# ADELIC DOUBLE COSETS OVER SEMI-GLOBAL FIELDS

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Abstract. Double coset spaces of adelic points on linear algebraic groups arise in the study of global fields; e.g., concerning local-global principles and torsors. A different type of double coset space plays a role in the study of semi-global fields such as $p$-adic function fields. This paper relates the two, by establishing adelic double coset spaces over semi-global fields; relating them to local-global principles and torsors; and providing explicit examples.

## 1. Introduction

Adelic double coset spaces play an important role in the study of linear algebraic groups $G$ over a global field $F$. In work beginning with Borel (see [Bor63]), these spaces have been used in understanding local-global principles, class groups (using $G = \mathbb{G}_\mathrm{m}$ or more generally $\mathrm{GL}_n$), and the genus of quadratic forms (using $G = \mathrm{O}(n)$), as well as other applications (see [Con12, Section 1.2], [CRR20]). For any finite set $S$ of places of $F$ including all the archimedean places (if any), the double coset space $G(F)\backslash G(\mathbb{A}(F,S))/G(\mathbb{A}^\infty(F,S))$ is an abelian group if $G$ is commutative, and in general is a pointed set. Because of the relationship to the class group of a number field or the Picard group of a curve over a finite field, this space is also referred to as the "class set" or "class group" of the group $G$ (see [PR94, Chapter 8], [RR21, Section 2]).

A different type of double coset space has arisen in the study of semi-global fields such as $\mathbb{Q}_p(x)$; i.e., one-variable function fields $F$ over a complete discretely valued field $K$. It involves only finitely many factors (unlike adelic double cosets), and provides the obstruction to a local-global principle for $G$-torsors over $F$ in the context of patching over fields (see [HHK15]). This patching obstruction approximates a more canonical local-global obstruction $Ш_X(F,G)$, which is taken with respect to points on the closed fiber $X$ of a projective model $\mathscr{X}$ of $F$ over the valuation ring $\mathscr{O}_K$; and that in turn can be used in studying the local-global obstruction $Ш(F,G)$ with respect to discrete valuations on $F$ (see [HHK15, Proposition 8.2], and [CHHKPS21, Theorem 3.2]).

This paper introduces an adelic double coset space in the context of semi-global fields that makes it possible to study $Ш_X(F,G)$ more directly, rather than via approximation by patching obstructions. It also provides a clearer link between the global and semi-global contexts. Via results that have already been proven about $Ш_X(F,G)$ and its patching analog, this double coset space can be computed in a number of situations. Its definition, however, is not quite what one might have expected, due to the fact that the intersection property for

*Date*: November 7, 2021.

*Mathematics Subject Classification* (2010): 11E72, 12G05, 14G05 (primary); 14H25, 20G15, 14G27 (secondary).

*Key words and phrases.* Linear algebraic groups, torsors, local-global principles, double cosets, class sets, Galois cohomology, semi-global fields, patching.

The author was supported in part by NSF grant DMS-2102987.

patching (asserting that $F$ is the intersection of the finitely many overfields in the double coset space; see [HHK15, Section 2]) turns out not to hold in the adelic situation. But the modified adelic double coset space introduced below nevertheless agrees with $Ш_X(F,G)$, as a consequence of Artin's Approximation Theorem [Art69]; see Corollary 3.5.

The above obstructions are contained in the Galois cohomology set $H^1(F,G)$; and if $G$ is commutative then this is a group and higher Galois cohomology groups $H^i(F,G)$ can also be considered. In that situation, we define higher double coset spaces that can be directly computed using higher obstructions $Ш^i_X(F,G) \subset H^i(F,G)$; see Corollary 3.8.

Whereas these double coset spaces are defined in terms of fields, closer analogs to the global field case, involving local rings, are introduced afterwards, in the context of one-dimensional schemes (generalizing the global field situation) and for projective models of semi-global fields. These analogs agree with the obstructions to local-global principles for torsors over such schemes in the context of étale cohomology. (See Corollaries 4.3 and 4.7.) In the case of $G = \mathbb{G}_{\mathrm{m}}$, this double coset space agrees with the Picard group of the scheme, and so generalizes the notion of the class set (or group) of $G$ in the theory of global fields. (For a different construction in a somewhat similar spirit, see [CRR20, Appendix A], which generalizes a result of Harder, [Har67, Section 2.3].)

This paper is organized as follows. Following a discussion of preliminaries in Section 2, double coset spaces for semi-global fields are introduced in Section 3, by building on results in the patching context. In that context, the agreement of the finite double coset space with the local-global obstruction had been proven using a six-term Mayer-Vietoris sequence (see [HHK15, Theorem 3.5]); here a limiting argument yields a version of that sequence that relates $Ш_X(F,G)$ to adelic double cosets. For groups $G$ that are commutative, we can similarly rely on a long exact Mayer-Vietoris sequence that was given in the patching context in [HHK14]. Afterwards, Section 4 establishes an adelic double coset space in terms of local rings, which is thus closer to the situation in the global field case. This is done for reasonable one-dimensional schemes and for models of semi-global fields, using a Tannakian approach to patching that was recently established in [HKL20]. In each of these situations, the double coset space is interpreted as a local-global obstruction, and examples are given that compute the double coset spaces explicitly.

I thank Julia Hartmann for comments on this manuscript.

## 2. Background on semi-global fields and patching

This section reviews the set-up established in [HH10], [HHK09], and [HHK15]. Let $T$ be a complete discrete valuation ring, with fraction field $K$, residue field $k$, and uniformizer $t$. A *semi-global field $F$ over $K$* (or *over $T$*) is a finitely generated field over $K$ of transcendence degree one in which $K$ is algebraically closed. A (projective) *model $\mathscr{X}$* of $F$ is an integral flat projective $T$-curve with function field $F$. For example, $\mathbb{P}^1_{\mathbb{Z}_p}$, and a blow up of $\mathbb{P}^1_{\mathbb{Z}_p}$ at a closed point, are both models of $\mathbb{Q}_p(x)$. We may in particular consider models that are normal or regular as schemes. A regular model whose closed fiber $X := \mathscr{X} \times_T k$ is a normal crossings divisor (i.e., each irreducible component is regular and the components meet at ordinary double points) is called a *normal crossings model*. (See [Liu02, Chapter 10] for more about such models.)

We may consider local-global principles for torsors under a linear algebraic group $G$ over a semi-global field $F$. Here and below, we require all linear algebraic groups to be smooth as schemes. Under that assumption, the Galois cohomology set $H^1(F,G)$ classifies isomorphism classes of $G$-torsors over $F$ (as in the global field case), and we may consider

$$Ш(F,G) := \ker(H^1(F,G) \to \prod_v H^1(F_v,G)),$$

where $v$ runs over the equivalence classes of non-archimedean absolute values on $F$; or equivalently, here, the discrete valuations on $F$. This Tate-Shafarevich set is the obstruction to a $G$-torsor over $F$ being trivial (or equivalently, having an $F$-point) whenever it is trivial over each $F_v$. (Some authors have restricted $v$ just to those discrete valuations that are trivial on $K$, in which case the obstruction is in general larger; e.g., see [CH15], [HSS15], [HS16].)

A variant local-global obstruction for a semi-global field $F$, also motivated by the global field case, concerns the closed fiber $X$ of a model $\mathscr{X}$ of $F$, and is given by

$$Ш_X(F,G) := \ker(H^1(F,G) \to \prod_{P\in X} H^1(F_P,G)),$$

where $P$ ranges over all the points of $X$ (including generic points) and where $F_P$ is the fraction field of the complete local ring $\widehat{R}_P := \widehat{\mathscr{O}}_{\mathscr{X},P}$.

These two obstructions are pointed sets, and are abelian groups if $G$ is commutative. By [HHK15, Proposition 8.2], $Ш_X(F,G) \subseteq Ш(F,G)$ if $\mathscr{X}$ is a regular model, and so the local-global principle with respect to discrete valuations can be shown to fail by proving the non-vanishing of $Ш_X(F,G)$. Moreover, in some situations it is known that $Ш_X(F,G) = Ш(F,G)$; e.g., see [CHHKPS21, Theorem 3.2]).

To study $Ш_X(F,G)$, one can approximate it by smaller obstruction sets $Ш_{\mathscr{P}}(F,G)$, where $\mathscr{P}$ ranges over non-empty finite sets of closed points of $X$ that contain all the singular points of $X$. These are defined using rings and fields that arise in the framework of patching, where they had earlier been used in inverse Galois theory over semi-global fields (e.g., [Har87], [Pri00], [Har03]). These rings and fields are motivated by the analogy between viewing the general fiber of $\mathscr{X} \to \operatorname{Spec}(T)$ as a $K$-curve under the $t$-adic topology (as in rigid geometry), and viewing a complex algebraic curve as a Riemann surface under the complex metric topology.

More precisely, if $U$ is a non-empty irreducible affine open subset of $X$, let $R_U \subset F$ be the subring of elements regular along $U$; and let $\widehat{R}_U$ be the $t$-adic completion of $R_U$, with fraction field $F_U$. Thus $\widehat{R}_U \subseteq \widehat{R}_{U'}$ and $F_U \subseteq F_{U'}$ if $U \subseteq U'$. If $P$ is a closed point of $X$, then a *branch* of $X$ at $P \in X$ is a height one prime $\wp$ of $\widehat{R}_P$ containing $t$. These are in bijection with the irreducible components of the restriction of $X$ to $\operatorname{Spec}(\widehat{R}_P)$. For such $P$ and $\wp$, let $R_\wp$ be the localization of $\widehat{R}_P$ at $\wp$; and let $\widehat{R}_\wp$ be the completion of $R_\wp$, with fraction field $F_\wp$. Now given a choice of a finite set $\mathscr{P}$ as above, let $\mathscr{U}$ be the set of connected components of $X \smallsetminus \mathscr{P}$, and let $\mathscr{B}$ be the set of branches of $X$ at the points of $\mathscr{P}$. We thus have a finite set of overfields of $F$; viz., $F_P$ (for $P \in \mathscr{P}$), $F_U$ (for $U \in \mathscr{U}$), and $F_\wp$ (for $\wp \in \mathscr{B}$). Here $F_\wp$ contains $F_P$ and $F_U$, if $\wp$ is a branch at $P$ on (the closure of) $U \in \mathscr{U}$; it also contains $F_\eta$, where $\eta$ is the generic point of $U$. We may view each $F_P$ and $F_U$ as the field of meromorphic functions on a $t$-adic neighborhood on the general fiber of $\mathscr{X}$, with the fields $F_\wp$ corresponding to the overlaps. Analogously to the behavior of meromorphic functions with respect to a metric open covering of a Riemann surface, the set of fields $F_P$ and $F_U$

satisfies an *intersection property*: their intersection (or strictly speaking, inverse limit) is equal to $F$, viewing $F_P, F_U \subset F_\wp$ as above.

The above fields $F_P, F_U, F_\wp$ have been useful in computing invariants of semi-global fields, especially concerning quadratic forms and Brauer groups (e.g., see [HHK09], [PS14]). They also give rise to the local-global obstruction

$$Ш_{\mathscr{P}}(F, G) := \ker\bigl(H^1(F, G) \to \prod_{\xi \in \mathscr{P} \cup \mathscr{U}} H^1(F_\xi, G)\bigr).$$

If $\mathscr{X}$ is a normal model of $F$, then $Ш_{\mathscr{P}}(F, G) \subseteq Ш_X(F, G)$ for each choice of $\mathscr{P}$ as above; moreover $Ш_X(F, G)$ is the union of its subsets $Ш_{\mathscr{P}}(F, G)$, if $\mathscr{P}$ ranges over all finite sets as above (see [HHK15, Corollary 5.9]).

One can identify $Ш_{\mathscr{P}}(F, G)$ with a (finite) double coset space via a six-term Mayer-Vietoris sequence in Galois cohomology. Namely, by [HHK15, Theorem 3.5], there is an exact sequence of pointed sets

$$\begin{tikzcd}[ampersand replacement=\&]
1 \arrow[r] \& H^0(F,G) \arrow[r, "\alpha^0"] \& \prod_{P\in\mathscr{P}} H^0(F_P,G) \times \prod_{U\in\mathscr{U}} H^0(F_U,G) \arrow[r, "\beta^0"] \& \prod_{\wp\in\mathscr{B}} H^0(F_\wp,G) \arrow[dll, rounded corners, "\delta^0"'] \\
\& H^1(F,G) \arrow[r, "\alpha^1"] \& \prod_{P\in\mathscr{P}} H^1(F_P,G) \times \prod_{U\in\mathscr{U}} H^1(F_U,G) \arrow[r, shift left, "\beta^1_1"] \arrow[r, shift right, "\beta^1_2"'] \& \prod_{\wp\in\mathscr{B}} H^1(F_\wp,G).
\end{tikzcd} \tag{1}$$

Here $H^0(F, G) = G(F)$, and similarly for the other terms in the first row. Thus we obtain a bijection of pointed sets

$$\prod_{U \in \mathscr{U}} G(F_U) \setminus \prod_{\wp \in \mathscr{B}} G(F_\wp) \; / \prod_{P \in \mathscr{P}} G(F_P) \cong Ш_{\mathscr{P}}(F, G), \tag{2}$$

between a double coset space and a Tate-Shafarevich set, in the patching context. A key ingredient is the intersection property.

In the case of a commutative group $A$, the Galois cohomology sets $H^i(F, A)$ are abelian groups and are defined for all $i \geq 0$. Under additional hypotheses (that $\mathrm{char}(k) = 0$ or $A$ has finite order not divisible by $\mathrm{char}(k)$), it was shown in [HHK14, Theorem 3.1.2] that (1) extends to a long exact Mayer-Vietoris sequence:

$$\begin{tikzcd}[ampersand replacement=\&]
0 \arrow[r] \& H^0(F,A) \arrow[r, "\Delta"] \& \prod_{P\in\mathscr{P}} H^0(F_P,A) \times \prod_{U\in\mathscr{U}} H^0(F_U,A) \arrow[r, "-"] \& \prod_{\wp\in\mathscr{B}} H^0(F_\wp,A) \arrow[dll, rounded corners] \\
\& H^1(F,A) \arrow[r, "\Delta"] \& \prod_{P\in\mathscr{P}} H^1(F_P,A) \times \prod_{U\in\mathscr{U}} H^1(F_U,A) \arrow[r, "-"] \& \prod_{\wp\in\mathscr{B}} H^1(F_\wp,A) \arrow[dll, rounded corners] \\
\& H^2(F,A) \arrow[r, "\Delta"] \& \cdots .
\end{tikzcd} \tag{3}$$

This provides information about higher local global principles and higher double coset spaces, in the patching context.

## 3. Adelic-type double cosets over semi-global fields

This section introduces an adelic-type double coset space in the context of a semi-global field, forming a bridge between the classical adelic double cosets for global fields and the finite double cosets over semi-global fields reviewed in Section 2. This double coset space is then interpreted in terms of local-global principles via a Mayer-Vietoris sequence, and explicit examples are computed. (Afterwards, Section 4 considers a closer analog to the global field case, using local rings rather than fields.)

The double coset space we introduce here will be in bijection with the elements of the Tate-Shafarevich obstruction set $\text{Ш}_X(F,G) := \ker(H^1(F,G) \to \prod_P H^1(F_P,G))$ that was recalled in Section 2. It is thus natural to expect that it would be obtained by modifying the bijection (2) by taking the products over $G(F_\wp)$ for all branches $\wp$ of the closed fiber $X$ of $\mathscr{X}$ at closed points of $X$; over $G(F_P)$ for all closed points $P \in X$; and over $G(F_\eta)$ for all generic points $\eta$ of $X$ (one for each irreducible component). But an unexpected issue arises; viz., the intersection property fails for these fields $F_P, F_\eta, F_\wp$:

**Example 3.1.** Let $k$ be a field of characteristic zero; let $T = k[[t]]$; and let $\mathscr{X}$ be the projective $x$-line over $T$. Then there is one generic point $\eta$ on the closed fiber $X = \mathbb{P}^1_k$. The corresponding function field is the semi-global field $F = k((t))(x)$. For each closed point $P$ of $X$, the field $F_\eta = k(x)((t))$ is contained in $F_\wp$, where $\wp$ is the branch of $X$ at $P$. In particular, if $P$ is the point $x = c$ on $X$ for some $c \in k$, then $\widehat{R}_P = k[[x-c,t]]$, $\widehat{R}_\wp = k((x-c))[[t]]$, and $F_\wp = k((x-c))((t)) \supset F_\eta$. Consider the element $a_\eta = \sum_{n=1}^\infty t^n/(x-n) \in F_\eta$. In the case that $P$ is the point $x = m$ on $X$ for some positive integer $m$, the series $(x-m)\sum_{n=1}^\infty t^n/(x-n)$ is an element of in $k[[x-m,t]] = \widehat{R}_P$; whereas for all other closed points $P \in X$, the series $\sum_{n=1}^\infty t^n/(x-n)$ lies in $\widehat{R}_P$. So in both cases, the element $a_\eta$, viewed as an element of $F_\wp$, is the image of an element $a_P \in F_P = \operatorname{frac}(\widehat{R}_P)$ given by the same expression. The elements $a_P$ and $a_\eta$ together define an element in the inverse limit of the fields $F_P$, $F_\eta$, and $F_\wp$ that is not induced by any element of $F$ (or even any element of some $F_U$, for $U$ an affine open subset of $X$), since it has infinitely many poles on the closed fiber $X$. Thus this inverse limit is strictly bigger than $F$. (See also Remark 3.4 below.)

Instead, the double coset space we consider, and the associated Mayer-Vietoris sequence, will be somewhat different from the naive expectation. To obtain it from the patching versions of those objects, we use the following lemma (where the "kernel" of a doubled map is the equalizer).

**Lemma 3.2.** *Let*

$$D^0 \xrightarrow{\alpha^0} A^0 \xrightarrow{\beta^0} B^0 \xrightarrow{\delta^0} D^1 \xrightarrow{\alpha^1} A^1 \underset{\beta^1_2}{\overset{\beta^1_1}{\rightrightarrows}} B^1 \tag{4}$$

*be an exact sequence of pointed sets. Let $C^0$ be a group, let $C^1$ be a pointed set, and for $i = 0, 1$ let $\epsilon^i : A^i \to C^i$ be a map of pointed sets.*

*(a) Then*

$$D^0 \xrightarrow{\alpha'^0} A^0 \times C^0 \xrightarrow{\beta'^0} B^0 \times C^0 \xrightarrow{\delta'^0} D^1 \xrightarrow{\alpha'^1} A^1 \times C^1 \underset{\beta'^1_2}{\overset{\beta'^1_1}{\rightrightarrows}} B^1 \times C^1 \tag{5}$$

*is also an exact sequence of pointed sets, where $\alpha'^i(d) = (\alpha^i(d), \epsilon^i\alpha^i(d))$ for $i = 0, 1$; where $\beta'^0(a,c) = (\beta^0(a), \epsilon^0(a)^{-1} \cdot c)$, $\beta'^1_1(a,c) = (\beta^1_1(a), c)$, and $\beta'^1_2(a,c) = (\beta^1_2(a), \epsilon^1(a)))$; and where $\delta'^0(b,c) = \delta^0(b)$.*

(b) *Suppose that $A^i, B^i, C^i, D^i$ are abelian groups; that $\alpha^i$, $\beta^0$, $\beta^1_j$, $\delta^0$, $\epsilon^i$ are group homomorphisms; and that (4) is an exact sequence of abelian groups if we replace the double arrow with the homomorphism $\beta^1$ given by $\beta^1(a) = \beta^1_2(a)^{-1}\beta^1_1(a)$. Then (5) is also an exact sequence of abelian groups if we similarly replace the double arrow with the map $\beta'^1$ given by $\beta'^1(a,c) = \beta'^1_2(a,c)^{-1}\beta'^1_1(a,c)$.*

*Proof.* In (a), the fact that (5) is a complex follows immediately from the definitions of the maps and the fact that (4) is a complex. For exactness at $B^0 \times C^0$, suppose that $\delta'^0(b,c)$ is the distinguished element of $D^1$. Then $b = \beta^0(a)$ for some $a \in A^0$, by the exactness of (4); and so $(b,c) = \beta'^0(a,c_0)$, where $c_0 = \epsilon^0(a)\cdot c \in C^0$. Exactness at the other terms of (5) is immediate from the definitions of the maps.

In (b), the maps $\alpha'^i, \delta'^0$ are clearly homomorphisms; and so are the maps $\beta'^i$ because $(g,h) \mapsto h^{-1}g$ is a group homomorphism in the case of abelian groups. Since a sequence of abelian groups is exact if and only if it is exact as a sequence of pointed sets, the assertion is immediate. □

We return to our geometric situation. Let $\mathscr{X}$ be a model of a semi-global field $F$ over $T$, with closed fiber $X$ over the residue field $k$ of $T$. For $i = 0, 1$, let $X_{(i)}$ denote the set of points of $X$ whose closure has dimension $i$. Thus $X_{(0)}$ is the set of closed points $P$ of $X$, and $X_{(1)}$ is the finite set of generic points of irreducible components of $X$. For any subset $S \subseteq X$, let $\mathscr{B}_S$ be the set of all branches of $X$ at closed points of $S$. For each $P \in X_{(0)}$ we have the associated complete local ring $\widehat{R}_P$ and its fraction field $F_P$; and similarly we have $\widehat{R}_\eta$ and $F_\eta$ for each $\eta \in X_{(1)}$, and $\widehat{R}_\wp$ and $F_\wp$ for $\wp \in \mathscr{B}_X$.

For any functor $\Phi$ from field extensions of $F$ to (pointed) sets, define the *restricted product* $\prod'_{\wp\in\mathscr{B}_X} \Phi(F_\wp)$ to be the direct limit of the family $\{\prod_{\wp\in\mathscr{B}_S} \Phi(F_\wp) \times \prod_{P\notin S} \Phi(F_P)\}$, indexed by finite subsets $S \subset X$. The natural map $\prod'_{\wp\in\mathscr{B}_X} \Phi(F_\wp) \to \prod_{\wp\in\mathscr{B}_X} \Phi(F_\wp)$ is injective provided that the maps $\Phi(F_P) \to \Phi(F_\wp)$ are injective for $\wp$ a branch at $P$. For example, for a linear algebraic group $G$ over $F$, there is the functor $H^i(-,G)$ for $i = 0,1$, and also for $i > 1$ if $G$ is commutative. Here $H^0(F_P,G) \to H^0(F_\wp,G)$ is injective. Hence the restricted product $\prod'_{\wp\in\mathscr{B}_X} G(F_\wp)$ is the subset of $\prod_{\wp\in\mathscr{B}_X} G(F_\wp)$ consisting of the families $(g_\wp)_{\wp\in\mathscr{B}_X}$ that satisfy $g_\wp \in G(F_P)$ for all but finitely many pairs $\wp, P$ where $\wp$ is a branch at $P$. (But caution: $\prod'_{\wp\in\mathscr{B}_X} H^1(F_\wp,G) \to \prod_{\wp\in\mathscr{B}_X} H^1(F_\wp,G)$ is not injective; see Remark 3.4.)

Similarly, if $G$ is a smooth affine group scheme of finite type over $\mathscr{X}$, we can define the *restricted product* $\prod'_{\wp\in\mathscr{B}_X} H^i(\widehat{R}_\wp,G)$ for $i = 0,1$, and also for $i > 1$ if $G$ is commutative. Namely, this is the direct limit of the family $\{\prod_{\wp\in\mathscr{B}_S} H^i(\widehat{R}_\wp,G) \times \prod_{P\notin S} H^i(\widehat{R}_P,G)\}$, indexed by finite subsets $S \subset X$.

The restricted product, as opposed to the full product, is one of the two ingredients that we need to introduce in order to obtain a well-behaved double coset space over a semi-global field. The other such ingredient is a field that will take the place of $F_\eta$, and which is described next.

For $\eta \in X_{(1)}$, let $\mathscr{U}_\eta$ be the collection of connected affine open neighborhoods of $\eta$ in $X$ all of whose closed points are unibranched on $X$; and let $R^{\mathrm{h}}_\eta \subseteq \widehat{R}_\eta$ be the subring $\bigcup_{U\in\mathscr{U}_\eta} \widehat{R}_U$. This is a Henselian discrete valuation ring with respect to the $\eta$-adic valuation, such that its fraction field $F^{\mathrm{h}}_\eta \subseteq F_\eta$ is the subfield $\bigcup_{U\in\mathscr{U}_\eta} F_U$, and its residue field is the residue field at $\eta$ (i.e., the function field of the corresponding irreducible component of $X^{\mathrm{red}}$); see [HHK14, Lemma 3.2.1]. Thus $R^{\mathrm{h}}_\eta$ contains the henselization $\widetilde{R}_\eta$ of the local ring $R_\eta$; and indeed, every element of $\widetilde{R}_\eta$ lies in $\widehat{R}_U$ for some $U \in \mathscr{U}_\eta$ (see the proof of [HHK15, Proposition 5.8]) and hence lies in $R^{\mathrm{h}}_\eta$.

Note that if a functor $\Phi$ as above is of finite presentation (i.e. if $\Phi(\varinjlim E_\nu) = \varinjlim(\Phi(E_\nu))$ for every direct system of $F$-algebras $\{E_\nu\}$), then there is a natural map $\prod_{\eta\in X_{(1)}} \Phi(F^{\mathrm{h}}_\eta) \to \prod'_{\wp\in\mathscr{B}_X} \Phi(F_\wp)$. In particular, if $G$ is a linear algebraic group over $F$ then for $i = 0, 1$ there is an induced map $\prod_{\eta\in X_{(1)}} H^i(F^{\mathrm{h}}_\eta, G) \to \prod'_{\wp\in\mathscr{B}_X} H^i(F_\wp, G)$; and similarly for $i > 1$ if $G$ is commutative.

We now obtain the desired Mayer-Vietoris sequence with respect to points on the closed fiber:

**Theorem 3.3.** *Let $F$ be a semi-global field over a complete discrete valuation ring $T$ with fraction field $K$, and let $\mathscr{X}$ be a normal model of $F$ over $T$, with closed fiber $X$. Then for any linear algebraic group $G$ over $F$, we have an exact sequence of pointed sets*

$$\begin{tikzcd} 1 \arrow[r] & H^0(F,G) \arrow[r] & \prod_{P\in X_{(0)}} H^0(F_P,G) \times \prod_{\eta\in X_{(1)}} H^0(F^{\mathrm{h}}_\eta,G) \arrow[r] & \prod'_{\wp\in\mathscr{B}_X} H^0(F_\wp,G) \arrow[dll, rounded corners] \\ & H^1(F,G) \arrow[r] & \prod_{P\in X_{(0)}} H^1(F_P,G) \times \prod_{\eta\in X_{(1)}} H^1(F^{\mathrm{h}}_\eta,G) \arrow[r, shift left] \arrow[r, shift right] & \prod'_{\wp\in\mathscr{B}_X} H^1(F_\wp,G). \end{tikzcd} \tag{6}$$

*Proof.* Let $\mathscr{P}$ be a non-empty finite set of closed points of $\mathscr{X}$ that includes all the points at which the closed fiber $X$ is not unibranched. Let $\mathscr{U}$ be the set of connected components of the complement of $\mathscr{P}$ in $X$, and let $\mathscr{B}$ be the set of branches of $X$ at the points of $\mathscr{P}$. As discussed in Section 2, by [HHK15, Theorem 3.5], there is the exact sequence (1) of pointed sets.

For $i = 0, 1$, let $A^i = \prod_{P\in\mathscr{P}} H^i(F_P, G) \times \prod_{U\in\mathscr{U}} H^i(F_U, G)$; let $B^i = \prod_{\wp\in\mathscr{B}} H^i(F_\wp, G)$; and let $D^i = H^i(F, G)$. Also, let $\mathscr{P}'$ be the complement of $\mathscr{P}$ in $X_{(0)}$, and let $C^i = \prod_{P\in\mathscr{P}'} H^i(F_P, G)$. Define $\epsilon^i : A^i \to C^i$ as the composition $A^i \to \prod_{U\in\mathscr{U}} H^i(F_U, G) \to C^i$, where the first map is the projection onto the second factor in the definition of $A^i$, and the second map is induced by the inclusions $F_U \hookrightarrow F_P$ for each closed point $P \in U$. (Note that each $P \in \mathscr{P}'$ lies in a unique $U \in \mathscr{U}$.)

For short, write $H^i_F := H^i(F, G)$ for $i = 1, 2$; and for any indexed set $\{F_s\}_{s\in S}$ of overfields of $F$ write $H^i_S := \prod_{s\in S} H^i(F_s, G)$ for $i = 0, 1$. Applying Lemma 3.2(a) to the exact sequence (1), we obtain the following expanded exact sequence:

$$1 \longrightarrow H^0_F \xrightarrow{\alpha_{0,\mathscr{U}}} H^0_{X_{(0)}} \times H^0_{\mathscr{U}} \xrightarrow{\beta_{0,\mathscr{U}}} H^0_{\mathscr{B}} \times H^0_{\mathscr{P}'} \xrightarrow{\delta_{0,\mathscr{U}}} H^1_F \xrightarrow{\alpha_{1,\mathscr{U}}} H^1_{X_{(0)}} \times H^1_{\mathscr{U}} \overset{\beta'^1_{1,\mathscr{U}}}{\underset{\beta'^1_{2,\mathscr{U}}}{\rightrightarrows}} H^1_{\mathscr{B}} \times H^1_{\mathscr{P}'}.$$

Here the initial 1, corresponding to the triviality of $\ker(\alpha_{0,\mathscr{U}})$, follows from the injectivity of $H^0(F, G) \to H^0(F_P, G)$ for any $P \in X_{(0)}$. Also, in the terms $H^i_{X_{(0)}} \times H^i_{\mathscr{U}}$, we use that $H^i_{X_{(0)}} = H^i_{\mathscr{P}} \times H^i_{\mathscr{P}'}$.

We can now let the above finite set $\mathscr{P}$ vary, with $\mathscr{U}$, $\mathscr{B}$, and $\mathscr{P}'$ varying along with it. The sets $\mathscr{P}$ form a direct system; and because of the compatibility of the maps in the above exact sequence, as $\mathscr{P}$ varies, we obtain a direct system of exact sequences. Here

$$\lim_{\to} H^i_{\mathscr{U}} = \prod_{\eta\in X_{(1)}} H^i(F^{\mathrm{h}}_\eta, G) \quad \text{and} \quad \lim_{\to} (H^i_{\mathscr{B}} \times H^i_{\mathscr{P}'}) = \prod_{\wp\in\mathscr{B}_X}' H^i(F_\wp, G).$$

Since direct limits preserve exactness, we obtain the asserted exact sequence of pointed sets. □

*Remark* 3.4. Theorem 3.3 would no longer hold if $F^{\mathrm{h}}_\eta$ were replaced by $F_\eta$. First note that such a replacement would also require replacing $\prod' H^i(F_\wp, G)$ by $\prod H^i(F_\wp, G)$, or else the last map in each row would no longer be defined. Doing both replacements would lead to the first row not being exact at the middle $H^0$ term even for the group $G = \mathbb{G}_a$ and $\mathscr{X} = \mathbb{P}^1_{k[[t]]}$ with $\mathrm{char}(k) = 0$, by Example 3.1. Moreover the second row would not be exact at its middle term with the same choice of $\mathscr{X}$ and taking $G$ to be cyclic of order two. Namely, for each positive integer $n$ let $P_n$ be the point on the closed fiber $X = \mathbb{P}^1_k$ where $x = n$, and take the $G$-torsor over $F_{P_n}$ given by the $G$-Galois extension defined by $y^2 = (x - n)(x - n - t)$. At all other points on $X$ (including $\eta$), take the trivial $G$-torsor. Then this element of the middle term of the second row is in the equalizer of the two maps to $\prod H^1(F_\wp, G)$ but is not in the image of $H^1(F, G)$, since a branched cover of $\mathscr{X}$ cannot have infinitely many components of its branch locus. (Note, though, that this element is not in the equalizer to $\prod' H^1(F_\wp, G)$, since that restricted product is defined as a direct limit, and at no finite level is the element induced by the above $G$-torsors trivial. This also shows that $\prod' H^1(F_\wp, G) \to \prod H^1(F_\wp, G)$ is not injective.) The same exactness issue on $H^1$ would arise if we retained $F^{\mathrm{h}}_\eta$ but still replaced $\prod' H^i(F_\wp, G)$ by $\prod H^i(F_\wp, G)$.

By analogy with the notion of a class set (or group) associated to a linear algebraic group over a global field being given by the adelic double coset space $G(F)\backslash G(\mathbb{A}(F, S))/G(\mathbb{A}^\infty(F, S))$, we write

$$\mathrm{Cl}_X(F, G) = \prod_{\eta\in X_{(1)}} G(F^{\mathrm{h}}_\eta) \setminus \prod_{\wp\in\mathscr{B}_X}' G(F_\wp) \,/ \prod_{P\in X_{(0)}} G(F_P)$$

in the situation of Theorem 3.3.

**Corollary 3.5.** *The coboundary map of the exact sequence in Theorem 3.3 induces a bijection of pointed sets* $\mathrm{Cl}_X(F, G) \to Ш_X(F, G)$.

*Proof.* By Theorem 3.3, we have a bijection from the above double coset space to the kernel of the map $H^1(F, G) \to \prod_{P\in X_{(0)}} H^1(F_P, G) \times \prod_{\eta\in X_{(1)}} H^1(F^{\mathrm{h}}_\eta, G)$. But by [HHK15, Proposition 5.8] (which relied on the Artin Approximation theorem, [Art69, Theorem 1.10]), if a $G$-torsor over $F$ is trivial over $F_\eta$ for some $\eta \in X_{(1)}$, then it is trivial over some $F_U \subset F_\eta$ and hence over $F^{\mathrm{h}}_\eta$. Moreover the converse also holds, since $F^{\mathrm{h}}_\eta \subseteq F_\eta$. Hence

$$\ker\left(H^1(F, G) \to \prod_{P\in X_{(0)}} H^1(F_P, G) \times \prod_{\eta\in X_{(1)}} H^1(F^{\mathrm{h}}_\eta, G)\right) = Ш_X(F, G),$$

yielding the corollary. □

**Example 3.6.** (a) If $G$ is a rational connected linear algebraic group, then $Ш_X(F, G)$ is trivial, by [HHK15, Theorem 5.10]. Hence $\mathrm{Cl}_X(F, G)$ is trivial, by Corollary 3.5. In particular, this holds for the groups $\mathrm{GL}_n$ and $\mathrm{PGL}_n$. The triviality of $\mathrm{Cl}_X(F, \mathrm{GL}_n)$ can also be seen from the exact sequence (6) because $H^1(F, \mathrm{GL}_n)$ vanishes by Hilbert 90. The case of $\mathrm{PGL}_n$ can then be deduced directly from the case of $\mathrm{GL}_n$. Namely, every element of $\mathrm{Cl}_X(F, \mathrm{PGL}_n)$ is represented by an element of $\prod'_{\wp \in \mathscr{B}_X} \mathrm{PGL}_n(F_\wp)$, and this can be lifted to an element of $\prod'_{\wp \in \mathscr{B}_X} \mathrm{GL}_n(F_\wp)$, representing an element of $\mathrm{Cl}_X(F, \mathrm{GL}_n)$. Since $\mathrm{Cl}_X(F, \mathrm{GL}_n)$ is trivial, that class is also represented by the trivial element of $\prod'_{\wp \in \mathscr{B}_X} \mathrm{GL}_n(F_\wp)$, and hence the given class in $\mathrm{Cl}_X(F, \mathrm{PGL}_n)$ is represented by the trivial element of $\prod'_{\wp \in \mathscr{B}_X} \mathrm{PGL}_n(F_\wp)$.

(b) If $G$ is a linear algebraic group over $F$ such that each connected component is a rational $F$-variety, then $Ш_X(F, G)$ need not be trivial, but it is finite. More explicitly, it is in bijection with $\mathrm{Hom}(\pi_1(\Gamma), G/G^0)/\sim$, where $\Gamma$ is the reduction graph of $\mathscr{X}$, which encodes the intersections of the irreducible components of the closed fiber $X$ (see [HHK15, Section 6]), and where $\sim$ denotes post-conjugation by elements of the finite group $G/G^0$. (Here $G^0$ is the identity component of $G$.) Thus in this situation, $\mathrm{Cl}_X(F, G)$ is finite, and its order is computed by the above expression.

(c) If $\mathrm{char}(F) \neq 2$ and $G$ is a linear algebraic group of type $G_2$, then $Ш_X(F, G)$ is trivial by [HHK15, Example 9.4] (corresponding to a local-global principle for octonion algebras). Hence $\mathrm{Cl}_X(F, G)$ is trivial for such groups.

(d) For each of the following types of linear algebraic groups $G$ over $F$, $Ш_X(F, G)$ is trivial by [HHK14, Corollaries 4.3.2, 4.3.3], and hence so is $\mathrm{Cl}_X(F, G)$: a quasi-split group of type $E_6$ or $E_7$; an almost simple group that is quasi-split of absolute rank at most 5; an almost simple group that is quasi-split of type $B_6$ or $D_6$; an almost simple group that is split of type $D_7$; $\mathrm{SL}_1(A)$ where $A$ is a central simple $F$-algebra whose degree is square free and not divisible by $\mathrm{char}(k)$.

(e) Suppose that $T = k[[t]]$ for some field $k$ of characteristic zero, and that $\mathscr{X}$ is a normal crossings model whose closed fiber $X$ is reduced. Suppose also that $G$ is defined as a linear algebraic group over $k$. If the reduction graph associated to the closed fiber is a tree, and if this property is preserved upon finite extension of $k$, then $Ш_X(F, G)$ is trivial by [CHHKPS21, Theorem 4.11]. Hence $\mathrm{Cl}_X(F, G)$ is also trivial.

(f) If $\mathscr{X}$ is a normal crossings model, and $G$ is a reductive group over $T$, then $Ш_X(F, G) = Ш(F, G) := \ker(H^1(F, G) \to \prod_v H^1(F_v, G))$, where $v$ ranges over the discrete valuations on $F$; see [CHHKPS21, Theorem 3.2]. In particular, this holds if $G$ is the semisimple group $\mathrm{SL}_1(D)$, where $D$ is a biquaternion division algebra over $k$, and $T = k[[t]]$. Assume in addition that the closed fiber $X$ of $\mathscr{X}$ consists of copies of $\mathbb{P}^1_k$ meeting at $k$-points. If $\mathrm{cd}_2(k) \leq 3$, then $Ш(F, G)$ is trivial by [CHHKPS21, Proposition 7.8], and hence so is $\mathrm{Cl}_X(F, G) = Ш_X(F, G)$. On the other hand, suppose we take $k = \mathbb{Q}(\sqrt{17})((x))((y))$, of cohomological dimension equal to four. Let $D = (-1, x) \otimes (2, y)$, and take $\mathscr{X} = \mathrm{Proj}(T[u, v, w]/(uvw - t(u + v + w)^3))$. Then $Ш(F, G) = \mathbb{Z}/2\mathbb{Z}$ by [CHHKPS21, Example 7.6], and so $\mathrm{Cl}_X(F, G) = \mathbb{Z}/2\mathbb{Z}$. Moreover, if we let $\tilde{k}/k$ be a suitable field extension (of infinite transcendence degree) and let $\tilde{F}$ be the base change of $F$ from $k((t))$ to $\tilde{k}((t))$, then $Ш(\tilde{F}, G)$ is infinite by that same example. Equivalently, $\mathrm{Cl}_X(\tilde{F}, G) = Ш_X(\tilde{F}, G)$ is infinite, in contrast to the situation for global fields, where the class group is always finite ([Bor63, Theorem 5.1]).

In the case that the group $G$ is commutative, the higher cohomology groups $H^i$ are defined, for all $i \geq 0$. In this situation, under a characteristic hypothesis, there is an associated long exact Mayer-Vietoris cohomology sequence that extends the one in [HHK15, Theorem 3.5]; see [HHK14, Theorem 3.1.3]. Using this, we obtain the following exact sequence of abelian groups (since $H^i$ with commutative coefficients is an abelian group):

**Theorem 3.7.** *Let $F$ be a semi-global field over a complete discrete valuation ring $T$ with fraction field $K$ and residue field $k$, and let $\mathscr{X}$ be a normal model of $F$ over $T$, with closed fiber $X$. Let $G$ be a commutative linear algebraic group over $F$, and assume that either $G$ has finite order not divisible by* $\mathrm{char}(k)$ *or that* $\mathrm{char}(k) = 0$. *Then there is the following long exact sequence of abelian groups:*

$$\begin{tikzcd}[column sep=small] 1 \arrow[r] & H^0(F,G) \arrow[r] & \prod_{P\in X_{(0)}} H^0(F_P,G) \times \prod_{\eta\in X_{(1)}} H^0(F_\eta^{\mathrm{h}},G) \arrow[r] & \prod'_{\wp\in\mathscr{B}_X} H^0(F_\wp,G) \arrow[dll, rounded corners] \\ & H^1(F,G) \arrow[r] & \prod_{P\in X_{(0)}} H^1(F_P,G) \times \prod_{\eta\in X_{(1)}} H^1(F_\eta^{\mathrm{h}},G) \arrow[r] & \prod'_{\wp\in\mathscr{B}_X} H^1(F_\wp,G) \arrow[dll, rounded corners] \\ & H^2(F,G) \arrow[r] & \cdots & \end{tikzcd} \tag{7}$$

*Proof.* We proceed analogously to the proof of Theorem 3.3. Let $\mathscr{P}, \mathscr{U}, \mathscr{B}, \mathscr{P}'$ be as in that proof. By [HHK14, Theorem 3.1.2], we have a long exact sequence (3) of abelian groups.

Define $A^i, B^i, C^i, D^i$ and $\epsilon^i : A^i \to C^i$ as in the proof of Theorem 3.3, but now considering all $i \geq 0$ (since $G$ is commutative). For each $j \geq 0$, by applying Lemma 3.2(b) to the sequence

$$D^j \to A^j \to B^j \to \ D^{j+1} \to A^{j+1} \to B^{j+1},$$

we obtain an exact sequence of abelian groups

$$D^j \to A^j \times C^j \to B^j \times C^j \to \ D^{j+1} \to A^{j+1} \times C^{j+1} \to B^{j+1} \times C^{j+1}.$$

Splicing these together yields a long exact sequence

$$\cdots D^j \to A^j \times C^j \to B^j \times C^j \ \to D^{j+1} \cdots,$$

which begins with $1 \to D^0$ since $D^0 \to A^0$ is injective. Taking direct limits as before then yields the asserted long exact sequence of abelian groups. $\square$

In the above situation, for $i \geq 0$ and $G$ commutative, write

$$Ш^i_X(F,G) := \ker\left(H^i(F,G) \to \prod_P H^i(F_P,G)\right),$$

where $P$ ranges over all the points of $X$ (including generic points), and

$$\mathrm{Cl}^i_X(F,G) := \prod_{\eta\in X_{(1)}} H^i(F_\eta^{\mathrm{h}},G) \setminus \prod_{\wp\in\mathscr{B}_X}' H^i(F_\wp,G) \ / \prod_{P\in X_{(0)}} H^i(F_P,G).$$

Thus $Ш^0_X(F,G) = 1$, $Ш^1_X(F,G) = Ш_X(F,G)$, and $\mathrm{Cl}^0_X(F,G) = \mathrm{Cl}_X(F,G)$.

**Corollary 3.8.** *For every $i \geq 0$, the coboundary map of the above exact sequence induces a group isomorphism $\mathrm{Cl}^i_X(F,G) \to Ш^{i+1}_X(F,G)$.*

*Proof.* This follows from Theorem 3.7 in the same way that Corollary 3.5 followed from Theorem 3.3, except that [HHK14, Proposition 3.2.2] is used in place of [HHK15, Proposition 5.8] to pass from $F_\eta$ to $F_\eta^{\mathrm{h}}$. $\square$

Under appropriate hypotheses, the coboundary maps in the long exact sequence in [HHK14, Theorem 3.1.3] are trivial, and so the sequence splits, yielding the triviality of $\mathrm{Cl}_X^i(F,G)$. Namely, we have the examples given in the following two corollaries, where $\mathbb{Z}/m\mathbb{Z}(r) := \mu_m^{\otimes r}$ for $m$ not divisible by the residue characteristic:

**Corollary 3.9.** *With $F$ as in Theorem 3.7 and $i \geq 1$, let $G = \mathbb{Z}/m\mathbb{Z}(r)$ for some integers $m, r$ with $m > 0$ not divisible by $\mathrm{char}(k)$, such that either $r = i$ or $[F(\mu_m) : F]$ is prime to $m$. Then the coboundary map $\prod'_{\wp \in \mathscr{B}_X} H^i(F_\wp, G) \to H^{i+1}(F, G)$ in Theorem 3.7 is trivial and $\mathrm{Cl}_X^i(F,G)$ is trivial.*

*Proof.* Under the above hypotheses, the map $H^{i+1}(F,G) \to \prod_{P \in X} H^{i+1}(F_P, G)$ is injective by [HHK14, Theorem 3.2.3(i)]. That is, $Ш_X^{i+1}(F,G)$ is trivial; and then so is $\mathrm{Cl}_X^i(F,G)$ by Corollary 3.8. The above injection factors through

$$\alpha_{i+1} : H^{i+1}(F,G) \to \prod_{P \in X_{(0)}} H^{i+1}(F_P, G) \times \prod_{\eta \in X_{(1)}} H^{i+1}(F_\eta^{\mathrm{h}}, G),$$

since $H^{i+1}(F,G) \to H^{i+1}(F_\eta, G)$ factors through $H^{i+1}(F,G) \to H^{i+1}(F_\eta^{\mathrm{h}}, G)$ for each $\eta \in X_{(1)}$. Thus $\alpha_{i+1}$ is also injective; and so the coboundary map $\prod'_{\wp \in \mathscr{B}_X} H^i(F_\wp, G) \to H^{i+1}(F,G)$ is trivial, by Theorem 3.7. $\square$

**Corollary 3.10.** *With $F$ as in Theorem 3.7, let $G = \mathbb{G}_{\mathrm{m}}$. Then the coboundary maps $\prod'_{\wp \in \mathscr{B}_X} H^i(F_\wp, G) \to H^{i+1}(F,G)$ in Theorem 3.7 are trivial for $i = 0, 1$, and $\mathrm{Cl}_X^i(F,G)$ is trivial. The same holds for all $i > 1$ provided that $\mathrm{char}(k) = 0$ and $K$ contains a primitive $m$-th root of unity for all $m \geq 1$.*

*Proof.* In the case $i = 0$, Example 3.6(a) showed that $\mathrm{Cl}_X(F, \mathbb{G}_{\mathrm{m}}) = \mathrm{Cl}_X^0(F, \mathbb{G}_{\mathrm{m}})$ is trivial. Hence so is the associated coboundary map, by the exactness of (7).

For the case $i = 1$, first note that $\mathrm{Cl}_X(F, \mathrm{PGL}_n)$ is also trivial by Example 3.6(a). Thus by Theorem 3.3 in the case $G = \mathrm{PGL}_n$, the coboundary map is trivial and so the map

$$H^1(F, \mathrm{PGL}_n) \to \prod_{P \in X_{(0)}} H^1(F_P, \mathrm{PGL}_n) \times \prod_{\eta \in X_{(1)}} H^1(F_\eta^{\mathrm{h}}, \mathrm{PGL}_n)$$

has trivial kernel. But for any field $E$, the pointed set $H^1(E, \mathrm{PGL}_n)$ classifies isomorphism classes of central simple $E$-algebras of degree $n$, with the distinguished element corresponding to the split algebra $\mathrm{M}_n(E)$. Thus the map $\mathrm{Br}(F) \to \prod_{P \in X_{(0)}} \mathrm{Br}(F_P) \times \prod_{\eta \in X_{(1)}} \mathrm{Br}(F_\eta^{\mathrm{h}})$ has trivial kernel. But this is just the map from $H^2(F, \mathbb{G}_{\mathrm{m}})$ in the exact sequence (7) with $G = \mathbb{G}_{\mathrm{m}}$. Hence the coboundary map to $H^2(F, \mathbb{G}_{\mathrm{m}})$ is trivial, and thus so is $\mathrm{Cl}_X^1(F, \mathbb{G}_{\mathrm{m}})$.

Finally, for the case $i > 1$, the map $H^{i+1}(F, \mathbb{G}_{\mathrm{m}}) \to \prod_{P \in X} H^{i+1}(F_P, \mathbb{G}_{\mathrm{m}})$ is injective by [HHK14, Theorem 3.2.3(ii)], under the given additional hypotheses. The remainder of the proof of the corollary is then the same as in the proof of Corollary 3.9. $\square$

## 4. Adelic double cosets for relative projective curves

The previous section considered a linear algebraic group $G$ defined over a semi-global field $F$, and worked with the Galois cohomology of $G$ over extension fields of $F$ that arose from looking locally on a model $\mathscr{X}$ of $F$. Below we consider an affine group scheme over $\mathscr{X}$, and work with étale cohomology with respect to completions of rings associated to the model. Again we obtain a Mayer-Vietoris sequence, which gives rise to an identification of a local-global obstruction to an adelic double coset space (now defined in terms of rings). This gives rise to a closer analog of the classical adelic double coset space, classifying locally trivial torsors over $\mathscr{X}$ rather than over $F$. Whereas Section 3 relied on results about patching over fields (from [HHK15] and [HHK14]), here we draw on patching results for torsors over schemes that were proven from a Tannakian point of view in [HKL20].

To emphasize the parallel with the classical case, we begin with the "toy model" of a connected (but possibly reducible) one-dimensional scheme $X$, which we can think of, in the projective case, as the closed fiber of a model of a semi-global field. The scheme $X$, however, is not required to be a projective curve, and in fact it can be the spectrum of the ring of integers of a number field. In that situation we recover the classical case as considered by Borel ([Bor63]).

Below, all our affine group schemes are required to be smooth and of finite type.

As indicated above, we first consider a connected reduced one-dimensional Noetherian scheme $X$, and let $X_{(0)}, X_{(1)}$ denote the set of points of dimension $0,1$ respectively. For each $P \in X_{(0)}$, let $\widehat{\mathscr{O}}_P$ be the completion of the local ring $\mathscr{O}_P$ of $X$ at $P$; this is a complete discrete valuation ring if $X$ is regular at $P$. For $P \in X_{(0)}$, let the *branches* at $P$ be the minimal primes $\wp$ in $\widehat{\mathscr{O}}_P$. For each branch $\wp$, let $\mathscr{K}_\wp$ be the localization of $\widehat{\mathscr{O}}_P$ at $\wp$; this is the fraction field of $\widehat{\mathscr{O}}_P$ if $X$ is regular at $P$. If $S \subseteq X$, let $\mathscr{B}_S$ be the set of branches at closed points in $S$; in particular, $\mathscr{B}_X$ is the set of all branches at points of $X_{(0)}$.

If $\eta \in X_{(1)}$, then the closure of $\eta$ is an irreducible component $X_0$ of $X$; and we let $\mathscr{K}_\eta$ be the function field of $X_0$. Thus $\mathscr{K}_\eta$ is the residue field at $\eta$, and it is the union of the coordinate rings $\mathscr{O}(U)$, where $U$ ranges over the nonempty affine open subsets of $X_0$. For $i = 0, 1$, consider the restricted product $\displaystyle\prod_{\wp\in\mathscr{B}_X}{}' H^i(\mathscr{K}_\wp, G) := \lim_{\to}\Big(\prod_{\wp\in\mathscr{B}_S} H^i(\mathscr{K}_\wp, G) \times \prod_{\wp\notin S} H^i(\widehat{\mathscr{O}}_P, G)\Big)$, where $S$ ranges over finite subsets of $X_{(0)}$.

**Theorem 4.1.** *Let $X$ be a connected reduced one-dimensional Noetherian scheme, and let $G$ be a smooth affine group scheme over $X$. Then there is an exact sequence in étale cohomology*

$$\begin{tikzcd}
1 \arrow[r] & H^0(X,G) \arrow[r] & \prod_{P\in X_{(0)}} H^0(\widehat{\mathscr{O}}_P, G) \times \prod_{\eta\in X_{(1)}} H^0(\mathscr{K}_\eta, G) \arrow[r] & \prod'_{\wp\in\mathscr{B}_X} H^0(\mathscr{K}_\wp, G) \arrow[dll, out=-90, in=90] \\
& H^1(X,G) \arrow[r] & \prod_{P\in X_{(0)}} H^1(\widehat{\mathscr{O}}_P, G) \times \prod_{\eta\in X_{(1)}} H^1(\mathscr{K}_\eta, G) \arrow[r, shift left] \arrow[r, shift right] & \prod'_{\wp\in\mathscr{B}_X} H^1(\mathscr{K}_\wp, G).
\end{tikzcd}$$

This follows from the following analogous assertion with respect to a finite set $\mathscr{P}$ of closed points, via Lemma 3.2(a), in the same way that the exact sequence (6) in Theorem 3.3 followed from the exact sequence (1).

**Proposition 4.2.** *Let $X$ be a connected reduced one-dimensional excellent scheme, and let $G$ be a smooth affine group scheme over $X$. Let $\mathscr{P}$ be a nonempty subset of $X_{(0)}$ that contains* *all the points at which $X$ is not regular. Let $\mathscr{U}$ be the set of connected components of the complement $X \smallsetminus \mathscr{P}$; each of these is affine. Let $\mathscr{B}$ be the set of branches of $X$ at the points of $\mathscr{P}$. Then there is a functorial exact sequence in étale cohomology*

$$\begin{tikzcd}
1 \arrow[r] & H^0(X,G) \arrow[r] & \prod_{P\in\mathscr{P}} H^0(\widehat{\mathscr{O}}_P,G) \times \prod_{U\in\mathscr{U}} H^0(\mathscr{O}(U),G) \arrow[r] & \prod_{\wp\in\mathscr{B}} H^0(\mathscr{K}_\wp,G) \arrow[dll, rounded corners, to path={ -- ([yshift=-2ex]\tikztostart.south) -| (\tikztotarget)}] \\
& H^1(X,G) \arrow[r] & \prod_{P\in\mathscr{P}} H^1(\widehat{\mathscr{O}}_P,G) \times \prod_{U\in\mathscr{U}} H^1(\mathscr{O}(U),G) \arrow[r, shift left] \arrow[r, shift right] & \prod_{\wp\in\mathscr{B}} H^1(\mathscr{K}_\wp,G).
\end{tikzcd}$$

Before proving this proposition, we first introduce some notation. For any scheme $X$, let $\mathfrak{M}(X)$ be the category of coherent sheaves of $\mathscr{O}_X$-modules. For any ring $R$, let $\mathfrak{M}(R) = \mathfrak{M}(\mathrm{Spec}(R))$, which may be identified with the category of finitely presented $R$-modules. Given functors $\phi_1 : \mathscr{C}_1 \to \mathscr{C}_0$ and $\phi_2 : \mathscr{C}_2 \to \mathscr{C}_0$, we write $\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_2$ for the 2-fiber product of $\mathscr{C}_1$ and $\mathscr{C}_2$ over $\mathscr{C}_0$. An object in this category consists of a triple $(A_1, A_2, \psi)$ with $A_i$ an object in $\mathscr{C}_i$ and $\psi : \phi_1(A_1) \to \phi_2(A_2)$ an isomorphism; and a morphism in this category consists of a pair of compatible morphisms in $\mathscr{C}_1$ and $\mathscr{C}_2$. Given morphisms of schemes $X_0 \to X_i \to X$ for $i = 1, 2$, we say that *patching holds for coherent sheaves* on $X$ with respect to $X_1, X_2, X_0$ if the base change functor $\mathfrak{M}(X) \to \mathfrak{M}(X_1) \times_{\mathfrak{M}(X_0)} \mathfrak{M}(X_2)$ is an equivalence of categories. This says that to give a coherent sheaf on $X$ is the same as giving coherent sheaves on $X_1$ and on $X_2$ together with an isomorphism between the sheaves they induce on $X_0$. For example, this holds if $X_1, X_2$ are Zariski open subsets of $X$ whose union is $X$ and whose intersection is $X_0$.

*Proof of Proposition 4.2 (and hence of Theorem 4.1).* Let $\widehat{\mathscr{P}} = \mathrm{Spec}(\prod_{P\in\mathscr{P}} \widehat{\mathscr{O}}_P)$ and let

$$\mathscr{P}^\circ = \widehat{\mathscr{P}} \times_X (X \smallsetminus \mathscr{P}) = \mathrm{Spec}(\prod_{\wp\in\mathscr{B}} \mathscr{K}_\wp).$$

Here $X \smallsetminus \mathscr{P} = \bigcup_{U\in\mathscr{U}} U$. Then patching holds for coherent sheaves on $X$ with respect to $\widehat{\mathscr{P}}, X \smallsetminus \mathscr{P}, \mathscr{P}^\circ$. In the case that $X$ is affine, this follows by [FR70, Proposition 4.2] or [Art70, Theorem 2.6]; and the case of a general $X$ is then a consequence, as shown in [Pri00, Theorem 3.2]. By [HKL20, Corollary 3.0.2], since patching holds for coherent sheaves, it follows that we have the exact sequence asserted in the proposition. $\square$

**Corollary 4.3.** *The coboundary map of the exact sequence in Theorem 4.1 induces a bijection of pointed sets*

$$\mathrm{Cl}(X,G) \to \text{Ш}(X,G),$$

*where* $\mathrm{Cl}(X,G) = \prod_{\eta\in X_{(1)}} G(\mathscr{K}_\eta) \backslash \prod'_{\wp\in\mathscr{B}_X} G(\mathscr{K}_\wp) \,/ \prod_{P\in X_{(0)}} G(\widehat{\mathscr{O}}_P)$ *and where* $\text{Ш}(X,G)$ *is the set of isomorphism classes of $G$-torsors over $X$ that are trivial over each complete local ring $\widehat{\mathscr{O}}_P$ and over each generic point of $X$.*

**Example 4.4.** In the situation that $X$ is a smooth connected curve over a finite field having function field $K$, or a dense open subset of $\mathrm{Spec}(\mathscr{O}_K)$ for some number field $K$, we recapture the classical situation. Namely, if we let $S$ be the (finite) set of places of $K$ that do not correspond to (closed) points of $X$, then $\prod'_{\wp\in\mathscr{B}_X} G(\mathscr{K}_\wp)$ is the same as the adèle group $\mathbb{A}(G(K,S)) = \{(g_v)_v \in \prod_{v\notin S} G(K_v) \mid g_v \in G(O_v) \text{ for almost all } v\}$, and $\prod_{P\in X_{(0)}} G(\widehat{\mathscr{O}}_P)$ equals $\mathbb{A}^\infty(G(K,S))$, the group of integral adèles with respect to the places of $K$ not in $S$. Also, there is a unique generic point $\eta$, with $\mathscr{K}_\eta = K$. Thus $\mathrm{Cl}(X,G) = G(K)\backslash G(\mathbb{A}(K,S))/G(\mathbb{A}^\infty(K,S))$. In particular, we have the following cases:

(a) If $G = \mathbb{G}_{\mathrm{m}}$ then Hilbert's Theorem 90 implies that $H^1(K,G)$ is trivial, as is each $H^1(\widehat{\mathscr{O}}_P,G)$ (since $\widehat{\mathscr{O}}_P$ is local); and thus from the exact sequence in Theorem 4.1 we recover the classical fact that $\mathrm{Cl}(X,G) = H^1(X,G) = \mathrm{Pic}(X)$, the divisor class group of $X$.

(b) If $G = \mathrm{PGL}_n$, then $H^1(X,G)$ classifies Azumaya algebras of degree $n$ over $X$, and $H^1(K,G)$ classifies central simple algebras of degree $n$ over $K$. Thus $H^1(X,G) \to H^1(K,G)$ is injective, since $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is injective by [Mil80, Chapter IV, Corollary 2.6]. Consequently, $\text{Ш}(X,G)$ is trivial, and hence so is $\mathrm{Cl}(X,G)$, by Corollary 4.3.

(c) In the equal characteristic case (where $X$ is a smooth curve over a finite field $k$), if $G$ is semisimple and simply connected over $k$, then $H^1(K,G)$ is trivial by [Har75, Satz A]; and so is $H^1(\widehat{\mathscr{O}}_{X,P},G)$, for $P$ a closed point of $X$ (e.g., see [BD09, Lemma 4.4]). So *every* $G$-torsor over $X$ corresponds to an element of $\text{Ш}(X,G)$ as defined above. We thus recover the classical fact that $\mathrm{Cl}(X,G) = G(K)\backslash G(\mathbb{A}(K,S))/G(\mathbb{A}^\infty(K,S))$ classifies $G$-torsors over $X$ in this situation.

We next turn to the case of a smooth affine group scheme $G$ over a model $\mathscr{X}$ of a semi-global field, obtaining an associated Mayer-Vietoris sequence in étale cohomology and a description of the associated double coset space as a local-global obstruction, paralleling the above case of a one-dimensional scheme. As in that case, we rely on [HKL20].

**Theorem 4.5.** *Let $T$ be a complete discrete valuation ring, let $\mathscr{X}$ be an integral flat projective $T$-curve with closed fiber $X$, and let $G$ be a smooth affine group scheme over $\mathscr{X}$. Then there is an exact sequence in étale cohomology*

$$\begin{tikzcd}[column sep=small] 1 \arrow[r] & H^0(\mathscr{X},G) \arrow[r] & \prod_{P\in X_{(0)}} H^0(\widehat{R}_P,G) \times \prod_{\eta\in X_{(1)}} H^0(R^{\mathrm{h}}_\eta,G) \arrow[r] & \prod'_{\wp\in\mathscr{B}_X} H^0(\widehat{R}_\wp,G) \arrow[dll, rounded corners] \\ & H^1(\mathscr{X},G) \arrow[r] & \prod_{P\in X_{(0)}} H^1(\widehat{R}_P,G) \times \prod_{\eta\in X_{(1)}} H^1(R^{\mathrm{h}}_\eta,G) \arrow[r, shift left] \arrow[r, shift right] & \prod'_{\wp\in\mathscr{B}_X} H^1(\widehat{R}_\wp,G). \end{tikzcd}$$

As in the case of Theorem 4.1, this follows via Lemma 3.2(a) from the analogous assertion with respect to a finite set, viz.:

**Proposition 4.6.** *Under the assumptions of Theorem 4.5, let $\mathscr{P}$ be a non-empty finite set of closed points of $\mathscr{X}$ that includes all the points at which the closed fiber $X$ is not unibranched, let $\mathscr{U}$ be the set of connected components of the complement of $\mathscr{P}$ in $X$, and let $\mathscr{B}$ be the set of branches of $X$ at the points of $\mathscr{P}$. Then there is a functorial exact sequence of pointed sets in étale cohomology*

$$\begin{tikzcd}[column sep=small] 1 \arrow[r] & H^0(\mathscr{X},G) \arrow[r] & \prod_{P\in\mathscr{P}} H^0(\widehat{R}_P,G) \times \prod_{U\in\mathscr{U}} H^0(\widehat{R}_U,G) \arrow[r] & \prod_{\wp\in\mathscr{B}} H^0(\widehat{R}_\wp,G) \arrow[dll, rounded corners] \\ & H^1(\mathscr{X},G) \arrow[r] & \prod_{P\in\mathscr{P}} H^1(\widehat{R}_P,G) \times \prod_{U\in\mathscr{U}} H^1(\widehat{R}_U,G) \arrow[r, shift left] \arrow[r, shift right] & \prod_{\wp\in\mathscr{B}} H^1(\widehat{R}_\wp,G). \end{tikzcd}$$

*Proof.* Proceeding as in the proof of Proposition 4.2, we let $\widehat{\mathscr{P}} = \mathrm{Spec}(\prod_{P\in\mathscr{P}} \widehat{R}_P)$, let $\widehat{\mathscr{U}} = \mathrm{Spec}(\prod_{U\in\mathscr{U}} \widehat{R}_U)$, and let $\widehat{\mathscr{B}} = \mathrm{Spec}(\prod_{\wp\in\mathscr{B}} \widehat{R}_\wp)$. Then patching holds for coherent sheaves on $\mathscr{X}$ with respect to $\widehat{\mathscr{P}}, \widehat{\mathscr{U}}, \widehat{\mathscr{B}}$ by [Pri00, Theorem 3.4] (see also [HKL20, Theorem 3.1.4]), as a consequence of [FR70, Proposition 4.2] or [Art70, Theorem 2.6] together with Grothendieck's Existence Theorem ([Gro61, Théorème 5.1.6]). By [HKL20, Corollary 3.0.2], it follows from this patching property that we have the exact sequence asserted in the proposition. □

**Corollary 4.7.** *The coboundary map of the exact sequence in Theorem 4.5 induces a bijection of pointed sets*

$$\mathrm{Cl}_X(\mathscr{X}, G) \to \text{Ш}_X(\mathscr{X}, G),$$

*where* $\mathrm{Cl}_X(\mathscr{X}, G) = \prod_{\eta\in X_{(1)}} G(R_\eta^{\mathrm{h}}) \setminus \prod'_{\wp\in\mathscr{B}_X} G(\widehat{R}_\wp) \,/ \prod_{P\in X_{(0)}} G(\widehat{R}_P)$ *and where* $\text{Ш}_X(\mathscr{X}, G)$ *is the set of isomorphism classes of $G$-torsors over $\mathscr{X}$ that are trivial over the complete local ring $\widehat{R}_P = \widehat{\mathscr{O}}_{\mathscr{X},P}$ at $P$ for every point $P$ of the closed fiber $X$ of $\mathscr{X}$ (including the generic points of $X$).*

*Proof.* By the exact sequence in Theorem 4.5, it suffices to show that $\text{Ш}_X(\mathscr{X}, G)$ is the kernel of

$$H^1(\mathscr{X}, G) \to \prod_{P\in X_{(0)}} H^1(\widehat{R}_P, G) \times \prod_{\eta\in X_{(1)}} H^1(R_\eta^{\mathrm{h}}, G).$$

For this, it suffices to show that a $G$-torsor over $\mathscr{X}$ that becomes trivial over $\widehat{R}_\eta$ must also become trivial over $R_\eta^{\mathrm{h}}$; i.e., if it has an $\widehat{R}_\eta$-point then it has an $R_\eta^{\mathrm{h}}$-point. Since $R_\eta^{\mathrm{h}}$ contains the henselization $\widetilde{R}_\eta$ of $R_\eta$, it suffices to show that a $G$-torsor over $\mathscr{X}$ with an $\widehat{R}_\eta$-point has an $\widetilde{R}_\eta$-point. This in turn follows from the Artin Approximation Theorem (see [Art69, Theorem 1.10 and Theorem 1.12]), which applies because the complete discrete valuation ring $T$ is excellent (by [Gro65, Scholie 7.8.3(iii)]) and because $R_\eta$ is the localization of a $T$-algebra of finite type at a prime ideal. □

**Example 4.8.** The analogs of Example 4.4(a),(b) hold in the situation of Theorem 4.5 and Corollary 4.7:

(a) If $G = \mathbb{G}_{\mathrm{m}}$ then Hilbert's Theorem 90 implies that $H^1(\widehat{R}_P, G)$ and $H^1(R_\eta^{\mathrm{h}}, G)$ are trivial, since those rings are local, and so the exact sequence in Theorem 4.5 yields that $\mathrm{Cl}_X(\mathscr{X}, G) = H^1(\mathscr{X}, G) = \mathrm{Pic}(\mathscr{X})$ (thereby providing a justification for the Cl notation).

(b) If $G = \mathrm{PGL}_n$, then we have the commutative diagram

$$\begin{tikzcd} H^1(\mathscr{X}, G) \arrow[r] \arrow[d] & \prod_{P\in X_{(0)}} H^1(\widehat{R}_P, G) \times \prod_{\eta\in X_{(1)}} H^1(R_\eta^{\mathrm{h}}, G) \arrow[d] \\ H^1(F, G) \arrow[r] & \prod_{P\in X_{(0)}} H^1(F_P, G) \times \prod_{\eta\in X_{(1)}} H^1(F_\eta^{\mathrm{h}}, G). \end{tikzcd}$$

Here the bottom horizontal arrow has trivial kernel by Example 3.6(a), since that kernel is the same as $\mathrm{Cl}_X(F, \mathrm{PGL}_n)$ by Theorem 3.3. Also, the left vertical arrow has trivial kernel by [Mil80, Chapter IV, Corollary 2.6], as in Example 4.4(b). So by commutativity of the diagram, the top horizontal arrow has trivial kernel; i.e.,

$\text{Ш}_X(\mathscr{X}, \mathrm{PGL}_n)$ is trivial (yielding a local-global principle for the Brauer group in this context). By Corollary 4.7, $\mathrm{Cl}_X(\mathscr{X}, \mathrm{PGL}_n)$ is also trivial.

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