Fundamental Groups of Curves in Characteristic p

David Harbater*
University of Pennsylvania, Philadelphia, PA, USA

Section 1: Introduction.

Consider the following general problem: Given a smooth affine curve U over an algebraically closed field k, find the fundamental group $\pi_1(U)$, and its set of (continuous) finite quotients $\pi_A(U)$. When $k = \mathbf{C}$, U is a Riemann surface, and π_1 can be computed using loops. If U is obtained by deleting $S = \{\xi_0, \ldots, \xi_r\}$ from a compact Riemann surface X of genus g, we thus obtain classically that π_1 has generators $a_1, \ldots, a_g, b_1, \ldots, b_g, c_0, \ldots, c_r$ subject to the single relation $\prod_{j=1}^{g}[a_j, b_j]\prod_{i=0}^{r} c_i = 1$. (Here $[a, b] = aba^{-1}b^{-1}$.) This is isomorphic to the free group on $2g + r$ generators, so $\pi_A(U)$ is the set of finite groups with $2g + r$ generators. Thus these are the Galois groups of finite unramified Galois covers of U, or equivalently of finite branched covers of X with branch locus disjoint from U.

Over other algebraically closed fields k, loops do not make sense. But it does make sense to speak of finite unramified covers of U, and of $\pi_A(U)$. So let $U = X - S$, where X is a smooth projective k-curve of genus $g \geq 0$, and $S = \{\xi_0, \ldots, \xi_r\}$ ($r \geq 0$); we call this an affine curve of *type* (g, r). The result over $\mathbf{C}$ no longer holds, if the characteristic of k is $p > 0$, e.g. because of Artin-Schreier covers of the affine line. In 1957, Abhyankar [Ab1] posed:

Abhyankar's Conjecture ("AC"). In characteristic p, if U is an affine curve of type (g, r), then a finite group G is in $\pi_A(U)$ if and only if every prime-to-p quotient of G has $2g + r$ generators.

Equivalently, writing $p(G)$ for the subgroup of G generated by the Sylow p-subgroups, AC asserts that $G \in \pi_A(U)$ if and only if $G/p(G)$ is in π_A of a complex curve of type (g, r).

Here Abhyankar allowed $p = 0$. Later, Grothendieck showed [Gr2, XIII, Cor. 2.12] that AC holds for $p = 0$ and that π_1 of a curve of type (g, r) is the same over all algebraically closed fields of characteristic 0. This was proven by specialization techniques, as was a weak form of AC in the $p > 0$ case: that the prime-to-p part of π_1 is the same in characteristic p and in characteristic 0, and that the tame fundamental group $\pi_1^t(U)$ over k is a quotient of π_1 of a complex curve of the same type. (If $U = X - S$, $\pi_1^t(U)$ is defined via branched covers of X that are unramified over U and tamely ramified over S.)

Grothendieck's results imply that the forward implication of AC holds; that a prime-to-p group G is in $\pi_A(U)$ if and only if it has $2g + r$ generators; and that not all groups conjectured to be in $\pi_A(U)$ can arise from branched covers of X that are tamely ramified over S and unramified elsewhere. This suggests:

Strong Abhyankar Conjecture ("SAC"). In characteristic p, if $U = X - \{\xi_0, \ldots, \xi_r\}$ with X of genus g and $r \geq 0$, and if each prime-to-p quotient of G lies in $\pi_A(U)$, then G is the Galois group of a Galois étale cover of U whose smooth completion is tamely ramified over X except possibly at ξ_0.

* Supported in part by NSA grant # MDA 904-92-H-3024.
Email: harbater@rademacher.math.upenn.edu

As a result of recent work of Raynaud and the author, we now have

Theorem. [Ra2 , Ha6] SAC (and hence AC) holds for all affine curves.

Abhyankar's Conjecture was stated in 1957, but evidence began to accumulate only about 1980. The case of $U = \mathbf{A}^1$ was considered first; there AC says that π_{A} consists of the *quasi-p-groups* (i.e. groups with $p(G) = G$). Nori (cf. [Ka]) and Abhyankar (cf. [Ab2]) showed that various finite groups, especially certain simple groups, lie in $\pi_{\mathrm{A}}(\mathbf{A}^1)$. Later Serre [Se1] proved AC over $\mathbf{A}^1$ for solvable quasi-p-groups. Raynaud [Ra2] then showed the full AC for $\mathbf{A}^1$ using rigid analytic patching and semi-stable reduction. The author's proof of SAC [Ha6] used another form of patching, involving formal schemes, as well as relying of [Ra2].

The structure of the rest of this paper is as follows: Section 2 describes formal and rigid patching, and Section 3 sketches the proof of AC. Finally, Section 4 discusses variants and open problems.

Section 2: Formal and rigid patching.

2.1: *Formal and rigid geometry.* Over the complex numbers, one can construct covers with desired properties by "cutting and pasting." In the proof of AC, analogous (formal or rigid) techniques are used to handle curves in characteristic p. The point is that the Zariski topology is too weak to use in mimicking complex constructions, since there are no "small" open sets. But the formal and rigid approaches provide smaller sets that can be cut and pasted usefully. Here we work over a complete field, e.g. $K = k((t))$, which in some ways is analogous to $\mathbf{C}$.

The formal approach is based on Grothendieck's formal schemes [Gr1, EGA I, sect. 10]. The rigid setting, due to Tate [Ta] and Kiehl [Ki], is more intuitive, but its foundations have not been worked out as thoroughly. The relationship between these two frameworks has been presented in [Ra1], [Me], [BL] and [BLR].

Consider a curve over $K = k((t))$. One can speak of metric open discs, and can attempt to do analytic geometry, in analogy with complex curves. Unfortunately, it is insufficient to use the naive approach of working with such discs and their rings of holomorphic functions, because the metric topology is totally disconnected, and so the geometry obtained would be "flabby." Instead, the rigid theory introduces a subtler notion of an affinoid set and its ring of functions. (See also [Ra2, sect. 3].) This enables cutting and pasting that behaves more as desired.

Meanwhile, in the formal context, we begin with a curve over k, and consider "thickenings" to $R = k[[t]]$. If X is a smooth projective k-curve, then such a thickening is $X_R = X \times_k R$, with generic fibre X_K. On the other hand if $U = \mathrm{Spec}(E)$ is an affine curve, then a thickening is $U^* = \mathrm{Spec}(E[[t]])$. This is "smaller" than $U_R = U \times_k R$, which is a Zariski open subset of X_R. For example, if $X = \mathbf{P}^1_k$ and $U = \mathbf{A}^1_k$, then $U^* = \mathrm{Spec}(k[x][[t]])$ and $U_R = \mathrm{Spec}(k[[t]][x])$. Since $1 - xt$ is a unit in $k[x][[t]]$ but not in $k[[t]][x]$, the point $(1 - xt)$ in U_R is missing from U^*. Geometrically, we can think of U_R as a "uniformly thick" tubular neighborhood of U, whereas U^* is a neighborhood that "pinches down" near points at infinity. (For projective curves X, there are no points at infinity, and $X^* = X_R$.) We can also consider thickenings of other subschemes of X, e.g. complete local neighborhoods $\mathrm{Spec}(\hat{\mathcal{O}}_{X,\xi})$ of any point ξ of X. In this case we obtain $\mathrm{Spec}(\hat{\mathcal{O}}_{X,\xi}[[t]])$.

For $U \subset X$, the thickening U^* is a surface whose closed fibre is U. Concerning the connection to rigid geometry, consider the generic fibre of U^*, obtained by deleting the closed fibre. This is an affine scheme $\mathrm{Spec}(A)$, where A is the ring of functions of an affinoid subset $\mathcal{U}$ of U_K. For example, if $X = \mathbf{P}^1_k$ and $U = \mathbf{A}^1_k$, then $\mathcal{U}$ is a disc about the origin. And if $X = \mathbf{P}^1_k$ and $U = \mathbf{A}^1_k - (x = 0)$, then $\mathcal{U}$ is a "corona" (annulus) whose complement has two components (one containing the point $(x = 0)$ and the other containing $(x = \infty)$). Under this correspondence, points of $\mathcal{U}$ correspond to curves in U^* not lying in the closed fibre, and two points of $\mathcal{U}$ are "close" if the corresponding curves have a high order of contact.

2.2: *Patching.* In the proof of Abhyankar's Conjecture, the main idea is to construct G-Galois covers over k by working inductively on the order of G, and to paste together Galois covers having smaller group. Given an affine k-curve U, if G-Galois covers are constructed over the induced K-curve U_K (where $K = k((t))$), then a specialization argument (the "Lefschetz principle") implies that there is a G-Galois cover of U. Thus it suffices to work over the complete field K.

Consider the following situation over $\mathbf{C}$, which we wish to mimic in characteristic p. We have a compact Riemann surface $\mathcal{X}$; a subset $\mathcal{U}_1$ obtained by deleting a small disc D; a disc $\mathcal{U}_2$ that is slightly larger than D; and the overlap $\mathcal{U}_0 = \mathcal{U}_1 \cap \mathcal{U}_2$, which is an annulus. Given a structure (e.g. a vector bundle, a branched cover, etc.) over $\mathcal{U}_1$ and $\mathcal{U}_2$ together with an agreement over $\mathcal{U}_0$, we wish to patch the data together to obtain such a structure over $\mathcal{X}$.

Analogs of these discs and annuli exist in the rigid setting. Meanwhile, in the formal setting, consider a point ξ on a smooth projective k-curve X. Let $U_1 = X - \{\xi\}$, $U_2 = \mathrm{Spec}(\hat{\mathcal{O}}_{X,\xi})$, and $U_0 = \mathrm{Spec}(\hat{\mathcal{K}}_{X,\xi})$, where $\hat{\mathcal{K}}_{X,\xi}$ is the fraction field of $\hat{\mathcal{O}}_{X,\xi}$. Then the formal analog is given by X^*, U_1^*, U_2^*, and U_0^*. Here, one can patch structures such as vector bundles or Galois covers. This is by a formal patching theorem [Ha5, Theorem 1] which is a variant on Grothendieck's Existence Theorem [Gr1, EGA III, 5.1.6], and can be regarded as a "formal GAGA":

Patching Theorem [Ha5, Thm. 1] *In the above situation, consider finite projective modules M_1 and M_2 over U_1^* and U_2^*, together with an isomorphism between the induced modules over U_0^*. Then up to isomorphism, there is a unique finite projective $\mathcal{O}_X$-module M inducing M_1 and M_2, compatibly with the identification over U_0^*. Moreover this association corresponds to an equivalence of categories, and so the result carries over to finite projective algebras, and to covers.*

This is proven by reducing to a local analog for modules over discrete valuation rings $\mathcal{O}$ (where projective modules are free). Set $\mathcal{K} = \mathrm{frac}(\mathcal{O})$ and $\hat{\mathcal{K}} = \mathrm{frac}(\hat{\mathcal{O}})$. The problem is to patch together free modules over $\hat{\mathcal{O}}[[t]]$ and $\mathcal{K}[[t]]$ with agreement over $\hat{\mathcal{K}}[[t]]$, and to obtain a free $\mathcal{O}[[t]]$-module inducing the given modules together with the identification. This is done by factoring the transition matrix $M \in \mathrm{GL}_N(\hat{\mathcal{K}}[[t]])$ as a product of change-of-basis matrices in $\mathrm{GL}_N(\hat{\mathcal{O}}[[t]])$ and $\mathrm{GL}_N(\mathcal{K}[[t]])$.

As an application of this patching theorem, we consider the following result, which permits the inductive construction of covers of curves. First we introduce a bit of terminology: Pick roots of unity $\{\zeta_n \mid \mathrm{char}(k) \text{ does not divide } n\} \subset k$ such that $\zeta_{nn'}^{n'} = \zeta_n$. Given a G-Galois cover of curves $Y \to X$, let $\eta \in Y$ be a ramification point lying over $\xi \in X$, with local uniformizers $y \in \hat{\mathcal{O}}_{Y,\eta}$ and $x \in \hat{\mathcal{O}}_{X,\xi}$

satisfying $y^n = x$. We call $g \in G$ the *inertial generator* at η if $g(y) = \zeta_n y$. (If $k = \mathbf{C}$, this can be interpreted via the lifting to Y of counterclockwise loops around ξ.)

Corollary. *Let H_1, H_2 be subgroups generating a finite group G; $Y \to X$ a connected H_1-Galois cover of k-curves with branch locus $B \subset X$; and $W \to \mathbf{P}^1$ a connected H_2-Galois cover with m branch points. Let $g \in G$ be the inertial generator at a tame point $\eta \in Y$ over $\xi \in B$, and suppose that g^{-1} is the inertial generator at a tame point $\omega \in W$ over one of the m branch points. Then there is a connected G-Galois cover $Z \to X$ that is branched at B and $m-2$ other points, and whose inertia groups over $B - \{\xi\}$ are the conjugates of those of $Y \to X$.*

To prove this result over $k = \mathbf{C}$, we induce each of the given covers up to G, by taking a disjoint union of copies of the cover, indexed by the cosets of H_i in G. We then cut out small discs around $\xi \in X$ and $\mu \in \mathbf{P}^1$, where ω lies over μ. The two disconnected G-Galois covers agree over the boundaries of the excised discs (because the two boundary orientations are opposite, and the inertial generators are g, g^{-1}); by pasting along the boundaries we obtain the desired cover $Z \to X$. Here, the base is still isomorphic to X, and the pasting can be done so that one of the new branch points coming from $W \to \mathbf{P}^1$ is now positioned at ξ.

For k of characteristic p, using formal geometry, consider the union X' of X and $\mathbf{P}^1$ crossing transversally (identifying $\xi \in X$ with $\mu \in \mathbf{P}^1$). By blowing up the point $(\xi, 0)$ on $X^* = X \times_k k[[t]]$ and pulling back by $t \mapsto t^n$ (where $n = \mathrm{ord}(g)$), we obtain an irreducible $k[[t]]$-thickening X'^* of X' with generic fibre $X_{k((t))}$, and given near the singular point by $xu = t^n$. By choosing a finite morphism $X'^* \to \mathbf{P}^1_{k[[t]]}$ and working over $\mathbf{P}^1$, we can apply the above formal patching theorem [Ha5, Thm. 1]. So there is a G-Galois cover of X'^* consisting of copies of thickenings of Y and W away from the node (first altering W to move a branch point to ∞), and copies of $\mathrm{Spec}\big(k[[y, w, t]]/(y^n - x, w^n - u, yw - t)\big)$ near the node. The generic fibre is a $k((t))$-cover with the desired properties. (Its connectivity follows from that of the closed fibre, which uses $G = H_1 H_2$.) Now apply the Lefschetz principle to obtain such a cover over k.

Section 3: Proof of Abhyankar's Conjecture.

3.1: *Outline of the proof of AC.* Let k be an algebraically closed field of characteristic p. In 1990, Serre proved the following result:

Theorem. [Se1, Thm. 1] *Let $1 \to N \to G \to H \to 1$ be an exact sequence of finite groups, with G quasi-p and N solvable. If $H \in \pi_{\mathrm{A}}(\mathbf{A}^1_k)$ then so is G.*

Taking $H = 1$, we obtain Abhyankar's Conjecture for solvable groups over $\mathbf{A}^1$. For the proof, induction reduces to the case of N an elementary abelian l-group on which H acts irreducibly. Since $\mathrm{cd}(\mathbf{A}^1) = 1$ [Se1, Prop. 1], we may replace H by a subgroup of G, and so assume that the exact sequence is split. The proof proceeds cohomologically. The most difficult case is that of $l \neq p$. There, the given H-Galois cover might not be dominated by any G-Galois cover of $\mathbf{A}^1$ (i.e. the corresponding embedding problem over $\mathbf{A}^1$ might have no solution). Instead the H-Galois cover may have to be altered, before obtaining a G-Galois cover dominating it.

Using Serre's result, together with rigid patching and semi-stable reduction, Raynaud [Ra2] proved Abhyankar's Conjecture over $\mathbf{A}^1$, in 1992. That is, he showed that if G is a finite quasi-p-group, then G is a Galois group over $\mathbf{A}^1$. The proof proceeds inductively on the order of G. For P a Sylow p-subgroup of G, let $G(P)$ be the subgroup of G generated by all the proper quasi-p-subgroups $H \subset G$ such that P contains a Sylow p-subgroup of H. There are three cases: (i) G has a non-trivial normal p-subgroup; (ii) $G(P) = G$ for some P; (iii) Otherwise.

Case (i) follows from Serre's result and the inductive hypothesis, since p-groups are solvable. Case (ii) uses rigid patching methods; cf. section 3.2 below. Case (iii) uses semi-stable reduction in mixed characteristic; cf. section 3.3.

Using Raynaud's result and formal patching, the author proved the general case of AC, including the stronger form SAC. This is discussed in section 3.4.

3.2: *Proof of AC for* $\mathbf{A}^1$ *in case (ii).* As discussed in section 2 above, it suffices to construct a G-Galois cover of the K-line, where $K = k((t))$. Let $G_1, \dots, G_r$ be the proper quasi-p-subgroups of G having Sylow p-subgroups contained in P. By the inductive hypothesis, each G_i is the Galois group of a cover $X_i \to \mathbf{A}^1$. Pulling back by a cover of the form $y^n = x$ and using Abhyankar's Lemma, we may assume that these G_i-Galois covers have p-groups $Q_i \subset P$ among the inertia groups over infinity. The restriction of X_i to a corona $\mathcal{C}_i$ centered at infinity is a disjoint union of copies of some Q_i-Galois cover $\mathcal{U}_i \to \mathcal{C}_i$.

Choose $r+1$ points $\sigma_1, \dots, \sigma_r, \infty \in \mathbf{P}^1_K$ together with copies of the r coronas $\mathcal{C}_i$ centered at the points σ_i. Also let $C = \mathbf{P}^1_K - \{\sigma_1, \dots, \sigma_r, \infty\}$. These points and coronas can be chosen so that the union $\mathcal{C} = \bigcup_i \mathcal{C}_i$ is disjoint and extends to a disjoint union on the corresponding discs, and so that $(C, \mathcal{C})$ is a *Runge pair* — i.e. so that $\mathbf{P}^1_K - C$ contains a point in each component of the complement of $\mathcal{C}$. Possibly after replacing K by a finite separable extension, there is a P-Galois cover $Y \to C$ whose restriction to each $\mathcal{C}_i$ is a disjoint union of copies of $\mathcal{U}_i \to \mathcal{C}_i$. (This is shown [Ra2, Cor. 4.2.6] using cohomology and induction on the order of the p-group P.) Now induce up to G, pasting each X_i to Y over $\mathcal{C}_i$. This yields a G-Galois cover, which is connected because we are in case (ii).

This case of the proof can also be shown using formal patching. See [Ha8, Application 2.2] for a discussion of this.

3.3: *Proof of AC for* $\mathbf{A}^1$ *in case (iii).* Since G is a quasi-p-group, there is a G-Galois cover $Y_K \to \mathbf{P}^1_K$ with p-power inertia groups, over a field K of characteristic 0. Here K can be chosen to be the fraction field of a complete discrete valuation ring R with residue field k. For suitable K and R, there is an R-model $Y \to X$ of this cover with semi-stable reduction and fibre $Y_k \to X_k$, such that X_k is a tree of $\mathbf{P}^1_k$'s; the inertia group I_s at each component s of Y_k is a p-group; and I_s is non-trivial unless s lies over a terminal component of the tree X_k.

Since X_k is a tree of $\mathbf{P}^1_k$'s, there is a natural partial order on the components, with the "base component" o' minimal and terminal components maximal. A partially ordered tree A of components of Y_k is constructed above it, with some o over o' minimal. It is chosen so that $G_o = G$, where for each component s of A, $G_s \subset G$ is the subgroup generated by $\{p(D_t) \mid t \text{ in } A,\ t \geq s\}$ (where D_t is the decomposition group at t and where $p(\cdot)$ is as in section 1).

Let s in A be maximal such that $G_s = G$. If $I_s \neq 1$ then a group theory argument (using that we are not in case (i)) shows $G_s \subset G(P)$ for some P —

contradicting $G_s = G$, since we are not in case (ii). So actually $I_s = 1$, and s is a terminal component, with $D_s = G$. Its image s' in X_k is a copy of $\mathbf{P}^1_k$. Since s is a terminal component of the tree A, s' meets the rest of the graph at only one point. Deleting this point yields a G-Galois cover of the affine line.

3.4: *Proof of SAC for general affine curves.* This proof relies on AC for $\mathbf{A}^1$ (which in that case is equivalent to SAC). The key step is to show the result for $\mathbf{A}^1 - \{0\}$. Once that is done, the general case can be shown as follows: Under the hypotheses of SAC, let $Q = p(G)$ and $F = G/Q$. By [Gr2, XIII, Cor. 2.12], there is an F-Galois cover $U \to X$ branched only at $\{\xi_0, \dots, \xi_r\}$. Let C be an inertia group over ξ_0, with inertial generator $g \in G$ (cf. 2.2). By group theory, we may assume that the exact sequence $1 \to Q \to G \to F \to 1$ splits and that $E = Q \cdot C$ is a semi-direct product. Using the case of SAC for $\mathbf{A}^1 - \{0\}$, we obtain an E-Galois cover of $\mathbf{A}^1 - \{0\}$ that is tamely ramified over 0, with C an inertia group there and (after pulling back by some $x \mapsto x^j$) inertial generator g^{-1}. Since E and F generate G, the result follows from the corollary to the patching theorem in section 2.2 above.

To prove SAC for $\mathbf{A}^1 - \{0\}$, let $Q = p(G)$, let $C = G/Q$, and let P be a Sylow p-subgroup of G. Thus C is cyclic of order n prime to p. By group theory we reduce to the case that $1 \to Q \to G \to C \to 1$ splits and $H = P \cdot C$ is a semi-direct product. By AC for $\mathbf{A}^1$, there is a Q-Galois cover $W \to \mathbf{A}^1 = \mathrm{Spec}(k[x])$. By enlarging inertia (e.g. by [Ha5, Theorem 2]), we may assume that P is an inertia group over $(x = \infty)$. By [Ha1, Cor. 2.4], there is a P-Galois cover $Y \to \mathbf{A}^1$ that agrees locally with $W \to \mathbf{A}^1$ over $\mathrm{Spec}\big(k((x^{-1}))\big)$. Using the moduli space of P-covers of the affine line [Ha1], one may construct a P-Galois cover $Z \to \mathbf{P}^1 \times \mathbf{P}^1$ of (x,t)-space that is étale over $\mathbf{A}^1 \times \mathbf{A}^1$ and totally ramified elsewhere; whose fibre over $\mathbf{A}^1 \times (t = 1)$ agrees with $Y \to \mathbf{A}^1$; and whose composition with $(x,t) \mapsto (x, t^n)$ is H-Galois [Ha6, Prop. 4.1] over (x,s)-space $\mathbf{P}^1 \times \mathbf{P}^1$ (where $s = t^n$).

For a suitable blow-up T of (x,s)-space, there is a covering morphism from T to (u,v)-space $\mathbf{P}^1 \times \mathbf{P}^1$ whose fibre over $(v = 0)$ consists of two lines X_1 (over $s = 1$) and X_2 (over $x = \infty$) crossing at a point τ. The restriction $T^* \to \mathbf{P}^1 \times \mathrm{Spec}\big(k[[v]]\big)$ has general fibre isomorphic to the s-line over $K = k((v))$. Pulling back the above H-Galois cover of (x,s)-space to T^* and normalizing, we obtain an H-Galois cover $B^* \to T^*$. Its fibre over $X_1' = X_1 - \{\tau\}$ is isomorphic to the disconnected H-Galois cover $\mathrm{Ind}_P^H\, Y \to \mathbf{A}^1$ induced by $Y \to \mathbf{A}^1$. The generic fibre $B^{*o} \to T^{*o}$ is branched precisely at $(s = 0)$ and $(s = \infty)$, with inertia groups C and H respectively [Ha6, Prop. 5.1]. So the cover is unramified over X_1', and the fibre over the thickening $X_1'^*$ (cf. section 2.1) is $\mathrm{Ind}_P^H\, Y^*$.

Since the covers $W \to \mathbf{A}^1$ and $Y \to \mathbf{A}^1$ agree locally over $\mathrm{Spec}\big(k((x^{-1}))\big)$, their thickenings W^* and Y^* agree locally over $\mathrm{Spec}\big(k((x^{-1}))\big)^*$; hence so do W^* and B^* (over $\mathrm{Spec}(\hat{\mathcal{K}}_{X_1,\tau})^*$). Since the base space T^* is fibred over $\mathbf{P}^1_{k[[v]]}$, we may apply the formal patching theorem in section 2 [Ha2, Thm. 1] to $\mathrm{Ind}_H^G B^*$ and $\mathrm{Ind}_Q^G W^*$, in order to cut out copies of Y^* from B^* and paste in copies of W^*. This yields an irreducible G-Galois cover of T^*. Its general fibre is an irreducible G-Galois cover of the s-line $\mathbf{P}^1_K$ that is branched only at $(s = 0)$ and $(s = \infty)$, with inertia groups C and H respectively. This solves the problem over K, and using the Lefschetz principle we obtain SAC for $\mathbf{A}^1_k - \{0\}$.

The above proof used formal patching, but it is also possible to prove SAC for $\mathbf{A}^1_k - \{0\}$ using rigid methods. Namely, Raynaud has observed that his result on Runge pairs discussed in section 3.2 above [Ra2, Cor. 4.2.6] can be generalized in a way that can yield the rigid analog of the above construction. See the Remark after [Ha6, Prop. 5.2] for a further discussion of this.

Section 4: Complements and open problems.

4.1: *Structure of* π_1. Abhyankar's Conjecture describes π_{A} of an affine curve of type (g,r) in characteristic p, and in particular shows that it depends only on the integers (g,r). But the fundamental group π_1 of an affine curve in characteristic p remains unknown, even for the affine line. Moreover π_1 depends on the cardinality of the field k, since covers in characteristic p can have "moduli" (e.g. consider the family $y^p - y = tx$ of p-cyclic covers of the affine x-line, parametrized by the t-line with $(t=0)$ removed.) And even for a fixed algebraically closed field k, π_1 does not simply depend on the type (g,r). Indeed, even two affine curves of the form $\mathbf{P}^1 - \{0,1,\infty,\lambda\}$ can have non-isomorphic π_1's [Ha7, Theorem 1.8].

Also, for $U = X - \{\xi_0, \ldots, \xi_r\}$ and $G \in \pi_{\mathrm{A}}(U)$, it is unknown which subgroups $G_i \subset G$ can be inertia groups over ξ_i of G-Galois branched covers of X that are étale over U. For $U = \mathbf{A}^1$, it is known that the inertia group can be taken to be a p-group (by Abhyankar's Lemma), and in general it is known that if a p-subgroup can be an inertia group then so can every larger p-subgroup [Ha5, Thm. 2]. Hence the Sylow p-subgroups can be inertia over infinity for covers of $\mathbf{A}^1$. There is also an obvious necessary condition on a subgroup of a quasi-p-group to arise as inertia over infinity [Ha7, Prop. 1.4]. But it is unclear if this is sufficient.

4.2: *Anabelian conjecture.* The discussion in 4.1 suggests the following problem: For given values of $g, r \geq 0$, consider the moduli space $M_{g,r+1}$ of smooth k-curves of genus g with $r+1$ points deleted. Is there a dense open subset of $M_{g,r+1}$ on which π_1 of the corresponding affine curves is constant? Or, at the other extreme, does $\pi_1(U)$ essentially determine the curve U? In particular, if $\pi_1(X_1) \approx \pi_1(X_2)$, where X_i is a curve of genus g_i with $r_i > 0$ points deleted, then must $g_1 = g_2$ and $r_1 = r_2$? Also, must X_1 and X_2 be isomorphic over the prime field? If k is the algebraic closure of a finite field, a more precise version of this question is given by Grothendieck's "anabelian conjecture," which here says:

Conjecture. [Gr3] Is an affine curve X over $\mathbf{F}_q$ determined up to $\mathbf{F}_q$-isomorphism by $\pi_1(X)$ together with the surjective homomorphism $\pi_1(X) \to \mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$?

An analogous result of Nakamura [Nm] provides support for this: Two open subsets of $\mathbf{P}^1_{\mathbf{Q}}$ are isomorphic if and only if their fundamental groups are isomorphic as $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-modules. Also, birational versions of the conjecture have been proven by Uchida [Uc] and Pop [Po1], and a birational version for number fields (rather than for function fields of curves, as above) is due to Neukirch [Ne].

4.3: *Projective case.* Although there is no conjecture describing π_{A} of a projective curve X of genus $g > 1$, Grothendieck [Gr2, XIII, Cor. 2.12] showed that $\pi_1(X)$ is some (unknown) quotient of π_1 of a complex curve of genus g, and he gave an explicit presentation of the maximal prime-to-p quotient of $\pi_1(X)$. Thus $\pi_1(X)$ is finitely generated, and so is determined by $\pi_{\mathrm{A}}(X)$ [FJ, Proposition 15.4].

For a given genus $g \geq 1$, curves with unequal p-rank (Hasse-Witt invariant) have distinct π_1's; while for $g \geq 2$, even the genus and p-rank do not determine π_1 (or π_{A}) [Kt], [Nj1]. Also, Nakajima [Nj2] has found a necessary condition for a group G to lie in $\pi_{\mathrm{A}}(g) = \{G \in \pi_{\mathrm{A}}(X) \mid \mathrm{genus}(X) = g\}$, viz. that the ideal $\{\sum_{\gamma \in G} a_\gamma \cdot \gamma \mid \sum a_\gamma = 0\} \subset k[G]$ has g generators.

Recently, formal and rigid patching methods (as in sections 2 and 3) have been used to obtain more information about $\pi_{\mathrm{A}}(g)$. In their 1994 theses, K. Stevenson [St] and M. Saïdi [Sa] have found quotients of π_1 that are "bigger" than the profinite group on g generators. In particular, $\pi_{\mathrm{A}}(g') \supset \pi_{\mathrm{A}}(g)$ whenever $g' \geq g$; and $\pi_{\mathrm{A}}(g)$ contains all finite groups that have g generators (e.g. all finite simple groups, if $g \geq 2$), among others.

4.4: *Embedding problems.* Given a finite group G, a quotient map $G \to H$, and an H-Galois unramified cover of k-curves $Y \to X$, we can ask if there is a G-Galois unramified cover $Z \to X$ inducing $Y \to X$. It is necessary that $G \in \pi_{\mathrm{A}}(X)$, but this is not sufficient; cf. the proof of Serre's result on AC for solvable groups (see section 3.1 above) in the split case with N an elementary abelian l-group, $l \neq p$.

On the other hand, if we instead permit *branched* covers, then this embedding problem can always be solved [Ha8], [Po3] using a patching construction (in fact, with some control on the additional branching). Moreover, for each such embedding problem, the cardinality of the set of solutions is equal to that of the base field k. As a result, the absolute Galois group of the function field of X is a free profinite group. This proves the function field version of a conjecture of Shafarevich: If K is a global field, then the absolute Galois group of its maximal cyclotomic extension is free profinite. This conjecture remains open in the number field case.

4.5: *Other base fields.* Let Φ be the class of fields K such that every finite group is the Galois group of a (geometrically irreducible) Galois branched cover of $\mathbf{P}^1_K$. It is classical that $\mathbf{C} \in \Phi$. By [Gr2, XIII, Cor. 2.12] and Abhyankar's Conjecture, every algebraically closed field is in Φ. Earlier [Ha2], this was shown (with less control on branching) by formal patching. Similarly [Ha3], the author showed that if R is the completion (or henselization) of a domain at a non-zero maximal ideal, then $K = \mathrm{frac}(R)$ is in Φ. In particular, $\mathbf{Q}_p$ and the algebraic p-adic field lie in Φ, as do $k((t))$ and the algebraic Laurent series field (for any field k).

Many other fields lie in Φ, including the fields $\mathbf{Q}^{\mathrm{tr}}$ of totally real [DF] and $\mathbf{Q}^{\mathrm{t}p}$ of totally p-adic [De] algebraic numbers, as well as PAC fields (see [FV] in the characteristic 0 case). More generally, $k \in \Phi$ (and even a stronger condition holds, concerning embedding problems [Po2, Thm. 1.5]) if k is *existentially closed* in $k((t))$, or equivalently if every geometrically irreducible k-variety with a $k((t))$-point has a k-point. (PAC fields are trivially existentially closed; $\mathbf{Q}^{\mathrm{tr}}$ and $\mathbf{Q}^{\mathrm{t}p}$ are by [GPR,1.4] and [Po2, Lemma 1.8].) The reason is that $k((t)) \in \Phi$, so there is a domain $A \subset k((t))$ of finite type over k and a G-Galois cover $Z \to \mathbf{P}^1_A$ whose k-fibres $Z_0 \to \mathbf{P}^1_k$ are irreducible. Since $A \subset k((t))$, the k-variety $\mathrm{Spec}(A)$ is geometrically irreducible, and taking a k-point yields that $k \in \Phi$.

Combining model theory with the above fact that PAC fields lie in Φ yields the following conclusion (observed by Jarden, Fried-Völklein, and Pop): If G is a finite group, then G is the Galois group of a branched cover of $\mathbf{P}^1_F$ for all but finitely many finite fields F. But it remains unknown whether finite fields lie in Φ.

Similarly, it is unknown if number fields lie in Φ. But by "rigidity," Matzat, Belyi, Thompson, Feit, Fried, Malle, Völklein et al. have realized many finite groups as Galois groups over $\mathbf{P}^1_{\mathbf{Q}}$ and hence over $\mathbf{Q}$. See [Se2, Chap. 8] for more details.

Another approach to the problem over $\mathbf{Q}$ [Ha4] used formal patching to find, for G any finite group, G-Galois (ramified) extensions of domains over $\mathbf{Z}[[t]]$ and $\mathbf{Z}\{t\} := \{f \in \mathbf{Z}[[t]] \mid f \text{ converges on } |t| < 1\}$. (These rings are analogous to $k[x][[t]]$ and $k[[t]][x]$.) Such a G-Galois extension of $\mathbf{Z}\{t\}$ induces G-Galois extensions of $\mathbf{Z}_{r+}[[t]] := \{f \in \mathbf{Z}[[t]] \mid f \text{ holomorphic on } |t| \le r\}$ for all $0 < r < 1$, and these descend to a compatible system of G-Galois extensions of the subrings $\mathbf{Z}_{r+}[[t]]^{\mathrm{h}}$ of algebraic power series. It is tempting to expect that these extensions are induced by a G-Galois extension of $\mathbf{Z}\{t\}^{\mathrm{h}}$, the ring of algebraic power series in $\mathbf{Z}\{t\}$. Since $\mathbf{Z}\{t\}^{\mathrm{h}}$ is a subring of $\mathbf{Q}(t)$, this would imply that $\mathbf{Q} \in \Phi$. Unfortunately not all such systems of extensions descend to $\mathbf{Z}\{t\}^{\mathrm{h}}$, but it would suffice to have at least one such system descend for each G. Cf. [Ha4].

Given the fields that are known to be in Φ, and the expectation that number fields and finite fields are in Φ, the following conjecture seems reasonable:

Conjecture. Every field lies in Φ. Hence every finite group is a Galois group over every field of the form $K(x)$, and also over every Hilbertian field.

References.

[Ab1] S. Abhyankar. Coverings of algebraic curves. Amer. J. Math. **79** (1957), 825-856.

[Ab2] S. Abhyankar. Galois theory on the line in nonzero characteristic. Bull. Amer. Math. Soc. (New Ser.) **27** (1992), 68-133.

[BL] S. Bosch, W. Lütkebohmert. Formal and rigid geometry. Part I: Math. Ann. **295** (1993), 291-317. Part II: Math. Ann. **296** (1993) 403-429.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud. Formal and rigid geometry. Parts III and IV. 1994 manuscripts.

[De] P. Debes. G-covers of $\mathbf{P}^1$ over the p-adics. 1993 preprint, to appear in "Recent Developments in the Inverse Galois Problem" (M. Fried, ed.), AMS Contemporary Math. series.

[DF] P. Debes, M. Fried. Nonrigid constructions in Galois theory. Pac. J. Math. **163** (1994), 81-122.

[FJ] M. Fried, M. Jarden. "Field Arithmetic." Springer-Verlag, Berlin-Heidelberg, 1986.

[FV] M. Fried, H. Völklein. The embedding problem over a Hilbertian PAC field. Annals of Math. **135** (1992), 469-481.

[GPR] B. Green, F. Pop, P. Roquette. On Rumely's local-global principle. 1993 manuscript, Heidelberg.

[Gr1] A. Grothendieck. Eléments de géométrie algébrique (EGA). Publ. Math. IHES **4, 11** (1960, 1961).

[Gr2] A. Grothendieck. Revêtements étales et groupe fondamental (SGA 1). Lecture Notes in Math. **224**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[Gr3] A. Grothendieck. Esquisse d'un programme. 1984 manuscript.

[Ha1] D. Harbater. Moduli of p-covers of curves. Communications in Algebra **8** (1980), 1095-1122.

[Ha2] D. Harbater. Mock covers and Galois extensions. J. Algebra **91** (1984), 281-293.

[Ha3] D. Harbater. Galois covers of the arithmetic line. In "Number Theory – New York 1984-85", Lecture Notes in Math. vol. 1240 (D.V. and G.V. Chudnovsky, eds.), 165-195, Springer-Verlag, New York, 1987.

[Ha4] D. Harbater. Galois covers of an arithmetic surface. Amer. J. Math. **110** (1988), 849-885.

[Ha5] D. Harbater. Formal patching and adding branch points. Amer. J. Math. **115** (1993), 487-508.

[Ha6] D. Harbater. Abhyankar's conjecture on Galois groups over curves. Invent. Math. **117** (1994), 1-25.

[Ha7] D. Harbater. Galois groups with prescribed ramification. 1993 preprint, to appear in "Arithmetic Geometry" (N. Childress and J. Jones, eds.), AMS Contemporary Math. series.

[Ha8] D. Harbater. Fundamental groups and embedding problems in characteristic p. 1993 preprint, to appear in "Recent Developments in the Inverse Galois Problem" (M. Fried, ed.), AMS Contemporary Math. series.

[Ka] T. Kambayashi. Nori's construction of Galois coverings in positive characteristics. Algebraic and Topological Theories, Tokyo (1985), 640-647.

[Kt] H. Katsurada. Generalized Hasse-Witt invariants and unramified Galois extensions of an algebraic function field. J. Math. Soc. Japan **31** (1979), 101-125.

[Ki] R. Kiehl. Der Endlichkeitssatz für eigentliche Abbildungen in der nichtarchimedischen Funktionentheorie. Invent. Math. **2** (1967), 191-214.

[Me] F. Mehlmann. Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoïder Algebren. Schriftenr. Math. Inst. Univ. Münster, 2. Ser. **19** (1981).

[Nj1] S. Nakajima. On generalized Hasse-Witt invariants of an algebraic curve. In "Galois groups and their representations," Advanced Studies in Pure Math. **2** (Y. Ihara, ed.), 69-88, North Holland, 1983.

[Nj2] S. Nakajima. On Galois module structure of the cohomology groups of an algebraic variety. Invent. Math. **75** (1984), 1-8.

[Nm] H. Nakamura. Galois rigidity of the étale fundamental groups of punctured projective lines. J. Reine Angew. Math. **411** (1990), 205-216.

[Ne] J. Neukirch. Kennzeichnung der p-adischen und endlichen algebraischen Zahlkörper. Invent. Math. **6** (1969), 269-314.

[Po1] F. Pop. On Grothendieck's conjecture of birational anabelian geometry. Annals of Math. **138** (1994), 145-182.

[Po2] F. Pop. Hilbertian fields with a universal local-global principle. 1993 manuscript, Heidelberg.

[Po3] F. Pop. Étale Galois covers of smooth affine curves. 1994 manuscript, Heidelberg.

[Ra1] M. Raynaud. Géométrie analytique rigide d'après Tate, Kiehl .... Mém. Soc. Math. Fr. **39-40** (1974), 319-327.

[Ra2] M. Raynaud. Revêtements de la droite affine en caractéristique $p>0$ et conjecture d' Abhyankar. Invent. Math. **116** (1994), 425-462.

[Sa] M. Saïdi. Revêtements étales et groupe fondamental de graphes de groupes. C. R. Acad. Sci. Paris **318** Série I (1994), 1115-1118.

[Se1] J.-P. Serre. Construction de revêtements étales de la droite affine en caractéristique p. Comptes Rendus **311** (1990), 341-346.

[Se2] J.-P. Serre. "Topics in Galois Theory." Jones and Bartlett, Boston-London, 1992.

[St] K. Stevenson. Fundamental groups of projective curves in characteristic p. University of Pennsylvania Ph. D. thesis, 1994.

[Ta] J. Tate. Rigid analytic spaces. Invent. Math. **12** (1971), 257-289.

[Uc] K. Uchida. Isomorphisms of Galois groups of algebraic function fields. Annals of Math. **106** (1977), 589-598.