Fields of definition of p-adic covers

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Abstract. This paper concerns fields of definition and fields of moduli of G-Galois covers of the line over p-adic fields, and more generally over henselian discrete valuation fields. We show that the field of moduli of a p-adic cover will be a field of definition provided that the residue characteristic p does not divide $|G|$ and that the branch points do not coalesce modulo p (or in the more general case, that the branch locus is smooth on the special fibre). Hence if p does not divide $|G|$, then a G-Galois cover of the $\overline{\mathbf{Q}}$-line with field of moduli $\mathbf{Q}$ will be defined over a number field contained in $\mathbf{Q}_p$ if the branch points do not coalesce modulo p. This provides an explicit global-to-local principle for p-adic covers.

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§1: Introduction

Suppose we are given a base field K, a Galois field extension L/K, and an object ξ over L (e.g. a curve, or a cover of curves). A subfield $F \subset L$ containing K is a *field of definition* for ξ if there is a "model" for ξ over F, i.e. an object ξ_F over F such that $\xi_F \times_F L \approx \xi$. The natural candidate for the smallest field of definition (assuming there is one) is the *field of moduli M* of ξ; this is the fixed field of

$$\{\omega \in \mathrm{Gal}(L/K) \mid \xi^\omega \approx \xi \text{ over } L\}.$$

(Here ξ^ω is the conjugate of ξ under ω, i.e. the result of applying ω to ξ.) This field M is contained in every field of definition of ξ, and so if it is a field of definition then it will in fact be the smallest one. And for many objects , the field of moduli is indeed a field of definition — e.g. for covers of curves that either have no non-trivial covering automorphisms (cf. [Fr], Cor. 5.3) or are Galois (cf. [CH], Prop. 2.5).

But for other objects, the field of moduli need not be a field of definition. In this paper we focus on G-*Galois* covers of curves (also called "G-covers with Galois group G") — by definition these are Galois covers together with a fixed isomorphism between their Galois group and a given finite group G. (Cf. §2 of [DeDo1] for precise definitions.) For these objects, the *absolute* field of moduli over $\mathbf{Q}$ (i.e. the field of moduli relative to the Galois field extension $\overline{\mathbf{Q}}/\mathbf{Q}$) need not be a field of definition ([CH], Example 2.6). In fact the obstruction lies in $H^2(G(L/K), Z(G))$ ([DeDo1]; cf. also [Bel], [CH]). Thus G-Galois covers will be defined over their field of moduli if the Galois group $\mathrm{Gal}(L/K)$ is projective (regardless of G) or if the center $Z(G)$ is trivial. In the case that $L = K^{\mathrm{s}}$ (the separable closure of K), $\mathrm{Gal}(L/K)$ is the *absolute Galois group* G_K, and projectivity is equivalent to the condition that $\mathrm{cd}\, K \leq 1$ ([FrJa], Lemma 10.18). In particular, if K is a finite field [Dew1] or $K = \mathbf{Q}^{ab}$, then the field of moduli of any G-Galois cover relative to the field extension $\overline{K}/K$ will be a field of definition. The latter case has been combined with "rigidity" methods in the context of the inverse Galois problem over $\mathbf{Q}^{ab}$ (e.g. [Bel], [Ma], [Th]; cf. [Se2], Chap. 7,8 for further discussion of rigidity).

The current paper focuses on fields of definition and absolute fields of moduli of G-Galois covers of the line over henselian (e.g. p-adic) fields K, i.e. relative to the field extension K^s/K. In [Dew1], it was conjectured that given a G-Galois cover of the line defined over $\overline{\mathbf{Q}}$, a number field F is a field of definition if and only if all of its completions F_v (including the infinite ones) are fields of definition of the induced covers over $\overline{\mathbf{Q}}_p$. This "local-to-global principle" was later proven to hold for "most" number fields F, including $\mathbf{Q}$, in [De], Theorem 7.1 (with the possible exceptional F's corresponding to the special case of the Grunwald-Wang Theorem). In fact, it is even true if "all F_v" is replaced by "all but possibly one F_v" ([DeDo2], §3.4). Moreover, without restriction on the number field, F is the absolute field of moduli (over F) if and only if all but finitely many completions F_v are fields of definition (the "global-to-local principle" of [De], Theorem 8.1).

In [Dew1], the question was asked as to whether every p-adic G-Galois cover is defined over its field of moduli. As explained in [De], §8.1, there is a difficulty; in fact the results just cited imply that this is not the case, as was pointed out to us by H. Lenstra. Namely, Example 2.6 of [CH] provides a G-Galois cover of $\mathbf{P}^1_{\overline{\mathbf{Q}}}$ with field of moduli $\mathbf{Q}$ that is not defined over $\mathbf{Q}$ or even over $\mathbf{R}$. Thus the induced cover over $\mathbf{C}$ (obtained by completing over the infinite prime) is not defined over $\mathbf{R}$, which is the field of moduli of this completion. Meanwhile, the completion of this cover at any finite place p has field of moduli $\mathbf{Q}_p$. If all of these p-adic completions were defined over their fields of moduli, then the field of moduli would be a field of definition at all but one place (the infinite completion); and thus [DeDo2], §3.4, would imply that $\mathbf{Q}$ would be a field of definition — a contradiction.

This paper shows, though (Cor. 4.3), that for completions at "good primes" (in the sense of [Bec1]), the field of moduli of a G-Galois cover will indeed be a field of definition. This can be regarded as an explicit version of (the forward implication of) the global-to-local principle for G-Galois covers ([De], Theorem 8.1); that result asserted that there are only finitely many exceptional primes, but did not say which they are (although a bound on the exceptional primes was given in [Sad]). Corollary 4.3 follows from the Main Theorem 3.1, which applies in the more general situation of G-Galois covers over henselian fields K. That result says that if $f : X \to \mathbf{P}^1_{K^s}$ is a G-Galois cover whose branch points are each defined over K^s and remain distinct on the closed fibre, and if $|G|$ is not divisible by the residue characteristic, then the field of moduli is a field of definition. In fact the result asserts somewhat more, concerning the existence of "stable" models.

One key tool in our proof is S. Beckmann's Good Models Theorem (cf. [Bec2], Prop. 2.4), and another is the first author's Stability Criterion ([De], Lemma 8.2). A possible alternative approach to proving the theorem has been suggested to us by Michel Emsalem and the referee. This would use a result of Grothendieck-Murre ([GrMu], Thm. 4.3.2) and Fulton ([Fu], Thms. 3.3, 4.10) to relate the fields of moduli over the fraction field and over the residue field.

An interesting consequence of our Main Theorem is the following (Cor. 4.4): Consider a G-Galois cover of the line defined over $\overline{\mathbf{Q}}$ and assume that its field of moduli is $\mathbf{Q}$ (or more generally a number field not corresponding to the special case of the Grunwald-Wang theorem, as in [De], Thm. 7.1). Then in order to verify that $\mathbf{Q}$ is a field of definition, it is sufficient to check that the G-Galois cover is defined over $\mathbf{Q}_p$ for each "bad" prime p (in the above sense).

This paper is structured as follows: Section 2 contains some results that will be needed in the proof of the Main Theorem. Section 3 states and proves the Main Theorem 3.1, and section 4 contains several corollaries (including those mentioned above and in the abstract). Finally, section 5 contains several open questions.

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§2: Some useful results

Consider a G-Galois cover $f : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ and a base field $K \subset \overline{\mathbf{Q}}$ (typically $\mathbf{Q}$). In [Bec1] and [Bec2], S. Beckmann introduced the notion of *bad primes* of K, relative to the cover. This set, denoted S_{bad}, consists of the finite primes $\wp$ of K such that either the residue characteristic of $\wp$ divides $|G|$ or at least two geometric branch points coalesce modulo $\wp$ (i.e. at a prime of K^{s} over $\wp$). The set S_{bad} is finite, and the remaining non-archimedean primes of K are regarded as "good". By [Bec1], Thm. 5.5, the absolute field of moduli M of the G-Galois cover (over K) is ramified over K only at primes in S_{bad}.

In [Bec2], a model $f_F : X_F \to \mathbf{P}^1_F$ (over a number field F containing K) was defined to be a *good model* for f if the corresponding cover of normal arithmetic surfaces $f^o_F : X_{O_F} \to \mathbf{P}^1_{O_F}$ has no vertical ramification except over primes in S_{bad}. (Here "vertical ramification" refers to branching over a divisor of $\mathbf{P}^1_{O_F}$ supported over a prime of O_F.) Under an assumption on the class group of F, she showed ([Bec2], Prop. 2.4) that there is a good model over F, or equivalently a model over the arithmetic curve $U = (\operatorname{Spec} O_F) - S_{\text{bad}}$ that has *no* vertical ramification. The proof of this Good Models Theorem carries over to more general base curves $U = \operatorname{Spec} R$ without change, if the result is stated as follows:

Proposition 2.1. *Let R be a Dedekind domain with fraction field F, let G be a finite group whose center Z has exponent m, and assume that F contains a primitive mth root of unity. Suppose also that the ideal class group of R is m-torsion, and that no residue characteristic of R divides the order of G. Let $f : X \to \mathbf{P}^1_{F^{\text{s}}}$ be a G-Galois cover of regular F^{s}-curves having a model over F, say with branch locus $D \subset \mathbf{P}^1_F$. Suppose that the closure $\overline{D} \subset \mathbf{P}^1_R$ of D is étale over R. Then there is a model of the G-Galois cover $f : X \to \mathbf{P}^1_{F^{\text{s}}}$ over R having no vertical ramification.*

Remarks. (a) For each prime ideal $\wp$ of R, the property that $\overline{D}$ is étale over R at $\wp$ is equivalent to the property that the following two conditions hold:

(i) Each geometric point of D is defined over F^{s}.

(ii) No two F^{s}-points of D coalesce at any prime over $\wp$; i.e. for any two F^{s}-points of D, their closures in $\mathbf{P}^1_{R^{\text{s}}}$ do not meet over any prime of R^{s} lying over $\wp$, where R^{s} is the integral closure of R in K^{s}.

Indeed, condition (i) is equivalent to being generically étale, and condition (ii) is equivalent to the cardinality of each geometric fibre over $\wp$ equaling that of the geometric generic fibre (which, for a generically étale cover, is equivalent to being étale at $\wp$).

Note that condition (ii) can be rephrased more explicitly as follows: View $\mathbf{P}^1_K$ as the x-line, and consider two geometric points $\alpha = (x = a)$ and $\alpha' = (x = a')$, where $a, a' \in \overline{K} \cup \{\infty\}$. Then α, α' coalesce at a prime $\mathcal{P}$ of R^{s} over $\wp$ if $|a|_{\mathcal{P}} \le 1$, $|a'|_{\mathcal{P}} \le 1$, and

$|a - a'|_{\mathcal{P}} < 1$, or else if $|a|_{\mathcal{P}} \geq 1$, $|a'|_{\mathcal{P}} \geq 1$, and $|a^{-1} - a'^{-1}|_{\mathcal{P}} < 1$. (Here we interpret "$\infty^{-1}$" to be 0.)

(b) In the situation of [Bec2], Prop. 2.4, the domain R would be taken to be the ring of functions on $(\operatorname{Spec} O_F) - S_{\mathrm{bad}}$. Note that the papers [Bec1] and [Bec2] refer explicitly only to condition (ii) above, but condition (i) is automatic there since those papers concern fields of characteristic 0, where $K^{\mathrm{s}} = \overline{K}$. The proof of the above proposition, though, uses (i) as well as (ii). Namely, [Bec1] uses that there is a finite Galois extension K of F over which the branch locus becomes rational, and in the general case this is guaranteed by (i). The construction in the proof of [Bec1], Prop. 5.3, is used in [Bec2], Prop. 2.3, and thus indirectly in [Bec2], Prop. 2.4. So it is necessary to include hypothesis (i) above in order to use the same argument as before. It is unclear whether this hypothesis is essential for the truth of the above proposition, however, or whether (ii) alone would suffice.

(c) Although the main result of [Bec2] (Theorem 1.2 there) assumes that the branch points are individually defined over the field K, this hypothesis is not used in the proof of [Bec2], Prop. 2.4, and so is not required above.

(d) In the applications below of Proposition 2.1, we will take the Dedekind domain R to be local (i.e. a discrete valuation ring). In this situation the ideal class group is trivial, and so the m-torsion hypothesis of Proposition 2.1 is automatically satisfied. □

After using Proposition 2.1 in order to perform a descent from the separable closure to a smaller field (viz. the maximal unramified extension; cf. below), a second descent will be accomplished via the following result, which uses E. Dew's notion of "stable models" [Dew2]: Given a base field K, a model $f_L : X_L \to \mathbf{P}^1_L$ of a G-Galois cover $f : X \to \mathbf{P}^1_{K^{\mathrm{s}}}$ is *stable* over K if the field of moduli of f_L, relative to the Galois extension L/K, is equal to the (absolute) field of moduli M of f over K. Note that as L grows (i.e. if L is replaced by an extension $L' \subset K^{\mathrm{s}}$ and f_L by $f_{L'} = f_L \times_L L'$), the relative fields of moduli of the induced covers will drop, and eventually will equal the absolute field of moduli M; and a sufficiently large base change of any given model is stable. Also note that if K is replaced by a larger field $K' \subset L$, then the field of moduli of f_L relative to K' is the compositum of K' with the field of moduli of f_L relative to K. Since M is contained in every relative field of moduli of f_L, by taking $K' = M$ it follows that f_L is stable over K if and only if it is stable over M.

Proposition 2.2. *Let K be a field, and let $f : X \to \mathbf{P}^1_{K^{\mathrm{s}}}$ be a G-Galois cover with field of moduli K. Let L be a Galois extension of K, and let $f_L : X_L \to \mathbf{P}^1_L$ be a model of the cover f (without the G-action) such that the fibre of f_L over some unramified K-rational point of $\mathbf{P}^1_K$ consists entirely of L-rational points.*

(a) Then the G-action on f is induced by such an action on f_L, and this G-Galois cover f_L is a stable model for f relative to K.

(b) If $\operatorname{Gal}(L/K)$ is projective, then the G-Galois covers f_L and f are induced by a G-Galois model $f_K : X_K \to \mathbf{P}^1_K$ over K.

Proposition 2.2 is a strengthening of the statement of [De], Lemma 8.2, which was used in the proof of [De], Theorem 8.1 (cited in §1). The proof in [De] showed more than was claimed there, though, and indeed it proves the above result *mutatis mutandis.* Specifically, the first part of Proposition 2.2 above is proven in 8.2.1 and 8.2.2 of [De], and the second part is proven in 8.2.3 of [De]. Although [De] restricted attention to number fields and p-adic fields, that was not used in the proof there. The only point to be careful about here is to consider fields of moduli of covers over K^{s}, rather than over $\overline{K}$ (which of course is equal to K^{s} in the characteristic 0 situation of [De]).

Our main result will be stated for henselian discrete valuation fields, and so will hold in particular for complete fields. Recall that a valuation ring (R, v) is *henselian* if it satisfies Hensel's Lemma, and in this case the fraction field F (together with the valuation v) is a *henselian* field. This is equivalent to the property that v has a unique extension to each algebraic field extension of F ([Ri], p.176). Thus any algebraic extension of a henselian field is henselian. In particular, if F is henselian then so is its maximal unramified extension F^{ur} (i.e. the fraction field of the maximal unramified extension of the corresponding local ring R). If K is henselian with residue field k, then k^{s} is the residue field of K^{ur}. Also, Hensel's Lemma implies that $\mathrm{Hom}_{K-\mathrm{alg}}(K', K'') \approx \mathrm{Hom}_{k-\mathrm{alg}}(k', k'')$ for any K-algebras K', K'' having residue fields k', k'' with K'/K unramified. Thus in particular $\mathrm{Gal}(K^{\mathrm{ur}}/K) = G_k$. We also have the following result concerning the cohomological dimension of henselian fields:

Proposition 2.3. *Let K be a henselian field whose residue field k is perfect. Then* $\mathrm{cd}(K) \leq \mathrm{cd}(k) + 1$.

Proof. Let ℓ be a prime number. If $\ell \neq \mathrm{char}\, K$ and $\mathrm{cd}(k) < \infty$, then $\mathrm{cd}_\ell(K) = \mathrm{cd}_\ell(k) + 1$ by [AGV], X, Theorem 2.2. If $\ell = \mathrm{char}\, K$ then $\mathrm{cd}_\ell(K) \leq 1$ by [Se1], II, §4.3, Prop. 12. So $\mathrm{cd}_\ell(K) \leq \mathrm{cd}_\ell(k) + 1$ for all ℓ, showing the result. □

For a given valuation field (F, v), its minimal separable algebraic extension that is henselian is the *henselization* of (F, v). For example, the henselization of $k(t)$ for the t-adic valuation is the field of algebraic Laurent series (i.e. the algebraic closure of $k(t)$ in $k((t))$), and the henselization of $\mathbf{Q}$ for the p-adic valuation is the field of algebraic p-adic numbers (i.e. the algebraic closure of $\mathbf{Q}$ in $\mathbf{Q}_p$). The Main Theorem will apply not only to complete fields like $\mathbf{Q}_p$ and $k((t))$, but also to these corresponding henselian subfields of algebraic elements.

§3: Main theorem

Let O be a Dedekind domain with fraction field K, let F be a field extension of K, and let R be the integral closure of O in F. If D is a proper closed subset of $\mathbf{P}^1_K$ and $\wp$ is a maximal ideal of O, we will say that D is *smooth* at $\wp$ if its closure $\overline{D} \subset \mathbf{P}^1_R$ is étale over R at each maximal ideal of R lying over $\wp$. As noted in the remarks after Proposition 2.1, this is equivalent to the branch points being defined over K^{s} and not coalescing modulo $\wp$.

Main Theorem 3.1. *Let K be the fraction field of a henselian discrete valuation ring $(O, \wp)$ whose residue field k is perfect. Let G be a finite group, and let $f : X \to \mathbf{P}^1_{K^{\mathrm{s}}}$ be a G-Galois cover of regular K^s-curves with field of moduli M. Assume that the degree of this cover is not divisible by* $\mathrm{char}\, k$*, and that its branch locus is smooth at $\wp$. Let M^{ur} be the maximal unramified extension of M in K^{s}.*

(a) Then the G-Galois cover f has a stable model $f_{M^{\mathrm{ur}}} : X_{M^{\mathrm{ur}}} \to \mathbf{P}^1_{M^{\mathrm{ur}}}$ relative to K.

(b) If $\mathrm{cd}\, k \leq 1$, then M is a field of definition of the G-Galois cover f and also of the model $f_{M^{\mathrm{ur}}}$.

Remark. After reducing to the case $M = K$, the proof will first use Proposition 2.1 and Proposition 2.2(a) in order to descend from K^{s} to K^{ur}. Then, if the extra hypothesis in 3.1(b) is satisfied (e.g. for p-adic fields), we will use Proposition 2.2(b) in order to descend to K. These steps correspond to invoking the projectivity of $\mathrm{Gal}(K^{\mathrm{s}}/K^{\mathrm{ur}})$ and of $\mathrm{Gal}(K^{\mathrm{ur}}/K)$, respectively. An extra step is needed in the case that the base space has no unramified rational points on the closed fibre.

Proof of 3.1. Since M is a finite separable extension of K, we have that $K^{\mathrm{s}} = M^{\mathrm{s}}$; that the residue field m of M is perfect; and that $\mathrm{cd}\, m \le 1$ provided that $\mathrm{cd}\, k \le 1$. Moreover a model of f will be stable over K if and only if it is stable over M (as observed just before the statement of Proposition 2.2). So replacing K by M, we may assume that $M = K$.

As remarked at the end of section 2, K^{ur} is henselian with residue field k^{s}. Since k is perfect, its separable closure k^{s} is equal to its algebraic closure $\overline{k}$. Thus $\overline{k}$ is the residue field of K^{ur}, and Proposition 2.3 yields $\mathrm{cd}(K^{\mathrm{ur}}) \le \mathrm{cd}(\overline{k}) + 1 = 1$. Thus $\mathrm{cd}(K^{\mathrm{ur}}) \le 1$, or equivalently $G_{K^{\mathrm{ur}}}$ is projective ([FrJa], Lemma 10.18). But K^{ur} is the field of moduli of the G-Galois cover f relative to the extension $K^{\mathrm{s}}/K^{\mathrm{ur}}$ (i.e. viewing K^{ur} as the base field). So by Cor. 3.3 of [DeDo1] (as discussed in §1 above), there is a G-Galois model of f over K^{ur}, say $f_{K^{\mathrm{ur}}} : X_{K^{\mathrm{ur}}} \to \mathbf{P}^1_{K^{\mathrm{ur}}}$. Thus there is also a finite unramified extension F of K over which f has a model.

Since $d = |G|$ is not divisible by $\mathrm{char}\, k$, it follows that $\overline{k}$ contains a primitive dth root of unity; hence so does K^{ur}, and thus we may assume (after enlarging F) that so does F. In particular, F contains a primitive mth root of unity, where m is the exponent of the center of G. Let R be the integral closure of O in F. Since O is a henselian discrete valuation ring, so is R (since there is a unique extension to R of the valuation of O by [Ri], p.186), and so its class group is trivial. Thus the hypotheses of Proposition 2.1 hold, and so there is a (normal) model $f_R : X_R \to \mathbf{P}^1_R$ having no vertical ramification over $\wp$. Let $f_F : X_F \to \mathbf{P}^1_F$ be the generic fibre of f_R, let $D \subset \mathbf{P}^1_F$ be the branch locus of f_F, and let $\overline{D} \subset \mathbf{P}^1_R$ be the closure of D in $\mathbf{P}^1_R$.

We now consider two cases:

Case 1: There is a K-point α of $\mathbf{P}^1$ such that $D \cup \{\alpha\}$ is smooth over $\wp$, where D is the branch locus of f_F. (That is, we assume that α does not meet any of the branch points residually over $\wp$.) Also, the model $f_R : X_R \to \mathbf{P}^1_R$ is generically unramified over the special fibre. So by Purity of Branch Locus ([Na], Theorem 41.1), the closure of α in $\mathbf{P}^1_R$ does not meet the branch locus of f_R. Thus (as in [Bec2], Lemma 3.1), we have that $f_F^{-1}(\alpha) \subset X(K^{\mathrm{ur}})$. Hence the model $f_{K^{\mathrm{ur}}}$ is a stable G-Galois cover, by Proposition 2.2(a) (with $L = K^{\mathrm{ur}}$). This proves (a) of the theorem in Case 1. Now if $\mathrm{cd}\, k \le 1$, then the absolute Galois group G_k is projective. But since K is henselian with residue field k, restriction to the closed fibre induces an isomorphism $\mathrm{Gal}(K^{\mathrm{ur}}/K) \approx G_k$. So Proposition 2.2(b) implies that the G-Galois covers f and $f_{K^{\mathrm{ur}}}$ descend to K, thereby proving (b) of the theorem in Case 1. (Alternatively, we may use [DeDo1], Cor. 3.3, instead of Proposition 2.2(b).)

Case 2: Otherwise. Then the closure of every K-point of $\mathbf{P}^1$ meets $\overline{D}$ over $\wp$. Hence $\mathbf{P}^1(k) = \Delta(k)$, where Δ is the intersection of $\overline{D}$ with the closed fibre. Thus $\mathbf{P}^1(k)$ is finite, and k is a finite field. Since $\overline{k}$ is infinite, there exists $\overline{\alpha}' \in \mathbf{P}^1(\overline{k})$ that does not lie on Δ. Since k is finite, there are infinitely many finite field extensions k'/k of degree relatively prime to $[k(\overline{\alpha}') : k]$, and each is generated by a primitive element. Thus there is an $\overline{\alpha}'' \in \mathbf{P}^1(\overline{k})$ that does not lie on Δ and such that $[k(\overline{\alpha}') : k]$ and $[k(\overline{\alpha}'') : k]$ are relatively prime. Since $\overline{k}$ is the residue field of K^{ur}, there exist $\alpha', \alpha'' \in \mathbf{P}^1(K^{\mathrm{ur}})$ that lift $\overline{\alpha}', \overline{\alpha}''$ respectively. Here $[K(\alpha') : K] = [k(\overline{\alpha}') : k]$ and similarly for α'', by Hensel's Lemma. So $[K(\alpha') : K]$ and $[K(\alpha'') : K]$ are relatively prime, and thus $K(\alpha') \cap K(\alpha'') = K$. The fields $K' = K(\alpha')$ and $K'' = K(\alpha'')$ are each contained in K^{ur}, since α' and α'' are. Thus $K'^{\mathrm{ur}} = K^{\mathrm{ur}} = K''^{\mathrm{ur}}$.

Now K' and K'' are henselian, since they are algebraic extensions of the henselian field K. Also, their residue fields k' and k'' are perfect, since they are finite separable extensions of the perfect field k. So we may let K' [resp. K''] play the role of K in part (a) of Case 1 of the theorem, with α' [resp. α''] playing the role of α. By the conclusion of Case 1 for K', the G-Galois cover f has a stable model $f_{K^{\mathrm{ur}}} : X_{K^{\mathrm{ur}}} \to \mathbf{P}^1_{K^{\mathrm{ur}}}$ relative to K' (using $K'^{\mathrm{ur}} = K^{\mathrm{ur}}$). Thus the field of moduli of $f_{K^{\mathrm{ur}}}$, relative to K, is contained in K'. Similarly this field of moduli is contained in K''. Since $K' \cap K'' = K$, the field of moduli of $f_{K^{\mathrm{ur}}}$ over K is K itself. That is, $f_{K^{\mathrm{ur}}}$ is a stable model, proving part (a).

For (b), suppose that $\operatorname{cd} k \le 1$. By part (a), the field of moduli of $f_{K^{\mathrm{ur}}}$ is K. Now $\mathrm{Gal}(K^{\mathrm{ur}}/K)$ is isomorphic to the projective group G_k, since K is henselian with residue field k. So again using [DeDo1], Cor. 3.3, there is a model f_K over K for the G-Galois cover $f_{K^{\mathrm{ur}}}$. Since $f_{K^{\mathrm{ur}}}$ is a model for f, it follows that f_K is also a model for f. This proves (b). □

Remarks. (a) By Remark (a) after Proposition 2.1, the smoothness hypothesis of Theorem 3.1 will be satisfied if the branch locus consists of K^{s}-points that do not coalesce over $\wp$. Moreover, by Lemma 3.3 of [LL], if X is *smooth* over K, and not merely regular, then the branch points are each automatically defined over K^{s}. Cf. also Remark (a) after Corollary 4.1 below.

(b) In part (b) of the Main Theorem, the fact that K is a field of definition corresponds to the vanishing of a certain explicit cocycle in $H^2(k, Z(G))$, where $Z(G)$ is the center of G, and where $G_k \approx \mathrm{Gal}(K^{\mathrm{ur}}/K)$ as above; cf. [DeDo1], Main Theorem II(e).

(c) In the above theorem, $\operatorname{cd} k \le 1$ if and only if $\operatorname{cd} K \le 2$, by Proposition 2.3. □

§4: Some corollaries

We consider some consequences of the Main Theorem 3.1. First, we consider the case of a *local field* K, in the sense of number theory – i.e. a non-trivial completion of a global field. Thus if the global field is a number field, then K is a finite extension of $\mathbf{Q}_p$ for some p, or else is $\mathbf{R}$ or $\mathbf{C}$; while if the global field is the function field of a curve over a finite field, then K is a finite extension of some $\mathbf{F}_p((t))$. In the case that the local field K is *non-archimedean* (so a finite extension of $\mathbf{F}_p((t))$ or of $\mathbf{Q}_p$), our Main Theorem applies:

Corollary 4.1. *Let K be a non-archimedean local field of residue characteristic p, let G be a finite group, and let $f : X \to \mathbf{P}^1_{K^{\mathrm{s}}}$ be a G-Galois cover of regular K^{s}-curves with field of moduli K. If p does not divide $|G|$ and if the branch locus consists of K^{s}-points that remain distinct modulo the maximal ideal of $O_{K^{\mathrm{s}}}$, then there is a G-Galois model $f_K : X_K \to \mathbf{P}^1_K$ of f.*

Proof. By Remark (a) after the proof of Theorem 3.1, the closure of the branch locus in $\mathbf{P}^1_{O_K}$ is étale over O_K; i.e. the branch locus is smooth at the maximal ideal of O_K.

Moreover, the residue field of O_K is perfect and has cd = 1. So the hypotheses of part (b) of Theorem 3.1 apply, and the conclusion follows. □

Remarks. (a) In the case that the local field K is p-adic, it has characteristic 0, and so every branch point of f is automatically defined over $K^{\mathrm{s}} = \overline{K}$. Thus this hypothesis need not be explicitly assumed in this case. But for local fields K of equal characteristic, it is possible for a cover over K^{s} to have branch points that are not defined over K^{s}; e.g. the C_2-Galois cover $y^2 = x^p - t$, with $K = \mathbf{F}_p((t))$ where $p \neq 2$. Observe that this cover is regular (indeed, it is the general fibre of a surface that is smooth over $\mathbf{F}_p$), but it is not smooth over K (and so Lemma 3.3 of [LL] does not apply; cf. Remark (a) after the proof of Theorem 3.1 above). This is possible because the field K is not perfect. Finally, note that this cover is already defined over K; and so it remains unclear if the hypothesis on K^{s}-valued branch points is essential in order for the conclusion of Corollary 4.1 to be valid.

(b) One can also consider the question of when the field of moduli is a field of definition in the case of *archimedean* local fields. The question is trivial for $K = \mathbf{C}$, but is interesting in the case of $K = \mathbf{R}$. Of course the hypotheses of Cor. 4.1 do not make sense in this situation (since there is no residue field), and the answer to the question is not always yes (as noted in the introduction to this paper). But for each cover the question can be decided by topological methods; see Thm. 1.1 and §3.5 of [DeFr]. □

A corollary about more global fields is the following:

Corollary 4.2. *Let Q be a field and $f : X \to \mathbf{P}^1_{Q^{\mathrm{s}}}$ a G-Galois cover over Q^{s}. Let M be its field of moduli relative to the extension Q^{s}/Q and let $\widetilde{M}_v$ denote the henselization of M at a discrete valuation v. Let $S_{M,\mathrm{bad}}$ be the set of discrete valuations v of M at which*

- the residue characteristic divides $|G|$, or,

- the branch locus of the cover is not smooth, or,

- the residue field of $\widetilde{M}_v$ is not perfect.

Then, for each $v \notin S_{M,\mathrm{bad}}$, the G-Galois cover f has a model $f_{\widetilde{M}_v^{\mathrm{ur}}}$ over $\widetilde{M}_v^{\mathrm{ur}}$ that is stable and has no vertical ramification over v. If in addition, the Galois group $\mathrm{Gal}(\widetilde{M}_v^{\mathrm{ur}}/\widetilde{M}_v)$ is projective, then $\widetilde{M}_v$ is a field of definition of the G-Galois covers $f_{\widetilde{M}_v^{\mathrm{ur}}}$ and f.

Proof. For each $v \notin S_{M,\mathrm{bad}}$, we may apply the Main Theorem to the field $K = \widetilde{M}_v$. □

Consider the classical case $Q = \mathbf{Q}$. Then $S_{M,\mathrm{bad}}$ is just the (finite) set of places of M that lie over S_{bad} (cf. section 2, taking the base field $K = \mathbf{Q}$), i.e. the places lying over prime numbers that divide $|G|$ or modulo which two geometric branch points coalesce. In this situation, we have the following result, which provides the explicit global-to-local principle:

Corollary 4.3. *Let $f : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ be a G-Galois cover with field of moduli M.*

(a) Then for every $p \notin S_{\mathrm{bad}}$, there is a number field $K(p)$ contained in $\mathbf{Q}_p$ such that the G-Galois cover f is defined over the compositum $MK(p)$.

(b) For every $p \notin S_{\mathrm{bad}}$ and every prime $\wp$ of M over p, the induced G-Galois cover $f_\wp : X_{\overline{\mathbf{Q}}_p} \to \mathbf{P}^1_{\overline{\mathbf{Q}}_p}$ is defined over the completion $M_\wp$, which is its field of moduli over $\mathbf{Q}_p$.

Proof. (a) Let $\widetilde{\mathbf{Q}}_p$ be the (abstract) henselization of $\mathbf{Q}$ at p, and choose an embedding $\widetilde{\mathbf{Q}}_p \hookrightarrow \overline{\mathbf{Q}}$. (The image is a henselization of $\mathbf{Q}$ inside $\overline{\mathbf{Q}}$ at p, and any two such choices are conjugate to each other under the action of $G_{\mathbf{Q}}$.) Then the compositum $M\widetilde{\mathbf{Q}}_p$ is the field of moduli of f relative to the extension $\overline{\mathbf{Q}}/\widetilde{\mathbf{Q}}_p$. By the Main Theorem (taking $K = M\widetilde{\mathbf{Q}}_p$), we have that f has a model over $M\widetilde{\mathbf{Q}}_p$. Since G-Galois covers are of finite type, it follows that f is actually defined over the compositum $MK(p)$ for some number field $K(p) \subset \widetilde{\mathbf{Q}}_p \subset \mathbf{Q}_p$.

(b) The field of moduli of $f_\wp$ is the compositum $M\mathbf{Q}_p = M_\wp \subset \overline{\mathbf{Q}}_p$. By part (a), f is defined over $MK(p) \subset M\mathbf{Q}_p = M_\wp$, and the conclusion follows. □

Remark. The proof of the above corollary actually shows more. Namely, the proof shows that the number field $K(p) \subset \overline{\mathbf{Q}}$ can be chosen to lie within any henselization of $\mathbf{Q}$ in $\overline{\mathbf{Q}}$ at p. Thus the G-Galois cover has a model over each of the henselizations of M in $\overline{\mathbf{Q}}$ at a given place of M over p. □

In the case of $Q = M = \mathbf{Q}$, we also have the following:

Corollary 4.4. *Let G be a finite group, and let $f : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ be a G-Galois cover with field of moduli $\mathbf{Q}$. Then $\mathbf{Q}$ is a field of definition of the G-Galois cover f if and only if $\mathbf{Q}_p$ is a field of definition of the induced G-Galois cover $f_p : X_{\overline{\mathbf{Q}}_p} \to \mathbf{P}^1_{\overline{\mathbf{Q}}_p}$ for each $p \in S_{\mathrm{bad}}$.*

Proof. The forward implication is clear: If $\mathbf{Q}$ is a field of definition of the G-Galois cover f, then it is automatic that $\mathbf{Q}_p$ is a field of definition of the induced G-Galois cover f_p for all primes p, and in particular for $p \in S_{\mathrm{bad}}$. For the converse, since $\mathbf{Q}$ is the field of moduli of f, it follows for every p that $\mathbf{Q}_p$ is the field of moduli of f_p (relative to the extension $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$). So for finite $p \notin S_{\mathrm{bad}}$, Corollary 4.1 asserts that $\mathbf{Q}_p$ is a field of definition of f_p. By hypothesis, this is also the case for $p \in S_{\mathrm{bad}}$, and hence for all finite primes p. Since there is only one infinite prime, and since the special case of Grunwald-Wang does not include $\mathbf{Q}$ (cf. [DeDo2], §3.2), we may apply the local-to-global principle in [DeDo2] (see Theorem 3.7(b) and §3.4) to conclude that $\mathbf{Q}$ is a field of definition of the G-Galois cover f. □

Corollary 4.5. *Let G be a finite group, and let $f : X \to \mathbf{P}^1_{\overline{\mathbf{Q}}}$ be a G-Galois cover. Suppose that $\mathbf{Q}_p$ is a field of definition of the induced G-Galois cover $f_p : X_{\overline{\mathbf{Q}}_p} \to \mathbf{P}^1_{\overline{\mathbf{Q}}_p}$ for each prime p in a set that contains S_{bad} and contains all but finitely many primes not in S_{bad}. Then $\mathbf{Q}$ is a field of definition of the G-Galois cover f.*

Proof. By the converse part of the global-to-local principle ([De], Theorem 8.1), the field of moduli of f is $\mathbf{Q}$. So the result follows from Corollary 4.4. □

§5: Questions and possible generalizations

§5A. Bad primes.

Given a G-Galois cover f over $\overline{\mathbf{Q}}$ with field of moduli M over $\mathbf{Q}$, denote the set of primes p for which the induced cover $f_\wp$ over $\overline{\mathbf{Q}}_p$ is not defined over $M_\wp$, for some $\wp | p$, by S_{obs} (the "obstructed set"). Corollary 4.3(b) says that $S_{\mathrm{obs}} \subset S_{\mathrm{bad}}$. From [Bec1], Theorem 5.5, the set S_{ram} of primes that ramify in $M/\mathbf{Q}$ is also contained in S_{bad}. Thus we have

$S_{\text{obs}} \cup S_{\text{ram}} \subset S_{\text{bad}}$. The following questions seem natural, but the examples below show that the answers are all negative.

Question 5.1. Under the above notation, is it always true that

(a) $S_{\text{obs}} \subset S_{\text{ram}}$?

(b) $S_{\text{ram}} \subset S_{\text{obs}}$?

(c) $S_{\text{obs}} = S_{\text{bad}}$?

(d) $S_{\text{bad}} \subset S_{\text{obs}} \cup S_{\text{ram}}$?

Examples. (a): Consider the cover of $\mathbf{P}^1$ in Example 2.6 of [CH] (already mentioned in the introduction). Its field of moduli M is $\mathbf{Q}$ but it is not defined over $\mathbf{R}$. The group of the cover is the quaternion group of order 8 with center $\{\pm 1\}$ and the branch points are 1, 2 and 3. In fact, the branch points could be taken to be any 3-point set $\{t_1, t_2, t_3\} \subset \mathbf{P}^1(\mathbf{Q})$ (using transitivity of $\mathrm{PGL}_2(\mathbf{Q})$ on such sets). Take $\{t_1, t_2, t_3\} = \{0, 1, \infty\}$. Then $S_{\text{bad}} = \{2\}$. Now according to §3.4 of [DeDo2], since the cover is not defined over M, nor over the completion of M at the infinite prime, there must be at least one other prime p over whose completion the cover is not defined. By our Main Theorem, that prime must lie in S_{bad}. Therefore $S_{\text{obs}} = \{2\}$. On the other hand $S_{\text{ram}} = \emptyset$. Thus $S_{obs} \not\subset S_{\text{ram}}$.

(d): Consider the same example as in (a) but take $\{t_1, t_2, t_3\} = \{0, 3, \infty\}$. Then $S_{\text{bad}} = \{2, 3\}$. As in (a), $S_{\text{obs}} = \{2\}$ and $S_{\text{ram}} = \emptyset$. Thus $S_{\text{bad}} \not\subset S_{\text{obs}} \cup S_{\text{ram}}$.

(b),(c): Example 8.3.2 of [Se2] is an A_5-cover of the projective line, branched at $\{0, 1, \infty\}$, whose field of moduli is $\mathbf{Q}(\sqrt{5})$ (which is also a field of definition). So here $S_{\text{bad}} = \{2, 3, 5\}$; $S_{\text{obs}} = \emptyset$; and $S_{\text{ram}} = \{5\}$. Thus (b) and (c) both fail. (This is also a counterexample to (d).)

Remark. (i) The counterexample to (b) above easily generalizes, viz. to any G-Galois cover with field of moduli $M \neq \mathbf{Q}$, provided that the given cover can be defined over M. This will happen, for example, if G is centerless (cf. [CH], Prop. 2.8(c)). It will also happen if the cover corresponds to an M-valued point on a Hurwitz space over which there is a Hurwitz family of G-Galois covers; and again this happens in particular if G is centerless (cf. [CH], Prop. 1.4(b)).

(ii) In connection with the above, it is natural to investigate what conditions would guarantee that for a given $p \in S_{\text{bad}}$, the p-completion of the field of moduli is indeed not a field of definition, i.e., $p \in S_{\text{obs}}$. Let $\mathcal{H}$ be a Hurwitz space parametrizing certain G-Galois covers of the projective line. If there is a Hurwitz family over $\mathcal{H}$, then (as just noted) the field of moduli of any cover in the family will be a field of definition. And whether or not there is such a Hurwitz family, if p does not divide the order of G, then there are members of the family for which p is not in S_{bad} — and hence the corresponding covers are defined over their fields of moduli (with respect to $\mathbf{Q}_p$). So if $p \in S_{\text{obs}}$ for each cover parametrized by $\mathcal{H}$, then $p \mid \#G$ and there is no Hurwitz family over $\mathcal{H}$. Conversely, we may ask: If $p \mid \#G$ and there is no Hurwitz family over $\mathcal{H}$, then is $p \in S_{\text{obs}}$ for some cover associated to a $\overline{\mathbf{Q}}$-point of $\mathcal{H}$?

Also, while Corollaries 4.2 and 4.3 treat primes one at a time, one may ask whether there is a single extension that works for all good primes at once:

Question 5.2. Let $f : X \to \mathbf{P}^1_{\mathbf{Q}}$ be a G-Galois cover. Must there exist a number field K ramified only over S_{bad}, such that there is a stable model for f over K?

In a similar vein, fixing a prime $p \notin S_{\text{bad}}$ of $\mathbf{Q}$, one may ask if the number field $K(p)$ in Corollary 4.3 can be chosen so as to be totally p-adic (i.e. so that every embedding of it into $\overline{\mathbf{Q}}_p$ actually has image in $\mathbf{Q}_p$). In particular, we may ask:

Question 5.3. Let $f : X \to \mathbf{P}^1_{\mathbf{Q}}$ be a G-Galois cover with field of moduli $\mathbf{Q}$ and say $p \notin S_{\text{bad}}$. Must there exist a model for f over some totally p-adic number field K?

In connection with this question, we note Pop's result ([Po], Theorem $\mathcal{S}$) that a smooth geometrically irreducible $\mathbf{Q}$-variety that has a p-adic point also has a totally p-adic point.

§5B. Other base spaces.

Above, we have been considering only G-Galois covers of the projective line. One possible way to generalize these results would be to allow other base curves, or even more general base schemes B. But some constraints on allowable base spaces B seem necessary:

(1) The K-scheme B should be given together with a *regular* model B^o over the ring of integers O, so that Purity of Branch Locus can be used. Purity was used above in the proof of Case 1 of the Theorem, and also in Proposition 2.1 – i.e. in the proof of the Good Models Theorem [Bec2], Prop. 2.4 (via its use in [Bec1], Prop. 5.3, which was used in [Bec2], Prop. 2.3). Also, B^o should presumably have good reduction, since that was used in [Bec1], Prop. 5.3 and thus in Proposition 2.1 above.

(2) For any scheme C over a field F, and any extension field E/F, we will say that (C, E) satisfies the *Intersection Property* if for any $F \subset F' \subset E$ there are infinitely many pairs of regular points $\alpha, \alpha' \in C(E)$ such that $F'(\alpha) \cap F'(\alpha') = F'$. In order to generalize our proof of the Main Theorem to a base scheme B over a henselian field K, the pair $(B_k, \overline{k})$ should satisfy the Intersection Property, where k is the residue field of K, and B_k is the closed fibre of B. Namely, in this case (B, K^{ur}) also satisfies the Intersection Property (as can be seen by lifting regular points of $B_k(\overline{k})$ $B_k(\overline{k})$ to points of $B(K^{\text{ur}})$), and this could be relied on in generalizing the strategy of the proof of Case 2 of the Theorem.

Note also that the Intersection Property holds for $(B_k, \overline{k})$ in each of the following situations:

- $B_k(k)$ is Zariski-dense in $B_k(\overline{k})$ (since there are then infinitely many pairs (α, α') with $\alpha = \alpha' \in B_k(k)$). This condition holds in particular
  - if k is algebraically closed (or more generally PAC — cf. [FrJa], Ch.10), or
  - if B_k is a k-rational variety and k is infinite.

- k is hilbertian. (Choose a covering morphism $B \to \mathbf{P}^s$ defined over k, where $s = \dim(B)$. Then apply the hilbertian property to get two points $\alpha, \alpha' \in B(\overline{k})$ such that $k(\alpha)$ and $k(\alpha')$ are linearly disjoint.)

- k is finite. (From the Riemann hypothesis, with $k = \mathbf{F}_q$, we have $B_k(\mathbf{F}_{q^h}) \neq \emptyset$ provided that h is suitably large, say $h > h_0$. Take $\alpha \in B_k(\mathbf{F}_{q^h})$ and $\alpha' \in B_k(\mathbf{F}_{q^{h'}})$ with h, h' bigger than h_0 and relatively prime.)

- k is a henselian field whose residue field κ is finite and B_k has good reduction (or, more generally, k is henselian, B_k has good reduction and the Intersection Property holds for $(B_\kappa, \overline{\kappa})$).

But the Intersection Property does not always hold for $(B_k, \overline{k})$. In particular, it fails to hold if $k = \mathbf{R}$ and B_k is a curve defined over $\mathbf{R}$ with no real points.

Based on the above, we make the following conjecture:

Conjecture 5.4. *Let K be the fraction field of a henselian discrete valuation ring $(O, \wp)$ with perfect residue field k. Let B^o be a regular projective O-scheme of relative dimension 1, with generic fibre B and smooth closed fibre B_k. Assume that $(B_k, \overline{k})$ satisfies the Intersection Property. Let $f : X \to B$ be a G-Galois cover with field of moduli M, whose degree is not divisible by* $\operatorname{char} k$*, and whose branch locus is smooth at $\wp$. Then conclusions (a) and (b) of the Main Theorem hold.*

Of course, in order to generalize the proof of the Main Theorem in order to prove this Conjecture, it would be necessary to generalize the proofs of Propositions 2.1 and 2.2 (i.e. of [Bec2], Prop. 2.4, and [De], Lemma 8.2) to covers of base curves other than $\mathbf{P}^1$. (Note that [DeDo2], Thm. 5.1, generalizes [De], Thm. 8.1, to other base spaces, and the proof of [De], Lemma 8.2, extends as well to more general base spaces.)

§*5C. Mere covers.*

We may also consider a generalization in another direction, viz. that of allowing consideration of "mere covers", i.e. covers that are not equipped with a given G-Galois action. Note that [De], Lemma 8.2, also applies to "mere covers", whereas the key results of [Bec1] and [Bec2] apply only to G-Galois covers. We may nevertheless consider the following question:

Question 5.5. Does the Main Theorem also hold for "mere covers" (under the assumption that char(k) does not divide the order of the Galois group of the Galois closure)?

Remark. In Question 5.5, the covers in question may or may not be Galois, but even when they are, their fields of moduli and of definition as "mere covers" will generally be smaller than the corresponding fields for the associated G-Galois cover (since the G-action is no longer part of the structure to be descended).

If Question 5.5 has an affirmative answer in the case of Galois mere covers, then there is the following application: Given a curve X of genus at least 2, consider the finite cover $X \to X/\operatorname{Aut} X$. By Theorem 3.1 of [DeEm], the obstruction to the field of moduli being a field of definition is the same for the curve X as it is for the "mere cover" $X \to X/\operatorname{Aut} X$. Hence we would obtain a criterion for fields of moduli of p-adic *curves* to be fields of definition. In fact, even more would follow, since (as observed in [DeEm], Remark 3.2) the conclusion of [DeEm], Theorem 3.1, holds for various other types of objects as well (e.g. marked curves) – and so our Main Theorem would also carry over there too if the answer to Question 5.5 is affirmative. □

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