FINITENESS OF FORMAL PUSHFORWARDS

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Abstract. Under mild hypotheses, given a scheme U and an open subset V whose complement has codimension at least two, the pushforward of a torsion-free coherent sheaf on V is coherent on U. We prove an analog of this result in the context of formal schemes over a complete discrete valuation ring. We then apply this to obtain a result about gluing formal functions, where the patches do not cover the entire scheme.

1. Introduction

If $j : V \to U$ is an inclusion of an open subscheme of a scheme U, then the map j_*, which carries sheaves of modules on V to sheaves of modules on U, preserves quasi-coherence but not necessarily coherence. For example, if U is the affine x-line over a field k, and V is the complement of the origin, then $j_*(\mathcal{O}_V)$ is not coherent because its global sections are $k[x, x^{-1}]$, which is not finite over $\mathcal{O}(U) = k[x]$.

But for a normal connected quasi-projective variety U, if the sheaf if torsion-free and the complement of V in U has codimension at least two, then coherence is preserved under pushforward (see Theorem 2.1, where the hypotheses on U are weaker). In this paper, we prove the following analogous result in the context of formal schemes over a complete discrete valuation ring T.

Theorem (see Theorem 6.6). *Let $\mathscr{X}$ be a normal connected quasi-projective T-scheme, and let $f : V \hookrightarrow U$ be an inclusion of non-empty open subsets of the reduced closed fiber of $\mathscr{X}$ such that the complement of V in U has codimension at least two in U. Write $\mathfrak{U}, \mathfrak{V}$ for the formal completions of $\mathscr{X}$ along U, V. If $\mathscr{F}$ is a torsion-free coherent sheaf on $\mathfrak{V}$, then $\widehat{f}_*(\mathscr{F})$ is a torsion-free coherent sheaf on $\mathfrak{U}$.*

A motivation for proving this result comes from patching problems for modules. Such problems arise, for example, in the context of an affine open cover of an affine scheme or formal scheme, where one gives compatible finite modules over the ring of functions on these subsets, and asks for a finite module over the ring of global functions that induces the data compatibly. Patching problems have been useful in obtaining results in Galois theory and local-global principles; e.g., see [Ha94], [HH10], [HHK09]. Those papers considered projective curves over complete discrete valuation rings and their function fields. In that situation, the closed fiber (which is the underlying topological space of the associated formal scheme) can be covered by just two affine open subsets. As a result, in patching formal modules on open subsets to obtain a global formal module, one can avoid the difficulty of having to satisfy cocycle conditions arising from triple overlaps. On the other hand, in higher dimensional

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cases, a quasi-projective variety need not have an open covering by just two affine open subsets. But on any quasi-projective variety, one can find two affine open subsets such that the complement of their union has codimension two. As a consequence, the result we prove here makes it possible to patch finite torsion-free formal modules on those affine open sets, thereby obtaining a global finite torsion-free module that restricts to the given formal modules on the affine open sets, without having to satisfy cocycle conditions. Namely, via Theorem 6.6, we prove in Corollary 8.3 that there is a unique maximum torsion-free solution to a patching problem of finite torsion-free formal modules defined away from codimension two, and that the solution is given by intersection. Moreover, in Corollary 8.4 we show that in the flat (or equivalently, locally free) case, this solution is unique.

Structure of the manuscript: We provide background and context in Section 2, followed by two commutative algebra results in Section 3, and general results on formal schemes and formal patches in Section 4. Using that material, in Section 5 we obtain a key result (Proposition 5.3) that asserts that the intersection of two finitely generated torsion-free formal modules is also finitely generated under a codimension two hypothesis on the complement of the union. In Section 6 we first show that for formal schemes, as for schemes, pushforward preserves quasi-coherence. Afterwards we obtain Theorem 6.6, mentioned above, in which the key property to prove is finiteness. A version of that result with a stronger conclusion is proven in Section 7 in the situation in which the modules are assumed to be flat, rather than just being torsion-free; see Theorem 7.4. Finally, in Section 8, patching problems are discussed, and Corollaries 8.3 and 8.4 shown.

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2. Background and context

We begin by fixing some terminology.

Following [EGA4, Partie 2, Proposition 5.1.2] and [EGA4, Partie 1, Chapter 0, Définition 14.2.1], if X is a scheme then the *codimension* of a closed subscheme $Y \subseteq X$ is the infimum $\operatorname{codim}_X(Y)$ of the Krull dimensions of the local rings $\mathcal{O}_{X,y}$ over $y \in Y$; this is also the infimum of the codimensions of the irreducible components of Y. Under this definition, the codimension of the empty set is infinite. Given closed subschemes $Z \subseteq Y \subseteq X$, we have $\operatorname{codim}_X(Z) \geq \operatorname{codim}_Y(Z) + \operatorname{codim}_X(Y)$.

Given a commutative ring R (not necessarily a domain), recall that an R-module M is *torsion-free* if no regular element of R annihilates any non-zero element of M; or equivalently, if $M \to M \otimes_R K$ is injective, where K is the total ring of fractions of R. E.g., see [Vas68, Section 1]. As in [EGA4, Partie 4, 20.1.5], a sheaf of modules $\mathcal{F}$ on a scheme X is *torsion-free* if the natural homomorphism $\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{M}_X$ is injective; here $\mathcal{M}_X$ is the sheaf of meromorphic functions on X. This is equivalent to the condition that $\mathcal{F}(U)$ is a torsion-free $\mathcal{O}_X(U)$-module for every affine open subset U of X; thus torsion-freeness is local. By [EGA4, Partie 4, Proposition 20.1.6], being torsion-free is also equivalent to the condition that every associated point of $\mathcal{F}$ is an associated point of $\mathcal{O}_X$. (Recall from [EGA4, Partie 2, Définition 3.1.1] that a point x of X is an *associated point* of $\mathcal{F}$ if the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is an associated prime of the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$; i.e., is the annihilator of an element of $\mathcal{F}_x$.)

Rings that one typically encounters tend to be excellent, meaning that several mild but technical conditions hold. Specifically, a *G-ring* is a Noetherian ring R such that the map $R_{\mathfrak{p}} \to \widehat{R_{\mathfrak{p}}}$ is regular for every prime ideal $\mathfrak{p}$ of R, where $\widehat{R_{\mathfrak{p}}}$ is the completion of the local ring $R_{\mathfrak{p}}$. A Noetherian G-ring with the J-2 property (see [Mat80, 32.B]) is called *quasi-excellent*, and a quasi-excellent ring that is universally catenary (see [Mat80, 14.B]) is called *excellent* (see [Sta25, Definition 07QT].) By [Mat80, Theorem 78]), quasi-excellent rings are Nagata rings (see the definition at [Mat80, 31.A]). Noetherian complete local rings are excellent, and excellence is preserved under localizing and under passage to a finitely generated algebra (see [Mat80, Section 34]). A scheme is *excellent* if it can be covered by affine open subsets U_i such that each of the rings $\mathcal{O}_X(U_i)$ is excellent (see [Liu02, Definition 8.2.35]); these are automatically locally Noetherian. One similarly defines schemes that are Nagata, are universally catenary, etc.

If $f : V \to U$ is a quasi-compact and quasi-separated morphism of schemes (e.g., an inclusion of Noetherian schemes), and if $\mathcal{F}$ is quasi-coherent on V, then $f_*(\mathcal{F})$ is quasi-coherent on U (see [Sta25, Lemma 01LC]). For coherent modules, there is the following result, which is known to the experts, and which is essentially a special case of [EGA4, Partie 2, Corollaire 5.11.4(ii)] and [Sta25, Lemma 0AWA] (as Johan de Jong pointed out to us). Note that this theorem holds in particu1lar in the case mentioned in the introduction, viz., of a normal connected quasi-projective scheme U, since normal (and integral) schemes are reduced, and since quasi-projective varieties are excellent (by [Mat80, Section 34]).

Theorem 2.1. *Let U be a reduced scheme that is excellent (or more generally, Nagata and universally catenary). Let $j : V \hookrightarrow U$ be the inclusion of an open subset such that the complement of V in U has codimension at least two in U. Then for any torsion-free coherent sheaf $\mathcal{F}$ on V, the pushforward $j_*(\mathcal{F})$ is a torsion-free coherent sheaf on U.*

Proof. As noted above, for $\mathcal{F}$ a torsion-free coherent sheaf on V, the associated points of $\mathcal{F}$ are also associated points of $\mathcal{O}_V$, or equivalently of V. Note that U is locally Noetherian, being Nagata. Since V is an affine open subset of the reduced scheme U, both V and its ring of functions $\mathcal{O}_U(V)$ are reduced, by [Sta25, Lemmas 01J1, 01J2]. So by [Sta25, Lemmas 0EMA, 05AR], the associated points of V are those of codimension zero. Thus this holds for the associated points of $\mathcal{F}$.

By hypothesis, the complement Z of V in U has codimension at least two in U. Also, by the previous paragraph, for every associated point x of $\mathcal{F}$, the closure $\overline{\{x\}}$ of $\{x\}$ in U is an irreducible component of U. It then follows that for every associated point x of $\mathcal{F}$, the codimension of $Z \cap \overline{\{x\}}$ in $\overline{\{x\}}$ is at least 2; or equivalently, $\dim(\mathcal{O}_{\overline{\{x\}},z}) \geq 2$ for every $z \in Z \cap \overline{\{x\}}$.

As a consequence, since U is Nagata and universally catenary, we obtain that $j_*(\mathcal{F})$ is coherent on U, by applying [Sta25, Lemma 0AWA] (or alternatively [EGA4, Partie 2, Corollaire 5.11.4(ii)]; see also [Sta25, Proposition 0334]).

Next, we show that $j_*(\mathcal{F})$ is torsion-free on U; i.e., $j_*(\mathcal{F})(O)$ is a torsion-free $\mathcal{O}_U(O)$-module for every affine open subset O of U. Since the torsion-free property is local, we may assume that U is the spectrum of a reduced ring R, and prove that $j_*(\mathcal{F})(U)$ is a torsion-free R-module. Note that by [Sta25, Lemmas 0EMA, 05C3], the set of zero-divisors in R is the union of the minimal primes of R; or equivalently, the set of elements of R that vanish at the generic point of some irreducible component of U.

Now let m be a non-zero element of $M := j_*(\mathcal{F})(U) = \mathcal{F}(V)$ and let r be a regular element of R. We wish to show that $rm \neq 0$. Since m is non-zero in $\mathcal{F}(V)$, there is a non-empty affine open subset $V' = \mathrm{Spec}(R') \subseteq V \subseteq U$ such that the restriction m' of m from V to V' is non-zero in $\mathcal{F}(V')$. Since r is regular in R (i.e., not a zero-divisor), it does not vanish at the generic point of any irreducible component of U. Thus the image r' of r in R' does not vanish at the generic point of any irreducible component of $V' = \mathrm{Spec}(R')$ (since the latter set of generic points is contained in the former set). As above, since U is reduced, the ring of functions R' on the affine open subset $V' \subseteq U$ is reduced. So r' is a regular element of R'. But $\mathcal{F}(V')$ is a torsion-free module over $R' = \mathcal{O}_V(V')$, since $\mathcal{F}$ is a torsion-free sheaf on V. Hence $r'm' \neq 0$ in $\mathcal{F}(V')$. Since $r'm'$ is the image of rm under the restriction map $M = \mathcal{F}(V) \to \mathcal{F}(V')$, it follows that $rm \neq 0$ in M, as needed. $\square$

To illustrate the role of the torsion-free condition on coherent sheaves here (or more generally, the condition on associated points), let V be the complement of the origin in the affine x, y-plane U over a field k, and let $\mathcal{F} = j^*(\mathcal{O}/\mathcal{I})$, where $\mathcal{I}$ is the sheaf of ideals on U induced by the ideal $(y) \subset k[x, y] = \mathcal{O}(U)$. The pushforward $j_*\mathcal{F}$ is not coherent on U, since again its global sections are $k[x, x^{-1}]$. Here the complement Z of V in U has codimension two, but $\mathcal{F}$ is not torsion-free, since it is y-torsion, with (y) an associated point. Moreover Z is of codimension one (not two) in the closure of the associated point (y).

In Section 6, we consider the analogous situation of pushforwards of quasi-coherent and coherent sheaves of $\mathcal{O}_{\mathfrak{U}}$-modules on a formal scheme $\mathfrak{U}$ over a complete discrete valuation ring T. See Proposition 6.1 for the quasi-coherent sheaf result and Theorem 6.6 for the coherent sheaf result. In the coherent formal situation, we again assume that the sheaf is torsion-free, meaning that its sections over each affine open set V of the underlying space U form a torsion-free module over $\mathcal{O}_{\mathfrak{U}}(V)$. Without the torsion-free assumption, one can construct counterexamples similar to the one above, by taking the t-adic completion of the base change of the above example from k to $k[[t]]$.

The proof for quasi-coherent formal sheaves parallels the proof for quasi-coherent sheaves on schemes. But the proof for coherent sheaves in the formal situation is more involved than the proof over schemes. Namely, suppose we are given a torsion-free coherent sheaf $\mathscr{F}$ on a formal scheme $\mathfrak{U}$ as above, with U_n being the n-th thickening of the reduced closed fiber. It is tempting to try to apply the scheme-theoretic result [Sta25, Lemma 0AWA] (or [EGA4, Partie 2, Corollaire 5.11.4(ii)]) to the pullback $\mathcal{F}_n$ of $\mathscr{F}$ to each U_n, and to use that a coherent sheaf on $\mathfrak{U}$ corresponds to an inverse system of coherent sheaves on the schemes U_n that has surjective transition functions (see [Sta25, Lemma 087W]). But the difficulty is that $\mathcal{F}_n$ need not be torsion-free, and may have new associated points of positive codimension in U_n; and this would prevent the use of the above results. (See also Remark 6.7.) Instead, in Section 5, we follow a strategy that relies on the commutative algebra lemmas proven in Section 3; and we build on that in proving Theorem 6.6.

3. Two general lemmas

Before turning to formal schemes, we prove some general results. The proof of the first lemma was outlined for us by Craig Huneke in the case that I is prime.

Lemma 3.1. *Let R be a G-ring that is a normal domain, let I be a proper ideal in R, and let M be a finitely generated torsion-free R-module. Let $P_1, \ldots, P_s$ be the minimal primes over I.*

(a) For every $i \geq 0$ there is an $n \geq 0$ such that $M \cap P_1^n M_{P_1} \cap \cdots \cap P_s^n M_{P_s} \subseteq I^i M$.
(b) In particular, for every integer $c \geq 0$ there is some $n \geq 0$ such that if $r \in R$ and $m \in M$ satisfy $rm \in I^n M$ then either $r \in P_j$ for some j or $m \in I^c M$.

Proof. The radical $\sqrt{I}$ of I is the ideal $P_1 \cap \cdots \cap P_s$, and by [AM69, Proposition 7.14] there is an integer α such that $\sqrt{I}^{\alpha} \subseteq I$. Thus for part (a), it suffices to prove the assertion with I replaced by $\sqrt{I}$. So we will assume that I is the intersection of the prime ideals P_i, and will proceed by induction on s.

If $s = 1$, then I is a prime ideal P. First consider the special case that $M = R$. In this situation, for each positive integer n, $M \cap P^n M_P$ is just the n-th symbolic power $P^{(n)} := R \cap P^n R_P$ of P. Since R is a normal G-ring, the completion $\widehat{R}_Q$ of R at each prime ideal $Q \subset R$ is also normal, by [Mat80, 33.I]. Since $\widehat{R}_Q$ is normal and local, it is a domain, and its only associated prime is (0). Since this holds for all Q, [Sch85, Theorem 1] asserts that the P-adic topology on R defined by the ideals P^n is equivalent to the P-symbolic topology defined by the ideals $P^{(n)}$. (Namely, the condition in part (ii) of that theorem holds because the annihilator ideals Q considered there properly contain P, and the only associated prime of the complete local ring at Q is (0).) Hence part (a) follows in this special case.

Next, still with $s = 1$ and $I = P$, consider a more general finitely generated torsion-free R-module M. By [Sta25, Lemma 0AUU], M is contained in a finitely generated free R-module E. By the Artin-Rees lemma (e.g., [Sta25, Lemma 00IN]), there is a positive integer d such that for every $e \geq d$, $M \cap P^e E = P^{e-d}(M \cap P^d E) \subseteq P^{e-d} M$. Take $i \geq 0$. By the previous paragraph, there exists $n \geq 0$ such that $R \cap P^n R_P \subseteq P^{i+d}$. Thus the free module E satisfies $E \cap P^n E_P \subseteq P^{i+d} E$. Here $M \subseteq E$ and so $M_P \subseteq E_P$. Hence

$$M \cap P^n M_P = M \cap E \cap P^n M_P \subseteq M \cap E \cap P^n E_P \subseteq M \cap P^{i+d} E \subseteq P^i M,$$

at the last step using Artin-Rees with $e = i + d$. This proves the case $s = 1$.

For the inductive step, take $I = P_1 \cap \cdots \cap P_s$, and assume that the assertion holds for $J := P_1 \cap \cdots \cap P_{s-1}$. Here $I = P_s \cap J$. We will prove that for every i there is an n such that $M \cap P_1^n M_{P_1} \cap \cdots \cap P_s^n M_{P_s} \subseteq I^i M$. So take some $i \geq 0$. By the inductive hypothesis, there is an $n' \geq 0$ such that $M \cap P_1^{n'} M_{P_1} \cap \cdots \cap P_{s-1}^{n'} M_{P_{s-1}} \subseteq J^i M$. By the above case of $s = 1$ applied to the finitely generated torsion free module $J^i M$ and the ideal P_s, there is some $m \geq 0$ such that $J^i M \cap P_s^m J^i M_{P_s} \subseteq P_s^i J^i M$. Since $P_1, \ldots, P_s$ are the (distinct) minimal primes over I, no P_j is contained in P_s for $j < s$. Thus $J = P_1 \cap \cdots \cap P_{s-1}$ is also not contained in P_s, by [AM69, Proposition 1.11(ii)]. Hence JR_{P_s} is the unit ideal of R_{P_s}, and $J^i M_{P_s} = M_{P_s}$. We now have $M \cap P_1^{n'} M_{P_1} \cap \cdots \cap P_{s-1}^{n'} M_{P_{s-1}} \cap P_s^m M_{P_s} \subseteq J^i M \cap P_s^m M_{P_s} = J^i M \cap P_s^m J^i M_{P_s} \subseteq P_s^i J^i M = (P_s J)^i M \subseteq (P_s \cap J)^i M = I^i M$. Let $n = \max(n', m)$. Thus $M \cap P_1^n M_{P_1} \cap \cdots \cap P_s^n M_{P_s} \subseteq M \cap P_1^{n'} M_{P_1} \cap \cdots \cap P_{s-1}^{n'} M_{P_{s-1}} \cap P_s^m M_{P_s} \subseteq I^i M$, and this concludes the inductive proof of part (a).

For part (b), let $n \geq 0$ be associated to the value $i = c$ as in part (a). Suppose $r \in R$ and $m \in M$ satisfy $rm \in I^n M$. Thus $rm \in P_j^n M_{P_j}$ for all j. If r does not lie in any P_j, then r is a unit in each R_{P_j} and so $m \in P_j^n M_{P_j}$ for all j. Hence $m \in M \cap P_1^n M_{P_1} \cap \cdots \cap P_s^n M_{P_s} \subseteq I^c M$ by part (a). $\square$

Lemma 3.2. *Let R be a normal G-ring that is complete with respect to a non-zero principal ideal $I = (t)$, and let M be a non-zero finitely generated torsion-free R-module. Let $P_1, \dots, P_s$ be the minimal primes over I, and for each $j = 1, \dots, s$ and $i \geq 1$ write $P_j R_i$ for the image of P_j in $R_i := R/I^i$. For each $i \geq 1$ also write $M_i = M/I^i M$, and let Q_i be the set of elements q of the R_i-module M_i such that $\operatorname{ann}(q)$ is not contained in any of the ideals $P_1 R_i, \cdots, P_s R_i$. Then the following hold.*

(a) Q_i is an R_i-submodule of M_i, and each $q \in Q_i$ satisfies $\operatorname{ann}(q) \not\subseteq P_1 R_i \cup \cdots \cup P_s R_i$.
(b) Every associated prime of the R_i-module $N_i := M_i/Q_i$ is of the form $P_j R_i$ with $1 \leq j \leq s$.
(c) The inverse system $\{M_i\}$ induces inverse systems $\{Q_i\}$ and $\{N_i\}$ by restriction and quotient.
(d) If $m_i \in M_i$ and $t^c m_i \in Q_i$ for some $c < i$, then the image of m_i in M_{i-c} lies in Q_{i-c}.
(e) There is a positive integer n such that for every i, $Q_{i-1+n} \to Q_i$ is the zero map.
(f) $\varprojlim M_i = M$, $\varprojlim Q_i = 0$, and $\varprojlim N_i = M$.

Proof. Recall that every Noetherian normal ring is a finite product of Noetherian normal domains; see [Sta25, Lemma 030C]. Hence we may write $R \cong R^{(1)} \times \cdots \times R^{(s)}$, where each factor is a Noetherian normal domain; and correspondingly, we have $M_i \cong M_i^{(1)} \times \cdots \times M_i^{(s)}$, $Q_i \cong Q_i^{(1)} \times \cdots \times Q_i^{(s)}$, and $N_i \cong N_i^{(1)} \times \cdots \times N_i^{(s)}$. Here the associated primes of N_i are the union of the associated primes of $N_i^{(1)}, \dots, N_i^{(s)}$. Thus in order to prove the lemma in general, it suffices to prove it in the special case in which R is a domain, by applying the special case to each factor. Here, for the proof of part (e), we can take n to be the maximum of the values $n^{(1)}, \dots, n^{(s)}$ corresponding to the factors.

So for the remainder of the proof we assume that R is a domain.

Since each ideal $P_j \subset R$ is prime and contains I, it follows that $P_j R_i \subset R_i = R/I^i$ is also prime. Thus if $q \in Q_i$ then $\operatorname{ann}(q)$ is not contained in $\Pi_i := P_1 R_i \cup \cdots \cup P_s R_i$, by prime avoidance. Now take $q_1, q_2 \in Q_i$. Since $\operatorname{ann}(q_i) \not\subseteq \Pi_i$, there exist elements $r_1, r_2 \in R_i \smallsetminus \Pi_i$ that annihilate q_1, q_2 respectively. So $r_1 r_2 \in R_i$ is not in Π_i and it annihilates $q_1 + q_2$. Also, for any $r \in R_i$, the above element $r_1 \in R_i \smallsetminus \Pi_i$ annihilates rq_1. Hence $q_1 + q_2$ and rq_1 lie in Q_i. So Q_i is an R_i-submodule of M_i, proving (a).

We claim that for every non-zero element $\bar{m} = m + Q_i \in N_i$, with $m \in M_i$, the annihilator of $\bar{m}$ is contained in one of the ideals $P_1 R_i, \cdots, P_s R_i \subset R_i$. Again by prime avoidance, this is equivalent to the assertion that this annihilator is contained in the above set Π_i. To prove that this containment holds, suppose that $\bar{m} \in N_i$ does not have this property; i.e., there exists $r \in R_i$ that is not in Π_i and such that $rm \in Q_i$. Thus, as in the previous paragraph, there exists $s \in R_i$ such that $srm = 0 \in M_i$ and s is not in Π_i. But then $sr \in R_i$ is also not in Π_i. So $m \in Q_i$ and thus $\bar{m} = 0$. This proves the claim.

So every associated prime of N_i is contained in some $P_j R_i$, $j = 1, \dots, s$. But each P_j is a minimal prime over I and hence over I^i; thus $P_j R_i$ is a minimal prime of R_i. Therefore every associated prime of the R_i-module N_i is among $P_1 R_i, \cdots, P_s R_i$, proving (b).

Since $(t) \subseteq P_i$, the surjection $M_i \to M_{i-1}$ restricts to a map $Q_i \to Q_{i-1}$, and so the modules Q_i form an inverse system. It follows that the maps $Q_i \to Q_{i-1}$ yield well-defined surjections $N_i \to N_{i-1}$, so that the modules N_i also form an inverse system. This proves (c).

By [Sta25, Lemma 00MA, (3)], we have $M = M \otimes_R R = M \otimes_R \varprojlim R_i = \varprojlim M_i$. This proves the first part of (f).

For part (d), by induction we are reduced to the case that $c = 1$, with $i \geq 2$. So suppose that $m_i \in M_i$ and $tm_i \in Q_i$. Let $m \in M$ be an element such that m_i is the image of m in M_i. By definition of Q_i, there exists $r_i \in R$ such that $r_i tm \in t^i M$ and $\bar{r}_i \notin P_1 R_i, \ldots, P_s R_i$, where $\bar{r}_i \in R_i$ is the image of r_i. Thus $r_i \notin P_1, \ldots, P_s$; and since each P_j contains t, the image of r_i in R_{i-1} is not in any $P_j R_{i-1}$. Write $r_i tm = t^i m'$ for some $m' \in M$. Since M is torsion-free, and since the non-zero element t is regular (because R is a domain), it follows that $r_i m = t^{i-1} m'$; hence the image of m_i in M_{i-1} lies in Q_{i-1}. This proves (d) in the case $c = 1$, and hence in the general case.

Next, we show that (e) holds for the integer n obtained by setting $c = 1$ in Lemma 3.1(b). We first treat the case of (e) where $i = 1$; i.e., we show that the image of $Q_n \to Q_1$ is trivial. Namely, given a non-zero element $m_n \in Q_n \subseteq M_n$, we may choose $m \in M$ lying over m_n; and then there exists $r \in R$ such that $rm \in t^n M$ and $r \notin P_1, \ldots, P_s$ (as in the previous paragraph). By the defining property of n, it follows that $m \in tM$. Hence the image of m in M_1 is trivial. But this element is the same as the image of m_n in Q_1; and so this proves (e) in the case $i = 1$.

For a more general value of i in the assertion of (e), suppose for the sake of contradiction that $m_{n+i-1} \in Q_{n+i-1} \subseteq M_{n+i-1}$ is an element whose image $m_i \in Q_i \subseteq M_i$ is non-zero. Pick a representative $m \in M$ of m_{n+i-1}. Then $m \notin t^i M$. So there is a maximum integer $d \geq 0$ such that $m \in t^d M$, and $d < i$. Thus we may write $m = t^d m'$ for some $m' \in M$ such that $m' \notin tM$. Let m'_{n+i-1} be the image of m' in M_{n+i-1}. Thus $t^d m'_{n+i-1} = m_{n+i-1} \in Q_{n+i-1} \subseteq M_{n+i-1}$; and so the image $m'_{n+i-1-d}$ of m'_{n+i-1} in $M_{n+i-1-d}$ lies in $Q_{n+i-1-d}$, by part (d). Let $m'_n \in Q_n \subseteq M_n$ and $m'_1 \in Q_1 \subseteq M_1$ be the images of $m'_{n+i-1-d}$. (Note that $n + i - 1 - d \geq n \geq 1$.) Thus m'_1 is the image of m'_n; and $m'_1 \neq 0 \in M_1 = M/tM$ because $m' \notin tM$. But by the previous paragraph, the image of $Q_n \to Q_1$ is trivial. This contradiction proves (e).

Part (e) implies that $\varprojlim Q_i = 0$, which is the second part of (f). For the third part of (f), note that part (e) implies that the inverse system $\{Q_n\}$ satisfies the Mittag-Leffler condition (see [Sta25, Section 0594]). Since $0 \to Q_i \to M_i \to N_i \to 0$ is exact, it then follows from [Sta25, Lemma 0598] that $0 \to \varprojlim Q_i \to M \to \varprojlim N_i \to 0$ is exact. Since $\varprojlim Q_i = 0$, the map $M \to \varprojlim N_i$ is an isomorphism, as asserted. This completes the proof in the case that R is a domain, and thus also in the general case. $\square$

4. Formal schemes and patches

Let T be a complete discrete valuation ring with uniformizer t, and let $\mathscr{X}$ be an integral normal T-scheme of finite type having function field F. Let $X := \mathscr{X}_s^{\mathrm{red}}$ be the reduced closed fiber of $\mathscr{X}$, where $\mathscr{X}_s$ is the fiber of $\mathscr{X}$ over the closed point s of $\operatorname{Spec}(T)$. Given an open subset $\mathscr{U} \subset \mathscr{X}$, we may consider the t-adic completion $\widehat{\mathcal{O}_{\mathscr{X}}(\mathscr{U})}$ of the ring $\mathcal{O}_{\mathscr{X}}(\mathscr{U})$. Also, given any subset U of X, we may take the subring $\mathcal{O}_{\mathscr{X},U} = \bigcap_{P \in U} \mathcal{O}_{\mathscr{X},P}$ of F consisting of the rational functions on $\mathscr{X}$ that are regular at every point of U; this is normal since each local ring $\mathcal{O}_{\mathscr{X},P}$ is. We write $\widehat{\mathcal{O}}_{\mathscr{X},U}$ for its t-adic completion.

Consider the formal scheme $\mathfrak{X} = \mathscr{X}_{/X}$ obtained by completing $\mathscr{X}$ along X, as in [EGA1, Section 10.8]. The underlying topological space of the ringed space $\mathfrak{X}$ is X; and the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is the inverse limit of the $\mathcal{O}_{\mathscr{X}}$-modules $\mathcal{O}_{X_n}$, where X_n is the fiber of $\mathscr{X}$ over $\operatorname{Spec}(T/(t^n))$. This inverse limit is defined because the morphisms $X \to X_n \to X_{n+1}$ are homeomorphisms by [EGA1, 5.1.2, 5.1.3], and so the underlying spaces may be identified. Similarly, we may identify the open subsets $U \subseteq X$ with the open subsets $\mathfrak{U} \subseteq \mathfrak{X}$ as topological spaces (though not as ringed spaces). With U corresponding to $\mathfrak{U}$, we will often write $\mathcal{O}_{\mathfrak{X}}(U)$ for the t-adically complete ring $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$. Similarly, we may write $\mathscr{F}(U)$ for $\mathscr{F}(\mathfrak{U})$, if $\mathscr{F}$ is a sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules. Here the structure sheaf of $\mathfrak{U}$ is the restriction of that of $\mathfrak{X}$; and so for an open subset $V \subseteq U$, we have $\mathcal{O}_{\mathfrak{U}}(V) = \mathcal{O}_{\mathfrak{X}}(V)$.

If U is an affine open subset of X, then the corresponding open subset $U_n \subseteq X_n$ is also affine, by [EGA1, Proposition 5.1.9]. Since the sheaf and presheaf inverse limits of sheaves coincide, $\mathcal{O}_{\mathfrak{X}}(U) = \varprojlim \mathcal{O}_{X_n}(U_n)$; here $\mathcal{O}_{X_n}(U_n) = \mathcal{O}_{\mathfrak{X}}(U)/(t^n)$. Similarly, $\mathcal{O}_X(U) = \mathcal{O}_{\mathfrak{X}}(U)/I$, where I is the radical of the ideal $t\mathcal{O}_{\mathfrak{X}}(U)$. In this situation, we will often write $\widehat{R}_U$ for the ring $\mathcal{O}_{\mathfrak{X}}(U)$. By part (c) of the next proposition, this generalizes the notation used in [HH10], [HHK09], and later papers, where $\widehat{R}_U$ was used for the ring $\widehat{\mathcal{O}}_{\mathscr{X},U}$ in the case of a projective normal T-curve $\mathscr{X}$.

Proposition 4.1. *Let T be a complete discrete valuation ring with uniformizer t, let $\mathscr{X}$ be a normal integral T-scheme of finite type, and let $\mathfrak{X}$ be the formal completion of $\mathscr{X}$ along its reduced closed fiber X. Let U be a non-empty affine open subset of X.*

(a) The natural map $\widehat{\mathcal{O}}_{\mathscr{X},U} \to \mathcal{O}_{\mathfrak{X}}(U)$ is injective.

(b) Suppose that $\mathscr{U}$ is an affine open subset of $\mathscr{X}$ such that $\mathscr{U} \cap X = U$. Then the natural maps $\widehat{\mathcal{O}_{\mathscr{X}}(\mathscr{U})} \to \widehat{\mathcal{O}}_{\mathscr{X},U} \to \mathcal{O}_{\mathfrak{X}}(U)$ are isomorphisms.

(c) If $\mathscr{X}$ is a normal projective T-curve, then such a $\mathscr{U}$ exists, and so the natural map $\widehat{\mathcal{O}}_{\mathscr{X},U} \to \mathcal{O}_{\mathfrak{X}}(U)$ is an isomorphism.

Proof. The assertion is trivial if $\mathscr{X}$ consists just of the fiber over the closed point of $\operatorname{Spec}(T)$, and so we may assume that t is a non-zero element of the function field F of $\mathscr{X}$.

Let X_n be the fiber of $\mathscr{X}$ over $\operatorname{Spec}(T/t^n)$, and as above let U_n be the affine open subset of X_n corresponding to U under the homeomorphism $X \to X_n$. Since taking inverse limits is left exact, in order to prove part (a), it suffices to show injectivity modulo t^n for all n. So take $f \in \widehat{\mathcal{O}}_{\mathscr{X},U}/(t^n) = \mathcal{O}_{\mathscr{X},U}/(t^n)$ that lies in the kernel of the map to $\mathcal{O}_{\mathfrak{X}}(U)/(t^n)$. Let $\widetilde{f} \in \mathcal{O}_{\mathscr{X},U} \subseteq F$ be an element that maps to f. Thus the restriction of $\widetilde{f}$ to U_n is zero. Hence for every generic point η of U_n, the image of $\widetilde{f}$ in $\mathcal{O}_{\mathscr{X},\eta}$ lies in the ideal (t^n), and so the element $g := \widetilde{f}/t^n \in F$ lies in $\mathcal{O}_{\mathscr{X},\eta} \subseteq F$. Now for every point $P \in U$, if $\mathfrak{p}$ is a height one prime of $\mathcal{O}_{\mathscr{X},P}$, then the localization $(\mathcal{O}_{\mathscr{X},P})_{\mathfrak{p}}$ is either of the form $\mathcal{O}_{\mathscr{X},\eta}$ for some generic point η of U_n as above (if $t \in \mathfrak{p}$), or else of the form $\mathcal{O}_{\mathscr{X},Q}$ for some codimension one point Q of $\mathscr{X}$ that is not a generic point η and whose closure meets U (if $t \notin \mathfrak{p}$). In either case g lies in $(\mathcal{O}_{\mathscr{X},P})_{\mathfrak{p}}$, in the latter case using that $\widetilde{f} \in \mathcal{O}_{\mathscr{X},P}$ and that t is a unit in $(\mathcal{O}_{\mathscr{X},P})_{\mathfrak{p}}$. Since $\mathcal{O}_{\mathscr{X},P}$ is a normal Noetherian domain, it follows from [Eis95, Corollary 11.4] that $g \in \mathcal{O}_{\mathscr{X},P}$ for each $P \in U$. Hence $g \in \bigcap_{P \in U} \mathcal{O}_{\mathscr{X},P} = \mathcal{O}_{\mathscr{X},U}$. Thus $\widetilde{f} = t^n g \in t^n \mathcal{O}_{\mathscr{X},U}$, and so $f = 0$, yielding part (a).

In part (b), since $\mathscr{U}$ is an affine open subset of $\mathscr{X}$ such that $\mathscr{U} \cap X = U$, we have that $\mathcal{O}_{\mathscr{X}}(\mathscr{U})/(t^n) = \mathcal{O}_{X_n}(U_n)$; and taking inverse limits yields that the map $\widehat{\mathcal{O}_{\mathscr{X}}(\mathscr{U})} \to \mathcal{O}_{\mathfrak{X}}(U)$ is an isomorphism. The inclusion $U_n \to \mathscr{U}$ induces a map $\mathcal{O}_{\mathscr{X}}(\mathscr{U})/(t^n) \to \mathcal{O}_{\mathscr{X},U}/(t^n) = \widehat{\mathcal{O}}_{\mathscr{X},U}/(t^n)$. Since $\mathcal{O}_{\mathscr{X}}(\mathscr{U})/(t^n) \to \mathcal{O}_{\mathfrak{X}}(U)/(t^n)$ factors through $\mathcal{O}_{\mathscr{X}}(\mathscr{U})/(t^n) \to \mathcal{O}_{\mathscr{X},U}/(t^n)$, by taking inverse limits we find that the isomorphism $\widehat{\mathcal{O}_{\mathscr{X}}(U)} \to \mathcal{O}_{\mathfrak{X}}(U)$ factors through

$\widehat{\mathcal{O}_{\mathscr{X}}(U)} \to \widehat{\mathcal{O}}_{\mathscr{X},U}$. Hence the map $\widehat{\mathcal{O}}_{\mathscr{X},U} \to \mathcal{O}_{\mathfrak{X}}(U)$ is surjective. So by part (a) this map is an isomorphism, concluding the proof of part (b).

To prove (c) we will show that an affine open subset $\mathscr{U} \subseteq \mathscr{X}$ as above exists when $\mathscr{X}$ is a normal projective T-curve. First consider the case where U is dense in X, so that its complement S in X is finite. Since U is affine, this complement meets each irreducible component of X. By [HHK15, Proposition 3.3], there is a finite morphism $\varphi : \mathscr{X} \to \mathbb{P}^1_T$ such that S is the inverse image of the point at infinity on the closed fiber $\mathbb{P}^1_k$ (where k is the residue field of T). Thus U is the inverse image of $\mathbb{A}^1_k$. We may then take $\mathscr{U} \subset \mathscr{X}$ to be the inverse image of $\mathbb{A}^1_T$. This is affine because the morphism φ is finite and hence an affine morphism.

For the proof of (c) in the more general case where U is not necessarily dense in X, let J be the set of irreducible components of X that do not meet U. Since U is non-empty, J does not contain every irreducible component of X. Thus by [BLR90, Section 6.7, Theorem 1, Corollary 3, Proposition 4], we may contract the components in J. That is, there is a proper birational morphism $\pi : \mathscr{X} \to \mathscr{Y}$, where $\mathscr{Y}$ is a projective normal T-scheme, such that the components of J each map to a point, and π is an isomorphism elsewhere. Thus U maps isomorphically onto its image V, which is dense in the reduced closed fiber Y of $\mathscr{Y}$. So by the above special case, there is an affine open subset $\mathscr{V} \subseteq \mathscr{Y}$ such that $\mathscr{V} \cap Y = V$. The inverse image $\mathscr{U} = \pi^{-1}(\mathscr{V})$ is isomorphic to $\mathscr{V}$, and so it is an affine open subset of $\mathscr{X}$. Moreover its intersection with X is U. So $\mathscr{U}$ is as asserted. □

Remark 4.2. In Proposition 4.1(c), once we reduce as above to the case that U is dense, we can construct $\mathscr{U}$ as follows (following the proof of the result [HHK15, Proposition 3.3] that was cited above): At each closed point $P \in S = X \smallsetminus U$, take an element r_P in the maximal ideal of the local ring $\mathcal{O}_{\mathscr{X},P}$ such that r_P does not vanish along any component of the closed fiber passing through P. This defines an effective Cartier divisor on $\mathrm{Spec}(\mathcal{O}_{\mathscr{X},P})$ whose support passes through P, and which is the restriction of an effective Cartier divisor $\mathscr{D}_P$ on $\mathscr{X}$ whose support meets X precisely at P. Here $\mathscr{D} := \sum_{P\in S} \mathscr{D}_P$ is an effective Cartier divisor on $\mathscr{X}$ whose support meets X precisely at S, and so in particular meets each irreducible component of X. Hence the restriction D of $\mathscr{D}$ is X is ample (by [Liu02, Chapter 7, Proposition 5.5]), and thus so is $\mathscr{D}$ (by [Liu02, Chapter 5, Corollary 3.24]). Hence some multiple of $\mathscr{D}$ is very ample, and so the complement of its support in $\mathscr{X}$ is affine. We may then take $\mathscr{U}$ to be that complement.

In the case where $\mathscr{X}$ has dimension greater than one over T, even if a given affine open set U is not of the form $\mathscr{U} \cap X$, one can still cover U by affine open subsets V of that form, since every point of U has such a neighborhood, by definition of the subspace topology. Here a finite set of such subsets V suffices, by quasi-compactness. The following lemma studies the behavior of the corresponding rings.

Lemma 4.3. *Let T be a complete discrete valuation ring with uniformizer t, let $\mathscr{X}$ be a normal integral T-scheme of finite type, and let $\mathfrak{X}$ be the formal completion of $\mathscr{X}$ along its reduced closed fiber X. Let U be an affine open subset of X.*

(a) The natural map $U \to \mathrm{Spec}(\widehat{R}_U)$ is a bijection on closed points.

(b) If $V \subseteq U$ is an affine open subset, then $\widehat{R}_V$ is flat over $\widehat{R}_U$.

(c) If $V_1, \dots, V_n \subseteq U$ are affine open subsets such that $\bigcup_{i=1}^n V_i = U$, then $\prod_{i=1}^n \widehat{R}_{V_i}$ is faithfully flat over $\widehat{R}_U$.

Proof. Let I be the radical of the ideal $t\mathcal{O}_{\mathfrak{X}}(U)$. Since $\widehat{R}_U/I = \mathcal{O}_X(U)$, the natural map $U \to \mathrm{Spec}(\widehat{R}_U)$ induces a bijection between the maximal ideals of $\mathcal{O}_X(U)$ and the maximal ideals of $\widehat{R}_U$ that contain I. But since $\widehat{R}_U$ is I-adically complete, the ideal I is contained in the Jacobson radical of $\widehat{R}_U$ (see [Mat80, Proposition 23.G]), and hence in every maximal ideal of $\widehat{R}_U$. So part (a) follows.

For (b), let X_n be the reduction of X modulo t^n, and let U_n, V_n be the homeomorphic images of U, V under $X \to X_n$. Then $V_n \subseteq U_n$ is an inclusion of affine open subsets of X_n by [EGA1, Proposition 5.1.9], and so $\mathcal{O}_{X_n}(V_n)$ is flat over $\mathcal{O}_{X_n}(U_n)$. Here $\mathcal{O}_{\mathfrak{X}}(U)/(t^n) = \mathcal{O}_{X_n}(U_n)$ and similarly for V and V_n. By [Sta25, Lemma 0912], $\widehat{R}_V = \mathcal{O}_{\mathfrak{X}}(V)$ is flat over $\widehat{R}_U = \mathcal{O}_{\mathfrak{X}}(U)$. So part (b) holds.

By part (a), every maximal ideal of $\widehat{R}_U$ is of the form $\mathfrak{m}_{U,P}$ for some closed point P of $U = \bigcup_{i=1}^n V_i$. Here P lies on some V_i, and so $\mathfrak{m}_{U,P}$ is the contraction of the maximal ideal $\mathfrak{m}_{V_i,P}$ of $\widehat{R}_{V_i}$. Thus every maximal ideal of $\widehat{R}_U$ is the contraction of a maximal ideal of $\prod_{i=1}^n \widehat{R}_{V_i}$. Also, $\prod_{i=1}^n \widehat{R}_{V_i}$ is flat over $\widehat{R}_U$ because each $\widehat{R}_{V_i}$ is, by part (b). Thus by [Bou72, Proposition I.3.5.9], $\prod_{i=1}^n \widehat{R}_{V_i}$ is faithfully flat over $\widehat{R}_U$; i.e., part (c) holds. $\square$

Lemma 4.4. *Let T be a complete discrete valuation ring with uniformizer t, let $\mathscr{X}$ be a normal integral T-scheme of finite type, and let $\mathfrak{X}$ be the formal completion of $\mathscr{X}$ along its reduced closed fiber X. Let U be an affine open subset of X.*

(a) The ring $\widehat{R}_U$ is quasi-excellent and normal (and in particular, Noetherian).

(b) If $U = \mathscr{U} \cap X$ for some affine open subset $\mathscr{U} \subseteq \mathscr{X}$, then $\widehat{R}_U$ is an excellent normal ring.

(c) The ring $\widehat{R}_U$ is a domain if and only if U is connected.

(d) If U is a disjoint union of affine open subsets U_i, then the natural map $\widehat{R}_U \to \prod_i \widehat{R}_{U_i}$ is an isomorphism.

Proof. Let I be the radical of $t\widehat{R}_U$, and let $U_n \subseteq X_n$ be as before. As noted before Proposition 4.1, $\widehat{R}_U = \mathcal{O}_{\mathfrak{X}}(U) = \varprojlim \mathcal{O}_{X_n}(U)$, with $\mathcal{O}_{X_n}(U) = \widehat{R}_U/(t^n)$ and $\mathcal{O}_X(U) = \widehat{R}_U/I$. Since $\mathcal{O}_X(U)$ is of finite type over k, it is excellent, and in particular quasi-excellent. Hence $\widehat{R}_U$ is quasi-excellent by a theorem of Gabber (see [KuSh21, Theorem 5.1]). This proves the first part of (a), that $\widehat{R}_U$ is quasi-excellent (and hence Noetherian). Note also that since $\mathcal{O}_X(U) = \widehat{R}_U/I$, we can identify U with the closed subset of $\mathrm{Spec}(\widehat{R}_U)$ defined by the ideal I.

Under the hypothesis of part (b), $\widehat{R}_U = \widehat{\mathcal{O}_{\mathscr{X}}(\mathscr{U})}$, by Proposition 4.1(b). Write $\mathscr{U} = \mathrm{Spec}(A) \subseteq \mathscr{X}$. The inclusion $\iota : U \hookrightarrow \mathscr{U}$ corresponds to a morphism $A \to \mathcal{O}_X(U)$ that factors through the t-adic completion $\widehat{A} = \widehat{\mathcal{O}_{\mathscr{X}}(\mathscr{U})}$ of A. That is, ι factors through $\mathrm{Spec}(\widehat{R}_U)$, corresponding to the natural embedding $U \to \mathrm{Spec}(\widehat{R}_U)$. By [Mat80, 34.B], T is excellent; hence so is $\mathcal{O}_{\mathscr{X}}(\mathscr{U})$, being a finitely generated T-algebra. So the t-adic completion $\widehat{R}_U$ of $\mathcal{O}_{\mathscr{X}}(\mathscr{U})$ is also excellent, by a theorem of Gabber (see [KuSh21, Main Theorem 2]). By [Mat80, 33.I, 34.A], $\widehat{R}_U$ is a normal ring, since it is the completion of the excellent normal ring $\mathcal{O}_{\mathscr{X}}(\mathscr{U})$. This proves part (b).

For the last part of (a), concerning normality, recall that any affine open subset U of X is the union of finitely many open subsets V_i of the form $\mathscr{V}_i \cap X$, with $\mathscr{V}_i$ an affine open subset of $\mathscr{X}$. By part (b), each $\widehat{R}_{V_i}$ is normal; hence so is $\prod_i \widehat{R}_{V_i}$. Also, by Lemma 4.3(c), $\prod_i \widehat{R}_{V_i}$

is faithfully flat over $\widehat{R}_U$. So by [Sta25, Lemma 030C], $\widehat{R}_U$ is normal, completing the proof of part (a).

Since $\mathrm{Spec}(\widehat{R}_U)$ is normal, it is in particular reduced. So $\mathrm{Spec}(\widehat{R}_U)$ is integral if and only if it is connected. But since every connected component of $\mathrm{Spec}(\widehat{R}_U)$ contains a closed point, it follows from Lemma 4.3(a) that $\mathrm{Spec}(\widehat{R}_U)$ is connected if and only if U is connected. Thus part (c) follows.

Part (d) is immediate from the definition of $\widehat{R}_U$ as $\mathcal{O}_{\mathfrak{X}}(U)$ together with the fact that $\mathcal{O}_{\mathfrak{X}}$ is a sheaf. $\square$

The next result further relates $\widehat{R}_U$ to $\widehat{R}_V$, where $V \subseteq U$ are affine open subsets of X.

Lemma 4.5. *Let T be a complete discrete valuation ring with uniformizer t, and let $\mathscr{X}$ be a normal integral T-scheme of finite type, with reduced closed fiber X. Let $V \subseteq U$ be an inclusion of non-empty affine open subsets of X.*

(a) The contraction of every minimal prime ideal of $\widehat{R}_V$ is a minimal prime ideal of $\widehat{R}_U$.

(b) Every regular element of $\widehat{R}_U$ has the property that its image is regular in $\widehat{R}_V$.

(c) The natural map $\widehat{R}_U \to \widehat{R}_V$ is injective if and only if V meets each connected component of U. In particular, it is injective if V is dense in U, or if U is connected.

(d) If the map $\widehat{R}_U \to \widehat{R}_V$ is injective, it induces a well-defined injection between the total rings of fractions of $\widehat{R}_U, \widehat{R}_V$.

Proof. Since $\widehat{R}_V$ is flat over $\widehat{R}_U$ by Lemma 4.3(b), the going down theorem holds for this ring extension by [Mat80, Theorem 5.D]. Hence the contraction of every minimal prime ideal of $\widehat{R}_V$ is a minimal prime ideal of $\widehat{R}_U$.

To prove part (b), we show that an element of $\widehat{R}_U$ that becomes a zero-divisor in $\widehat{R}_V$ is already a zero-divisor in $\widehat{R}_U$. By Lemma 4.4(a), the rings $\widehat{R}_V$ and $\widehat{R}_U$ are normal and in particular reduced. Hence by [Sta25, Lemmas 0EMA, 05C3], the set of zero-divisors in $\widehat{R}_V$ (resp. $\widehat{R}_U$) is the union of the minimal primes of that ring. So if the image $r' \in \widehat{R}_V$ of some $r \in \widehat{R}_U$ is a zero-divisor in $\widehat{R}_V$, then r' lies in a minimal prime of $\widehat{R}_V$. By part (a), r lies in a minimal prime of $\widehat{R}_U$, and so is a zero-divisor in $\widehat{R}_U$, as needed.

In part (c), the second assertion is immediate from the first. For the forward direction of the first assertion, in the special case that U is connected, $\widehat{R}_U$ is a domain by Lemma 4.4(c), and hence every non-zero element $r \in \widehat{R}_U$ is regular. Thus by part (b) above, the image of r in $\widehat{R}_V$ is regular and hence non-zero. Thus the map is injective. For the more general case, let $U_1, \dots, U_n$ be the connected components of U, and let $V_i = U_i \cap V$. Thus each U_i and V_i is an affine open set, with $\widehat{R}_U \cong \prod_i \widehat{R}_{U_i}$ and $\widehat{R}_V \cong \prod_i \widehat{R}_{V_i}$ by Lemma 4.4(d). By the above special case, each $\widehat{R}_{U_i} \to \widehat{R}_{V_i}$ is injective. Hence so is $\widehat{R}_U \to \widehat{R}_V$, showing the forward direction. For the reverse direction, if V does not meet some connected component U_j of U, let $r \in \widehat{R}_U \cong \prod_i \widehat{R}_{U_i}$ be the element given by 1 in $\widehat{R}_{U_j}$ and by 0 in every other $\widehat{R}_{U_i}$. Then the image of r in $\widehat{R}_V$ is 0, and so the map $\widehat{R}_U \to \widehat{R}_V$ is not injective.

For (d), let S_U, S_V be the sets of regular elements in $\widehat{R}_U, \widehat{R}_V$. Thus the total rings of fractions of these rings are $K_U = S_U^{-1}\widehat{R}_U$ and $K_V = S_V^{-1}\widehat{R}_V$. By part (b), the injection $\widehat{R}_U \to \widehat{R}_V$ restricts to an injection $S_U \to S_V$. Thus $\widehat{R}_U \to \widehat{R}_V$ induces a map $K_U = S_U^{-1}\widehat{R}_U \to S_V^{-1}\widehat{R}_V = K_V$. This map factors through $S_U^{-1}\widehat{R}_V$. Here $S_U^{-1}\widehat{R}_U \to S_U^{-1}\widehat{R}_V$ is injective because localization is exact; and $S_U^{-1}\widehat{R}_V \to S_V^{-1}\widehat{R}_V$ is injective because the elements of S_V are regular in $\widehat{R}_V$. This proves (d). $\square$

The next lemma controls the behavior of the principal ideal (t) in the rings corresponding to different patches.

Lemma 4.6. *Let T be a complete discrete valuation ring with uniformizer t, and let $\mathscr{X}$ be a normal integral T-scheme of finite type. Let U be an affine open subset of the reduced closed fiber X of $\mathscr{X}$, and let $U' \subseteq U$ be an affine dense open subset. Write $\widehat{R}$ and $\widehat{R}'$ for $\widehat{R}_U$ and $\widehat{R}_{U'}$, respectively. For $i \geq 1$, let R_i, R_i' denote the quotients of $\widehat{R}, \widehat{R}'$ by the ideals generated by t^i in the respective rings. Let $\{P_1, \ldots, P_s\}$ be the set of minimal primes over $t\widehat{R}$, and write $P_j\widehat{R}', P_jR_i, P_jR_i'$ for the extension of P_j to $\widehat{R}', R_i, R_i'$, respectively. Then*

- *(a) The minimal primes over $t\widehat{R}'$ are the ideals $P_j\widehat{R}'$ $(j = 1, \ldots, s)$.*
- *(b) P_jR_i is the contraction of P_jR_i' to R_i.*
- *(c) The ideal $J_i \subset R_i$ defining the complement of $\operatorname{Spec}(R_i')$ in $\operatorname{Spec}(R_i)$ has the property that J_iR_i' is the unit ideal. Moreover, it is generated by (finitely many) elements that are not in $\bigcup_{j=1}^{s} P_jR_i$.*

Proof. The natural map $\widehat{R} \to \widehat{R}'$ is an inclusion, by Lemma 4.5(c). The irreducible components of the reduced closed fiber of $\operatorname{Spec}(\widehat{R})$ are the integral schemes $Y_j := \operatorname{Spec}(\widehat{R}/P_j)$ for $j = 1, \ldots, s$. The irreducible components of the reduced closed fiber of $\operatorname{Spec}(\widehat{R}')$ are the intersections $Y_j' = U' \cap Y_j \subseteq U'$, each of which is non-empty because U' is dense in U. Here Y_j' is the closed subset of $\operatorname{Spec}(\widehat{R}')$ defined by the ideal $P_j\widehat{R}'$, for $j = 1, \ldots, s$. So these are the minimal primes of $\widehat{R}'$ over $t\widehat{R}'$, showing (a).

Fix j. Since Y_j' is a dense open subset of the integral scheme Y_j, the natural map $\mathcal{O}_X(Y_j) \to \mathcal{O}_X(Y_j')$ is an inclusion of subrings of the function field of Y_j (or equivalently of Y_j'). But $\mathcal{O}_X(Y_j) = \widehat{R}/P_j = (\widehat{R}/t^i\widehat{R})/(P_j/t^i\widehat{R}) = R_i/P_jR_i$, and similarly $\mathcal{O}_X(Y_j') = R_i'/P_jR_i'$. So the map $R_i/P_jR_i \to R_i'/P_jR_i'$ is an inclusion, for all i. Hence $\ker(R_i \to R_i'/P_jR_i')$, which is the contraction of P_jR_i' to R_i, is equal to $\ker(R_i \to R_i/P_jR_i) = P_jR_i$; showing (b).

The first part of (c) is immediate because J_iR_i' defines the empty subscheme of $\operatorname{Spec}(R_i')$. To prove the second part of (c), first choose any finite set of generators $\{s_1, \ldots, s_d\} \subset R_i$ of J_i. We will modify these generators so that none of them lie in $\bigcup_{j=1}^{s} P_jR_i$.

Since the ideals P_j are minimal over $t\widehat{R}$, no P_j contains any P_k for $k \neq j$. By [AM69, Proposition 1.11(ii)], P_j does not contain $\bigcap_{k\neq j} P_k$; i.e., there exists $\rho_j \in \widehat{R}$ such that ρ_j is not contained in P_j but is contained in every other P_k. Hence its image $\bar{\rho}_j \in R_i$ is not contained in P_jR_i (using that $t \in P_j$) but is contained in P_kR_i for every other k.

Since U' is dense in U, the ideal J_i is not contained in any of the ideals P_jR_i (each of which defines an irreducible component of $U_i := \operatorname{Spec}(R_i)$ and hence of $U = U_i^{\mathrm{red}}$). By prime avoidance (see [AM69, Proposition 1.11(i)]), J_i is not contained in $\bigcup_j P_jR_i$; i.e., there exists $r_0 \in J_i$ that is not in any P_jR_i. For $h = 1, \ldots, d$, let $S_h = \{j \mid s_h \in P_jR_i\}$, and let $r_h = s_h + \sum_{j\in S_h} r_0\bar{\rho}_j$. Then r_h does not lie in any P_jR_i, and the ideal J_i is generated by $r_0, r_1, \ldots, r_d$. This proves the second part of (c). $\square$

5. Modules on patches

In this section, we build on the previous results to obtain Proposition 5.3, a key step in the proof of our main theorem.

Lemma 5.1. *Let T be a complete local domain and let $\mathscr{X}$ be a normal integral T-scheme of finite type. Let U be a non-empty affine open subset of the reduced closed fiber X of $\mathscr{X}$, and let V be an affine dense open subset of U. Then for every finitely generated torsion-free $\widehat{R}_U$-module M, the natural map $\iota_V : M \to M \otimes_{\widehat{R}_U} \widehat{R}_V$ is injective.*

Proof. Let $U_1, \dots, U_s$ be the connected components of U. So $\widehat{R}_U = \prod_i \widehat{R}_{U_i}$ by Lemma 4.4(d); and each $\widehat{R}_{U_i}$ is a Noetherian normal domain by Lemma 4.4(a,c). Since M is a finitely generated torsion-free $\widehat{R}_U$-module, it follows that $M = \prod_i M_i$, where M_i is a finitely generated torsion-free $\widehat{R}_{U_i}$-module for each i. For every i, the intersection $V_i := V \cap U_i$ is an affine open dense subset of U_i; and V is their disjoint union. Thus $\widehat{R}_V = \prod_i \widehat{R}_{V_i}$, and ι_V decomposes as a product of maps $\iota_{V,i} : M_i \to M_i \otimes_{\widehat{R}_{U_i}} \widehat{R}_{V_i}$. So by considering each pair $\widehat{R}_{U_i}$, $\widehat{R}_{V_i}$, we are reduced to the case where U is connected and $\widehat{R}_U$ is a Noetherian domain.

Let K_U, K_V be the total rings of fractions of $\widehat{R}_U$ and $\widehat{R}_V$; thus K_U is the fraction field of $\widehat{R}_U$. Since V is dense in U, we have a natural injection $K_U \hookrightarrow K_V$ by Lemma 4.5(c,d); and so K_V is a K_U-module. The composition $M \to M \otimes_{\widehat{R}_U} \widehat{R}_V \to M \otimes_{\widehat{R}_U} \widehat{R}_V \otimes_{\widehat{R}_V} K_V = M \otimes_{\widehat{R}_U} K_V$ also factors as $M \to M \otimes_{\widehat{R}_U} K_U \to M \otimes_{\widehat{R}_U} K_U \otimes_{K_U} K_V = M \otimes_{\widehat{R}_U} K_V$. Here the map $M \to M \otimes_{\widehat{R}_U} K_U$ is injective because M is torsion-free over $\widehat{R}_U$; and the map $M \otimes_{\widehat{R}_U} K_U \to M \otimes_{\widehat{R}_U} K_U \otimes_{K_U} K_V$ is injective since $M \otimes_{\widehat{R}_U} K_U$ is flat over the field K_U. So the composition of these maps is injective. But the above two compositions are equal, hence the map $M \to M \otimes_{\widehat{R}_U} \widehat{R}_V$ is injective. $\square$

Lemma 5.2. *Let K be a complete discretely valued field with valuation ring T and uniformizer t. Let $\mathscr{X}$ be a normal integral T-scheme of finite type, and let U be an affine open subset of the reduced closed fiber X of $\mathscr{X}$. Consider an affine open subset $U' \subseteq U$ that is dense in U. Write $\widehat{R}$ and $\widehat{R}'$ for $\widehat{R}_U$ and $\widehat{R}_{U'}$, respectively. Let M be a finitely generated $\widehat{R}$-module, let $M' = M \otimes_{\widehat{R}} \widehat{R}'$, and let R_i, M_i, Q_i, N_i (resp., R_i', M_i', Q_i', N_i') be the rings and modules given by Lemma 3.2 for these two modules, with respect to the ideal $t\widehat{R}$ (resp., $t\widehat{R}'$). Then the natural map $R_i \to R_i'$ induces a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & Q_i \otimes_{R_i} R_i' \arrow[r] \arrow[d, "\cong"] & M_i \otimes_{R_i} R_i' \arrow[r] \arrow[d, "\cong"] & N_i \otimes_{R_i} R_i' \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & Q_i' \arrow[r] & M_i' \arrow[r] & N_i' \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Proof. First note that $\widehat{R}$ and $\widehat{R}'$ are quasi-excellent normal rings by Lemma 4.4(a), and in particular they are G-rings. So Lemma 3.2 does in fact provide us with the data R_i, M_i, Q_i, N_i and R_i', M_i', Q_i', N_i' as in the above assertion. Moreover, $\widehat{R} \hookrightarrow R'$ by Lemma 4.5(c).

For each i, $R_i = \widehat{R}/t^i\widehat{R} = \mathcal{O}(U_i)$ and $R_i' = \widehat{R}'/t^i\widehat{R}' = \mathcal{O}(U_i')$, where U_i, U_i' are the homeomorphic images of $U, U' \subseteq X$ in the mod t^i reduction X_i of $\mathscr{X}$. Since $U_i' = \mathrm{Spec}(R_i')$ is an open subset of $U_i = \mathrm{Spec}(R_i)$, the ring R'_i is flat over R_i. Thus the exact sequence

$$0 \to Q_i \to M_i \to N_i \to 0$$

from Lemma 3.2 yields an exact sequence

$$0 \to Q_i \otimes_{R_i} R'_i \to M_i \otimes_{R_i} R'_i \to N_i \otimes_{R_i} R'_i \to 0$$

as in the top row in the diagram above. Similarly, the bottom row is exact by Lemma 3.2. Using the definition of M', we have isomorphisms

$$M_i \otimes_{R_i} R'_i \xrightarrow{\sim} M \otimes_{\widehat{R}} R_i \otimes_{R_i} R'_i \xrightarrow{\sim} M \otimes_{\widehat{R}} R'_i \xrightarrow{\sim} M \otimes_{\widehat{R}} \widehat{R}' \otimes_{\widehat{R}'} R'_i \xrightarrow{\sim} M' \otimes_{\widehat{R}'} R'_i \xrightarrow{\sim} M'_i.$$

Let $P_1, \dots, P_s$ denote the minimal primes over the ideal (t) in $\widehat{R}$. Thus the minimal primes over $t\widehat{R}'$ are the ideals $P_j\widehat{R}'$, by Lemma 4.6(a). For any element $m \in M_i$ that lies in Q_i, the annihilator of m in R_i is not contained in P_jR_i for any j, by definition of Q_i. Thus by Lemma 4.6(b), this annihilator is also not contained in $P_jR'_i$ for any j. Hence the image of the inclusion $Q_i \otimes_{R_i} R'_i \to M_i \otimes_{R_i} R'_i$ is contained in Q'_i. This gives the left hand vertical arrow $Q_i \otimes_{R_i} R'_i \to Q'_i$, which is then injective; and it also gives the right hand vertical arrow $N_i \otimes_{R_i} R'_i \to N'_i$ such that the diagram commutes. We claim that the map $Q_i \otimes_{R_i} R'_i \to Q'_i$ is surjective, and hence an isomorphism. Since the middle vertical arrow is an isomorphism as observed above, this claim will imply that the right hand vertical arrow is an isomorphism, and thus will finish the proof.

We begin with the case in which U' is a basic open subset of U; i.e., it is the complement of the zero set of some element $\bar{f} \in \mathcal{O}_X(U)$. We may lift $\bar{f}$ to some $\widetilde{f} \in \widehat{R}_U = \mathcal{O}_{\mathfrak{X}}(U)$. Since U' is dense in U, the element $\widetilde{f}$ (and similarly, $\bar{f}$) does not vanish at the generic point of any irreducible component of U. Fixing i, we write f for the image of $\widetilde{f}$ in R_i. Thus $R'_i = R_i[f^{-1}]$, and for $j = 1, \dots, s$ the element f does not lie in the ideal P_jR_i.

Let $m' \in Q'_i \subseteq M'_i = M_i \otimes_{R_i} R'_i$; we wish to show that m' is in the image of the map $Q_i \otimes_{R_i} R'_i \to Q'_i$. Since $m' \in Q'_i$, there exists $r' \in R'_i$ with $r'm' = 0 \in M'_i$, such that r' does not lie in any of the primes $P_jR'_i$. For $j = 1, \dots, s$, we have $f \notin P_jR_i$; and the image $f' \in R'_i$ of f is a unit in R'_i, say with inverse g.

The homomorphism $R_i \to R'_i = R_i[f^{-1}] = S^{-1}R_i$ induces the homomorphism $M_i \to M'_i = M_i \otimes_{R_i} R'_i = S^{-1}M_i$, where $S \subset R_i$ is the multiplicative set generated by f. We may write $r' = r/f^a \in S^{-1}R_i$ and $m' = m/f^b \in S^{-1}M_i$, for some $r \in R_i$, some $m \in M_i$, and some $a, b \geq 0$. Thus the image of $m \in M_i$ in M'_i is $(f')^b m'$. Since $f' \in R'_i$ is a unit and since r' lies in no $P_jR'_i$, the element $(f')^a r' = r/1 \in R'_i$ also lies in no $P_jR'_i$. The element $r \in R_i$ maps to $r/1 = (f')^a r' \in R'_i$, hence it lies in no $P_jR_i \subset R_i$.

Now $rm/f^{a+b} = r'm' = 0 \in M'_i$, and so by definition of localization we have $f^c rm = 0 \in M_i$ for some $c \geq 0$. But $f^c r \in R_i$ lies in no P_jR_i, since this is true for the elements $f, r \in R_i$ and since P_jR_i is prime. Thus $m \in Q_i$, and $m \otimes g^b \in Q_i \otimes_{R_i} R'_i \subseteq M_i \otimes_{R_i} R'_i$. The image of $m \otimes g^b$ in $Q'_i \subseteq M'_i$ is $(f')^b g^b m' = m'$, proving the claim in this case.

For the general case, let $J_i \subset R_i$ be the ideal defining the complement of $\mathrm{Spec}(R'_i)$ in $\mathrm{Spec}(R_i)$ as in Lemma 4.6(c); and let $f_1, \dots, f_d$ be generators of J_i given by that lemma, with no f_h lying in P_jR_i for any j. For $h = 1, \dots, d$, the element f_h vanishes along the complement of $\mathrm{Spec}(R'_i)$ in $\mathrm{Spec}(R_i)$; and so for every $r \in R'_i$ there is some non-negative integer c such that $f_h^c r \in R_i$. Thus $R_i \subseteq R'_i \subseteq R_{h,i} := R_i[f_h^{-1}]$, and so $R_{h,i} = R'_i[f_h^{-1}]$. The ring $R_{h,i}$ is flat over R'_i, being a localization; hence the product ring $\prod_h R_{h,i}$ is also flat over

R_i'. Moreover $U_{h,i} := \mathrm{Spec}(R_{h,i})$ is a basic open subset of $U_i = \mathrm{Spec}(R_i)$ that is contained in $U_i' = \mathrm{Spec}(R_i')$, and such that $\bigcup_h U_{h,i} = U_i'$. Thus $\mathrm{Spec}(\prod_h R_{h,i})$, which is the disjoint union of the open sets $U_{h,i}$, maps surjectively to $U_i' = \mathrm{Spec}(R_i')$. Hence $\prod_h R_{h,i}$ is faithfully flat over R_i', by [Mat80, 4D, Theorem 3].

Let $M_{h,i} = M_i \otimes_{R_i} R_{h,i} = M_i' \otimes_{R_i'} R_{h,i}$, and let $Q_{h,i}$ be the submodule of $M_{h,i}$ given as in Lemma 3.2. By the above special case, the maps $Q_i \otimes_{R_i} R_{h,i} \to Q_{h,i}$ and $Q_i' \otimes_{R_i'} R_{h,i} \to Q_{h,i}$ are isomorphisms. Let Q_i'' be the image of the injective map $Q_i \otimes_{R_i} R_i' \to Q_i'$. Then $Q_i \otimes_{R_i} R_{h,i} = (Q_i \otimes_{R_i} R_i') \otimes_{R_i'} R_{h,i}$, and so the image of $Q_i'' \otimes_{R_i'} R_{h,i} \to Q_{h,i}$ is $Q_{h,i}$. Thus for each h, the quotient R_i'-module Q_i'/Q_i'' becomes trivial upon tensoring with $R_{h,i}$. So Q_i'/Q_i'' also becomes trivial upon tensoring with the faithfully flat R_i'-module $\prod_h R_{h,i}$. Hence Q_i'/Q_i'' is already trivial; i.e., $Q_i'' = Q_i'$ and so the map $Q_i \otimes_{R_i} R_i' \to Q_i'$ is indeed surjective, as claimed. □

Proposition 5.3. *Let T be a complete discrete valuation ring with uniformizer t, and let $\mathscr{X}$ be a normal integral T-scheme of finite type. Let U_0, U_1, U_2, U be affine open subsets of the reduced closed fiber X of $\mathscr{X}$, with $U_1, U_2 \subseteq U$ dense, and with $U_0 = U_1 \cap U_2$, such that the complement of $W := U_1 \cup U_2$ in U has codimension at least two. Let M_e be a finitely generated torsion-free $\widehat{R}_{U_e}$-module for $e = 0, 1, 2$. For $e = 1, 2$, consider the natural map $\iota_e : M_e \to M_e \otimes_{\widehat{R}_{U_e}} \widehat{R}_{U_0}$, and let $\alpha_e : M_e \otimes_{\widehat{R}_{U_e}} \widehat{R}_{U_0} \to M_0$ be an isomorphism. Then $\alpha_e \iota_e$ is injective for $e = 1, 2$, and the intersection $M := \alpha_1\iota_1(M_1) \cap \alpha_2\iota_2(M_2) \subseteq M_0$ is a finitely generated torsion-free $\widehat{R}_U$-module.*

Proof. For short, write $\widehat{R}_e = \widehat{R}_{U_e}$ for $e = 0, 1, 2$. Since U_1, U_2 are each dense in U, the intersection $U_0 = U_1 \cap U_2$ is dense in U_1, U_2. So we may apply Lemma 5.1 and obtain that each ι_e is injective. Since α_e is an isomorphism, the composition $\alpha_e\iota_e$ is injective. Because of this injectivity, we may identify M_e with its image under $\alpha_e\iota_e : M_e \to M_0$ for $e = 1, 2$, and thus regard M_e as contained in M_0. Here $\widehat{R}_U, M$ are respectively contained in $\widehat{R}_e, M_e$, and every regular element of $\widehat{R}_U$ is regular over $\widehat{R}_e$ by Lemma 4.5(b). Thus M is torsion-free over $\widehat{R}_U$, since M_e is torsion-free over $\widehat{R}_e$.

With respect to the above identifications, the goal of the proof is then to show that $M := M_1 \cap M_2$ is finitely generated over $\widehat{R}_U$.

For $e = 0, 1, 2$ and $i \geq 1$, write $R_{e,i} = \widehat{R}_e / t^i \widehat{R}_e$. The irreducible components of the reduced closed fiber of $\mathrm{Spec}(\widehat{R}_U)$ are $\mathrm{Spec}(\widehat{R}_U/P_j)$ for $j = 1, \ldots, s$, where $P_1, \ldots, P_s$ are the minimal primes over $t\widehat{R}_U$. For $e = 0, 1, 2$, the minimal primes over $t\widehat{R}_e$ are the ideals $P_j\widehat{R}_e$ for $j = 1, \ldots, s$, by Lemma 4.6(a). By Lemma 4.4(a), each $\widehat{R}_e$ is a quasi-excellent t-adically complete normal ring, and hence a G-ring. So Lemma 3.2 applies, with $\widehat{R}_e, M_e, P_j\widehat{R}_e$ playing the roles of R, M, P_j there. Let $M_{e,i}, Q_{e,i}, N_{e,i}$ be the modules given in Lemma 3.2 in that situation. Thus $M_{e,i}$ and its quotient $N_{e,i}$ are finitely generated modules over $R_{e,i}$ and over $\widehat{R}_e$, and $\varprojlim M_{e,i} = \varprojlim N_{e,i} = M_e$, for $e = 1, 2$. Also by that lemma, for $e = 0, 1, 2$ and $i \geq 1$, the associated primes of $N_{e,i}$ are among $P_1R_{e,i}, \ldots, P_sR_{e,i}$. Here the support of $P_jR_{e,i}$ is dense in the corresponding irreducible component of $\mathrm{Spec}(\widehat{R}_U/(t^i))$.

Since U_0 is dense in U_e for $e = 1, 2$, we may apply Lemma 5.2 to $U_0 \subseteq U_e$, and obtain isomorphisms of finite modules $N_{e,i} \otimes_{R_{e,i}} R_{0,i} = N_{0,i}$. By [Sta25, Lemma 00AM], these modules and isomorphisms define a coherent sheaf $\mathscr{N}_i$ of $\mathcal{O}_{W_i}$-modules on W_i, where we write $W_i := \mathrm{Spec}(R_{1,i}) \cup \mathrm{Spec}(R_{2,i}) \subseteq \mathrm{Spec}(\widehat{R}_U/(t^i))$. Since the complement of W in U has codimension at least two, the same holds for the complement of W_i in $\mathrm{Spec}(\widehat{R}_U/(t^i))$. Thus each point z of that latter complement has codimension at least two in each irreducible component of $\mathrm{Spec}(\widehat{R}_U/(t^i))$ on which it lies, and in particular in the closed subset defined by any of the associated primes of $N_{e,i}$ (each of which is of the form $P_j R_{e,i}$, corresponding to one of these irreducible components). Since $\widehat{R}_U/(t^i)$ is of finite type over T, it is excellent. So [Sta25, Lemma 0AWA] (or equivalently, [EGA4, Partie 2, Corollaire 5.11.4(ii)]) applies and shows that $(f_i)_*\mathscr{N}_i$ is coherent over $\mathrm{Spec}(\widehat{R}_U/(t^i))$, where $f_i : W_i \to \mathrm{Spec}(\widehat{R}_U/(t^i))$ is the natural inclusion. Its module of global sections, which is $N_i' := N_{1,i} \times_{N_{0,i}} N_{2,i}$, is thus finite over $\widehat{R}_U/(t^i)$.

For every $i \geq 1$, let $M_i' = M_{1,i} \times_{M_{0,i}} M_{2,i}$. For every $i \geq 1$, the maps $M = M_1 \cap M_2 \to M_e \to M_{e,i} = M_e/t^i M_e$ for $e = 0,1,2$ together induce a map $M \to M_i'$ that descends to a map $M/t^iM \to M_i'$. We claim that this latter map is injective. To see this, let $m \in M/t^iM$ lie in the kernel, and pick a representative $\widetilde{m} \in M$ for m. For $e = 1,2$, we may view $\widetilde{m} \in M_e$, and the image of m in $M_{e,i} = M_e/t^iM_e$ is trivial. Hence there exist $m_e' \in M_e$ such that $\widetilde{m} = t^i m_e'$ in M_e, for $e = 1,2$. The elements $t^i m_e'$, for $e = 1,2$, have the same image in M_0; and thus the element $m_1' - m_2' \in M_0$ is t^i-torsion. But M_0 is torsion free, and so $m_1' = m_2' \in M_0$. That is, the two elements $m_e' \in M_e$ define an element $m' \in M$. But $t^i m' = \widetilde{m} \in M$, since the two sides have the same image $t^i m_1'$ in M_1 and since $M \to M_1$ is injective. So $\widetilde{m} \in t^iM$, and thus $m \in M/t^iM$ is trivial, as needed to prove the claim.

Say $h \geq i \geq 1$ is an integer. Then the mod t^i reduction maps $M_{e,h} \to M_{e,i}$, for $e = 0,1,2$, together define a map $M_h' \to M_i'$. With respect to the injections $M/t^hM \to M_h'$ and $M/t^iM \to M_i'$, this restricts to the surjection $M/t^hM \to M/t^iM$ given by reduction modulo t^i. Hence the image of $M_h' \to M_i'$ contains M/t^iM, viewed as a submodule of M_i'.

For every $i \geq 1$, write $Q_i' = Q_{1,i} \times_{Q_{0,i}} Q_{2,i}$. For $e = 0,1,2$, we have a short exact sequence $0 \to Q_{e,i} \to M_{e,i} \to N_{e,i} \to 0$. Since taking fiber products is left exact, we obtain a left exact sequence $0 \to Q_i' \to M_i' \to N_i'$ for each i, where as above $N_i' = N_{1,i} \times_{N_{0,i}} N_{2,i}$. Thus $N_i := M_i'/Q_i'$ is a submodule of the finitely generated $\widehat{R}_U/(t^i)$-module N_i'; and so N_i is also finitely generated over $\widehat{R}_U/(t^i)$, since $\widehat{R}_U/(t^i)$ is Noetherian.

We want to show that M is a finitely generated $\widehat{R}_U$-module. By [Sta25, Lemma 087W], it suffices to show that M/t^iM is a finitely generated $\widehat{R}_U/(t^i)$-module for all i. For $e = 0,1,2$, let n_e be the integer given in Lemma 3.2(e) for the modules $\{Q_{e,i}\}$. Let $n = \max(n_0, n_1, n_2)$. Thus $Q_{e,i-1+n} \to Q_{e,i}$ is trivial for $e = 0,1,2$, and so the map $M_{e,i-1+n} \to M_{e,i}$ restricts to the trivial map on $Q_{e,i-1+n}$. Hence the restriction of $M_{i-1+n}' \to M_i'$ to Q_{i-1+n}' is also trivial. Thus the map $M_{i-1+n}' \to M_i'$ induces a map $N_{i-1+n} \to M_i'$ that has the same image. This image is finitely generated because N_{i-1+n} is. But as noted above (taking $h = i-1+n$), the image of $M_{i-1+n}' \to M_i'$ contains M/t^iM. Thus M/t^iM is finitely generated over $\widehat{R}_U/(t^i)$, completing the proof. $\square$

6. Formal pushforwards

Recall that if $(Z, \mathcal{O}_Z)$ is any ringed space, and M is a module over $R := \Gamma(Z, \mathcal{O}_Z)$, then there is a functorially associated quasi-coherent sheaf $\mathcal{F}_M$ on Z whose presentation is induced by that of M; see [Sta25, Lemma 01BH, Definition 01BI]. In the case of a Noetherian affine formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ and a finite A-module M, the sheaf $\mathcal{F}_M$ on $\mathfrak{X}$ is the formal sheaf

M^{Δ} associated to the coherent sheaf of modules $\widetilde{M}$ on the scheme $\mathrm{Spec}(A)$; see [EGA1, Section 10.10.1]. This sheaf M^{Δ} is coherent as an $\mathcal{O}_{\mathfrak{X}}$-module and it satisfies $\Gamma(\mathfrak{X}, M^{\Delta}) = M$, by [EGA1, Propositions 10.10.5, 10.10.2(i)]. Moreover, every coherent $\mathcal{O}_{\mathfrak{X}}$-module is uniquely of the form M^{Δ}, by [EGA1, Proposition 10.10.5].

Consider a normal integral scheme $\mathscr{X}$ of finite type over a complete discrete valuation ring T, with reduced closed fiber X, and let $V \subseteq U$ be open subsets of X. Since $X \subseteq \mathscr{X}$ has the subspace topology, there exist (not necessarily affine) open subsets $\mathscr{V} \subseteq \mathscr{U}$ of $\mathscr{X}$ meeting X at V, U. The inclusion map $g : \mathscr{V} \hookrightarrow \mathscr{U}$ restricts to the inclusion $g : V \hookrightarrow U$; and it also pulls back to compatible inclusions $g_n : V_n \hookrightarrow U_n$ on the reductions of $\mathscr{V}, \mathscr{U}$ modulo (t^n) for all $n \geq 1$. As in [EGA1, 10.9.1], the morphisms g_n together yield a morphism $\widehat{g} : \mathfrak{V} \to \mathfrak{U}$ between the induced formal schemes $\mathfrak{V} = \mathscr{V}_{/V}$ and $\mathfrak{U} = \mathscr{U}_{/U}$. Note that g_n and hence $\widehat{g}$ are independent of the choice of $\mathscr{V}$ and $\mathscr{U}$, and depend just on the inclusion $g : V \hookrightarrow U$ (and on the T-scheme $\mathscr{X}$).

The proof of the following result parallels that of [Hts77, Proposition II.5.8(c)] and [Sta25, Lemma 01LC], which make the corresponding assertion in the context of schemes.

Proposition 6.1. *Let $\mathfrak{X}, \mathfrak{Y}$ be locally Noetherian formal schemes, and let $\mathscr{M}$ be a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module. If $f : \mathfrak{X} \to \mathfrak{Y}$ is a quasi-compact and quasi-separated morphism, then $f_*(\mathscr{M})$ is a quasi-coherent $\mathcal{O}_{\mathfrak{Y}}$-module. This holds in particular if $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism that defines an open inclusion of the underlying topological spaces.*

Proof. The assertion is local on $\mathfrak{Y}$, so we are reduced to the case that $\mathfrak{Y}$ is an affine formal scheme; i.e., of the form $\mathrm{Spf}(E)$. Thus $\mathfrak{X}, \mathfrak{Y}$ are quasi-compact. Since $\mathfrak{X}$ is locally Noetherian, every point of $\mathfrak{X}$ has a fundamental system of quasi-compact neighborhoods. Hence by [Sta25, Lemma 01BK], for every point x of $\mathfrak{X}$ there is an open affine neighborhood $\mathfrak{U}_x = \mathrm{Spf}(A_x)$ of x such that $\mathscr{M}|_{\mathfrak{U}_x}$ is the sheaf of $\mathcal{O}_{\mathfrak{U}_x}$-modules associated to some A_x-module. Since $\mathfrak{X}$ is quasi-compact, there is a finite set $\{x_1, \ldots, x_n\}$ of points of $\mathfrak{X}$ such that $\mathfrak{X}$ is the union of the open subsets $\mathfrak{U}_i := \mathfrak{U}_{x_i}$. By [Sta25, Lemma 01KO], for every pair i, j the intersection $\mathfrak{U}_i \cap \mathfrak{U}_j$ is a finite union of affine open subsets $\mathfrak{U}_{ij\ell}$, since f is quasi-separated. Write $f_i = f|_{\mathfrak{U}_i}$ and $f_{ij\ell} = f|_{\mathfrak{U}_{ij\ell}}$, and also write $\mathscr{M}_i = \mathscr{M}|_{\mathfrak{U}_i}$ and $\mathscr{M}_{ij\ell} = \mathscr{M}|_{\mathfrak{U}_{ij\ell}}$. For any open subset $\mathfrak{V} \subseteq \mathfrak{Y}$,

$$\begin{aligned}
f_*\mathscr{M}\,(\mathfrak{V}) &= \mathscr{M}(f^{-1}(\mathfrak{V})) \\
&= \mathscr{M}\,\big(\bigcup_i (f^{-1}(\mathfrak{V}) \cap \mathfrak{U}_i)\big) \\
&= \ker\big(\bigoplus_i \mathscr{M}(f^{-1}(\mathfrak{V}) \cap \mathfrak{U}_i) \to \bigoplus_{i,j,\ell} \mathscr{M}(f^{-1}(\mathfrak{V}) \cap \mathfrak{U}_{ij\ell})\big) \\
&= \ker\big(\bigoplus_i f_{i,*}(\mathscr{M}_i)(\mathfrak{V}) \to \bigoplus_{i,j,\ell} f_{ij\ell,*}(\mathscr{M}_{ij\ell})(\mathfrak{V})\big) \\
&= \ker\big(\bigoplus_i f_{i,*}(\mathscr{M}_i) \to \bigoplus_{i,j,\ell} f_{ij\ell,*}(\mathscr{M}_{ij\ell})\big)(\mathfrak{V}),
\end{aligned}$$

where the maps in the third to fifth lines of the display are given by taking differences on the overlaps. Thus we have an exact sequence of formal sheaves

$$0 \to f_*(\mathscr{M}) \to \bigoplus_i f_{i,*}(\mathscr{M}_i) \to \bigoplus_{ij\ell} f_{ij\ell,*}(\mathscr{M}_{ij\ell}).$$

Since $\mathscr{M}$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module, $\mathscr{M}_i, \mathscr{M}_{ij\ell}$ are quasi-coherent sheaves of modules over $\mathcal{O}_{\mathfrak{U}_i}$ and $\mathcal{O}_{\mathfrak{U}_{ij\ell}}$, respectively. Recall that $\mathfrak{Y} = \mathrm{Spf}(E)$. By construction, $\mathfrak{U}_i = \mathrm{Spf}(A_i)$ and $\mathfrak{U}_{ij\ell} = \mathrm{Spf}(A_{ij\ell})$ for some rings $A_i, A_{ij\ell}$; and $\mathscr{M}_i$ and $\mathscr{M}_{ij\ell}$ are the sheaves of modules associated to some modules M_i and $M_{ij\ell}$ over A_i and $A_{ij\ell}$, respectively. So $f_{i,*}(\mathscr{M}_i)$ is the sheaf associated to $(M_i)_E$, the E-module obtained from M_i by restricting scalars to E. Similarly, $f_{ij\ell,*}(\mathscr{M}_{ij\ell})$ is the sheaf associated to $(M_{ij\ell})_E$. Thus $f_{i,*}(\mathscr{M}_i)$ and $f_{ij\ell,*}(\mathscr{M}_{ij\ell})$ are quasi-coherent; hence so are $\bigoplus_i f_{i,*}(\mathscr{M}_i)$ and $\bigoplus_{i,j,\ell} f_{ij\ell,*}(\mathscr{M}_{ij\ell})$, by [Sta25, Lemma 01BF]. So $f_*(\mathscr{M})$ is the kernel of a morphism between quasi-coherent sheaves on a locally Noetherian formal scheme, and thus is itself quasi-coherent by [AJL99, Corollary 3.1.6(a)]. This proves the assertion.

Finally, to check that the quasi-compact and quasi-separated conditions hold when f gives an open inclusion of underlying topological spaces, note that those two properties are local and depend only on the underlying spaces. Since f is locally affine, those properties hold by [Sta25, Lemma 01S7]. $\square$

Proposition 6.2. *Let T be a complete discrete valuation ring with uniformizer t, and let $\mathscr{X}$ be a normal integral T-scheme of finite type, with reduced closed fiber X and formal completion $\mathfrak{X}$. Let $g : V \to U$ be an inclusion of open subsets of X, with inclusion $\widehat{g} : \mathfrak{V} \to \mathfrak{U}$ of the associated formal open subschemes of $\mathfrak{X}$.*

(a) If $\mathscr{M}$ is a coherent (resp. torsion-free) $\mathcal{O}_{\mathfrak{U}}$-module, then $\widehat{g}^*(\mathscr{M})$ has the same property on $\mathfrak{V}$.
(b) If U and V are affine, and $\mathscr{M}$ is a coherent $\mathcal{O}_{\mathfrak{U}}$-module, then there is a natural isomorphism $\mathscr{M}(V) \xrightarrow{\sim} \mathscr{M}(U) \otimes_{\widehat{R}_U} \widehat{R}_V$.
(c) If $\mathscr{N}$ is a torsion-free $\mathcal{O}_{\mathfrak{V}}$-module, then $\widehat{g}_*(\mathscr{N})$ is a torsion-free $\mathcal{O}_{\mathfrak{U}}$-module.

Proof. Part (a) is immediate from the fact that the properties of being coherent and torsion-free are each local.

For part (b), write $\mathfrak{U} = \mathrm{Spf}(R)$ and $\mathfrak{V} = \mathrm{Spf}(S)$. Then $\mathscr{M}(\mathfrak{U})$ is the unique finitely generated R-module M such that $\mathscr{M} = M^{\Delta}$. By [EGA1, Proposition 10.10.8], there is a canonical isomorphism $\widehat{g}^*(\mathscr{M}) = \widehat{g}^*(M^{\Delta}) \xrightarrow{\sim} (M \otimes_R S)^{\Delta}$. Hence the composition $\mathscr{M}(\mathfrak{V}) = \widehat{g}^*(\mathscr{M})(\mathfrak{V}) \to (M \otimes_R S)^{\Delta}(\mathfrak{V}) = M \otimes_R S = \mathscr{M}(\mathfrak{U}) \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{V})$ defines an isomorphism $\mathscr{M}(V) \xrightarrow{\sim} \mathscr{M}(U) \otimes_{\widehat{R}_U} \widehat{R}_V$, where as before we identify the underlying spaces of $\mathfrak{U}, \mathfrak{V}$ with those of U, V.

For part (c), we prove the contrapositive. If $\widehat{g}_*(\mathscr{N})$ is not torsion-free, then there is an affine open subset $U' \subseteq U$ such that $\widehat{g}_*(\mathscr{N})(U')$ has torsion as a module over $\mathcal{O}_{\mathfrak{U}}(U') = \widehat{R}_{U'}$. That is, there exists a non-zero element $m \in \widehat{g}_*(\mathscr{N})(U') = \mathscr{N}(U' \cap V)$ and a regular element $r \in \widehat{R}_{U'}$ such that $rm = 0 \in \mathscr{N}(U' \cap V)$. Since $m \neq 0$, there is an affine open subset $V' \subseteq U' \cap V$ such that the image $m' \in \mathscr{N}(V')$ of m is non-zero. Let $r' \in \widehat{R}_{V'}$ be the image of $r \in \widehat{R}_{U'}$. Thus $r'm' = 0 \in \mathscr{N}(V')$ since $rm = 0$; and r' is regular in $\widehat{R}_{V'}$ by Lemma 4.5(b) applied to the inclusion $V' \subseteq U'$ of affine open sets. Hence $\mathscr{N}(V')$ is not a torsion-free module over $\widehat{R}_{V'} = \mathcal{O}_{\mathfrak{V}}(V')$, and so $\mathscr{N}$ is not a torsion-free $\mathcal{O}_{\mathfrak{V}}$-module. $\square$

Although Proposition 6.2(a) holds both for the properties of being torsion-free and coherent, Proposition 6.2(c) does not carry over in general to the coherent property (e.g., if

U is affine and $V \subset U$ is the complement of a principal divisor). But as we show in Theorem 6.6 below, coherence is preserved under pushforward if the complement of V in U has codimension at least two. First, we obtain the following special case of Theorem 6.6.

Lemma 6.3. *Let $\mathscr{X}, U_0, U_1, U_2, U, W$ be as in Proposition 5.3, let $\mathfrak{X}$ be the formal completion of $\mathscr{X}$, and let $\widehat{g} : \mathfrak{W} \to \mathfrak{U}$ be the inclusion of the formal open subschemes of $\mathfrak{X}$ that are associated to W, U. Let $\mathscr{N}$ be a torsion-free coherent sheaf on the formal scheme $\mathfrak{W}$ that is associated to W. Then $\widehat{g}_*(\mathscr{N})$ is a torsion-free coherent sheaf on $\mathfrak{U}$.*

Proof. By Proposition 6.1, $\mathscr{M} := \widehat{g}_*(\mathscr{N})$ is quasi-coherent on the formal scheme $\mathfrak{U}$. So by [Sta25, Lemma 01BK], for every point P of U there is an open neighborhood V of P such that $\mathscr{M}|_V$ is isomorphic to the $\mathcal{O}_{\mathfrak{V}}$-module associated to some $\Gamma(V, \mathcal{O}_{\mathfrak{V}})$-module M_V, where $\mathfrak{V}$ is the formal scheme associated to V. After shrinking V, we may assume that it is an affine open subset of U containing P. For $e = 0, 1, 2$, let $V_e = V \cap U_e$. Then $\mathscr{N}(V_e) \subseteq \mathscr{N}(V_0)$ for $e = 1, 2$ by Lemma 5.1, and $M_V = \mathscr{M}(V) = \widehat{g}_*(\mathscr{N})(V) = \widehat{g}_*(\mathscr{N}|_{V_1 \cup V_2})(V) = \mathscr{N}(V_1) \cap \mathscr{N}(V_2)$. Since $\mathscr{N}$ is coherent and since V_e is affine, $\mathscr{N}(V_e)$ is a finite module over $\mathcal{O}_{\mathfrak{U}}(V_e) = \widehat{R}_{V_e}$. Now V_1, V_2 are dense in V, since U_1, U_2 are assumed dense in U. So we may apply Proposition 5.3 to V_0, V_1, V_2, V, and conclude that M_V is a finitely generated torsion-free module over $\widehat{R}_V = \mathcal{O}_{\mathfrak{V}}(V)$. Thus $\mathscr{M}|_V = M_V^{\triangle}$ is coherent over $\mathcal{O}_{\mathfrak{V}}$, by [EGA1, Proposition 10.10.5]. Since this holds in a neighborhood of an arbitrary point P of U, it follows that $\mathscr{M}$ is coherent over $\mathfrak{U}$. It is also torsion-free, by Proposition 6.2(c). $\square$

Lemma 6.4. *Let V be a quasi-projective variety over a field.*

(a) There are affine dense open subsets $U_0, U_1, U_2 \subseteq V$ with $U_0 = U_1 \cap U_2$, such that the complement of $U_1 \cup U_2$ in V has codimension at least two.

(b) If V is connected, then we may choose U_0, U_1, U_2 in (a) such that for every connected open subset $O \subseteq V$ the intersection $O \cap U_e$ is connected for $e = 0, 1, 2$. In particular, U_0, U_1, U_2 are connected in this case.

Proof. It suffices to prove the lemma under the hypothesis that V is connected (i.e., proving part (b)), since part (a) then follows by considering the connected components of V.

Let $V_1, \ldots, V_n$ be the irreducible components of V, with generic points $\eta_1, \ldots, \eta_n$. For $i \neq j$, consider the irreducible components $V_{i,j,\ell}$ of $V_i \cap V_j$, and write $\eta_{i,j,\ell}$ for the generic point of $V_{i,j,\ell}$.

We first give a criterion for a non-empty open subset $O \subseteq V$ to be connected. Given O, let $S_O \subseteq \{1, \ldots, n\}$ be the set of indices i such that $\eta_i \in O$ (or equivalently, $O \cap V_i$ is non-empty). Thus the closure of O is the union of the irreducible components V_i for $i \in S_O$. So O is connected if and only if for every pair $i, j \in S_O$, there exists a chain of indices $i_0, \ldots, i_r \in S_O$ with $i_0 = i$ and $i_r = j$, such that for every $h = 0, \ldots, r-1$ the set O contains $\eta_{i_h, i_{h+1}, \ell}$ for some ℓ.

In particular, if an open subset $O \subseteq V$ contains each η_i (for $i = 1, \ldots, n$) and each of the points $\eta_{i,j,\ell}$ (for all i, j, ℓ), then O is connected and dense in V.

We now construct the open sets U_e asserted in the lemma. Since V is a quasi-projective variety over a field, by [Liu02, Proposition 3.3.36(b)] there exists an affine open subset $U_1 \subseteq V$ that contains each η_i and each $\eta_{i,j,k}$. Thus U_1 is a connected affine dense open subset of V, by the above criterion; and so the complement Z of U_1 in V has codimension at least one in V. Similarly, there exists an affine open subset $U_2 \subseteq V$ that contains each η_i, each

$\eta_{i,j,k}$, and the generic points of each irreducible component of Z. Thus U_2 is also a connected dense affine open subset of V. Hence so is $U_0 := U_1 \cap U_2$, which contains each of the points η_i, $\eta_{i,j,k}$. The intersection $U_2 \cap Z$ is dense in Z, since U_2 contains the generic points of Z. Thus the complement Y of $U_2 \cap Z$ in Z has codimension at least one in Z. Hence Y, which is also the complement of $U_1 \cup U_2$ in V, has codimension at least two in V, as asserted.

Finally, let $O \subseteq V$ be an arbitrary (non-empty) connected open subset; let $O_e = O \cap U_e$ for $e = 0, 1, 2$; and let the set S_O be as in the third paragraph of this proof. Thus for every pair $i, j \in S_O$, there is a chain of indices in S_O connecting i to j as above. Since U_e contains all the points η_i and $\eta_{i,j,\ell}$, it follows that $S_{O_e} = S_O$ for $e = 0, 1, 2$. Thus O_e also satisfies the above chain criterion, and hence it is connected. □

Lemma 6.5. *Let T be a complete discrete valuation ring with residue field k, and let $\mathscr{X}$ be a quasi-projective normal integral T-scheme with reduced closed fiber X and formal completion $\mathfrak{X}$. Let $f : V \to U$ be an inclusion of open subsets of X such that V is dense in U, and write $\mathfrak{V}, \mathfrak{U}$ for the formal open subschemes of $\mathfrak{X}$ associated to V, U. Let $\mathscr{F}$ be a torsion-free coherent sheaf on $\mathfrak{U}$. Then $\mathscr{F}$ is a subsheaf of $\widehat{f}_* \widehat{f}^*(\mathscr{F})$ via the natural morphism $\mathscr{F} \to \widehat{f}_* \widehat{f}^*(\mathscr{F})$.*

Proof. For every open subset $U' \subseteq U$, we have a restriction map $\mathscr{F}(U') \to \mathscr{F}(U' \cap V) = \widehat{f}_* \widehat{f}^*(\mathscr{F})(U')$, and these are compatible as U' varies. These maps define a morphism $\mathscr{F} \to \widehat{f}_* \widehat{f}^*(\mathscr{F})$. Since injectivity of sheaves is local, in order to show that $\mathscr{F}$ is a subsheaf of $\widehat{f}_* \widehat{f}^*(\mathscr{F})$ via this morphism, it suffices to show that if U' is affine then $\mathscr{F}(U') \to \mathscr{F}(U' \cap V)$ is injective. Let V' be an affine dense open subset of V. By Proposition 6.2(b), the natural map $\mathscr{F}(U' \cap V') \to \mathscr{F}(U') \otimes_{\widehat{R}_{U'}} \widehat{R}_{U' \cap V'}$ is an isomorphism. Since $U' \cap V'$ is dense in U', Lemma 5.1 then yields that the map $\mathscr{F}(U') \to \mathscr{F}(U' \cap V')$ is injective. But this map factors through $\mathscr{F}(U') \to \mathscr{F}(U' \cap V)$; and so that map is injective as well. Thus the torsion-free coherent sheaf $\mathscr{F}$ is a subsheaf of $\widehat{f}_* \widehat{f}^*(\mathscr{F})$, as asserted. □

We now come to our main theorem, which generalizes Lemma 6.3, and provides an analog for formal schemes of the assertion in Theorem 2.1.

Theorem 6.6. *Let T be a complete discrete valuation ring and let $\mathscr{X}$ be a quasi-projective normal integral T-scheme with reduced closed fiber X and formal completion $\mathfrak{X}$. Let U be a non-empty open subset of X, and let V be an open subset of U whose complement in U has codimension at least two. Let $f : V \to U$ be the inclusion map, and write $\mathfrak{U}, \mathfrak{V}$ for the formal open subschemes of $\mathfrak{X}$ associated to U, V. Let $\mathscr{F}$ be a torsion-free coherent sheaf on $\mathfrak{V}$. Then $\widehat{f}_*(\mathscr{F})$ is a torsion-free coherent sheaf on $\mathfrak{U}$.*

Proof. Let k be the residue field of T. Since $\mathscr{X}$ is quasi-projective over T, the k-scheme V is quasi-projective over k. By Lemma 6.4(a), we may choose affine open dense subsets $U_1, U_2 \subseteq V$ such that the complement of $W := U_1 \cup U_2$ in V has codimension at least two. Hence the complement of W in U also has codimension at least two. Let $g : W \to V$ be the inclusion map. By Proposition 6.2(a), pullbacks with respect to open inclusions preserve the property of being a torsion-free coherent sheaf; so $\widehat{g}^*(\mathscr{F})$ is a torsion-free coherent sheaf on the formal scheme $\mathfrak{W}$ associated to W. By Lemma 6.3 applied to $\widehat{g}^*(\mathscr{F})$ and the inclusion $fg : W \to U$, we have that $(\widehat{fg})_* \widehat{g}^*(\mathscr{F})$ is a torsion-free coherent sheaf on $\mathfrak{U}$.

By Lemma 6.5, $\mathscr{F}$ is a subsheaf of $\widehat{g}_* \widehat{g}^*(\mathscr{F})$ on $\mathfrak{V}$. Thus $\widehat{f}_*(\mathscr{F})$ is a subsheaf of $\widehat{f}_* \widehat{g}_* \widehat{g}^*(\mathscr{F}) = (\widehat{fg})_* \widehat{g}^*(\mathscr{F})$. Also, $\widehat{f}_*(\mathscr{F})$ is quasi-coherent by Proposition 6.1. Since $(\widehat{fg})_* \widehat{g}^*(\mathscr{F})$ is coherent on the locally Noetherian formal scheme $\mathfrak{U}$, it follows from [AJL99, Corollary 3.1.6(c)] that its quasi-coherent subsheaf $\widehat{f}_*(\mathscr{F})$ is coherent. It is also torsion-free, being a subsheaf of the torsion-free sheaf $(\widehat{fg})_*\widehat{g}^*(\mathscr{F})$ (or by Proposition 6.2(c)). $\square$

Remark 6.7. The proof of Theorem 6.6 relies in particular on Lemma 6.3, whose proof uses the technical results in Section 5 and therefore also builds on those in Section 3. As mentioned at the end of Section 2, it would be tempting to try to prove Theorem 6.6 more directly by using Theorem 2.1 or the ingredients used in its proof; viz., by applying such assertions about schemes to the reductions of the given sheaf modulo powers of the uniformizer of T. The difficulty with that approach is that these reductions need not be torsion-free. For example, take $T = k[[t]]$ for some field k and take $\mathscr{X} = \mathbb{A}^1_T$. Let U be the closed fiber $\mathbb{A}^1_k$, so that $\widehat{R}_U = k[x][[t]]$. Let M be the torsion-free $\widehat{R}_U$-module with two generators m, n and the single relation $xm - tn = 0$. In the reduction M_i of M modulo t^i, the element $t^{i-1}m$ is x-torsion, and x is regular. So M_i is not torsion-free, and (x, t^i) is a non-minimal associated prime of M_i (being the annihilator of $t^{i-1}m$), with support of codimension one in $X_i = \mathbb{A}^1_{T/(t^i)}$. Hence one cannot apply [Sta25, Lemma 0AWA] (or [EGA4, Partie 2, Corollaire 5.11.4(ii)]) to M_i. Note also that as in Theorem 2.1, the torsion-free hypothesis cannot simply be dropped; see the discussion after that assertion.

Corollary 6.8. *In the situation of Theorem 6.6, up to isomorphism, $\widehat{f}_*(\mathscr{F})$ is the maximum torsion-free coherent sheaf on $\mathfrak{U}$ whose restriction to $\mathfrak{V}$ is $\mathscr{F}$; i.e., every other such sheaf is a subsheaf of $\widehat{f}_*(\mathscr{F})$.*

Proof. By Theorem 6.6, $\widehat{f}_*(\mathscr{F})$ is a torsion-free coherent sheaf on $\mathfrak{U}$. Also, the restriction $\widehat{f}^*\widehat{f}_*(\mathscr{F})$ of $\widehat{f}_*(\mathscr{F})$ to $\mathfrak{V}$ is $\mathscr{F}$, since V is an open subset of U. Suppose that $\mathscr{G}$ is also a torsion-free coherent sheaf on $\mathfrak{U}$ whose restriction to $\mathfrak{V}$ is $\mathscr{F}$. Thus $\widehat{f}^*(\mathscr{G}) = \mathscr{F}$. By Lemma 6.5, $\mathscr{G} \subseteq \widehat{f}_*\widehat{f}^*(\mathscr{G}) = \widehat{f}_*(\mathscr{F})$. $\square$

Remark 6.9. In Theorem 6.6 and Corollary 6.8, it would suffice to assume that $\mathscr{X}$ is a normal integral T-scheme and that V is quasi-projective over k, rather than requiring $\mathscr{X}$ to be quasi-projective over T, because the proofs use only the weaker assumption.

7. The flat case

Proposition 5.3 says in particular that $\widehat{R}_{U_1} \cap \widehat{R}_{U_2}$ is a finitely generated torsion-free module over $\widehat{R}_U$, in the situation of affine dense open subsets where the complement of $U_1 \cup U_2$ has codimension at least two. In fact, more is true:

Proposition 7.1. *Let T be a complete discrete valuation ring, and let $\mathscr{X}$ be a normal integral T-scheme of finite type. Let U_0, U_1, U_2, U be connected affine open subsets of the reduced closed fiber X of $\mathscr{X}$, with $U_1, U_2 \subseteq U$ dense, and with $U_0 = U_1 \cap U_2$, such that the complement of $W := U_1 \cup U_2$ in U has codimension at least two. Then $\widehat{R}_{U_1} \cap \widehat{R}_{U_2} = \widehat{R}_U$, where the intersection takes place in $\widehat{R}_{U_0}$.*

Proof. Observe that $\widehat{R}_{U_e} \to \widehat{R}_{U_0}$ is injective for $e = 1, 2$ by Lemma 4.5(c), because U_0 is dense in U_e. Viewing $\widehat{R}_{U_e}$ as a subring of $\widehat{R}_{U_0}$, we let $A = \widehat{R}_{U_1} \cap \widehat{R}_{U_2} \subseteq \widehat{R}_{U_0}$. By Lemma 4.4(c), $\widehat{R}_U, \widehat{R}_{U_1}, \widehat{R}_{U_2}, \widehat{R}_{U_0}$ are domains; hence so is their subring A. We wish to show that $A = \widehat{R}_U$.

Let t be a uniformizer of T. For $n \geq 1$, and for $e = 0, 1, 2$, $\mathrm{Spec}(\widehat{R}_{U_e}/(t^n))$ has the same underlying topological space as U_e. So $\mathrm{Spec}(\widehat{R}_{U_0}/(t^n))$ is a Zariski dense open subset of $\mathrm{Spec}(\widehat{R}_{U_i}/(t^n))$ for $i = 1, 2$; and the map $\widehat{R}_{U_i}/(t^n) \to \widehat{R}_{U_0}/(t^n)$ is injective. We may thus form the intersection $A_n := \widehat{R}_{U_1}/(t^n) \cap \widehat{R}_{U_2}/(t^n)$ in $\widehat{R}_{U_0}/(t^n)$. We claim that the natural map $\alpha_n : A/t^n A \to A_n$ is injective. To see this, let $a \in \ker(\alpha_n)$, and pick a representative $\widetilde{a} \in A$ for a. Thus we may view $\widetilde{a} \in \widehat{R}_{U_i}$ for $i = 1, 2$; and the image of $\widetilde{a}$ in $\widehat{R}_{U_i}/(t^n)$ is trivial. Hence there exist $b_i \in \widehat{R}_{U_i}$ such that $\widetilde{a} = t^n b_i$ in $\widehat{R}_{U_i}$, for $i = 1, 2$. The elements $t^n b_i$, for $i = 1, 2$, have the same image in $\widehat{R}_{U_0}$; and thus the element $b_1 - b_2 \in \widehat{R}_{U_0}$ is t^n-torsion. But $\widehat{R}_{U_0}$ is a domain, and so $b_1 = b_2 \in \widehat{R}_{U_0}$. That is, the two elements $b_i \in \widehat{R}_{U_i}$ define an element $b \in A$. But $t^n b = \widetilde{a}$, since they have the same image in $\widehat{R}_{U_1}$ and since A is a subring of $\widehat{R}_{U_1}$. So $\widetilde{a} \in t^n A$, and a is trivial in $A/t^n A$, as claimed.

By Proposition 5.3, A is finite over $\widehat{R}_U$ as an extension of normal domains, say of generic degree $d \geq 1$. So tensoring the ring extension $\widehat{R}_U \subseteq A$ with the fraction field K_U of $\widehat{R}_U$, we obtain a finite field extension $K_U \subseteq A \otimes_{\widehat{R}_U} K_U = \mathrm{frac}(A)$ of degree d. It remains to show that $d = 1$, since then $\widehat{R}_U \subseteq A$ is a finite extension of normal domains having the same fraction field, and this inclusion is then an equality as desired.

Let $A' := A \otimes_{\widehat{R}_U} \widehat{R}_{U_1}$. By [Sta25, Lemma 00MA, (3)], $A' = A \otimes_{\widehat{R}_U} \varprojlim \widehat{R}_{U_1}/(t^n) = \varprojlim A'/t^n A'$. Now $A'/t^n A' = A/t^n A \otimes_{\widehat{R}_U/(t^n)} \widehat{R}_{U_1}/(t^n)$. But $A/t^n A \subseteq A_n$ via the injection α_n. Also, $\mathrm{Spec}(\widehat{R}_{U_1}/(t^n))$ is an affine open subset of $\mathrm{Spec}(\widehat{R}_U/(t^n))$ by [EGA1, Proposition 5.1.9], and so $\widehat{R}_{U_1}/(t^n)$ is flat over $\widehat{R}_U/(t^n)$. We thus obtain an inclusion

$$A'/t^n A' = A/t^n A \otimes_{\widehat{R}_U/(t^n)} \widehat{R}_{U_1}/(t^n) \subseteq A_n \otimes_{\widehat{R}_U/(t^n)} \widehat{R}_{U_1}/(t^n).$$

Let $W_n := \mathrm{Spec}(\widehat{R}_{U_1}/(t^n)) \cup \mathrm{Spec}(\widehat{R}_{U_2}/(t^n)) \subseteq \mathrm{Spec}(\widehat{R}_U/(t^n))$ and write $f_n : W_n \to \mathrm{Spec}(\widehat{R}_U/(t^n))$ for the natural inclusion. Thus $(f_n)_*(\mathcal{O}_{W_n})$ is a quasi-coherent sheaf on $\mathrm{Spec}(\widehat{R}_U/(t^n))$, and its module of global sections is

$$\mathcal{O}_{W_n}(\mathrm{Spec}(\widehat{R}_{U_1}/(t^n)) \cap \mathcal{O}_{W_n}(\mathrm{Spec}(\widehat{R}_{U_2}/(t^n)) = \widehat{R}_{U_1}/(t^n) \cap \widehat{R}_{U_2}/(t^n) = A_n.$$

Hence $A_n \otimes_{\widehat{R}_U/(t^n)} \widehat{R}_{U_1}/(t^n) = \Gamma(\mathrm{Spec}(\widehat{R}_{U_1}/(t^n)), (f_n)_*(\mathcal{O}_{W_n})) = \Gamma(\mathrm{Spec}(\widehat{R}_{U_1}/(t^n)), \mathcal{O}_{W_n}) = \widehat{R}_{U_1}/(t^n)$. That is, we have an inclusion $A'/t^n A' \subseteq \widehat{R}_{U_1}/(t^n)$. Since taking inverse limits is left exact, it follows that $A' \subseteq \widehat{R}_{U_1}$. Thus $A \otimes_{\widehat{R}_U} K_{U_1} = A' \otimes_{\widehat{R}_{U_1}} K_{U_1} \subseteq K_{U_1}$, where K_{U_1} is the fraction field of $\widehat{R}_{U_1}$.

Hence $K_{U_1}^d = K_U^d \otimes_{K_U} K_{U_1} = (A \otimes_{\widehat{R}_U} K_U) \otimes_{K_U} K_{U_1} = A \otimes_{\widehat{R}_U} K_{U_1} \subseteq K_{U_1}$, as K_{U_1}-vector spaces. Thus $d = 1$, completing the proof. □

Remark 7.2. (a) If the normality assumption is dropped from the hypotheses of the proposition, then the conclusion need not hold. For example, suppose that $\mathscr{X}$ is an affine integral T-variety with closed fiber $X = U$, with U_0, U_1, U_2 as before, such that $\mathscr{X}$ is normal at the points of $W = U_1 \cup U_2$ but not at all the points of U. Then $\widehat{R}_{U_1}, \widehat{R}_{U_2}$ are normal, and hence so is their intersection. But $\widehat{R}_U$ is not normal, and hence is strictly smaller than $\widehat{R}_{U_1} \cap \widehat{R}_{U_2}$.

(b) Suppose that $\mathscr{X}$ is a normal integral projective T-variety such that the reduced closed fiber X is a union of two copies of $\mathbb{P}^2_k$ meeting at a single point P. Let $U \subseteq X$ be the union of two copies of $\mathbb{A}_k^2$ meeting at P, and let U_1, U_2 respectively be the union of the complements of the x-axes (resp., y-axes) in the two copies of $\mathbb{A}_k^2$. Then $\widehat{R}_{U_1} = k[x,y,y^{-1}][[t]]^{\oplus 2}$ and $\widehat{R}_{U_2} = k[x,x^{-1},y][[t]]^{\oplus 2}$. So $\widehat{R}_{U_1} \cap \widehat{R}_{U_2} = k[x,y][[t]]^{\oplus 2}$, which is strictly larger than $\widehat{R}_U$, the difference being that the spectrum of the former consists of two disjoint copies of a thickened $\mathbb{A}_k^2$. This would not contradict the assertion of Proposition 7.1, because in this situation the connectivity hypothesis of the proposition does not hold.

Recall that by Proposition 6.5, given an inclusion $f : V \to U$ of open subsets of the reduced closed fiber of a quasi-projective normal integral T-scheme, with V dense in U, if $\mathscr{F}$ is a torsion-free coherent sheaf on the formal scheme $\mathfrak{U}$ associated to U, then $\mathscr{F}$ is a subsheaf of $\widehat{f}_*\widehat{f}^*(\mathscr{F})$. In the context of Proposition 7.1 (with $W = V$), this containment is an equality in the case that $\mathscr{F} = \mathcal{O}_{\mathfrak{X}}$. But in general for a torsion-free coherent sheaf $\mathscr{F}$, the containment $\mathscr{F} \subseteq \widehat{f}_*\widehat{f}^*(\mathscr{F})$ need not be an equality, as the following example shows, even in the situation of Proposition 7.1, where the complement of V in U has codimension at least two.

Example 7.3. Let k be a field, let $T = k[[t]]$, and let $\mathscr{X} = \mathbb{P}_T^2$, the projective x,y-plane over T, with closed fiber $X = \mathbb{P}_k^2$. Let $U = \mathbb{A}_k^2 \subset X$ and let $V \subset U$ be the complement of the origin, with inclusion morphism $f : V \to U$. Thus $V = U_1 \cup U_2$, where $U_1, U_2 \subset U$ are the complements in U of the x- and y-axes, respectively. Let $\mathfrak{U}, \mathfrak{V}, \mathfrak{U}_i$ be the formal schemes associated to U, V, U_i. Thus $\mathfrak{U} = \mathrm{Spf}(k[x,y][[t]])$. Let I be the ideal $(x,y) \subset k[x,y][[t]]$, and let $\mathscr{F} = I^\Delta$ be the coherent formal $\mathcal{O}_{\mathfrak{U}}$-module associated to I (see the beginning of Section 6). Note that $\mathscr{F}$ is torsion-free, but not flat (since it is not locally free). The pullback $\widehat{f}^*(\mathscr{F})$ to $\mathfrak{V}$ is the structure sheaf on $\mathfrak{V}$, and its pushforward $\widehat{f}_*\widehat{f}^*(\mathscr{F})$ is the structure sheaf on $\mathfrak{U}$. Thus $\mathscr{F} \subseteq \widehat{f}_*\widehat{f}^*(\mathscr{F})$ is a strict containment.

In contrast to Example 7.3, suppose that the coherent formal sheaf $\mathscr{F}$ in Lemma 6.5 is assumed to be flat (or equivalently, locally free), and not just torsion-free. If V is connected, then the containment $\mathscr{F} \subseteq \widehat{f}_*\widehat{f}^*(\mathscr{F})$ is an equality, as the following result shows.

Theorem 7.4. *Let T be a complete discrete valuation ring with residue field k, and let $\mathscr{X}$ be a quasi-projective normal integral T-scheme with reduced closed fiber X and formal completion $\mathfrak{X}$. Let $f : V \hookrightarrow U$ be an inclusion of connected open subsets of X such that the complement of V in U has codimension at least two, and write $\mathfrak{V}, \mathfrak{U}$ for the formal open subschemes of $\mathfrak{X}$ associated to V, U. Let $\mathscr{F}$ be a flat coherent sheaf on $\mathfrak{U}$. Then $\widehat{f}_*\widehat{f}^*(\mathscr{F}) = \mathscr{F}$.*

Proof. We first consider the special case in which $V = U_1 \cup U_2$, where U_1, U_2 and $U_0 := U_1 \cap U_2$ are connected affine dense open subsets of V with the property that the intersection of each U_e with every non-empty connected open subset of V is connected. To prove the assertion in this case, it suffices to show that for every non-empty connected affine open subset $O \subset U$, the map $\mathscr{F}(O) \to \mathscr{F}(O \cap V) = \mathscr{F}(O_1 \cup O_2)$ is an isomorphism, where $O_e = O \cap U_e$ for $e = 1, 2$. Here each O_e is also a connected affine open set, because we are in this special case and because O, U_e are affine. Let $\widehat{f}_e : \mathfrak{O}_e \to \mathfrak{O}$ be the inclusion map between the formal schemes $\mathfrak{O}_e, \mathfrak{O}$ associated to O_e, O, and let $M = \mathscr{F}(O)$. Thus M is the finitely generated flat $\widehat{R}_O$-module such that $\mathscr{F}|_{\mathfrak{O}} = M^\Delta$ (see the beginning of Section 6); and we have an induced isomorphism $\mathscr{F}(O_e) \xrightarrow{\sim} M \otimes_{\widehat{R}_O} \widehat{R}_{O_e}$, by Proposition 6.2(b).

Let $O_0 = O \cap U_0 = O_1 \cap O_2$. The connected affine open subsets O, O_0, O_1, O_2 satisfy the hypotheses of Proposition 7.1, because U, U_0, U_1, U_2 do, and because $O_e = O \cap U_e$. That proposition then implies that $\widehat{R}_O = \widehat{R}_{O_1} \cap \widehat{R}_{O_2} \subseteq \widehat{R}_{O_0}$. Thus we have a short exact sequence $0 \to \widehat{R}_O \to \widehat{R}_{O_1} \times \widehat{R}_{O_2} \to \widehat{R}_{O_0}$ of $\widehat{R}_O$-modules, where the maps are respectively given by diagonal inclusion and difference. Tensoring with the flat $\widehat{R}_O$-module M, we obtain a left exact sequence $0 \to M \to \mathscr{F}(O_1) \times \mathscr{F}(O_2) \to \mathscr{F}(O_0)$, using the isomorphism $\mathscr{F}(O_e) \xrightarrow{\sim} M \otimes_{\widehat{R}_O} \widehat{R}_{O_e}$ given in the previous paragraph. Since $M = \mathscr{F}(O)$, and since $\mathscr{F}(O_1 \cup O_2)$ is the intersection of $\mathscr{F}(O_1)$ and $\mathscr{F}(O_2)$ in $\mathscr{F}(O_0)$, we conclude that $\mathscr{F}(O) \to \mathscr{F}(O_1 \cup O_2)$ is an isomorphism, completing the proof in this special case.

For the general case, by Lemma 6.4(b) there is an open subset $W \subseteq V$ of the form $U_1 \cup U_2$ where each U_e is an affine dense open subset of V; where the set $O \cap U_e$ is connected for every connected open subset $O \subseteq V$ and each $e = 0, 1, 2$ (with $U_0 := U_1 \cap U_2$); and where the complement $V \smallsetminus W$ of W in V has codimension at least two in V. Thus the closure Z of $V \smallsetminus W$ in U has codimension at least two in U. Hence the complement of W in U also has codimension at least two in U, since this complement is the union of Z with the complement of V in U (which has codimension at least two by hypothesis).

Let $g : W \hookrightarrow V$ be the inclusion map. By the above special case applied to the sheaf $\mathscr{F}$ on $\mathfrak{U}$ and the inclusion $gf : W \hookrightarrow U$, we have that $(\widehat{f}\widehat{g})_*(\widehat{f}\widehat{g})^*(\mathscr{F}) = \mathscr{F}$. The sheaf $\widehat{f}^*(\mathscr{F})$ is torsion-free and coherent, by Proposition 6.2(a). So we can apply the above special case to the sheaf $\widehat{f}^*(\mathscr{F})$ on $\mathfrak{V}$ and the inclusion $g : W \hookrightarrow V$, obtaining $\widehat{f}^*(\mathscr{F}) = \widehat{g}_*\widehat{g}^*\widehat{f}^*(\mathscr{F})$. Hence $\widehat{f}_*\widehat{f}^*(\mathscr{F}) = \widehat{f}_*\widehat{g}_*\widehat{g}^*\widehat{f}^*(\mathscr{F}) = (\widehat{f}\widehat{g})_*(\widehat{f}\widehat{g})^*(\mathscr{F}) = \mathscr{F}$, as asserted. □

Remark 7.5. Concerning the necessity of the connectivity hypothesis in Theorem 7.4, consider the situation in Remark 7.2(b), and let $V = U_1 \cup U_2$ there, with inclusion morphism $f : V \to U$. Then the complement of V in U (viz., the point P) has codimension at least two in U, and V is disconnected. Let $\mathfrak{U}, \mathfrak{V}$ be the formal completions of U, V, with inclusion map $\widehat{f} : \mathfrak{V} \to \mathfrak{U}$. Then $\widehat{f}_*\widehat{f}^*(\mathcal{O}_{\mathfrak{U}})$ is strictly bigger than $\mathcal{O}_{\mathfrak{U}}$, since the former sheaf is "doubled" at the point P (corresponding to the strict containment $\widehat{R}_U \subset \widehat{R}_{U_1} \cap \widehat{R}_{U_2}$ in Remark 7.2(b)).

Let $f : V \hookrightarrow U$ be an inclusion of connected open subsets of the reduced closed fiber $X \subset \mathscr{X}$ as in Theorem 7.4, where the complement of V in U has codimension at least two. If $\mathscr{G}$ is a flat coherent sheaf on the formal scheme $\mathfrak{V}$ associated to V, the pushforward $\widehat{f}_*(\mathscr{G})$ is a torsion-free coherent sheaf on $\mathfrak{U}$ by Theorem 6.6, and we can ask whether it is flat. But in fact, flatness for such formal pushforwards need not hold in dimension at least three, even in the regular case, and similarly in the context of schemes (rather than formal schemes). In particular, there is the following example.

Example 7.6. (a) We first consider the scheme case. In [OSS11, Example 1.1.13], the authors give an example of a coherent sheaf F on $X = \mathbb{P}^3_{\mathbb{C}}$ that is reflexive (i.e., agrees with its double dual $F^{\vee\vee}$) but is not locally free (so not flat), though it is locally free away from a certain closed point x_0. Let V be the complement of x_0 in X, and let $G = F|_V$. Thus G is locally free. Write $f : V \to X$ for the inclusion map, so that $G = f^*F$. Since F is reflexive, and since the complement of V has codimension at least two (in fact, three), it follows from [Hts80, Proposition 1.6] that $F \cong f_*G$. Thus G is a flat coherent sheaf on V, but f_*G is not flat.

(b) We use the above example to produce an example in the case of formal schemes. Preserving the above notation, take $\mathscr{X} = \mathbb{P}^3_{\mathbb{C}[[t]]}$, and consider the coherent sheaf $\mathscr{F} := \pi^*(F)$ on $\mathscr{X}$, where $\pi : \mathscr{X} \to X = \mathbb{P}^3_{\mathbb{C}}$ is the morphism induced by the inclusion $\mathbb{C} \to \mathbb{C}[[t]]$. Observe first that $\mathscr{F}$ is a reflexive sheaf on $\mathscr{X}$; this follows from the fact that if M, N are modules over a ring R with M finitely presented, and if S is a flat R-algebra, then the natural map $S \otimes_R M^\vee \to (S \otimes_R M)^\vee$ is an isomorphism (see [Eis95, Proposition 2.10]). Note also that $\mathscr{F}$ is not locally free, since if it were then it would still be locally free (hence flat) modulo (t). Moreover the restriction $\mathscr{G}$ of $\mathscr{F}$ to $\mathscr{V} := V \times_{\mathbb{C}} \mathbb{C}[[t]]$ is locally free on $\mathscr{V}$, since the pullback of a free module is free. Let $\mathfrak{X}$ be the formal scheme associated to $\mathscr{X}$, let $\mathfrak{V}$ be the formal subscheme of $\mathfrak{X}$ associated to $V \subset X$, and let $\widehat{\mathscr{F}}, \widehat{\mathscr{G}}$ be the induced formal coherent sheaves on $\mathfrak{X}, \mathfrak{V}$. Then $\widehat{\mathscr{G}} = \widehat{f}^*(\widehat{\mathscr{F}})$ is flat because it is the completion of the finitely generated flat $\mathcal{O}_{\mathscr{X}}$-module $\mathscr{G}$; and $\widehat{\mathscr{F}}$ is not flat because it is not locally free (since its closed fiber is not locally free).

By Lemma 6.5, $\widehat{\mathscr{F}}$ is a subsheaf of $\widehat{f}_*\widehat{f}^*(\widehat{\mathscr{F}})$. To show that $\widehat{\mathscr{G}}$ is an example of a flat formal coherent sheaf on $\mathfrak{V}$ whose pushforward $\widehat{f}_*(\widehat{\mathscr{G}})$ to $\mathfrak{X}$ is not flat, it remains to prove that $\widehat{\mathscr{F}} = \widehat{f}_*\widehat{f}^*(\widehat{\mathscr{F}}) = \widehat{f}_*(\widehat{\mathscr{G}})$. To do this, it suffices to show that for every affine open subset U of X, we have $\widehat{\mathscr{F}}(U) = \widehat{f}_*\widehat{f}^*\widehat{\mathscr{F}}(U)$. This is trivial if $x_0 \notin U$; so we assume $x_0 \in U$, and write $U' = U \cap V = f^{-1}(U)$, the complement of x_0 in U. The restriction map $\widehat{\mathscr{F}}(U) \to \widehat{\mathscr{F}}(U')$ is given by the inclusion $\widehat{\mathscr{F}}(U) \subseteq \widehat{f}_*\widehat{f}^*\widehat{\mathscr{F}}(U) = \widehat{\mathscr{F}}(U')$; and our goal is now to show that this is an isomorphism.

To do this, first note that the inclusion $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \hookrightarrow \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}')$ is an isomorphism, by applying Theorem 7.4 to the sheaf $\mathcal{O}_{\mathfrak{U}} = \mathcal{O}_{\mathfrak{X}}|_U$. Since U is affine and $\widehat{\mathscr{F}}$ is a coherent formal sheaf, the restriction $\widehat{\mathscr{F}}|_U$ is of the form M^Δ for some finitely generated $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$-module M (by [EGA1, Propositions 10.10.5, 10.10.2(i)]); and so the natural map $\widehat{\mathscr{F}}(U) \to F(U)$ via reduction modulo (t) is surjective. But $F(U) = F(U')$, since $F \cong f_*G$ by part (a). Thus the reduction map $\widehat{\mathscr{F}}(U') \to F(U')$ modulo (t) is surjective, using that $\widehat{\mathscr{F}}(U) \subseteq \widehat{f}_*\widehat{f}^*\widehat{\mathscr{F}}(U) = \widehat{\mathscr{F}}(U')$. So $\widehat{\mathscr{F}}(U) + (t)\widehat{\mathscr{F}}(U') = \widehat{\mathscr{F}}(U')$. Also, the module $\widehat{\mathscr{F}}(U')$ is finitely generated over $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}') = \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$, and the ideal (t) is contained in the radical of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}')$. So by a version of Nakayama's Lemma (see [Mat80, Corollary to Lemma 1.M]), $\widehat{\mathscr{F}}(U) = \widehat{\mathscr{F}}(U') = \widehat{f}_*\widehat{f}^*\widehat{\mathscr{F}}(U)$, as desired. So indeed $\widehat{\mathscr{F}} = \widehat{f}_*\widehat{f}^*\widehat{\mathscr{F}}$, and $\widehat{\mathscr{G}}$ provides the asserted example.

But as we now show, if we restrict attention to the dimension two case, then there is a flatness assertion for pushforwards, in both the scheme and formal scheme situations.

Proposition 7.7. *(a) Let X be an excellent (e.g., quasi-projective) regular scheme of dimension two, and let $V \subseteq X$ be an open subset whose complement has codimension two in X. Write $f : V \to X$ for the inclusion map. If G is a flat coherent sheaf on V, then $f_*(G)$ is a flat coherent sheaf on X.*

(b) Let T be a complete discrete valuation ring, and let $\mathscr{X}$ be a two-dimensional regular quasi-projective flat T-scheme with reduced closed fiber X and formal completion $\mathfrak{X}$. Let $f : V \hookrightarrow U$ be an inclusion of open subsets of X such that the complement of V in U has codimension two, and write $\mathfrak{V}, \mathfrak{U}$ for the formal open subschemes of $\mathfrak{X}$ associated to V, U. Let $\mathscr{G}$ be a flat coherent sheaf on $\mathfrak{V}$. Then $\widehat{f}_*(\mathscr{G})$ is a flat coherent sheaf on $\mathfrak{U}$.

Proof. In part (a), $F := f_*(G)$ is a torsion-free coherent sheaf on X by Theorem 2.1, since G is a torsion-free coherent sheaf on V. Also, F is flat over V since G is; and so it remains to check flatness at the (isolated) points of X in the complement of V. Since flatness is local, we may assume that X is the spectrum of a two-dimensional regular local ring, and that V is the complement of the closed point P. Here $V \subset X$ is the only strict inclusion $U' \subset U$ of open subsets of X such that the complement of U' in U has codimension two. Moreover $f_*(F|_V) = f_*f^*F = f_*f^*f_*(G) = f_*(G) = F$, since $f^*f_*(G) = G$. So by [Hts80, Proposition 1.6], F is reflexive. Since X is regular of dimension two, it then follows from [Hts80, Corollary 1.4] that F is flat.

For part (b), Theorem 6.6 says that $\mathscr{F} := \widehat{f}_*(\mathscr{G})$ is a torsion-free coherent sheaf on $\mathfrak{U}$. We wish to show that it is flat. Take a closed point $P \in U$; we will show that $\mathscr{F}$ is free on an open neighborhood of P in $\mathfrak{U}$. Let G, F be the restrictions of $\mathscr{G}, \mathscr{F}$ to the reduced closed fiber X. Thus G is flat on V, and $F = f_*(G)$. By part (a), F is flat, and hence free on an affine open neighborhood W of P in U, say of rank n. After shrinking W, we may assume that there is an affine open neighborhood $\mathscr{W} = \operatorname{Spec}(R)$ of P in $\mathscr{X}$ that meets X in W. Since $\mathcal{O}_{\mathscr{X},P}$ is a regular local ring, it is a UFD, and every height one prime is principal. So after shrinking W again, we may assume that the closed subscheme $W \subset \mathscr{W}$ is defined by a principal ideal $I = (s)$ for some element $s \in R$; and this element is regular because $\mathscr{X}$ is flat over T. Let $\mathfrak{W}$ be the formal open subscheme of $\mathfrak{X}$ associated to W. Thus $\mathcal{O}_{\mathfrak{X}}(W) = \mathcal{O}_{\mathfrak{X}}(\mathfrak{W})$ is the I-adic completion $\widehat{R}$ of R. Also, $\mathcal{O}_X(W) = \bar{R} := R/I = \widehat{R}/I\widehat{R}$ is reduced, since X is.

By the freeness of F on W, we may choose an isomorphism $\bar{R}^n \to F(W) = \mathscr{F}(\mathfrak{W})/I\mathscr{F}(\mathfrak{W})$. Let $\bar{a}_1, \ldots, \bar{a}_n$ be the images of the standard basis elements of $\bar{R}^n$, and choose lifts $a_1, \ldots, a_n$ of the elements $\bar{a}_i$ to $\mathscr{F}(\mathfrak{W})$. We thus obtain a lift of the above isomorphism to a homomorphism $\widehat{R}^n \to \mathscr{F}(\mathfrak{W})$, taking the standard basis elements of $\widehat{R}^n$ to the lifts a_i. We claim that this map is injective. To see this, take some non-zero element $(r_1, \ldots, r_n) \in \widehat{R}^n$ in the kernel. Since $\mathscr{F}$ is torsion-free, we may divide the elements r_i by any common factor that is a power of s, and still have an element in the kernel. So we may assume that some r_i is not divisible by s. But then the image $(\bar{r}_1, \ldots, \bar{r}_n) \in \bar{R}^n$ of $(r_1, \ldots, r_n)$ is non-zero but is in the kernel of the isomorphism $\bar{R}^n \to F(W)$. This is a contradiction, proving the claim.

Thus the image of $\widehat{R}^n \to \mathscr{F}(\mathfrak{W})$ is a free R-submodule N of $\mathscr{F}(\mathfrak{W})$ with the property that $N + I\mathscr{F}(\mathfrak{W}) = \mathscr{F}(\mathfrak{W})$. Since $I\widehat{R}$ is contained in the radical of $\widehat{R}$ and $\mathscr{F}(\mathfrak{W})$ is a finitely generated $\widehat{R}$-module, $N = \mathscr{F}(\mathfrak{W})$ by [Mat80, Corollary to Lemma 1.M]. Thus $\mathscr{F}(\mathfrak{W})$ is free, as desired. $\square$

8. Patching problems

Given a ring R and overrings $R_0, R_1, R_2 \supseteq R$ with $R_1, R_2 \subseteq R_0$ and $R = R_1 \cap R_2 \subseteq R_0$, a *patching problem* for these rings consists of finitely generated R_e-modules M_e for $e = 0, 1, 2$, together with isomorphisms $\alpha_e : M_e \otimes_{\widehat{R}_e} R_0 \to M_0$ for $e = 1, 2$. A *solution* to the patching problem consists of a finitely generated R-module M together with isomorphisms

$\gamma_e : M \otimes_R R_e \to M_e$, for $e = 1, 2$, such that the diagram

$$\begin{tikzcd}
 & (M \otimes_R R_1) \otimes_{R_1} R_0 \arrow[r, "\gamma_1 \otimes \mathrm{id}"] & M_1 \otimes_{R_1} R_0 \arrow[dr, "\cong", "\alpha_1"'] & \\
M \otimes_R R_0 \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & & M_0 \\
 & (M \otimes_R R_2) \otimes_{R_2} R_0 \arrow[r, "\gamma_2 \otimes \mathrm{id}"] & M_2 \otimes_{R_2} R_0 \arrow[ur, "\alpha_2", "\cong"'] &
\end{tikzcd}$$

commutes. That is, the R-module M induces the modules M_e, for $e = 1, 2$, compatibly with the maps α_e.

For example, let $U = \mathrm{Spec}(R)$ be an affine scheme with affine dense open subsets $U_e = \mathrm{Spec}(R_e)$ for $e = 0, 1, 2$, such that $U_0 = U_1 \cap U_2$ and $U = U_1 \cup U_2$. Thus $R = R_1 \cap R_2 \subseteq R_0$. In this situation, every patching problem for the rings R, R_0, R_1, R_2 has a solution, by Zariski patching (gluing) of coherent sheaves and the correspondence between coherent sheaves on an affine scheme $\mathrm{Spec}(A)$ and finitely presented A-modules (e.g., see [Sta25, Lemmas 00AN, 01I9(1), 01IA]).

An analogous statement holds for formal schemes. Namely, let $\mathfrak{X}$ be a formal scheme with reduced closed fiber X, and let U, U_0, U_1, U_2 be affine dense open subsets of X such that $U_0 = U_1 \cap U_2$ and $U = U_1 \cup U_2$. Let $\mathfrak{U}, \mathfrak{U}_e$ be the formal schemes associated to U, U_e (i.e., the restrictions of $\mathfrak{X}$ to those subsets). As before, we write $\widehat{R}_U = \mathcal{O}_{\mathfrak{X}}(U)$ and $\widehat{R}_{U_e} = \mathcal{O}_{\mathfrak{X}}(U_e)$ for $e = 0, 1, 2$. Then $\widehat{R}_U = \widehat{R}_{U_1} \cap \widehat{R}_{U_2} \subseteq \widehat{R}_{U_0}$ since $\mathcal{O}_{\mathfrak{X}}$ is a sheaf; and every patching problem for the rings $\widehat{R}_U, \widehat{R}_{U_0}, \widehat{R}_{U_1}, \widehat{R}_{U_2}$ has a solution. To see this, recall that by [EGA1, Proposition 10.10.5], every coherent sheaf on $\mathfrak{U}$ is of the form M^Δ for some finitely generated $\widehat{R}_U$-module, and similarly for each $\mathfrak{U}_e$. Since the underlying space U of $\mathfrak{U}$ is the union of the underlying spaces of $\mathfrak{U}_1, \mathfrak{U}_2$, with intersection being the underlying space of $\mathfrak{U}_0$, a coherent sheaf on $\mathfrak{U}$ is given by finitely generated modules over $\widehat{R}_{U_1}, \widehat{R}_{U_2}$ together with an agreement over $\widehat{R}_{U_0}$. Hence every patching problem for the rings $\widehat{R}_U, \widehat{R}_{U_0}, \widehat{R}_{U_1}, \widehat{R}_{U_2}$ has a solution. Moreover, in these two situations (schemes and formal schemes), the solution is unique up to isomorphism, because there is an equivalence of categories between patching problems and finitely generated modules over the base ring (R or $\widehat{R}_U$, respectively).

The next result shows that if we restrict to torsion-free coherent formal sheaves, then patching problems have solutions even if $U_1 \cup U_2$ is strictly contained in U, with complement having codimension at least two.

Recall the situation of Proposition 5.3: We have a complete discrete valuation ring T with uniformizer t, and a normal integral T-scheme $\mathscr{X}$ of finite type. We consider affine open subsets U_0, U_1, U_2, U of the reduced closed fiber X of $\mathscr{X}$, with U_e a dense subset of U for $e = 1, 2$, and with $U_0 = U_1 \cap U_2$, such that the complement of $W := U_1 \cup U_2$ in U has codimension at least two. In this situation, we still have $\widehat{R}_U = \widehat{R}_{U_1} \cap \widehat{R}_{U_2} \subseteq \widehat{R}_{U_0}$, by Proposition 7.1. Let M_e be a finitely generated torsion-free $\widehat{R}_{U_e}$-module for $e = 0, 1, 2$. For $e = 1, 2$, we consider the natural map $\iota_e : M_e \to M_e \otimes_{\widehat{R}_{U_e}} \widehat{R}_{U_0}$, which is injective by Lemma 5.1; and we let $\alpha_e : M_e \otimes_{\widehat{R}_{U_e}} \widehat{R}_{U_0} \to M_0$ be an isomorphism. Then $\alpha_e \iota_e : M_e \to M_0$ is injective, mapping M_e isomorphically onto its image in M_0. As in Proposition 5.3, the intersection $M := \alpha_1\iota_1(M_1) \cap \alpha_2\iota_2(M_2) \subseteq M_0$ is a finitely generated torsion-free $\widehat{R}_U$-module.

Proposition 8.1. *In the above situation, for $e = 1, 2$ let $\gamma_e : M \otimes_{\widehat{R}_U} \widehat{R}_{U_e} \to M_e$ be the map induced by $M \hookrightarrow \alpha_e \iota_e(M_e) \cong M_e$. Then γ_e is an isomorphism for $e = 1, 2$. Moreover the finitely generated torsion-free module $\widehat{R}_U$-module M, together with the maps γ_e, defines a solution to the patching problem given by M_e, α_e.*

Proof. Let $\mathfrak{U}$, $\mathfrak{U}_e$ be the formal schemes associated to U, U_e (for $e = 0, 1, 2$), with inclusions $\widehat{f}_e : \mathfrak{U}_e \to \mathfrak{U}$. Let M_e^Δ be the coherent formal sheaf on $\mathfrak{U}_e$ associated to M_e. The isomorphisms α_e induce isomorphisms $M_e^\Delta \otimes_{\mathcal{O}_{\mathfrak{U}_e}} \mathcal{O}_{\mathfrak{U}_0} \to M_0^\Delta$, and so by [Sta25, Lemma 00AM] we may glue the sheaves M_e^Δ to obtain a sheaf of modules $\mathscr{N}$ on the formal scheme $\mathfrak{W}$ associated to $W := U_1 \cup U_2$. Here $\mathscr{N}(U_e) = \alpha_e \iota_e(M_e) \cong M_e$, where the isomorphism follows from the injectivity of $\alpha_e \iota_e$ shown in Proposition 5.3. Moreover $\mathscr{N}$ is coherent, since this is a local condition and since M_e^Δ is coherent. Let $\widehat{f} : \mathfrak{W} \to \mathfrak{U}$ be the natural inclusion, and let $\mathscr{M} = \widehat{f}_*(\mathscr{N})$. Then $\mathscr{M}$ is a coherent $\mathcal{O}_{\mathfrak{U}}$-module by Lemma 6.3.

Since U is affine, $\mathscr{M}$ is the formal sheaf associated to some finite $\widehat{R}_U$-module M', by [EGA1, Proposition 10.10.5]. Thus

$$M' = \mathscr{M}(U) = \mathscr{N}(W) = \mathscr{N}(U_1) \cap \mathscr{N}(U_2) = \alpha_1 \iota_1(M_1) \cap \alpha_2 \iota_2(M_2) = M,$$

and so $\mathscr{M} = M'^\Delta = M^\Delta$. By Proposition 6.2(b), we have a natural isomorphism between $\mathscr{M}(U_e)$ and $\mathscr{M}(U) \otimes_{\widehat{R}_U} \widehat{R}_{U_e}$. Since $\mathscr{M}(U_e) = \mathscr{N}(U_e) = M_e^\Delta(U_e) = M_e$ and $\mathscr{M}(U) = M$, we conclude that the natural map $M \otimes_{\widehat{R}_U} \widehat{R}_{U_e} = \mathscr{M}(U) \otimes_{\widehat{R}_U} \widehat{R}_{U_e} \to \alpha_e \iota_e(M_e) \cong M_e$ is an isomorphism. This proves the first assertion.

For the second assertion, observe that via the sheaf $\mathscr{M}$, and for $e = 1, 2$, the inclusions $U_0 \subseteq U_e \subseteq U$ induce the module homomorphisms α_e and γ_e. Here the compositions $\alpha_e \circ (\gamma_e \otimes \mathrm{id})$, for $e = 1, 2$, are both the homomorphism similarly induced by the inclusion $U_0 \subseteq U$. Thus M, together with α_1, α_2, defines a solution to the given patching problem. $\square$

In the above situation, though, the solution to the given patching problem need not be unique, as the next example shows.

Example 8.2. In the notation of Example 7.3, let $U_0 = U_1 \cap U_2$, and let $M_e = \widehat{R}_e$ for $e = 0, 1, 2$, with associated isomorphism $\alpha_e : M_e \otimes_{\widehat{R}_e} R_0 \to M_0$ for $e = 1, 2$. Then both $M = \widehat{R}_U = k[x, y][[t]]$ and the ideal $I = (x, y) \subset \widehat{R}_U$ are solutions to the patching problem, with I strictly contained in M. Here I is torsion-free but not flat; whereas M, which is the module given in Proposition 5.3, is flat.

As in this example, it is true more generally that the solution given in Proposition 8.1 is the maximum torsion-free solution:

Corollary 8.3. *In the situation of Propositions 5.3 and 8.1, suppose that $\mathscr{X}$ is quasi-projective over T. Then the module M, together with the maps γ_e for $e = 1, 2$, defines the maximum torsion-free solution to the patching problem given by the maps α_1, α_2 in Proposition 5.3.*

Proof. As in the proof of Proposition 8.1, the isomorphisms α_e and coherent formal sheaves M_e^Δ define a coherent formal sheaf $\mathscr{N}$ on the formal scheme $\mathfrak{W}$ associated to W. Here $\mathscr{N}$ and $\mathscr{M} := M^\Delta$ are torsion-free since M is. Moreover $\widehat{g}_*(\mathscr{N}) = \mathscr{M}$ and $\widehat{g}^*(\mathscr{M}) = \mathscr{N}$ by the definition of M. Under the correspondence $M \mapsto M^\Delta$ between finitely generated $\widehat{R}_U$-modules and coherent sheaves over the associated formal scheme $\mathfrak{U}$, a solution to the given module patching problem corresponds to a coherent sheaf $\mathscr{F}$ on $\mathfrak{U}$ whose restriction to $\mathfrak{W}$ is $\mathscr{N}$. Here the solution M corresponds to the sheaf $\mathscr{M}$. Now given any torsion-free solution to the embedding problem, the corresponding sheaf $\mathscr{F}$ on $\mathfrak{U}$ is also torsion-free. Since $\mathscr{F}$ restricts to $\mathscr{N}$ on $\mathfrak{W}$, and since $\mathscr{X}$ is quasi-projective, it follows from Corollary 6.8 that $\mathscr{F} \subseteq \widehat{g}_*(\mathscr{N}) = \mathscr{M}$, proving that M is maximum. $\square$

Note that maximality can fail without the torsion-free hypothesis. For example, any torsion $\widehat{R}_U$-module that is supported on the complement of W in $U \subset \operatorname{Spec}(\widehat{R}_U)$ is a solution to the trivial patching problem (i.e., the one defined by the zero modules over the rings $\widehat{R}_{U_e}$). In the flat case there is the following stronger assertion.

Corollary 8.4. *In Proposition 8.1, if $\mathscr{X}$ is quasi-projective over T and M is a flat $\widehat{R}_U$-module, then up to isomorphism M defines the unique flat solution to the patching problem given by α_1, α_2.*

Proof. We identify M_e with its isomorphic image $\alpha_e \iota_e(M_e) \subseteq M_0$. By Proposition 8.1, the module $M = M_1 \cap M_2$ and the maps γ_e define a solution to the patching problem. Suppose that M' is also a solution, and that M' is also flat. Let $\mathscr{F}$ be the formal coherent $\mathcal{O}_{\mathfrak{U}}$-module $(M')^{\Delta}$, and let $f : U_1 \cup U_2 \hookrightarrow U$ be the natural inclusion map. By Theorem 7.4, $M' = \mathscr{F}(U) = \widehat{f}_* \widehat{f}^* \mathscr{F}(U) = M_1 \cap M_2 = M$. $\square$

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