Abhyankar's Local Conjecture on Fundamental Groups

David Harbater* and Katherine F. Stevenson

Abstract: This paper proves the remaining open case of Abhyankar's higher dimensional conjecture on local fundamental groups in characteristic p ([Ab2], [Ab3]). This conjecture, which is analogous to Abhyankar's conjectures on global fundamental groups, proposed that a finite group G is a Galois group over $k[[x_1,\ldots,x_n]][(x_1\cdots x_r)^{-1}]$ if and only if its maximal prime-to-p quotient is, provided $n \geq 2$ and $1 \leq r \leq n$. For $r > 1$, this conjecture was disproven in [HP]. Here we prove that the conjecture is true in the case $r = 1$. So the Galois groups over $k[[x_1,\ldots,x_n]][x_1^{-1}]$ are precisely the cyclic-by-quasi-p groups.

Section 1. Introduction.

In 1957, Abhyankar made a conjecture [Ab1] concerning the fundamental group of an affine curve X over an algebraically closed field k of characteristic p. Specifically, his conjecture stated what the finite quotients G of $\pi_1(X)$ are — or equivalently, which finite groups are Galois groups of finite unramified connected covers of X. Namely, if X is obtained by deleting r points from a smooth projective curve of genus $g \geq 0$, then G is such a Galois group if and only if its maximal prime-to-p quotient $G/p(G)$ can be generated by a set of at most $2g + r - 1$ elements. (Here $p(G)$ is the subgroup of G generated by the Sylow p-subgroups of G.) As Grothendieck later showed [Gr2], a *prime-to-p* group is a Galois group over X if and only if it has such a set of generators. Thus Abhyankar's curve conjecture was equivalent to the assertion that a finite group G is a Galois group over X if and only if $G/p(G)$ is. This was proven in the case that $X = \mathbb{A}^1$ in [Ra], and was then proven for general affine curves in [Ha1].

Generalizing the statement of this conjecture, Abhyankar has proposed that the same principle should govern Galois groups of affine k-varieties in higher dimensions, in both local and global situations. (This was stated implicitly in [Ab2] and explicitly in [Ab3].) In the global case, he considered the fundamental group of an affine variety X that is the complement of a normal crossing divisor D in $\mathbb{P}^n$, with $n > 1$. Say D has irreducible components $D_1, \ldots, D_r$ of degrees $d_1, \ldots, d_r$. Then Abhyankar's global conjecture says that a finite group G is a Galois group over X if and only if $G/p(G)$ is an abelian group that is generated by elements $g_1, \ldots, g_r$ satisfying $g_1^{d_1} \cdots g_r^{d_r} = 1$. For a *prime-to-$p$* group, G is indeed a Galois group over X if and only if it has such a set of generators [Ab2], [F]. (The corresponding case over $\mathbb{C}$ had previously appeared in work of Zariski [Z1], [Z2], [F].)

* Supported in part by NSF Grant DMS99-70481.

Thus Abhyankar's higher dimensional global conjecture was again equivalent to saying that a finite group G is a Galois group over X if and only if $G/p(G)$ is.

Similarly, in the local case, Abhyankar proposed his

Local conjecture. (Abhyankar) *Let $n > 1$ and $1 \le r \le n$. Then a finite group G is the Galois group of a finite unramified extension of $\hat{R}_{n,r} := k[[x_1, \dots, x_n]][(x_1 \cdots x_r)^{-1}]$ if and only if its maximal prime-to-p quotient $G/p(G)$ is abelian and has a generating set of at most r elements.*

(Here, as elsewhere in this paper, it is understood that the extension is required to be a domain, or equivalently that its spectrum is (reduced and) irreducible.)

A *prime-to-p* group is a Galois group over $R_{n,r}$ if and only if it is abelian with such a generating set (because of Abhyankar's Lemma; see [Ab2], [HP, Prop. 3.1]). So as before, the conjecture says that a finite group G is a Galois group over $X = \operatorname{Spec} \hat{R}_{n,r}$ if and only if $G/p(G)$ is.

Of course, for G to be a Galois group over any space X it is necessary for $G/p(G)$ to be a Galois group over X. So in these conjectures, the issue is whether this condition is also sufficient. In [HP], it was shown by the first author and by M. van der Put that this condition is *not* sufficient in either the local and global cases in dimension > 1. Namely, if $1 \le r \le n$, and if $\hat{X}_{n,r} = \operatorname{Spec} \hat{R}_{n,r}$, then the short exact sequence

$$1 \to p(\pi_1(\hat{X}_{n,r})) \to \pi_1(\hat{X}_{n,r}) \to \pi_1(\hat{X}_{n,r})/p(\pi_1(\hat{X}_{n,r})) \to 1$$

has a splitting [HP, Cor. 3.4(a)]; and from this it follows that an analogous splitting exists for the fundamental group of $X_{n,r} := \mathbb{A}^n - (x_1 \cdots x_r = 0)$ [HP, Cor. 4.7(a)]. This splitting imposes an additional non-trivial condition on Galois groups over X and $\hat{X}$, if $1 < r \le n$. So the local and global conjectures do not hold in those cases [HP, Examples 5.2, 5.3], although several possible variants are suggested by [HP, §5].

This leaves open the question of whether Abhyankar's higher dimensional conjectures hold for $X_{n,1}$ and for $\hat{X}_{n,1}$. In these situations, the cokernel of the above exact sequence is free of rank 1 (as a pro-prime-to-p group), hence it automatically splits. So in these two cases, the splitting condition does not impose any new restrictions for a finite group to be a Galois group. And in the case of $X_{n,1}$, it is easy to see from Abhyankar's original conjecture for $\mathbb{A}^1$ that the higher dimensional global conjecture does in fact hold here [HP, Ex. 5.3]. The case of $\hat{X}_{n,1}$ has remained open, though; and the purpose of the present paper is to prove that Abhyankar's higher dimensional local conjecture does hold for this space:

Main Theorem. *For $n > 1$, a finite group G is the Galois group of an unramified extension of $\hat{R}_{n,1} = k[[x_1, \dots, x_n]][x_1^{-1}]$ if and only if $G/p(G)$ is cyclic.*

Note that the case $n = 1$ clearly does not hold, since only cyclic-by-p groups can occur over the Laurent series field $k((x)) = k[[x]][x^{-1}]$ (using that k is algebraically closed). Of course this case was not part of Abhyankar's conjectures.

As in [Ha1], the proof will rely on Abhyankar's original conjecture for the affine line [Ra], together with results about embedding problems and methods of formal patching. Since the formal patching methods apply to global objects, not to local ones, we will first use blowings-up in order to introduce exceptional divisors and thereby pass from a local situation to a global one.

Section 2 of this paper contains some related results about Galois covers in a global situation. These are combined with a blowing-up construction in Section 3 in order to prove our Main Theorem above (rephrased as Theorem 3.3 below), saying that Abhyankar's local conjecture holds if (and only if) $r = 1$. We provide another viewpoint on this result in Section 4.

Throughout this paper, if X is a connected scheme, then a *cover* $f : Y \to X$ will be a morphism that is finite and generically separable. A *Galois* cover $f : Y \to X$ is a connected cover whose covering group $\mathrm{Aut}_X(Y)$ acts simply transitively on each geometric generic fibre of f. If G is a finite group, then a *G-Galois* cover is a (possibly disconnected) cover $f : Y \to X$ together with a homomorphism $G \to \mathrm{Aut}_X(Y)$ with respect to which G acts simply transitively on each generic geometric fibre.

Section 2. Global results.

This section contains an extension of a result in [Ha3] related to embedding problems over curves in characteristic p. Recall that a finite group Q is a *quasi-p group* if $Q = p(Q)$. The following proposition is a special case of [Ha3, Prop. 2.3]:

Proposition 2.1. *Let $\Gamma = Q \rtimes G$ be the semi-direct product of a quasi-p group Q with a finite group G such that G normalizes a Sylow p-subgroup P of Q. Let $Y \to X$ be a G-Galois cover of smooth connected projective k-curves, and let $\xi_0 \in X$. Then there is a normal absolutely irreducible Q-Galois cover $Z_t \to Y_t := Y \times_k k((t))$ that is étale away from the fibre over ξ_0, such that $Z_t \to X_t := X \times_k k((t))$ is Γ-Galois.*

In fact, as we show below, even more is true:

Proposition 2.2. *In Proposition 1, let $\delta_1, \dots, \delta_m \in X - \{\xi_0\}$ be distinct non-branch points of $Y \to X$. For $i = 1, \dots, m$ let $W_i \to S$ be (possibly disconnected) smooth Q-Galois covers of the projective s-line S, branched only at $s = 0$, where they have p-group inertia. Let $\hat{X}_t$ be the blow-up of $X \times_k k[[t]]$ at the points $\delta_1, \dots, \delta_m$ on the closed fibre, and let $\hat{Z}_t$ be the normalization of $\hat{X}_t$ in Z_t. Then the cover Z_t in Proposition 1 may be chosen so that the fibre of $\hat{Z}_t \to \hat{X}_t$ over the exceptional divisor at δ_i is a disjoint union* *of copies of $W_i \to S$ away from its branch point, and such that fibre of $\hat{Z}_t \to \hat{X}_t$ over the proper transform of $(t = 0)$ is connected.*

Here we prove these two results together, essentially following the construction in the proof of [Ha3, Prop. 2.3]. As in that paper, there are three main ingredients: Abhyankar's Conjecture over the affine line [Ra], formal patching [HS], and the existence of solutions to p-embedding problems in characteristic p [Ha2].

Proof of Propositions 2.1 and 2.2. After adding an additional δ_i if necessary and renumbering, we may assume that W_1 is the trivial (disconnected) Q-Galois cover of the projective s-line S. Let $\tilde{G} = P \rtimes G$ be the subgroup of Γ generated by P and G. Let $\mathcal{K}_i$ be the local field of a point of $W_i \to S$ over $(s = 0)$ whose inertia group P_i is contained in P. Meanwhile, by [Ra], there is a smooth connected Q-Galois cover $W \to S$ branched only over $(s = 0)$, where its inertia groups are the Sylow p-subgroups of Q. Let $\mathcal{K}$ be the local field of a ramification point of $W \to S$ with inertia group P; this is a P-Galois extension of $k((s))$. Pick a non-branch point $\delta_0 \neq \xi_0$ distinct from $\delta_1, \ldots, \delta_m$, and let $W_0 = W$ and $P_0 = P$. By [Ha2, Theorem 5.6], there is an irreducible $\tilde{G}$-Galois cover $\tilde{Y}_s \to X_s := X \times_k k((s))$ that dominates the G-Galois cover $Y_s := Y \times_k k((s)) \to X_s$, such that $\tilde{Y}_s \to Y_s$ is étale away from $\xi_{0,s} := \xi_0 \times_k k((s))$, and the fibre over $\delta_{i,s} := \delta_i \times_k k((s))$ consists of a disjoint union of copies of the P_i-Galois cover $\operatorname{Spec} \mathcal{K}_i$ (for $i = 0, \ldots, m$). The normalization $\tilde{\mathcal{Y}}_s$ of $\mathcal{X}_s := X \times_k k[[s]]$ in $\tilde{Y}_s$ is an irreducible $\tilde{G}$-Galois cover of $\mathcal{X}_s$, proper over $k[[s]]$. So by [Gr1, Proposition 5.5.1], the closed fibre of $\tilde{\mathcal{Y}}_s$ is connected.

Let $\mathcal{X}_t$ be the blow-up of $X \times_k k[[t]]$ at the points $\delta_0, \ldots, \delta_m$ on the closed fibre $(t = 0)$. We may identify the proper transform of $(t = 0)$ with X, and regard each exceptional divisor S_i as a copy of S, meeting X at the point δ_i (corresponding to the point $s = 0$ on S). So $\mathcal{X}_t$ is a projective $k[[t]]$-curve whose general fibre is $X_t := X \times_k k((t))$ and whose closed fibre consists of X and the exceptional divisors S_i. By formal patching [HS, Cor. to Theorem 1], there is a Γ-Galois cover $\mathcal{Z}_t \to \mathcal{X}_t$ whose formal completion along X is a disjoint union of copies of the $\tilde{G}$-Galois cover $\tilde{\mathcal{Y}}_s \to \mathcal{X}_s$, indexed by the cosets of $\tilde{G}$; whose fibre over the exceptional divisor S_i (for $i = 0, \ldots, m$) is a union of copies of the Q-Galois cover $W_i \to S$ that are indexed by the cosets of Q and are disjoint away from δ_i; and such that the identity copies of $\tilde{Y}_s$ and W_i meet at a point ζ_i over δ_i on X. Let $\mathcal{Z}_0$ be the inverse image of $\mathcal{X}_0 := X \cup S_0$ under $\mathcal{Z}_t \to \mathcal{X}_t$. Then the connected component of ζ_0 in $\mathcal{Z}_0$ is all of $\mathcal{Z}_0$, since Q and $\tilde{G}$ generate Γ. That is, $\mathcal{Z}_0$ is connected.

Recall that $\mathcal{X}_t$ is the blow up of $X \times_k k[[t]]$ at $\delta_0, \delta_1, ..., \delta_m$ (on the closed fibre). Now consider the blow up $\hat{X}_t$ of $X \times_k k[[t]]$ at $\delta_1, ..., \delta_m$ (omitting δ_0). In $\hat{X}_t$, identify the proper transform of $(t = 0)$ under $\hat{X}_t \to X \times_k k[[t]]$ with $X \subset \hat{X}_t$. Then $\mathcal{X}_t$ is the blow-up of $\hat{X}_t$ at δ_0. Let $\hat{Z}_t$ be the normalization of $\hat{X}_t$ in $\mathcal{Z}_t$. Thus $\mathcal{Z}_t \to \mathcal{X}_t$ is the normalized pullback of $\hat{Z}_t \to \hat{X}_t$. So for $i = 1, \ldots, m$, the fibre of $\hat{Z}_t \to \hat{X}_t$ over S_i is the same as that of $\mathcal{Z}_t \to \mathcal{X}_t$ over S_i, viz. a union of copies of $W_i \to S$ as asserted in Proposition 2.2.

Under the blow-up morphism $\mathcal{X}_t \to \hat{X}_t$, the closed set $\mathcal{X}_0 \subset \mathcal{X}_t$ is the inverse image of $X \subset \hat{X}_t$. Also, $\mathcal{Z}_0$ is the inverse image under $\mathcal{Z}_t \to \hat{Z}_t$ of the fibre of $\hat{Z}_t \to \hat{X}_t$ over $X \subset \hat{X}_t$. But $\mathcal{Z}_0$ is connected. Hence so is the fibre over $X \subset \hat{X}_t$, the proper transform of $(t = 0)$. Thus the closed fibre of $\hat{Z}_t \to \hat{X}_t$ is connected, and hence so is the general fibre $Z_t \to X_t = X \times_k k((t))$. The G-Galois cover $\mathcal{Z}_t/Q$ of $\mathcal{X}_t$ is just $\mathcal{Y}_t := Y \times_X \mathcal{X}_t$, since this is true along the formal completion along X and along the S_i's; and $\mathcal{Z}_t \to \mathcal{Y}_t$ is étale away from $\xi_0 \times_k k[[t]]$ and the closed fibre. So $Z_t \to X_t$ factors through Y_t, and $Z_t \to Y_t$ is étale away from $\xi_{0,t} := \xi \times_k k((t))$. So it is a Γ-Galois cover. Moreover Z_t is normal since $\hat{Z}_t$ is. So Z_t is irreducible, and $\hat{Z}_t$ is the normalization of $\hat{X}_t$ in Z_t (and so agrees with the definition of $\hat{Z}_t$ in the statement of Proposition 2.2). Finally, if K is a non-trivial finite extension of $k((t))$, then the normalization of $\hat{X}_t$ in $X_K := X \times_k K$ has the property that its fibre over the generic point of S_1 is totally ramified; whereas the fibre of $\hat{Z}_t$ there is totally split. So Z_t is linearly disjoint from $X_K = X_t \times_{k((t))} K$ over X_t, and thus the $k((t))$-curve Z_t is absolutely irreducible. □

Corollary 2.3. *In Proposition 2.1, let $\delta \in X$ be a non-branch point of $Y \to X$, and let $\hat{X}_t$ be the blow up of $X \times_k k[[t]]$ at the point δ on the closed fibre. Then we may choose Z_t such that the normalization $\hat{Z}_t$ of $\hat{X}_t$ in Z_t is unramified over the exceptional divisor away from where it meets the proper transform of $(t = 0)$, and such that the fibre of $\hat{Z}_t$ over this proper transform is connected.*

Proof. In Proposition 2.2, take $m = 1$ and $\delta_1 = \delta$. Take $W_1 \to S$ to be an arbitrary Q-Galois cover (e.g. the trivial cover, composed of a disjoint union of copies of S). Then Proposition 2.2 gives us a choice of $Z_t \to X_t$ satisfying the conclusion there. In particular, the fibre of $\hat{Z}_t \to \hat{X}_t$ over the exceptional divisor is a union of copies of $W_1 \to S$ away from the point δ. So this fibre is étale there. □

Remark. In the above proof of Propositions 2.1 and 2.2, the point δ_0 is what holds the Γ-Galois cover together, since a $\tilde{G}$-Galois cover and a Q-Galois cover meet at a point ζ_0 over it. On the other hand, the locus $\{\delta_1, \ldots, \delta_m\}$ (or just δ in the Corollary) is where the constructed cover from Proposition 2.1 is blown up (for use in Lemma 3.2 below).

Section 3. Proof of the Main Theorem.

In this section we prove the main theorem of the paper, that Abhyankar's local conjecture holds in dimension > 1 if only one coordinate hyperplane is deleted. The proof relies on the previous section together with a formal patching construction applied to a blow-up of a local scheme. It suffices to work in dimension 2, and afterwards to pass to dimension n.

Specifically, using Corollary 2.3, we prove

Theorem 3.1. *Let k be an algebraically closed field of characteristic $p > 0$. Let Γ be a finite group, let $Q = p(\Gamma)$, and suppose that $C := \Gamma/Q$ is cyclic. Then there is a Galois étale cover of* $\operatorname{Spec} k[[x, y]][1/y]$ *with Galois group Γ.*

The key step in the proof of Theorem 3.1 is the following

Lemma 3.2. *Under the hypotheses of Theorem 3.1, suppose that $\Gamma = Q \rtimes C$, and that $C \subset \Gamma$ normalizes a Sylow p-subgroup of Q. Then there is a normal connected Γ-Galois cover $Z \to \operatorname{Spec} k[[x, y]]$ which is étale away from $(xy = 0)$; such that C is an inertia group over the generic point of $(x = 0)$; and such that the C-Galois cover $Z/Q \to \operatorname{Spec} k[[x, y]]$ is totally ramified over the generic point of $(y = 0)$.*

Proof of Lemma 3.2. Let $R = k[[z]]$, and let $P_t = \mathbb{P}^1_R$, the projective t-line over R. Thus we may regard $\mathbb{P}^1_K$ as the generic fibre of P_t, where $K = k((z))$. Let $\tilde{P}$ be the blow-up of P_t at the point $\xi \in P_t$ where $(t = z = 0)$; let $E \subset \tilde{P}$ be the exceptional divisor of the blow-up; and let $T \subset \tilde{P}$ be the proper transform of the closed fibre of P_t. Also, let $\tilde{\xi} \in \tilde{P}$ be the point where E meets T (and which lies over $\xi \in P_t$). Consider the parameter $x = z/t$ on E. Thus $x = 0$ at the point $\tilde{\xi}$; and we may identify E with the projective x-line $\mathbb{P}^1_k$ over k, or equivalently with the closed fibre of the projective x-line P_x over R. Note here that the blow-up $\tilde{P}$ is a closed subset of $P_t \times_k \mathbb{P}^1_k = P_t \times_R P_x$. Here the second projection $\pi : \tilde{P} \to P_x$ is a birational isomorphism which is an isomorphism away from $T \subset \tilde{P}$ (and blows down T).

Let n be the order of C. Let $W_0 \to \mathbb{P}^1_k$ be the C-Galois cover of the projective t-line that is given generically by $w^n = t - 1$. This cover is branched precisely over $t = 1, \infty$, where it is totally ramified. Since C normalizes a Sylow p-subgroup of the quasi-p group Q, we may apply Proposition 2.1 above to this C-Galois cover, to the point $\xi_0 : (t = 1)$ on $\mathbb{P}^1_k$, and to the group Γ, using the Laurent series field $K = k((z))$. The conclusion is that there is a normal absolutely irreducible Q-Galois cover $U^\circ \to W_0 \times_k K$ that is étale away from the fibre over $(t = 1)$, and such that $U^\circ \to P^\circ := \mathbb{P}^1_K$ is Γ-Galois. In particular, C is an inertia group over $t = \infty$, and the C-Galois cover $U^\circ/Q \approx W_0 \times_k K$ of P° is totally ramified over the point $(t = 1)$. Let U be the normalization of P_t in U°. So $U \to P_t$ is a connected normal Γ-Galois cover which is étale away from $(z = 0)$, $(t = 1)$, and $(t = \infty)$, and whose inertia groups over the generic point of $(t = \infty)$ are n-cyclic.

Moreover, using Corollary 2.3 above with δ taken as the point $(t = 0)$ on $\mathbb{P}^1_k$, we may choose U° above with two additional properties. Namely, let $\tilde{U} \to \tilde{P}$ be the normalized pullback of $U \to P_t$ with respect to $\tilde{P} \to P_t$, and let $\tilde{U}_0$ be the fibre of $\tilde{U}$ over T. Then we may choose U° such that $\tilde{U}_0$ is connected and such that $\tilde{U}$ is étale over E away from $\tilde{\xi}$. So $\tilde{U} \to \tilde{P}$ is ramified only over $t = 1, \infty$ and over T.

We next consider a Γ-Galois cover of P_x whose local behavior at $(x = z = 0)$ will enable us to obtain the local cover asserted in the statement of the lemma. Namely, let V be the normalization of P_x in $\tilde{U}$, relative to the morphism $\tilde{U} \to \tilde{P} \to P_x$. This space is the same as the normalization of P_x in the general fibre of $\tilde{U}$, which is an irreducible normal Γ-Galois cover of the generic fibre of P_x. Thus V is an irreducible normal Γ-Galois cover of P_x. This cover is branched only over $t = 1, \infty$, since $\tilde{U} \to \tilde{P}$ is ramified only there and over T, and since T maps to the closed point $(x = z = 0)$ under $\tilde{P} \to P_x$. Here the locus of $t = 1$ on P_x is the locus of $x = z$, and the locus of $t = \infty$ on P_x is the locus of $x = 0$. (Note that both of these loci contain the closed point $(x = z = 0)$.) So the general fibre of $V \to P_x$ is an irreducible Γ-Galois cover of the projective x-line over K, branched only over $x = 0, z$. Moreover this cover is tamely ramified over the generic point of $x = 0$ with C as an inertia group, and the C-Galois subcover V/Q is totally ramified over the generic point $x = z$, by the corresponding facts for U and hence for $\tilde{U}$ over $t = 1, \infty$.

Now under the surjection $\tilde{U} \to V$, the connected closed set $\tilde{U}_0$ is the inverse image of the fibre of V over the point $(x = z = 0)$. So that fibre is connected; i.e. the irreducible normal Γ-Galois cover $V \to P_x$ is totally ramified over $(x = z = 0)$. So the pullback of V under the morphism $\operatorname{Spec} k[[x, z]] \to P_x$ is also an irreducible normal Γ-Galois cover, branched only over $x = 0$ and $x = z$, with C as an inertia group over $x = 0$ and with the quotient modulo Q being a C-Galois cover with total ramification over $x = z$. Setting $y = z - x$ completes the proof of the lemma. □

Using Lemma 3.2, we complete the proof of Theorem 3.1:

Proof of Theorem 3.1. Let n be the order of C. Thus n is prime to p. By [Ha1, Lemma 5.3] (or by using the Schur-Zassenhaus Theorem in group theory [Go]), there is a prime-to-p cyclic supplement $C' \subset \Gamma$ to Q in Γ such that C' normalizes a Sylow p-subgroup of Q. Thus Γ is a quotient of the semidirect product $\Gamma' := Q \rtimes C'$, and $Q = p(\Gamma')$. So it suffices to prove the result for Γ'. Replacing Γ, C by Γ', C', we may assume that $\Gamma = Q \rtimes C$, and view C as a subgroup of Γ that normalizes a Sylow p-subgroup of Q.

So the lemma applies. Thus there is a normal connected Γ-Galois cover $Z \to S :=$ $\operatorname{Spec} k[[s, y]]$ which is étale away from $(sy = 0)$, such that C is an inertia group over the generic point of $(s = 0)$, and such that $W := Z/Q \to S$ is totally ramified over $(y = 0)$. Here $W \to S$ is a normal connected C-Galois cover of S, branched only over $(sy = 0)$; and $Z \to W$ is a normal connected Q-Galois cover.

Let $X = \operatorname{Spec} k[[x, y]]$, and consider the morphism $X \to S$ given by $x^n = s$. Let $Z' \to X$ [resp. $W' \to X$] be the normalized pullback of $Z \to S$ [resp. of $W \to S$] with respect to this morphism. Thus $W' = Z'/Q$, and Z' is the normalized fibre product of W' and Z over W. Since $X \to S$ is étale over the generic point of $(y = 0)$, whereas $W \to S$ is totally ramified there, it follows that X and W are linearly disjoint over S, and hence W' is irreducible.

By Abhyankar's Lemma, the cover $Z' \to X$ is étale over the generic point of $(x = 0)$, and so this cover is branched only over $(y = 0)$. Now $W' \to W$ is C-Galois, where C is of order prime-to-p. But $Z \to W$ is Q-Galois, where Q is a quasi-p group. So C and Q have no non-trivial common quotients, and the two covers $Z \to W$ and $W' \to W$ dominate no non-trivial common subcover of W. Since these two covers are also Galois, it follows that they are linearly disjoint. Hence their normalized fibre product Z' is irreducible. The restriction of the Γ-Galois cover $Z' \to X$ to $y \neq 0$ is then the asserted cover of $\operatorname{Spec} k[[x, y]][1/y]$. □

Theorem 3.1 is the case of dimension 2 in the Main Theorem. The general case is now immediate:

Main Theorem 3.3. *Abhyankar's Local Conjecture on Galois groups over* $\hat{R}_{n,r} = k[[x_1, \ldots, x_n]][(x_1 \cdots x_r)^{-1}]$ *(with* $n > 1$ *and* $1 \leq r \leq n$*) holds if and only if* $r = 1$.

Proof. As noted in the introduction, the case of $r > 1$ was disproven in [HP]. So it remains to prove the case $r = 1$; i.e. that a finite group G is the Galois group of an unramfied extension of $\hat{R}_{n,1} = k[[x_1, \ldots, x_n]][x_1^{-1}]$ if and only if $G/p(G)$ is cyclic. Theorem 3.1 is the case $n = 2$. For the general case, let G be a finite group with $G/p(G)$ cyclic, and let S be a Galois étale extension of $R_{n,1}$ with group G, given by Theorem 3.1. Then $R_{n,1}$ is algebraically closed in $R_{n,r}$, and $R_{n,r}$ is separable over $R_{n,1}$; so $R_{n,r}$ is linearly disjoint from S over $R_{n,1}$ [FJ, Lemma 9.7]. Thus $R_{n,r} \otimes_{R_{n,1}} S$ is a domain, and is an unramified Galois extension of $R_{n,r}$ with group G. □

Thus we conclude that the Galois groups over $k[[x_1, \ldots, x_n]][x_1^{-1}]$ are precisely the cyclic-by-quasi-p groups:

Corollary 3.4. *If* $1 \to Q \to G \to C \to 1$ *is an exact sequence of finite groups, with* Q *a quasi-*p *group and* C *a cyclic group, then* G *is the Galois group of an unramified extension of* $\hat{R}_{n,1} = k[[x_1, \ldots, x_n]][x_1^{-1}]$ *for all* $n > 1$*. Conversely, every Galois group* G *over* $\hat{R}_{n,1}$ *is of this form.*

Proof. Since Q is a quasi-p group, it is contained in $p(G)$. Hence $G/p(G)$ is a quotient of C, and thus is cyclic. So G is a Galois group over $k[[x_1, \ldots, x_n]][x_1^{-1}]$ by the Main Theorem. This proves the forward direction. The converse is also immediate from the Main Theorem, by taking $Q = p(G)$ and $C = G/p(G)$. □

Section 4. Another viewpoint.

In this section, we sketch an alternative construction to prove Theorem 3.1 and hence the Main Theorem. The presentation here, which relies on a series of blowings-up, may be viewed as geometrically more intuitive than the earlier argument; but filling in all the details here (particularly concerning the use of the patching result of [HS]) would produce a longer argument than the one in the previous section.

The basic idea of the proof here is to start with a cyclic cover of the base space $\operatorname{Spec} k[[x, y]]$ branched only along $(x = 0)$; to blow up the base, obtaining an exceptional divisor E; and then to use Abhyankar's Conjecture for the affine line [Ra] together with formal patching [HS] on E and the existence of solutions to p-embedding problems [Ha2] in order to enlarge the Galois group of the cover by a quasi-p group. The difficulty is that doing this would cause the resulting cover of $\operatorname{Spec} k[[x, y]]$ (after blowing back down) to be ramified along both axes; so instead we blow up a series of times, to avoid obtaining extra ramification. (The corresponding trick in the first proof of Theorem 3.1, given in Section 3, was to consider the morphism $X \to S$ given by $x^n = s$.)

We begin with a definition and a lemma. As before, let k be an algebraically closed field of characteristic $p \geq 0$. In k, fix a compatible system $\{\zeta_n\}_{(p,n)=1}$ of roots of unity in k (i.e. such that $\zeta_{mn}^m = \zeta_n$).

Definition 4.1 (a) Let $V \to X$ be a G-Galois cover of normal k-schemes. Let α be a point of X of codimension 1 at which X is regular and over which the cover is tamely ramified, say of ramification index m. Let $x \in \hat{\mathcal{O}}_{X,\alpha}$ be a uniformizer at α. If β is a point of V over α, and $v \in \hat{\mathcal{O}}_{V,\beta}$ is a uniformizer at β such that $v^m = x$, then the *canonical generator of inertia of* $V \to X$ *at* β is the unique element $g \in G$ of the inertia group at β such that $g(v) = \zeta_m v$. An element $g \in G$ is a *canonical generator of inertia of* $V \to X$ *over* α if it is the canonical generator of inertia at some point β over α.

(b) Let X be a regular k-scheme, let V be a normal k-scheme, and let $V \to X$ be a G-Galois cover. Let $\xi \in X$ be a closed point in the branch locus B, at which B has at most normal crossings. Let ν be a point of V over ξ, and let $D_1, \ldots, D_r$ be the irreducible components of the branch locus of $\hat{V} = \operatorname{Spec} \hat{\mathcal{O}}_{V,\nu} \to \hat{X} = \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi}$. Let $g_i \in G$ be the canonical generator of inertia of $\hat{V} \to \hat{X}$ at the generic point of D_i. Then $g_1, \ldots, g_r$ are the *canonical generators of inertia of* $V \to X$ *at* ν. □

Note that in (a) of the definition, once one chooses x, a uniformizer v as above always exists. Moreover the canonical generator of inertia at β depends only on β, not on the choice of x or v. The canonical generators of inertia over α form a conjugacy class in G. So in (b), the g_i's are each determined up to conjugation in the inertia group at ν (which is the Galois group of $\hat{\mathcal{O}}_{V,\nu}$ over $\hat{\mathcal{O}}_{X,\xi}$). But that group is abelian, of order prime to p, and of rank at most r. (This standard fact follows from Abhyankar's Lemma; e.g. see [HP, Proposition 3.1].) So actually the g_i's in (b) are uniquely determined by ν.

Lemma 4.2 *Let $V \to X$ be a G-Galois cover of k-schemes, with X regular and V normal. Let ξ be a closed point of X at which the branch locus B has at most normal crossings, and suppose $V \to X$ is tamely ramified at the generic point of each branch component containing ξ. Let $\nu \in V$ be a point lying over ξ, with canonical generators of inertia $g_1, \ldots, g_r$. Let $\hat{X}$ be the blow-up of X at ξ, and let $\hat{V}$ be the normalization of $V \times_X \hat{X}$.*

(a) Then $\hat{V} \to \hat{X}$ is tamely ramified over the exceptional divisor E.

(b) Let ϵ be the generic point of $E \subset \hat{X}$, and let $\phi \in \hat{V}$ be a point lying over ϵ that maps to ν under $\hat{V} \to V$. Then the canonical generator of inertia at ϕ is $g_1 \cdots g_r$.

Note that the element $\prod g_i$ in (b) of the lemma is independent of the order in which the g_i's are taken. This is because the g_i's lie in an abelian group (viz. the inertia group at ν), by the comment preceding the statement of the lemma.

Proof of Lemma 4.2. Since the result is local, we are reduced to the special case that $X = \operatorname{Spec} k[[x_1, x_2, \ldots, x_n]]$, with ξ being the closed point $(x_1, x_2, \ldots, x_n)$, and where $V \to X$ is branched only at $D_1 \cup \ldots \cup D_r$ for some $r \le n$, where $D_i := (x_i = 0)$, and where g_i is the canonical generator of inertia over D_i. This cover is totally ramified over ξ since X is completely local and since k is algebraically closed; so its Galois group G is the inertia group over ξ.

First consider the case that $r = n$ and that $V \to X$ is the cover $S \to X$ given by $s_i^m = x_i$ $(1 \le i \le n)$, for some m prime to p. This cover is Galois with group $A = (C_m)^n = \langle a_1, a_2, \ldots, a_n \rangle$, where the action is given by $a_i(s_i) = \zeta_m s_i$ and $a_i(s_j) = s_j$ for $i \neq j$. Thus a_i is the canonical generator of inertia of $S \to X$ over D_i. Let $\hat{X}$ be the blow-up of X at ξ, with exceptional divisor $E \approx \mathbb{P}^{n-1}$. The function field of E is generated by $z_j = x_j/x_1$, for $j = 2, \ldots, n$. Let $\hat{S}$ be the normalized pullback of $S \to X$ via $\hat{X} \to X$; thus $\hat{S} \to \hat{X}$ is an A-Galois cover. Since A is of order prime to p, all ramification is tame. Let F be a component of the fibre of $\hat{S} \to \hat{X}$ over E. Thus the function field of F is generated by $w_j = s_j/s_1$, for $j = 2, \ldots, n$. Here $w_j^m = z_j$. Now $a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}(w_j) = \zeta_m^{i_j - i_1} w_j$. Thus the parameter w_j is fixed exactly when $i_j = i_1$, so the inertia group at the generic point ϕ of F is the cyclic group $\langle a_1 a_2 \cdots a_n \rangle$. The element s_1 is a local uniformizer at the generic point ϵ of E in $\hat{X}$; and for each d in the inertia group $\langle a_1 a_2 \cdots a_n \rangle$, we have $d = a_1^i a_2^i \cdots a_n^i$ and $d(s_1) = a_1^i a_2^i \cdots a_n^i(s_1) = \zeta_m^i s_1$. Thus the canonical generator of inertia is $d = a_1 a_2 \cdots a_n$. This proves the lemma for the cover $S \to X$.

Now consider the case of a more general cover $V \to X = \operatorname{Spec} k[[x_1, x_2, \ldots, x_n]]$, branched over $D_1, \ldots, D_r$, with $r \le n$, and with g_i being the canonical generator of inertia over D_i for $1 \le i \le n$. (Here $g_i = 1$ for $r < i \le n$.) By [HP, Proposition 3.1], G is prime-to-p and abelian of rank r, and $V \to X$ is dominated by the A-Galois cover $S \to X$ of the previous paragraph, for some m prime to p. For $1 \leq i \leq n$, the canonical generator of inertia of $V \to X$ over the generic point of D_i [resp. of E] is the image, under the quotient map $\pi : A \to G$, of the canonical generator of inertia of $S \to X$ there — i.e. the image of a_i [resp. of $a_1 a_2 \cdots a_n$] in G. So $\pi(a_i) = g_i$; and hence $g_1 g_2 \cdots g_r = g_1 g_2 \cdots g_n = \pi(a_1 a_2 \cdots a_n)$ is the canonical generator of inertia of $V \to X$ over ϵ as claimed. $\square$

We now use this lemma to provide the alternative construction to prove Theorem 3.1 and hence the Main Theorem.

Alternative proof sketch of Theorem 3.1. As in the previous proof of this theorem, we immediately reduce to the case that $\Gamma = Q \rtimes C$, where Q is a quasi-p group, C is a cyclic group of order n prime to p, and $C \subset \Gamma$ normalizes a Sylow p-subgroup P of Γ (or equivalently, of Q).

Let X be the projective x-line over k, and $\mathcal{X} = X \times_k k[[y]]$. Identify X with the closed fiber of $\mathcal{X}$. Consider the C-Galois cover $\mathcal{W} \to \mathcal{X}$ defined by $w^n = x$. This is ramified only at the point τ where $x = 0$ and at the point σ where $x = \infty$; and over both points the cover is totally ramified. Let c be the canonical generator of inertia over $x = 0$. So $C = \langle c \rangle$. Let $Y = \tau \times_k k[[y]] \subset \mathcal{X}$ and $S = \sigma \times_k k[[y]] \subset \mathcal{X}$. Let $\mathcal{X}_0 = \mathcal{X}$ and for each $i = 1, 2, ..., n$, let $\mathcal{X}_i$ be the blow-up of $\mathcal{X}_{i-1}$ at τ with exceptional divisor E_i. Here we inductively identify X, $E_1, ..., E_{i-1}, Y, S \subset \mathcal{X}_{i-1}$ with their proper transforms in $\mathcal{X}_i$, and identify τ with the point of $\mathcal{X}_i$ where Y meets E_i. Let $\mathcal{W}_i \to \mathcal{X}_i$ be the normalized pullback of W_{i-1} with respect to $\mathcal{X}_i \to \mathcal{X}_{i-1}$. Then this is a C-Galois cover ramified only over $E_1, E_2, ..., E_i, Y, S$. The canonical generator of inertia over Y is c; and by the lemma (and induction on i), the canonical generators of inertia over E_i is c^i. Notice that the canonical generator of inertia of E_n is $c^n = 1$, so $W_n \to X_n$ is unramified over the generic point of E_n.

Let δ be the point where E_n meets E_{n-1}, and let ξ be a point of $E_n - \{\tau, \delta\}$. Let $\mathcal{X}_{n+1}$ be the blow-up of $\mathcal{X}_n$ at ξ, with exceptional divisor E_{n+1}. Identify ξ with the point of $\mathcal{X}_{n+1}$ where E_{n+1} meets E_n and identify X, Y, S, E_i with their proper transforms in $\mathcal{X}_{n+1}$. By [Ra] there exists a Q-Galois cover $Z \to E_{n+1}$ branched only at the point ξ such that the inertia groups over ξ are conjugate to P.

Let $\mathcal{E}$ be the formal completion of $\mathcal{X}_{n+1}$ along the closed subset $E_1 \cup E_2 \cup ... \cup E_n$, and let $\mathcal{W}_{\mathcal{E}} \to \mathcal{E}$ be the pullback of $\mathcal{W}_{n+1} \to \mathcal{X}_{n+1}$ via $\mathcal{E} \to \mathcal{X}_{n+1}$. By applying [Ha2, Theorem 5.6] to the open fibre of the C-Galois cover $\mathcal{W}_{\mathcal{E}} \to \mathcal{E}$, we obtain a $G = P \rtimes C$-Galois cover $\mathcal{U} \to \mathcal{E}$ that dominates $\mathcal{W}_{\mathcal{E}}$ and is ramified only at $E_1, E_2, ..., E_n, Y$, and such that there is agreement locally between the generic fibre of $\mathcal{U} \to \mathcal{E}$ and the Q-Galois cover $Z \to E_{n+1}$ near the point ξ (i.e. over the generic point of Spec $\hat{\mathcal{O}}_{E_{n+1},\xi}$, which is a closed point of the generic fibre of $\mathcal{E}$). Moreover, according to [Ha2, Theorem 5.6], we can require that the G-Galois cover $\mathcal{U} \to \mathcal{E}$ also agrees locally with the original C-Galois cover $W \to X$ near the point χ at which E_1 meets X. Using these local agreements, we may apply formal patching [HS] to obtain a Γ-Galois cover $\mathcal{V}_{n+1} \to \mathcal{X}_{n+1}$ whose restrictions agree with the covers $Z \to E_{n+1}$, $\mathcal{U} \to \mathcal{E}$, and $W \to X$, where the first two meet over ξ and the latter two meet over χ. In particular, this cover is unramified over the generic point of X. Let $\mathcal{V}$ be the normalization of $\mathcal{X}$ in $\mathcal{V}_{n+1}$ (relative to the blow-up morphism $\mathcal{X}_{n+1} \to \mathcal{X}$). The natural morphism $\mathcal{V}_{n+1} \to \mathcal{V}$ is a birational isomorphism, and $\mathcal{V} \to \mathcal{X}$ is a Γ-Galois cover ramified only along Y and totally ramified at τ. Let $\mathcal{X}_\tau = \operatorname{Spec} k[[x, y]]$ be the spectrum of the complete local ring of $\mathcal{X}$ at τ, and let $\mathcal{V}_\tau$ be the pullback of $\mathcal{V}$ to $\mathcal{X}_\tau$. Thus $\mathcal{V}_\tau \to \mathcal{X}_\tau = \operatorname{Spec} k[[x, y]]$ is a Γ-Galois cover that is branched totally at the closed point τ, and is unramified away from $(x = 0)$. This completes the construction, yielding Theorem 3.1 and hence the Main Theorem in general. □

References.

[Ab1] S.S. Abhyankar. Coverings of algebraic curves. Amer. J. Math. **79** (1957), 825-856.

[Ab2] S.S. Abhyankar. Tame coverings and fundamental groups of algebraic varieties, Part 1. Amer. J. Math. **84** (1959), 46-94.

[Ab3] S.S. Abhyankar. Local fundamental groups of algebraic varieties. Proc. Amer. Math. Soc. **125** (1997), 1635-1641.

[F] W. Fulton. On the fundamental group of the complement of a node curve. Annals of Math. **111** (1980), 407-409.

[Go] D. Gorenstein. “Finite Groups”. Chelsea Publishing Co., New York, 1980.

[Gr1] A. Grothendieck. EGA III, 1^{e} partie, Publ. Math. IHES, no. 11, 1961.

[Gr2] A. Grothendieck. “Revêtements Etales et Groupe Fondamental” (SGA1). Lect. Notes in Math. **224**, Springer Verlag, 1971.

[Ha1] D. Harbater. Abhyankar’s conjecture on Galois groups over curves. Inventiones Math., **117** (1994), 1-25.

[Ha2] D. Harbater. Embedding problems with local conditions. Israel Journal of Mathematics, **118** (2000), 317-355.

[Ha3] D. Harbater. Abhyankar’s Conjecture and embedding problems. 2000 manuscript, to appear in Crelle’s Journal. Available at <http://www.math.upenn.edu/~harbater/qp.dvi>.

[HP] D. Harbater, M. van der Put; appendix by R. Guralnick. Valued fields and covers in characteristic p. In “Valuation Theory and its Applications”, Fields Institute Communications, vol. 32, ed. by F.-V. Kuhlmann, S. Kuhlmann and M. Marshall, 2002, pp.175-204.

[HS] D. Harbater, K. Stevenson. Patching and thickening problems. J. Alg. **212** (1999), 272-304.

[Ra] M. Raynaud. Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d’Abhyankar. Invent. Math. **116** (1994), 425-462.

[Z1] O. Zariski. On the problem of existence of algebraic functions of two variables possessing a given branch curve. Amer. J. of Math., **51** (1929), 305-328.

[Z2] O. Zariski. "Algebraic surfaces". New York, Chelsea Pub. Co., 1948.

Author information:

David Harbater: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395.
E-mail address: `harbater@math.upenn.edu`

Katherine F. Stevenson: Department of Mathematics, California Institute of Technology, Pasadena, CA 91125. *E-mail address*: `kfs@vision.caltech.edu`
Current address: Department of Mathematics, California State University at Northridge, Northridge, CA 91330.