rho_{C_2} : \text{Mat}_n(F_2) \xrightarrow{\simeq} A_2$, where ρ_B denotes right conjugation by a matrix B . Also let $\psi'_{i,0} : \text{Mat}_n(F_0) \xrightarrow{\simeq} A_0$ be the isomorphism induced from ψ'_i by base change to F_0 . Then $\psi'_{2,0}^{-1} \circ \psi'_{1,0} = \rho_{C_2^{-1}} \rho_C \rho_{C_1^{-1}}$ is the identity on $\text{Mat}_n(F_0)$; the common isomorphism $\psi'_{1,0} = \psi'_{2,0}$ will be denoted by ψ'_0 . Thus the three isomorphisms $\psi'_i : \text{Mat}_n(F_i) \xrightarrow{\simeq} A_i$ (for $i = 0, 1, 2$) are compatible with the natural isomorphisms $\text{Mat}_n(F_i) \otimes_{F_i} F_0 \xrightarrow{\simeq} \text{Mat}_n(F_0)$ and $A_i \otimes_{F_i} F_0 \xrightarrow{\simeq} A_0$ for $i = 1, 2$. Equivalently, letting $\text{CSA}(K)$ denote the category of finite dimensional central simple K -algebras for a field K , the triples (A_1, A_2, A_0) and $(\text{Mat}_n(F_1), \text{Mat}_n(F_2), \text{Mat}_n(F_0))$, along with the associated natural base change isomorphisms as above, represent isomorphic objects in the category $\text{CSA}(F_1) \times_{\text{CSA}(F_0)} \text{CSA}(F_2)$. Using the equivalence of categories in part (vi) of the above theorem, there is up to isomorphism a unique central simple F_U -algebra inducing these objects. But D and $\text{Mat}_n(F_U)$ are both such algebras. Hence they are isomorphic. So $n = 1$ and $D = F_U$, as desired. \square

These ideas are pursued further in [16], in the context of studying Galois groups of maximal subfields of division algebras.

7.2 Inverse Galois Theory

We can use our results on patching over fields to recover results in inverse Galois theory that were originally proven (by the first author and others) using patching over rings. The point is that if F is the fraction field of a ring R , then Galois field extensions of F are in bijection with irreducible normal Galois branched covers of $\text{Spec } R$, by considering generic fibres and normalizations. So one can pass back and forth between the two situations.

In particular, we illustrate this by proving the result below, on realizing Galois groups over the function field of the line over a complete discrete valuation ring T . This result was originally shown in [10] (Theorem 2.3 and Corollary 2.4) using formal patching, and afterwards reproven in [22] using rigid patching. We first fix some notation and terminology.

Let G be a finite group, let H be a subgroup of G , and let E be an H -Galois F -algebra for some field F . The **induced** G -Galois F -algebra $\text{Ind}_H^G E$ is defined as follows:

Fix a set $C = \{c_1, \dots, c_m\}$ of left coset representatives of H in G , with the identity coset being represented by the identity element. Thus for every $g \in G$ and every $i \in \{1, \dots, m\}$ there is a unique j such that $gc_j \in c_i H$. Let $\sigma^{(g)} \in S_m$ be the associated permutation given by $\sigma_i^{(g)} = j$. Thus for each i , the element $h_{i,g} := c_i^{-1} g c_{\sigma_i^{(g)}}$ lies in H .

As an F -algebra, let $\text{Ind}_H^G E$ be the direct product of m copies of E indexed by C . For $g \in G$ and $(e_1, \dots, e_m) \in \text{Ind}_H^G E$, set $g \cdot (e_1, \dots, e_m) \in \text{Ind}_H^G E$ equal to the element whose i th entry is $h_{i,g}(e_{\sigma_i^{(g)}})$. This defines a G -action on $\text{Ind}_H^G E$, whose fixed ring is F (embedded diagonally). For all $i, j \in \{1, \dots, m\}$, the elements of $c_i H c_j^{-1}$ define isomorphisms $E_j \rightarrow E_i$, where E_i denotes the i th factor of $\text{Ind}_H^G E$. In particular, $c_i H c_i^{-1}$ is the stabilizer of E_i for each i . One checks that up to isomorphism, this construction does not depend on the choice of left coset representatives.

Note that $\text{Ind}_1^G F$ is just the direct product of copies of F that are indexed by G and are permuted according to the left regular representation; i.e. $g \cdot (e_1, \dots, e_n) = (e'_1, \dots, e'_n)$ is given by $e'_i = e_j$ where $gc_j = c_i$. (Here $n = |G|$.) Also, $\text{Ind}_G^G E = E$ if E is a G -Galois F -algebra. If $H \leq J \leq G$ and E is an H -Galois F -algebra, we may identify $\text{Ind}_J^G \text{Ind}_H^J E$ with $\text{Ind}_H^G E$ as G -Galois F -algebras. If A is any G -Galois F -algebra, and E is a maximal subfield of A containing F , then E is a Galois field extension of F whose Galois group $H := \text{Gal}(E/F)$ is a subgroup of G , and A is isomorphic to $\text{Ind}_H^G E$ as a G -Galois F -algebra.

As in the proof in [10] of the result below, we will patch together “building blocks” which are Galois and cyclic and which induce trivial extensions over the closed fibre $t = 0$ (though here we will consider extensions of fields rather than rings). For example, if F contains a primitive n th root of unity, then an n -cyclic building block may be given by $y^n = f(f-t)^{n-1}$, for some f . If there is no primitive n th root of unity in F but n is prime to the characteristic, then one can descend some n -cyclic extension of the above form from $F[\zeta_n]$ to F ; while if n is a power of the characteristic, building blocks can be constructed using Artin-Schreier-Witt extensions. See [10], Lemma 2.1, for an explicit construction.

Theorem 7.3. *Let K be the fraction field of a complete discrete valuation ring T and let G be a finite group. Then G is the Galois group of a Galois field extension A of $K(x)$ such that K is algebraically closed in A .*

Proof. Let g_1, \dots, g_r be generators for G that have prime power orders, and let $H_i \leq G_i$ be the cyclic subgroup generated by g_i . Let k be the residue field of T , and pick distinct monic irreducible polynomials $f_1(x), \dots, f_n(x) \in k[x]$; these define distinct closed points P_1, \dots, P_n of the projective x -line \mathbb{P}_k^1 . For each i let $\hat{f}_i(x) \in T[x]$ be some (irreducible) monic polynomial lying over $f_i(x)$. This defines a lift \hat{P}_i of P_i to a reduced effective divisor on \mathbb{P}_T^1 .

According to [10], Lemma 2.1, there is an irreducible H_i -Galois branched cover $Y_i \rightarrow \mathbb{P}_T^1$ whose special fibre is unramified away from P_i , and such that its fibre over the generic point η of the special fibre is trivial (corresponding to a mock cover, in the terminology there). That is, there is an isomorphism $\phi_i : \text{Spec}(\text{Ind}_1^{H_i} k(x)) \rightarrow Y_i \times_{\mathbb{P}_T^1} \eta$ as H_i -Galois covers of η . Replacing Y_i by its normalization in its function field, we may assume that Y_i is

normal. Necessarily, $Y_i \rightarrow \mathbb{P}_T^1$ is totally ramified over the closed point P_i . Namely, if $I \leq H_i$ is the inertia group at P_i then $Y_i/I \rightarrow \mathbb{P}_T^1$ is unramified and hence purely arithmetic (i.e. of the form $\mathbb{P}_S^1 \rightarrow \mathbb{P}_T^1$ for some finite extension S of T); but generic triviality on the special fibre then implies that $S = T$ and so $I = H_i$. (The fact that it is totally ramified at P_i can also be deduced from the explicit expressions in the proof of [10], Lemma 2.1.)

Let t be a uniformizer for T , and for $i = 1, \dots, r$ let \hat{R}_i be the t -adic completion of the local ring of \mathbb{P}_T^1 at P_i , with fraction field F_i . The pullback of $Y_i \rightarrow \mathbb{P}_T^1$ to $\text{Spec } \hat{R}_i$ is finite and totally ramified. Hence it is irreducible, of the form $\text{Spec } \hat{S}_i$ for some finite extension \hat{S}_i of \hat{R}_i that is a domain. Thus the fraction field E_i of \hat{S}_i is an H_i -Galois field extension of F_i . Let \hat{R}_0 be the completion of the local ring of \mathbb{P}_T^1 at η , with fraction field F_0 . Let R_{r+1} be the subring of $F := K(x)$ consisting of the rational functions on \mathbb{P}_T^1 that are regular on the special fibre \mathbb{P}_k^1 away from P_1, \dots, P_r . Let \hat{R}_{r+1} be the t -adic completion of R_{r+1} and let F_{r+1} be the fraction field of \hat{R}_{r+1} . Also let $H_0 = H_{r+1} = 1 \leq G$ and write $E_0 = F_0$, $E_{r+1} = F_{r+1}$. For $i = 0, 1, \dots, r+1$, we consider the G -Galois F_i -algebra $A_i := \text{Ind}_{H_i}^G E_i$.

We claim that there is an isomorphism $E_i \otimes_{F_i} F_0 \rightarrow \text{Ind}_1^{H_i} F_0$ of H_i -Galois F_0 -algebras along with compatible F_i -algebra inclusions $E_i \hookrightarrow E_0 = F_0$ and $A_i \hookrightarrow A_0$, for $i = 1, \dots, r+1$. In the case $i = r+1$ this is clear from the definitions of H_{r+1} and E_{r+1} , via the inclusion $F_{r+1} \hookrightarrow F_0$. For $1 \leq i \leq r$, the asserted isomorphisms are induced by ϕ_i . Namely, by Hensel's Lemma applied to \hat{R}_0 , there is a unique isomorphism $\hat{\phi}_i : \text{Spec}(\text{Ind}_1^{H_i} \hat{R}_0) \rightarrow Y_i \times_{\mathbb{P}_T^1} \text{Spec } \hat{R}_0$ of H_i -Galois covers of $\text{Spec } \hat{R}_0$ that lifts ϕ_i . Using the natural identifications

$$Y_i \times_{\mathbb{P}_T^1} \text{Spec } \hat{R}_0 = Y_i \times_{\mathbb{P}_T^1} \text{Spec } \hat{R}_i \times_{\text{Spec } \hat{R}_i} \text{Spec } \hat{R}_0 = \text{Spec } \hat{S}_i \times_{\text{Spec } \hat{R}_i} \text{Spec } \hat{R}_0 = \text{Spec}(\hat{S}_i \otimes_{\hat{R}_i} \hat{R}_0),$$

the isomorphism $\hat{\phi}_i$ corresponds on the ring level to an isomorphism $\hat{S}_i \otimes_{\hat{R}_i} \hat{R}_0 \rightarrow \text{Ind}_1^{H_i} \hat{R}_0$ of H_i -Galois \hat{R}_0 -algebras. Since E_i is the fraction field of the finite \hat{R}_i -algebra \hat{S}_i , and since F_i is the fraction field of \hat{R}_i , there is a natural identification of E_i with $\hat{S}_i \otimes_{\hat{R}_i} F_i$. So tensoring the above \hat{R}_0 -algebra isomorphism with F_0 yields an isomorphism $E_i \otimes_{F_i} F_0 = \hat{S}_i \otimes_{\hat{R}_i} F_i \otimes_{F_i} F_0 = \hat{S}_i \otimes_{\hat{R}_i} F_0 \rightarrow \text{Ind}_1^{H_i} F_0$ of H_i -Galois F_0 -algebras; and hence also an F_i -algebra inclusion $E_i \hookrightarrow F_0$, using the projection onto the identity component. The functor $\text{Ind}_{H_i}^G$ then induces an isomorphism $\text{Ind}_{H_i}^G (E_i \otimes_{F_i} F_0) \rightarrow \text{Ind}_{H_i}^G \text{Ind}_1^{H_i} F_0 = \text{Ind}_1^G F_0 = A_0$ of G -Galois F_0 -algebras. Tensoring the inclusion $F_i \hookrightarrow F_0$ with the F_i -algebra A_i yields an F_i -algebra inclusion $A_i = \text{Ind}_{H_i}^G E_i \hookrightarrow (\text{Ind}_{H_i}^G E_i) \otimes_{F_i} F_0 = \text{Ind}_{H_i}^G (E_i \otimes_{F_i} F_0) \rightarrow A_0$, concluding the verification of the claim.

Thus we may apply Theorem 7.1(v), in the case of Theorem 4.14, to the fields F_i and the G -Galois F_i -algebras A_i , for $i = 0, 1, \dots, r+1$. We then obtain a G -Galois F -algebra A that induces the A_i 's compatibly. Moreover, as observed after Theorem 4.14, A is the intersection of the algebras A_1, \dots, A_r, A_{r+1} inside A_0 . Note that K is algebraically closed in A because it is algebraically closed in F_0 and hence in A_0 .

It remains to show that the G -Galois F -algebra A is a field. For $i = 0, 1, \dots, r+1$ let $I_i \subset A_i$ be the kernel of the projection of $A_i = \text{Ind}_{H_i}^G E_i$ onto the identity copy of E_i (i.e. the copy of E_i indexed by the identity element of G), and identify this identity copy with A_i/I_i . Then $I_i \subset A_i$ is the inverse image of $I_0 \subset A_0$ under $A_i \hookrightarrow A_0$, since the inclusions

$A_i \hookrightarrow A_0$ and $E_i \hookrightarrow E_0 = F_0$ are compatible with the projections $A_i \rightarrow E_i$ onto their identity components. Let $I \subseteq A$ be the inverse image of $I_0 \subset A_0$ under $A \hookrightarrow A_0$, and let $E = A/I$. Thus I is also the inverse image of $I_i \subset A_i$ under $A \hookrightarrow A_i$, since $A \hookrightarrow A_0$ factors through A_i and $I_i \subset A_i$ is the inverse image of $I_0 \subset A_0$ under $A_i \hookrightarrow A_0$. Hence I is a prime ideal of A , and E is an integral domain. But E is finite over the field F , since the G -Galois F -algebra A is. Thus E is a field. Now the above inclusions are compatible with the G -Galois actions. So using the identification $E_i = A_i/I_i$ and the fact that $H_i = \text{Gal}(E_i/F_i) \subseteq G$ is the stabilizer of I_i in G , we have that every element of H_i restricts to an element of $H := \text{Gal}(E/F) \subseteq G$, the stabilizer of I in G . That is, H contains H_i for all i . But H_1, \dots, H_r generate G . So $H = G$. Thus I is stabilized by all of G ; and since the identity component of each element of I is zero (regarding $I \subseteq I_0 \subset A_0 = \text{Ind}_1^G F_0$), it follows that $I = (0)$. Hence $E = A$ and A is a field. \square

- Remark 7.4.** (a) The above proof can be extended to more general smooth curves \hat{X} over a complete discrete valuation ring T . Namely, Theorem 4.14 permits patching on such curves; and the same expressions used for building blocks in the case of the line can be used for other curves, since they remain n -cyclic and totally ramified. This latter fact can be seen directly from the construction in [10]. It can also be seen by choosing a parameter x for a point P on the closed fibre X of \hat{X} ; constructing the building blocks for the x -line over T ; and then taking a base change to the local ring at P (which, being étale, preserves total ramification). This contrasts with the strategy in [7], Proposition 1.4, which is to map a curve to the line; perform a patching construction there; and then deduce a result about the curve.
- (b) Alternatively, the above proof can be extended to more general smooth curves over T by using Theorem 5.10 instead of Theorem 4.14 (where the complete local ring is independent of which smooth curve is taken). It can also be extended to the case of a singular normal T -curve whose closed fibre is generically smooth, by instead using Theorem 6.4.
- (c) In [10], Section 2, more was shown: that the theorem remains true if we replace T by any complete local domain that is not a field. But in fact this more general assertion follows from the above theorem because every such domain contains a complete discrete valuation ring; see [21], Lemma 1.5 and Corollary 1.6.
- (d) One can similarly recover other results in inverse Galois theory within our framework of patching over fields; e.g., the freeness of the absolute Galois group of $k(x)$, for k algebraically closed (the “Geometric Shafarevich Conjecture” [12], [26]). But the above result is merely intended to be illustrative, to show how patching over fields can be used in geometric Galois theory.

7.3 Differential Modules

The main interest in patching vector spaces is of course that we can also patch vector spaces with additional structure. This was done for various types of algebras in Section 7.1 above.

The following application is another example of this sort.

Suppose that F is a field of characteristic zero equipped with a derivation ∂_F . A **differential module** over F is a finite dimensional F -vector space M together with an additive map $\partial_M : M \rightarrow M$ such that $\partial_M(f \cdot m) = \partial_F(f) \cdot m + f \cdot \partial_M(m)$ for all $f \in F, m \in M$ (Leibniz rule). A **homomorphism of differential modules** is a homomorphism of the underlying vector spaces that respects the differential structures. It is well known that differential modules over a differential field F form a tensor category $\partial\text{-Mod}(F)$ (in fact a Tannakian category over F ; e.g. see [23], §1.4).

We will state only the simplest version of patching differential modules, a consequence of Theorem 4.12. There are respective versions of Theorem 4.14, and of the patching results in Section 5.

Theorem 7.5. *Let T be a complete discrete valuation ring with fraction field K of characteristic zero and residue field k , and let \hat{X} be a smooth connected projective T -curve with closed fibre X and function field F . Let $U_1, U_2 \subseteq X$, and let $U := U_1 \cup U_2$, $U_0 := U_1 \cap U_2$. Equip F_U, F_{U_i} with the derivation $\frac{d}{dx}$ for some rational function x on \hat{X} that is not contained in K .*

Then the base change functor

$$\partial\text{-Mod}(F_U) \rightarrow \partial\text{-Mod}(F_{U_1}) \times_{\partial\text{-Mod}(F_{U_0})} \partial\text{-Mod}(F_{U_2})$$

is an equivalence of categories, with inverse given by intersection.

Proof. Recall that $F_U = F_{U_1} \cap F_{U_2}$ (Theorem 4.9). By Theorem 4.12, base change is an equivalence of categories on the level of vector spaces; so for every object $(M_1, M_2; \phi)$ in $\partial\text{-Mod}(F_{U_1}) \times_{\partial\text{-Mod}(F_{U_0})} \partial\text{-Mod}(F_{U_2})$, there is an F_U -vector space M that induces $(M_1, M_2; \phi)$ as an object in $\text{Vect}(F_{U_1}) \times_{\text{Vect}(F_{U_0})} \text{Vect}(F_{U_2})$. Moreover, as noted after that result, M is given by $M_1 \cap M_2$. Consequently, the derivations on M_1 and M_2 restrict compatibly to M ; i.e., M is a differential module under that common restricted derivation. By Corollary 2.2, $\dim_{F_U} M = \dim_{F_{U_i}} M_i$ for $i = 1, 2$; in particular, M contains a basis of M_i as a vector space over F_{U_i} ($i = 1, 2$). But the derivation on each M_i is already determined when given on such a basis (by the Leibniz rule). Thus M induces the M_i 's as differential modules, compatibly with ϕ .

So the base change functor gives a bijection on isomorphism classes. Similarly, morphisms between corresponding objects in the two categories are in bijection on the level of vector spaces, and hence also on the level of differential modules (using that the derivations are related by taking base change and restriction). Thus the functor is an equivalence of categories. \square

After choosing a basis of each M_i in the above proof, one can also explicitly define the derivation on M using the matrix representations of the derivations and a factorization of the matrix defining ϕ given by Theorem 4.10.

Remark 7.6. As noted in the proof of Theorem 7.1, the equivalence of the categories of vector spaces is in fact an equivalence of tensor categories; the same remains true for differential modules.

There is a Galois theory for differential modules that mimics the usual Galois theory of finite field extensions. A natural question to ask is whether one can control the differential Galois group of a differential module obtained by patching. This question (along with its implications for the inverse problem in differential Galois theory) is the subject of [15] (see also [18]), which provides applications of the above theorem.

References

- [1] Nicolas Bourbaki, *Commutative Algebra*, Addison-Wesley Publishing Co., 1972.
- [2] Henri Cartan, Sur les matrices holomorphes de n variables complexes, *Journal de Mathématiques Pures et Appliquées* **19** (1940), 1–26.
- [3] Michael D. Fried and Moshe Jarden, *Field Arithmetic*, Springer, 2nd edition, 2005.
- [4] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, 1978.
- [5] Alexander Grothendieck, *Éléments de Géométrie Algébrique (EGA) III*, 1^e partie, volume 11 of *Publ. Math. IHES*, 1961.
- [6] Dan Haran and Moshe Jarden, Regular split embedding problems over complete valued fields, *Forum Math.* **10 (3)** (1998), 329–351.
- [7] Dan Haran and Moshe Jarden, Regular split embedding problems over function fields of one variable over ample fields, *J. Algebra* **208** (1998), 147–164.
- [8] Dan Haran and Helmut Völklein, Galois groups over complete valued fields, *Israel J. Math.* **93** (1996), 9–27.
- [9] David Harbater, Convergent arithmetic power series, *Amer. J. Math.* **106** (1984), 801–846.
- [10] David Harbater, Galois coverings of the arithmetic line. In Chudnovsky, D.V. and Chudnovsky, G.V., editors, *Number Theory (New York, 1984–1985)*, volume 1240 of *Lecture Notes in Math.*, pages 165–195, Springer, 1987.
- [11] David Harbater, Formal patching and adding branch points, *Amer. J. Math.* **115** (1993), 487–508.
- [12] David Harbater, Fundamental groups and embedding problems in characteristic p . In M. Fried, editor, *Recent Developments in the Inverse Galois Problem*, volume 186 of *Contemporary Math. Series*, pages 353–369, American Math. Society, 1995.
- [13] David Harbater, Patching and Galois theory. In L. Schneps, editor, *Galois Groups and Fundamental Groups*, volume 41 of *MSRI Publications series*, pages 313–424, Cambridge University Press, 2003.

- [14] David Harbater, Patching over fields (joint work with Julia Hartmann). In *Arithmetic and Differential Galois Groups*, volume 4, no.2 of *Oberwolfach reports*, European Mathematical Society, 2007.
- [15] David Harbater and Julia Hartmann, Patching and differential Galois groups, preprint, 2007.
- [16] David Harbater, Julia Hartmann, and Daniel Krashen, Patching subfields of division algebras, preprint, 2007.
- [17] David Harbater and Katherine F. Stevenson, Patching and thickening problems, *J. Algebra* **212** (1) (1999), 272–304.
- [18] Julia Hartmann, Patching and differential Galois groups (joint work with David Harbater), In *Arithmetic and Differential Galois Groups*, volume 4, no.2 of *Oberwolfach reports*, European Mathematical Society, 2007.
- [19] Robin Hartshorne, *Algebraic Geometry*, Springer, 1977.
- [20] Israel N. Herstein, *Noncommutative Rings*, Math. Assoc. of America, 1968.
- [21] Moshe Jarden, The inverse Galois problem over formal power series fields, *Israel J. Math.* **85** (1994), 263–275.
- [22] Qing Liu, Tout groupe fini est un groupe de Galois sur $\mathbb{Q}_p(T)$, d’après Harbater. In M. Fried, *Recent Developments in the Inverse Galois Problem*, volume 186 of *Contemporary Math. Series*, pages 261–265, American Math. Society, 1995.
- [23] B. Heinrich Matzat and Marius van der Put, Constructive differential Galois theory. In L. Schneps, *Galois Groups and Fundamental Groups*, volume 41 of *MSRI Publications series*, pages 425–467, Cambridge University Press, 2003.
- [24] David Mumford, *The Red Book of Varieties and Schemes*, volume 1358 of *Lecture Notes in Math.*, Springer, 1988 and 1999.
- [25] Richard Pierce, *Associative Algebras*, Springer, 1982.
- [26] Florian Pop, Étale Galois covers of smooth affine curves, *Invent. Math.* **120** (1995), 555–578.
- [27] Rachel Pries, Construction of covers with formal and rigid geometry. In J.-B. Bost, F. Loeser, and M. Raynaud, editors, *Courbes Semi-stables et Groupe Fondamental en Géométrie Algébrique* volume 187 of *Progress in Mathematics*, pages 157–167, Birkhäuser, 2000.
- [28] Michel Raynaud, Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d’Abhyankar, *Invent. Math.* **116** (1994), 425–462.

- [29] Jean-Pierre Serre, Géométrie algébrique et géométrie analytique, *Annales de l'Institut Fourier* **6** (1956), 1–42.
- [30] Jean-Pierre Serre, *Algebraic Groups and Class Fields*, Springer, 1988.
- [31] Helmut Völklein, *Groups as Galois Groups*, Cambridge University Press, 1996.

Author information:

David Harbater: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA; email: harbater@math.upenn.edu

Julia Hartmann: IWR, University of Heidelberg, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany; email: Julia.Hartmann@iwr.uni-heidelberg.de