Abhyankar's Conjecture on Galois Groups over Curves

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Section 1. Introduction.

This paper contains a proof of Abhyankar's Conjecture [Ab1] concerning the fundamental group of affine curves U in finite characteristic. The conjecture gives a necessary and sufficient condition for a finite group G to be a quotient of $\pi_1(U)$, or equivalently for G to be the Galois group of a Galois unramified cover of U. In fact, we show more, viz. that all but one of the branch points can be taken to be tame.

Roughly speaking, Abhyankar's Conjecture asserts that a group G occurs over a given affine curve in characteristic p if and only if every prime-to-p quotient of G occurs over the "analogous complex curve." More precisely, if g and r are non-negative integers, let $\Gamma_{g,r}$ be the group generated by elements $a_1, \ldots, a_g, b_1, \ldots, b_g, c_0, \ldots, c_r$ subject to the single relation $\prod_{j=1}^{g}[a_j, b_j] \prod_{i=0}^{r} c_i = 1$, where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. Thus if X is a compact Riemann surface of genus g, and U is the complement in X of a set of $r+1$ points $\xi_0, \ldots, \xi_r$, then $\Gamma_{g,r}$ may be identified with the fundamental group of U. Under this identification, a_j and b_j correspond to loops around the jth "hole," and c_i corresponds to a loop around ξ_i. Thus a finite group G is the Galois group of a connected finite Galois covering space $V \to U$ if and only if G is a quotient of $\Gamma_{g,r}$; and in this case the image of c_i in G generates the inertia group of a ramification point over ξ_i. Note that $\Gamma_{g,r}$ is isomorphic to the free group on $2g + r$ generators, $a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_r$; here $c_0 = (\prod_{i=1}^{r} c_i \ \prod_{j=1}^{g}[a_j, b_j])^{-1}$.

Let k be an algebraically closed field of characteristic $p \neq 0$.

Conjecture 1.1. (Abhyankar's Conjecture) *Let X be a smooth connected projective curve of genus g over k, let $\xi_0, \ldots, \xi_r$ ($r \geq 0$) be distinct points of X, and let $U = X - \{\xi_0, \ldots, \xi_r\}$. Then for any finite group G, G is the Galois group of a connected finite étale Galois cover of U if and only if every prime-to-p quotient of G is a quotient of $\Gamma_{g,r}$.*

The conjecture can also be rephrased a bit differently. If G is a finite group, we define the *quasi-p-part* of G to be the subgroup $p(G)$ generated by the p-subgroups of G. This can also be described as the subgroup generated by the Sylow p-subgroups of G, or alternatively as the subgroup generated by the elements of p-power order. A group G is said to be a *quasi-p-group* if $G = p(G)$. For any finite group G, $p(G)$ is a characteristic subgroup of G (and in particular is normal), and $G/p(G)$ has order prime to p. Moreover a quotient G/N of G has order prime to p if and only if $p(G) \subset N$. Abhyankar's Conjecture is thus equivalent to the assertion that the groups G that occur as Galois groups over U are precisely those such that $G/p(G)$ occurs over "the analogous complex curve," i.e. such that $G/p(G)$ is a quotient of $\Gamma_{g,r}$ (where g and r are as in 1.1).

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In the case of groups G of order prime to p, the conjecture simply states that G occurs over U if and only if G is a quotient of $\Gamma_{g,r}$; this has been shown by Grothendieck [Gr, XIII, Cor.2.12]. That result yields one implication of Conjecture 1.1 in the general case: if a finite group G occurs over U then $G/p(G)$ (which thus also occurs over U, and which is prime-to-p) must be a quotient of $\Gamma_{g,r}$. The other direction of Conjecture 1.1 has recently been proven in the case of $g = r = 0$, i.e. for the affine line, by Raynaud [Ra], following partial results by Nori (cf. [Ka] or [Se3]), Abhyankar (e.g. [Ab2]), and Serre [Se2].

Grothendieck has shown [Gr, XIII, Cor.2.12] that if a group G occurs as the Galois group of a connected Galois étale cover of U whose smooth completion over X is tamely ramified, then G itself must be a quotient of $\Gamma_{g,r}$. For example, since $\Gamma_{0,0}$ is trivial, there are no tamely ramified covers of $\mathbf{P}^1$ branched only over one point. Thus Conjecture 1.1 cannot be strengthened so as to assert that the cover may be chosen to have tamely ramified completion over X. But one may still conjecture that only one wildly ramified branch point is needed:

Conjecture 1.2. *Let X be a smooth connected projective curve of genus $g \geq 0$ over k, and let $\xi_0, \ldots, \xi_r$ ($r \geq 0$) be distinct points of X. Let G be a finite group such that $G/p(G)$ is a quotient of $\Gamma_{g,r}$. Then there is a smooth connected G-Galois branched cover of X branched only over the points ξ_i, and tamely ramified except possibly over ξ_0.*

Partial results in the case of a general affine curve have been obtained by the present author [Ha3], using formal patching. In the present paper, formal patching is combined with Raynaud's result to prove Conjectures 1.1 and 1.2; cf. Theorem 6.2.

The paper is organized as follows: Section 2 contains the formal patching and deformation results which will be needed; these depend on [Ha3]. Section 3 proves some results about the moduli space of p-covers of the affine line, and these are used in section 4 to obtain a family of cyclic-by-p covers. Section 5 proves the two conjectures in the special case of $\mathbf{P}^1 - \{0, \infty\}$. It does so by patching a cyclic-by-p cover of $\mathbf{P}^1_{k((x^{-1}))} - \{0, \infty\}$, obtained from section 4, together with a quasi-p-cover of $\mathbf{A}^1$, obtained from [Ra]. Then in section 6, the general case is handled by patching a general cover of $\mathbf{P}^1 - \{0, \infty\}$, obtained from section 5, together with a tame cover of X, obtained from [Gr].

Conventions:

Throughout this paper, we will work over a base field k. Beginning in section 3, k will be assumed algebraically closed and of characteristic $p \neq 0$. Unless otherwise adorned, $\mathbf{P}^1$ and $\mathbf{A}^1$ will denote the affine and projective lines over the field k.

If X is a scheme and ξ is a point of X such that the complete local ring $\hat{\mathcal{O}}_{X,\xi}$ is a domain, then we let $\hat{\mathcal{K}}_{X,\xi}$ be the fraction field of $\hat{\mathcal{O}}_{X,\xi}$.

For any ring R let $\mathrm{frac}(R)$ be the total ring of fractions of R. If R is a domain and $R \subset S$ is an extension of rings, then we say that S is *generically separable* as an R-algebra if $\mathrm{frac}(S)$ is a separable $\mathrm{frac}(R)$-algebra and if no non-zero element of R becomes a zero-divisor of S. For any finite group G, a *G-Galois* R-algebra consists of a finite generically separable R-algebra S together with a group homomorphism $\rho : G \to \mathrm{Aut}_R(S)$ with respect to which G acts simply transitively on a generic geometric fibre of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.

We say that a morphism of schemes $\phi : Y \to X$ is *generically separable* [resp. *G-Galois*, with respect to a homomorphism $\rho : G \to \mathrm{Aut}_X(Y)$] if X can be covered by affine open subsets $U = \mathrm{Spec}(R)$ such that the ring extensions $R \subset \mathcal{O}(\phi^{-1}(U))$ have that property. A morphism $\phi : Y \to X$ which is finite and generically separable [resp. finite and G-Galois] will be called a *cover* [resp. a *G-Galois cover*]. Its *branch locus* is the complement in X of the locus where ϕ is étale. As usual, if $Y \to X$ is a G-Galois cover, and η is a point of Y, then the *decomposition group* at η is the subgroup of G consisting of elements taking η to itself, and the *inertia group* is the subgroup of the decomposition group which induces the identity on the residue field. Ramification at η is *tame* if the order of the inertia group is prime to the residue characteristic; otherwise it is *wild*. A cover is *tamely ramified* if all of its ramification is.

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Section 2. Formal patching and deformation.

In [Ha3] some formal patching results were proven, and these were then used in the proofs of several results about Galois groups over affine curves in finite characteristic. This section contains variants and applications of those patching results, for use in sections 5 and 6 of the present paper. In this section, k will denote an arbitrary base field.

As in [Ha3], for any scheme X, let $\mathcal{P}(X)$ [resp. $\mathcal{AP}(X)$] denote the category of coherent sheaves of projective $\mathcal{O}_X$-modules [resp. projective $\mathcal{O}_X$-algebras]. Also, let $\mathcal{SP}(X)$ denote the subcategory of $\mathcal{AP}(X)$ consisting of sheaves which are generically separable; and for any finite group G let $G\mathcal{P}(X)$ be the subcategory of $\mathcal{SP}(X)$ consisting of G-Galois sheaves. For any ring R, let $\mathcal{P}(R) = \mathcal{P}(\mathrm{Spec}(R))$, and similarly for $\mathcal{AP}$, $\mathcal{SP}$, and $G\mathcal{P}$. Given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and functors $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$, let $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ denote the category of triples (A, B, C) of objects in $\mathcal{A}, \mathcal{B}, \mathcal{C}$ together with isomorphisms between C and the images of A and of B in $\mathcal{C}$.

Proposition 2.1. *Let L be a regular connected projective k-curve, let λ be a closed point of L, and let $\mathrm{Spec}(S) = L - \{\lambda\}$. Let $L^* = L \times_k \mathrm{Spec}(k[[v]])$, let $\phi : T^* \to L^*$ be a cover, and assume that ϕ is flat (e.g. if T^* is normal). Let $\mathcal{B}$ be the category*

$$\mathcal{P}(\phi^*(S[[v]])) \times_{\mathcal{P}(\phi^*(\hat{\mathcal{K}}_{L,\lambda}[[v]]))} \mathcal{P}(\phi^*(\hat{\mathcal{O}}_{L,\lambda}[[v]])).$$

Then the base change functor $\mathcal{P}(T^*) \to \mathcal{B}$ is an equivalence of categories. Moreover this remains true if $\mathcal{P}$ is replaced by $\mathcal{AP}$, $\mathcal{SP}$, or $G\mathcal{P}$ for any finite group G.

Proof. Since L is a regular curve, it follows that L^* is a regular surface. So if T^* is normal, then it is flat over L^* (using [AB], as in the proof of [Ha3,Proposition 4(b)]). Also, a flat cover is finite and hence corresponds to a coherent sheaf of projective algebras. So under the hypotheses of the proposition, every finite projective module over T^* is projective over L^*. Thus giving an object in the category $\mathcal{P}(T^*)$ is equivalent to giving an object $\mathcal{V}$ in the category $\mathcal{P}(L^*)$, together with a morphism $\mathcal{O}_{T^*} \to \mathcal{E}nd(\mathcal{V})$ in $\mathcal{P}(L^*)$. Similarly, since flatness is preserved under pullback, giving an object in the category $\mathcal{B}$ is equivalent to giving an object $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_0)$ in the category

$$\mathcal{D} = \mathcal{P}(S[[v]]) \times_{\mathcal{P}(\hat{\mathcal{K}}_{L,\lambda}[[v]])} \mathcal{P}(\hat{\mathcal{O}}_{L,\lambda}[[v]]),$$

together with a morphism $(\phi^*(S[[v]]), \phi^*(\hat{\mathcal{O}}_{L,\lambda}[[v]]), \phi^*(\hat{\mathcal{K}}_{L,\lambda}[[v]])) \to \mathcal{E}nd(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_0)$ in $\mathcal{D}$. So the result for $\mathcal{P}$ follows from the fact that the base change functor $\mathcal{P}(L^*) \to \mathcal{D}$ is an equivalence of categories [Ha3,Theorem 1]. The other analogs, for $\mathcal{AP}$, $\mathcal{SP}$, and $G\mathcal{P}$, follow formally from the result for $\mathcal{P}$ as in the proof of [Ha3,Theorem 1].   []

Remark. Proposition 2.1 says that projective modules or algebras that are given over "pieces" of T^* can be patched to give a global module or algebra over all of T^*. In the assertion, the auxiliary space L^* appears only in order to permit the use of [Ha3, Theorem 1]. It seems likely, though, that Proposition 2.1 and [Ha3, Theorem 1] hold in a more general context. Namely, suppose that R is a complete discrete valuation ring, T^* is a proper flat R-scheme of relative dimension 1, and F is a finite set of closed points of T^*. Let T'^* be the completion of $T^* - F$ along its closed fibre, and let $\hat{T}^* = \bigcup_{\tau \in F} \mathrm{Spec}(\hat{\mathcal{O}}_{T^*,\tau})$. Then compatible projective modules or algebras over T'^* and $\hat{T}^*$ can presumably be patched to yield such a global object over all of T^*. But such a general patching assertion is not essential here, and we do not prove it in this paper.

Corollary 2.2. *Under the hypotheses of Proposition 2.1, assume that the closed fibre of ϕ has two irreducible components X_1, X_2, each of which is a regular curve; that these meet at a unique point $\tau \in T^*$; and that $\phi(\tau) = \lambda$. For $i = 1,2$ let R_i be the ring of functions on the affine curve $X'_i = X_i - \{\tau\}$; let $X'^*_i = \mathrm{Spec}(R_i[[v]])$, and let $\hat{X}'^*_i = \mathrm{Spec}(\hat{\mathcal{K}}_{X_i,\tau}[[v]])$. Also, let $\hat{T}^* = \mathrm{Spec}(\hat{\mathcal{O}}_{T^*,\tau})$. Let $\mathcal{C}$ be the category*

$$\mathcal{P}(X'^*_1 \cup X'^*_2) \times_{\mathcal{P}(\hat{X}'^*_1 \cup \hat{X}'^*_2)} \mathcal{P}(\hat{T}^*).$$

Then the base change functor $\mathcal{P}(T^*) \to \mathcal{C}$ is an equivalence of categories. Moreover this remains true if $\mathcal{P}$ is replaced by $\mathcal{AP}$, $\mathcal{SP}$, or $G\mathcal{P}$ for any finite group G.

Proof. Viewing $L \subset L^*$, let $D = \phi^{-1}(\lambda)$, $D_i = D \cap X_i$, $D'_i = D \cap X'_i$, and $L' = L - \{\lambda\} = \mathrm{Spec}(S)$. For $i = 1,2$, let $T'_i = X_i - D_i$ and let S_i be the ring of functions on the affine curve T'_i. Let $Y^* = \mathrm{Spec}(\phi^*(\hat{\mathcal{O}}_{L,\lambda}[[v]])) = \bigcup_{\delta \in D} \mathrm{Spec}(\hat{\mathcal{O}}_{\hat{T}^*,\delta})$ and $Y'^* = \mathrm{Spec}(\phi^*(\hat{\mathcal{K}}_{L,\lambda}[[v]])) = \bigcup_{i=1,2;\delta \in D_i} \mathrm{Spec}(\hat{\mathcal{K}}_{X_i,\delta}[[v]])$. For $i = 1,2$, let $Y^*_i = \bigcup_{\delta \in D'_i} \mathrm{Spec}(\hat{\mathcal{O}}_{\hat{T}^*,\delta})$, $Y'^*_i = \bigcup_{\delta \in D'_i} \mathrm{Spec}(\hat{\mathcal{K}}_{X_i,\delta}[[v]])$, and $T'^*_i = \mathrm{Spec}(S_i[[v]])$. Let $\mathcal{B}$ be as in Proposition 2.1. Thus $\mathcal{B} = \mathcal{P}(T'^*_1 \cup T'^*_2) \times_{\mathcal{P}(Y'^*)} \mathcal{P}(Y^*)$, and the base change functor $\mathcal{P}(T^*) \to \mathcal{B}$ is an equivalence of categories.

By [Ha3, Prop.3] (which is the affine analog of [Ha3, Theorem 1]) and induction on $\#(D'_i)$, base change induces an equivalence of categories

$$\mathcal{P}(X'^*_i) \tilde{\to} \mathcal{P}(T'^*_i) \times_{\mathcal{P}(Y'^*_i)} \mathcal{P}(Y^*_i)$$

for $i = 1, 2$. So base change induces an equivalence of categories $\mathcal{P}(X_1'^* \cup X_2'^*) \tilde{\to} \mathcal{E}$, where $\mathcal{E} = \mathcal{P}(T_1'^* \cup T_2'^*) \times_{\mathcal{P}(Y_1'^* \cup Y_2'^*)} \mathcal{P}(Y_1^* \cup Y_2^*)$, and hence also induces an equivalence $\mathcal{C} \tilde{\to} \mathcal{E} \times_{\mathcal{P}(\hat{X}_1'^* \cup \hat{X}_2'^*)} \mathcal{P}(\hat{T}^*)$. The latter category is canonically equivalent to $\mathcal{B}$, because of the disjoint unions $Y^* = Y_1^* \cup Y_2^* \cup \hat{T}^*$ and $Y'^* = Y_1'^* \cup Y_2'^* \cup \hat{X}_1'^* \cup \hat{X}_2'^*$. Thus the base change functors $\mathcal{P}(T^*) \to \mathcal{B}$ and $\mathcal{C} \to \mathcal{B}$ are equivalences of categories, and hence so is the base change functor $\mathcal{P}(T^*) \to \mathcal{C}$. This proves the result for $\mathcal{P}$. Replacing $\mathcal{P}$ throughout by $\mathcal{AP}$, $\mathcal{SP}$, or $G\mathcal{P}$ yields the proofs in those cases. $\square$

If H is a subgroup of a finite group G, and $Y \to X$ is an H-Galois cover, then there is an induced G-Galois cover $\mathrm{Ind}_H^G Y \to X$. It is a union of $(G : H)$ disjoint copies of Y, indexed by the left cosets of H in G. The stabilizer of the identity copy is $H \subset G$, and the stabilizers of the other copies are the conjugates of H in G. Similarly, if $\mathcal{F}$ is a coherent sheaf of H-Galois $\mathcal{O}_X$-algebras, then there is an induced coherent sheaf $\mathrm{Ind}_H^G \mathcal{F}$ of G-Galois $\mathcal{O}_X$-algebras, which as a sheaf of modules is isomorphic to $\mathcal{F}^{\oplus(G:H)}$.

Proposition 2.3. *Under the hypotheses of Corollary 2.2, let G be a finite group; let G_1, G_2, I be subgroups which generate G; and for $i = 1, 2$ let I_i be a subgroup of $I \cap G_i$. For $i = 1, 2$ let $W_i'^* \to X_i'^*$ be an irreducible normal G_i-Galois cover; and let $\hat{W}_i'^*$ be an irreducible component of $W_i'^* \times_{X_i'^*} \hat{X}_i'^*$ such that $\mathrm{Gal}(\hat{W}_i'^*/\hat{X}_i'^*) = I_i \subset G_i$. Also, let $\hat{N}^* \to \hat{T}^*$ be an irreducible normal I-Galois cover. Suppose that for $i = 1, 2$ there is an isomorphism $\hat{N}^* \times_{\hat{T}^*} \hat{X}_i'^* \tilde{\to} \mathrm{Ind}_{I_i}^I \hat{W}_i'^*$ of I-Galois covers of $\hat{X}_i'^*$. Then there is an irreducible normal G-Galois cover $V^* \to T^*$ such that $V^* \times_{T^*} X_i'^* \approx \mathrm{Ind}_{G_i}^G W_i'^*$ as G-Galois covers of $X_i'^*$ for $i = 1, 2$, and $V^* \times_{T^*} \hat{T}^* \approx \mathrm{Ind}_I^G \hat{N}^*$ as G-Galois covers of $\hat{T}^*$.*

Proof. We preserve the notation of the statements of 2.1 and 2.2. As in the proof of Proposition 2.1, $W_i'^* \to X_i'^*$ and $\hat{N}^* \to \hat{T}^*$ are flat and hence define projective modules, since the total spaces are normal surfaces. So $\mathcal{V}_2 = \mathrm{Ind}_I^G \mathcal{O}_{\hat{N}^*}$ is an object in $G\mathcal{P}(\hat{T}^*)$ and $\mathrm{Ind}_{G_i}^G \mathcal{O}_{W_i'^*}$ is an object in $G\mathcal{P}(X_i'^*)$, and so $\mathcal{V}_1 = \mathrm{Ind}_{G_1}^G \mathcal{O}_{W_1'^*} \times \mathrm{Ind}_{G_2}^G \mathcal{O}_{W_2'^*}$ is an object in $G\mathcal{P}(X_1'^* \cup X_2'^*)$. Similarly $\mathcal{V}_0 = \mathrm{Ind}_{I_1}^G \mathcal{O}_{\hat{W}_1'^*} \times \mathrm{Ind}_{I_2}^G \mathcal{O}_{\hat{W}_2'^*}$ is an object in $G\mathcal{P}(\hat{X}_1'^* \cup \hat{X}_2'^*)$. For $i = 0, 1, 2$, let $V_i = \mathbf{Spec}(\mathcal{V}_i)$.

By definition of induced modules, we have an isomorphism

$$\mathcal{O}_{W_i'^*} \otimes_{R_i[[v]]} \mathcal{O}_{\hat{X}_i'^*} \tilde{\to} \mathrm{Ind}_{I_i}^{G_i} \mathcal{O}_{\hat{W}_i'^*}$$

of modules over $\hat{X}_i'^*$. Using $\mathcal{O}_{\hat{X}_i'^*} \approx \hat{\mathcal{K}}_{X_i,\tau}[[v]]$, this induces an isomorphism

$$\mathcal{V}_1 \otimes_{R_1[[v]] \times R_2[[v]]} (\hat{\mathcal{K}}_{X_1,\tau}[[v]] \times \hat{\mathcal{K}}_{X_2,\tau}[[v]]) = \prod_{i=1}^{2} (\mathrm{Ind}_{G_i}^G \mathcal{O}_{W_i'^*} \otimes_{R_i[[v]]} \mathcal{O}_{\hat{X}_i'^*}) \tilde{\to} \mathcal{V}_0$$

in $G\mathcal{P}(\hat{X}_1'^* \cup \hat{X}_2'^*)$. Here the left hand side is the object in $G\mathcal{P}(\hat{X}_1'^* \cup \hat{X}_2'^*)$ induced by $\mathcal{V}_1$. Meanwhile, the given isomorphisms $\hat{N}^* \times_{\hat{T}^*} \hat{X}_i'^* \tilde{\to} \mathrm{Ind}_{I_i}^I \hat{W}_i'^*$ induce an isomorphism

$$\mathcal{V}_2 \otimes_{\hat{\mathcal{O}}_{T^*,\tau}} (\hat{\mathcal{K}}_{X_1,\tau}[[v]] \times \hat{\mathcal{K}}_{X_2,\tau}[[v]]) = \prod_{i=1}^{2} (\mathrm{Ind}_I^G \mathcal{O}_{\hat{N}^*} \otimes_{\mathcal{O}_{\hat{T}^*}} \mathcal{O}_{\hat{X}_i'^*}) \tilde{\to} \mathcal{V}_0$$

in $G\mathcal{P}(\hat{X}_1'^* \cup \hat{X}_2'^*)$. Again, the left hand side is the object in $G\mathcal{P}(\hat{X}_1'^* \cup \hat{X}_2'^*)$ induced by $\mathcal{V}_2$.

Let $G\mathcal{C}$ be the category which is the analog of the category $\mathcal{C}$ in Corollary 2.2, but with $\mathcal{P}$ replaced by $G\mathcal{P}$. Then the triple $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_0)$, together with the above isomorphisms, defines an object in $G\mathcal{C}$. So by Corollary 2.2 this object is induced (up to isomorphism) by an object $\mathcal{V}$ in $G\mathcal{P}(T^*)$. Thus $V^* = \mathbf{Spec}_{T^*}(\mathcal{V})$ is a G-Galois cover of T^* inducing V_j on the jth patch; i.e. inducing $\mathrm{Ind}_{G_i}^G W_i'^* \to X_i'^*$ and $\mathrm{Ind}_I^G \hat{N}^* \to \hat{T}^*$, as G-Galois covers.

For $i = 1, 2$, the identity component of $\mathrm{Ind}_{I_i}^G \hat{W}_i'^*$ maps to the identity component of $\mathrm{Ind}_{G_i}^G W_i'^*$ under $V_0 \to V_1$, and it maps to the identity component of $\mathrm{Ind}_I^G \hat{N}^*$ under $V_0 \to V_2$. Since the stabilizers of these latter components are respectively G_i and I, and since the subgroups G_1, G_2, I generate G, it follows that V^* is irreducible.

To verify that V^* is normal, it suffices to show that for every closed point σ in the closed fibre of T^*, $\hat{V}_\sigma^* = V^* \times_{T^*} \mathrm{Spec}(\hat{\mathcal{O}}_{T^*,\sigma})$ is normal. If $\sigma = \tau$ then $\hat{V}_\sigma^* = V^* \times_{T^*} \hat{T}^* \stackrel{\sim}{=} \mathrm{Ind}_I^G \hat{N}^*$, which is normal. Otherwise, we may identify σ with some other point on the closed fibre of T^*, i.e. a point on the closed fibre X_i' of $X_i'^*$, for $i = 1$ or 2. Choosing $\tilde{\sigma} \in W_i^*$ lying over $\sigma \in X_i'^*$, we have that $\hat{V}_\sigma^* = \mathrm{Ind}_{G_i}^G W_i^* \times_{X_i'^*} \mathrm{Spec}(\hat{\mathcal{O}}_{X_i'^*,\sigma})$ is a union of copies of $\mathrm{Spec}(\hat{\mathcal{O}}_{W_i^*,\tilde{\sigma}})$; this is normal since W_i^* is. []

Lemma 2.4. *Let S be a regular scheme, let V be irreducible and normal, and let $\pi : V \to S$ be a proper surjective morphism such that the fibre over every closed point is generically smooth.*

(a) Then there is a finite étale cover $\phi : S' \to S$ and a proper morphism $\pi' : V \to S'$ whose closed fibres are non-empty and connected, such that $\pi = \phi \circ \pi'$.

(b) If S is the spectrum of a complete local ring having algebraically closed residue field, then the closed fibre of π is connected.

Proof. (a) By Stein factorization, if $S' = \mathbf{Spec}\, \pi_*(\mathcal{O}_V)$, then π factors as $V \to S' \to S$, where the first morphism is proper with non-empty connected fibres and the second morphism is finite and surjective. Here S' is irreducible and normal since V is. By Purity of Branch Locus, in order to show that $S' \to S$ is étale it suffices to show this property holds in codimension one. So localizing at an irreducible hypersurface in S reduces us to the case that S is of dimension one and has only one closed point σ. Now S' is irreducible, $S' \to S$ is finite and surjective, and S is regular and of dimension one; so $S' \to S$ is torsion-free and hence flat. Let $\mathbf{m}$ be the maximal ideal in the local ring $\mathcal{O}_{S,\sigma}$, and let $k_\sigma = \mathcal{O}_{S,\sigma}/\mathbf{m}$ be the residue field at σ. It remains to show that if $\sigma' \in S'$ lies over σ, then $k' = \mathcal{O}_{S',\sigma'}/\mathbf{m}\mathcal{O}_{S',\sigma'}$ is étale over k_σ.

Now k' is local; it is a finite extension of k_σ; and it is a subalgebra of the algebra of global functions on the fibre $V_{\sigma'}$ of $V \to S'$ over σ'. Since the fibre over σ is generically smooth, it is also generically geometrically smooth, and hence geometrically reduced. Thus so is k', which is thus étale over k_σ, as desired.

(b) Let S' be as in (a). Then S' is connected, and the residue field of S is algebraically closed; so S' defines a connected étale neighborhood of the closed point of S. By Hensel's Lemma, $S' \to S$ is an isomorphism. So $\pi = \pi'$, and the closed fibre of π is connected. []

Lemma 2.5. *Let k be algebraically closed, let E and Y be smooth irreducible k-schemes, and let $\eta_0, \ldots, \eta_r \in Y$, and write $Y_E = Y \times_k E$. Let V_E be irreducible and normal, let $V_E \to E$ be surjective and proper, let G be a finite group, and let $V_E \to Y_E$ be a G-Galois cover. Assume that $V_E \to Y_E$ is étale away from $\eta_0^E, \ldots, \eta_r^E$, where $\eta_i^E = \eta_i \times_k E$. For each i let C_i be an inertia group over the generic point of η_i^E, and assume that the conjugates of the subgroups C_i together generate G. Then for all closed points e in a non-empty open subset of E, the fibre V_e is irreducible.*

Proof. By the ramification hypothesis, for each closed point $e \in E$ the fibre $V_e \to Y_e$ of $V_E \to Y_E$ is generically étale. Since $Y_e \approx Y$ is smooth, it follows that V_e is generically smooth. So by Lemma 2.4(a), we may factor $V_E \to E$ as $V_E \to D \to E$, where the first morphism is proper with non-empty connected fibres and the second morphism is finite étale.

Since the compositions $V_E \to Y_E \to E$ and $V_E \to D \to E$ agree, they together induce a morphism $V_E \to Y_D = Y_E \times_E D$, through which $V_E \to Y_E$ factors. Since k is algebraically closed, and since Y and D are connected, the product $Y_D = Y \times_k D$ is also connected. And since $D \to E$ is finite étale, so is $Y_D \to Y_E$. Thus $Y_D \to Y_E$ is an étale subcover of $V_E \to Y_E$, and we may write $Y_D = V_E/N$ for some normal subgroup N of G. Since the inertia groups of $V_E \to Y_E$ over η_i^E are the conjugates of C_i (for $i = 0, \ldots, r$), and since $Y_D \to Y_E$ is étale, it follows that N contains all the conjugates of the subgroups C_i. But the conjugates of the subgroups C_i generate G; so $N = G$. Hence $Y_D = V_E/G = Y_E$ and so $D = E$. Thus by definition of D, the fibres of the projective morphism $V_E \to E$ are non-empty and connected.

Now pick any k-point ε on E. The fibre V_ε of $V_E \to E$ over ε is (geometrically) connected and generically smooth. Since V_E is a normal k-scheme, it is geometrically unibranched along V_ε. Since the morphism $V_E \to E$ is projective, Proposition 5 of [Ha3] applies. That result asserts that the desired conclusion holds. □

Proposition 2.6. *Assume k is algebraically closed. Let X be a smooth connected projective k-curve; $\xi_0, \ldots, \xi_r \in X$; $K = k((t))$; $X^o = X \times_k K$; and $\xi_i^o = \xi_i \times_k K \in X^o$ for each i. Let G be a finite group, let $\pi^o : V^o \to X^o$ a regular G-Galois cover of K-curves with branch locus $\{\xi_0^o, \ldots, \xi_r^o\}$, and for each i let C_i be an inertia group over ξ_i^o. Let $I \subset \{0, \ldots, r\}$, and for each $i \in I$ assume V^o is K-smooth at $(\pi^o)^{-1}(\xi_i^o)$. Suppose either*

(i) V^o is irreducible, and G is generated by the conjugates of $C_0, \ldots, C_r$; or

(ii) There is a projective $k[[t]]$-curve T^*, an irreducible normal scheme V^* with generically smooth closed fibre, and a G-Galois cover $V^* \to T^*$ with generic fibre $V^o \to X^o$.

Then there is a k-subalgebra $A \subset K$ of finite type, and a regular G-Galois cover $\pi_E : V_E \to X_E = X \times_k E$ (where $E = \mathrm{Spec}(A)$), such that

(a) $V_E \to X_E$ is branched only over $\xi_0^E, \ldots, \xi_r^E$, where $\xi_i^E = \xi_i \times_k E$;

(b) For each $i \in I$, C_i is an inertia group over ξ_i^E, and $V_E \to E$ is smooth at $\pi_E^{-1}(\xi_i^E)$;

(c) The fibre of $V_E \to X_E$ over each closed point of E is irreducible and non-empty;

(d) $V_E \times_E K$ is isomorphic to V^o as a G-Galois cover of X^o.

Proof. (i) Since a G-Galois cover is of finite presentation, it follows that $V^o \to X^o$ descends, along with its G-action, to a regular k-subalgebra $A \subset K$ of finite type over k, having (smooth) connected spectrum $E = \operatorname{Spec} A$. That is, there is an irreducible regular E-scheme V_E such that $V_E \to E$ is surjective, and there is a G-Galois cover $V_E \to X_E = X \times_k E$, satisfying (a), (b), and (d). It remains to show that (c) can also be satisfied. By Lemma 2.5 (taking $Y = X$ and $\eta_i = \xi_i$), the fibre V_e of $V_E \to E$ over e is irreducible, for all closed points e in a non-empty open subset $E' \subset E$. We may assume that E' is a basic open subset $\operatorname{Spec}(A')$ of E, where $A' = A[f^{-1}]$, for some non-zero $f \in A$. Replacing A by A', and $V_E \to X_E$ by the pullback over E', the result follows.

(ii) By hypothesis, the closed fibre of $V^* \to \operatorname{Spec}(k[[t]])$ is generically smooth. So applying Lemma 2.4(b) to $V^* \to \operatorname{Spec}(k[[t]])$, we deduce that the closed fibre is connected.

Since the connected normal G-Galois cover $V^* \to T^*$ is of finite presentation, it descends to a regular $k[t]$-algebra $R \subset k[[t]]$ of finite type over $k[t]$. That is, for some such algebra R, if we let $A = R[t^{-1}]$ and $E = \operatorname{Spec}(A)$, then there is a connected normal projective R-scheme X_R such that $X_E = X_R \times_R E$ is isomorphic to $X \times_k E$; and there is an irreducible normal projective R-scheme V_R together with a G-Galois covering morphism $V_R \to X_R$ which induces $V^* \to T^*$ over $k[[t]]$, and such that $V_E = V_R \times_R E$ is regular and satisfies (a), (b), and (d).

It remains to verify (c). Since V_R induces V^*, the fibre of V_R over $(t = 0)$ is connected and generically smooth. Moreover V^* is normal. Applying [Ha3,Proposition 5] to $V_R \to \operatorname{Spec}(R)$, and letting ε be the point $(t = 0)$, it follows that for all k-points e in a dense open subset of $\operatorname{Spec}(R)$ (and hence in a dense open subset of $E = \operatorname{Spec}(R) - (t = 0)$), the fibre V_e is irreducible. So as in the case of (i), after shrinking E we obtain condition (c) as well. []

Corollary 2.7. *Under the hypotheses of Proposition 2.6, there is a smooth connected G-Galois cover $V_0 \to X$, branched only over $\xi_0, \dots, \xi_r$, and with C_i an inertia group over ξ_i for $i \in I$.*

Proof. Let E and $V_E \to X_E$ be as in Proposition 2.6. Take any closed point $e \in E$, and let V_0 be the normalization of the fibre V_e of $V_E \to E$ over e. Then $V_0 \to X$ is as desired. []

Section 3. Some results on p-covers.

This section contains some results concerning the moduli space of p-covers of an affine curve in characteristic p. These will used to obtain a family of cyclic-by-p covers of the line in section 4, for use in section 5. In this section, and for the remainder of the paper, we will work over a fixed base field k which is algebraically closed and of characteristic p.

Let (U, u) be a smooth pointed affine curve over k, and let P be a finite p-group. By [Ha1], there is a fine moduli space M_P for (not necessarily connected) pointed P-Galois étale covers of (U, u) – viz. a certain direct limit of affine spaces $\mathbf{A}_k^{m_i}$, where the transition maps are $\mathbf{Z}/p\mathbf{Z}$-linear injections. Actually, there are several ways to make this assertion precise, depending on which functor is asserted to be represented by M_P. In particular, define the contravariant functors $\mathcal{F}'_P$ and $\mathcal{F}^*_P$: (k-schemes) $\to$ (sets) by $\mathcal{F}'_P(S) = \{$étale P-Galois covers $Z \to U \times S$, together with a section of $Z_u \to \{u\} \times S\}$ and $\mathcal{F}^*_P(S) = \mathcal{F}'_P(S)/\sim$, where $\sim$ is the equivalence relation under which two covers are considered equivalent if they agree after pullback to $U \times \tilde{S}$, for some finite étale cover $\tilde{S} \to S$. Each of these functors can be extended to a contravariant functor on the category of ind-schemes, i.e. direct limits of schemes. Namely, define the extended $\mathcal{F}'_P$: (k-ind-schemes) $\to$ (sets) by

$$\mathcal{F}'_P(\lim_{\rightarrow} S_i) = \lim_{\leftarrow} \mathcal{F}'_P(S_i)$$

and similarly for $\mathcal{F}^*_P$. According to [Ha1, 1.7], the ind-scheme M_P represents the functor $\mathcal{F}^*_P$. But as shown below (Proposition 3.2), M_P also represents $\mathcal{F}'_P$. First we prove a lemma:

Lemma 3.1. *Let (U, u) be a smooth pointed affine curve over k, let S be an irreducible k-scheme, and let P be a p-group. Let $X, Y \to U \times S$ be P-Galois étale covers with sections $\xi : S \to X$, $\eta : S \to Y$ over $\{u\} \times S$. Let $S' \to S$ be a finite étale cover, let $X' = X \times_S S'$ and $Y' = Y \times_S S'$, and let ξ', η' be the induced sections of $X', Y' \to U \times S'$ over $\{u\} \times S'$. Suppose that there is an isomorphism $X' \to Y'$ of P-Galois covers of $U \times S'$, taking ξ' to η'. Then there is an isomorphism $X \to Y$ of P-Galois covers of $U \times S$, taking ξ to η.*

Proof. Let $S'' = S' \times_S S'$ and let $X'' = X \times_S S''$, $Y'' = Y \times_S S''$. It suffices to show that the natural map

$$\mathrm{Hom}_{U \times S}(X, Y) \to \mathrm{Hom}_{U \times S'}(X', Y')$$

is surjective. By [Gr,IX, Prop. 3.2], this is the equalizer of the two induced maps

$$\mathrm{Hom}_{U \times S'}(X', Y') \to \mathrm{Hom}_{U \times S''}(X'', Y'').$$

So it suffices to show that these two induced maps are equal. Now a morphism of covers $X' \to Y'$ is determined by its restriction to the fibre X'_u over $\{u\} \times S'$, and similarly for $X'' \to Y''$. So it suffices to show that the two induced maps

$$\mathrm{Hom}_{\{u\} \times S'}(X'_u, Y'_u) \to \mathrm{Hom}_{\{u\} \times S''}(X''_u, Y''_u)$$

are equal. But since the sections ξ and η exist, and since the covers are P-Galois, the fibres X_u and Y_u are each trivial covers of $\{u\} \times S$. So any $\phi' \in \mathrm{Hom}_{\{u\}\times S'}(X', Y')$ is actually induced by some $\phi \in \mathrm{Hom}_{\{u\}\times S}(X_u, Y_u)$, and thus

$$\mathrm{Hom}_{\{u\}\times S}(X, Y) \to \mathrm{Hom}_{\{u\}\times S'}(X', Y')$$

is surjective. Again using [Gr,IX, Prop. 3.2], the conclusion follows. []

Proposition 3.2. *With notation as above, M_P represents $\mathcal{F}'_P$.*

Proof. There is a natural map $\mathcal{F}'_P \to \mathcal{F}^*_P$ which sends each cover to its equivalence class. By definition this is surjective, and it is injective by Lemma 3.1. So the functors $\mathcal{F}'_P$ and $\mathcal{F}^*_P$ are isomorphic, proving the result. []

Thus for every k-scheme S, the morphisms $S \to M_P$ are in natural bijection with the isomorphism classes of Galois étale P-covers $Z \to U \times S$, together with a section over $\{u\} \times S$. In the case that $S = k$, this says that the k-points of M_P are in bijection with isomorphism classes of P-Galois étale covers $Z \to U$, together with base point over u. If P is an elementary abelian p-group (i.e. of type $(p, p, \ldots, p)$), then $M_P(k)$ may be functorially identified with the $\mathbf{Z}/p$-vector space $H^1(U, P)$ [Ha1, Theorem 1.2].

Remark. If the analogous functors without section were considered, then they would not be isomorphic, i.e. the equivalence relation $\sim$ would not be trivial. For example take the covers of the (x, t)-plane $\mathbf{A}^1 \times \mathbf{A}^1$, given respectively by $y^p - y = x$ and $y^p - y = x + t$. Then for each $\alpha \in k$, the covers have isomorphic fibres over $\mathbf{A}^1 \times (t = \alpha)$, but the covers are not themselves isomorphic. Note also that the second cover does not have a section over $(x = 0) \times \mathbf{A}^1$.

If P and Q are p-groups, and $\phi : P \to Q$ is a group homomorphism, then there is an induced morphism of functors $\mathcal{F}'_P \to \mathcal{F}'_Q$, and hence an induced morphism $M_P \to M_Q$. In particular, taking $Q = P$, if $\phi : P \to P$ is an automorphism then there is an induced automorphism $M_P \to M_P$. This association is functorial; so if a group C acts on P, then C also acts on M_P. More explicitly, in terms of the k-points of M_P, we may describe the induced C-action as follows: For any $\xi \in M_P$, consider the corresponding pointed P-Galois cover $Z_\xi \to U$. Then for $c \in C$, $c_*(\xi)$ is the point ξ' of M_P corresponding to the pointed P-Galois cover $Z_{\xi'} \to U$ whose underlying pointed cover is the same as that of $Z_\xi \to U$, but whose P-action is defined so that $c \cdot g$ acts on $Z_{\xi'}$ in the same manner that g acts on Z_ξ, for all $g \in P$.

According to [Ha1, Theorem 1.2], M_P also represents the functor $\mathcal{F}_P$: (pointed ind-schemes over k) $\to$ (sets) given by $\mathcal{F}_P(S) = \{$equivalence classes of P-Galois covers of $S \times U$, with a base point over $(s, u)\}$, where s is the base point of S. Here, two such covers are declared to be equivalent if and only if they agree after pullback by some finite étale cover $T \to S$. So $\mathcal{F}_P(S)$ is the quotient of $\mathrm{Hom}(\pi_1(S \times U), P)$ by this equivalence relation.

In particular, the set $M_P(k)$ is in bijection with $\mathrm{Hom}(\pi_1(U), P)$, which parametrizes the pointed P-Galois covers of U (which, as always, come equipped with a fixed group action). Here, the surjective homomorphisms correspond to the connected covers. Given such a cover, if the choice of base point over $\{u\}$ is changed by $g \in P$, then the unique isomorphism taking the old base point to the new one will in general not preserve the given group action (which will be conjugated by g); and so the isomorphism class of the object will in general change. Observe similarly that the quotient $\mathrm{Hom}(\pi_1(U), P)/\mathrm{Inn}(P)$ parametrizes P-Galois covers of U without a chosen base point.

By [Ha1, Theorem 1.2], if P is elementary abelian, then M_P is a direct limit of commutative group schemes. Also by that result, the corresponding group structure on the set k-points $M_P(k)$ is the same as that on $H^1(U,P) = \mathrm{Hom}(\pi_1(U), P)$, under its identification with $M_P(k)$. In particular, this group is p-torsion.

If $A \subset P$ is a central elementary abelian subgroup of a p-group P, then M_A acts on M_P, via compatible actions of $M_A(S)$ on $M_P(S)$ for all k-schemes S. Namely, using the fact that M_P represents $\mathcal{F}_P$ and that $\mathcal{F}_P(S) = \mathrm{Hom}(\pi_1(S \times U), P)/\sim$, each such action is induced by the action of $\mathrm{Hom}(\pi_1(S \times U), A)$ on $\mathrm{Hom}(\pi_1(S \times U), P)$ given by $(\eta \cdot \psi)(\xi) = \eta(\xi)\psi(\xi) \in P$, where $\xi \in \pi_1(S \times U)$, $\eta \in \mathrm{Hom}(\pi_1(S \times U), A)$, and $\psi, \eta \cdot \psi \in \mathrm{Hom}(\pi_1(S \times U), P)$. In particular, if $P = A$, the action of M_A on itself agrees with the group structure on M_A (which is written additively).

With $A \subset P$ as above, let $\overline{P} = P/A$. Then the quotient map $P \to \overline{P}$ induces a morphism $\nu : M_P \to M_{\overline{P}}$. Observe that the fibres of ν are the orbits of the action of M_A on M_P. Now since $M_{\overline{P}}$ represents $\mathcal{F}_{\overline{P}}$, for any scheme S and any morphism $\overline{\gamma} : S \to M_{\overline{P}}$ there is an induced family of $\overline{P}$-Galois covers of U parametrized by S, and this in turn corresponds to a homomorphism $\overline{\psi} : \pi_1(S \times U) \to \overline{P}$. Since $\pi_1(S \times U)$ has p-cohomological dimension ≤ 1 [AGV, X 5.1], $\overline{\psi}$ lifts to a homomorphism $\psi : \pi_1(S \times U) \to P$, by [Se1, I Prop. 16]. Since M_P represents $\mathcal{F}_P$, we thus obtain a lifting $\gamma : S \to M_P$ of $\overline{\gamma}$, i.e. a morphism such that $\nu \circ \gamma = \overline{\gamma}$.

Now fix a $\overline{\gamma} : S \to M_{\overline{P}}$ as above, and let $H_{\overline{\gamma}}$ be the set of liftings of $\overline{\gamma}$ to $\mathrm{Hom}(S, M_P)$. For each choice of a (base) lifting $\gamma : S \to M_P$ of $\overline{\gamma}$, there is an induced structure on $H_{\overline{\gamma}}$ as a p-torsion abelian group, such that γ is the zero element. Namely, if $\gamma_1, \gamma_2 \in H_{\overline{\gamma}}$, then define $\gamma_1 + \gamma_2$ as follows: Since $\gamma \in H_{\overline{\gamma}}$, for $i = 1, 2$ there is an $\alpha_i \in M_A(S)$ such that $\gamma_i = \alpha_i \cdot \gamma$ for some $\alpha_i \in M_A(S)$. Now take $\gamma_1 + \gamma_2 = (\alpha_1 + \alpha_2) \cdot \gamma$. This group law is not canonical, however, since it depends on the choice of a base lifting γ. Nevertheless, we do have the following result:

Proposition 3.3. *Let P be a p-group, let $A \subset P$ be a central elementary abelian subgroup, and let $\overline{P} = P/A$. Let $\overline{\gamma} : S \to M_{\overline{P}}$ be a morphism, and let $\gamma_1, \dots \gamma_n \in M_P(S)$ be lifts of $\overline{\gamma}$. Let $a_1, \dots, a_n \in \mathbf{Z}/p\mathbf{Z}$ be elements satisfying $\sum_{i=1}^n a_i = 1$.*

(a) Then $\sum_{i=1}^n a_i \gamma_i$ is independent of the choice of base lifting γ of $\overline{\gamma}$.

(b) Let $\sigma \in \mathrm{Aut}(P)$ satisfy $\sigma(A) = A$. Let σ_* denote the induced automorphisms on M_P and on $M_{\overline{P}}$. Then $\sigma_*(\alpha\gamma) = \sigma_*(\alpha)\sigma_*(\gamma)$ for each lift γ of $\overline{\gamma}$ and each $\alpha \in M_A(S)$; σ_* preserves the fibres of $\nu : M_P \to M_{\overline{P}}$; and $\sigma_*(\sum_{i=1}^n a_i \gamma_i) = \sum_{i=1}^n a_i \sigma_*(\gamma_i)$.

Proof. (a) Let γ, $\gamma' : S \to M_P$ be any two lifts of $\overline{\gamma}$. Thus $\gamma' = \alpha \cdot \gamma$, for some $\alpha : S \to M_A$. Write $\gamma_i = \alpha_i \cdot \gamma$, for some $\alpha_i \in M_A(S)$. Thus $\gamma_i = \alpha_i' \cdot \gamma'$, where $\alpha_i' = \alpha_i - \alpha : S \to M_A$. With respect to the lift γ, $\sum_{i=1}^n a_i \gamma_i = (\sum_{i=1}^n a_i \alpha_i) \cdot \gamma$. Meanwhile, respect to the lift γ', we have

$$\sum_{i=1}^n a_i\gamma_i = (\sum_{i=1}^n a_i\alpha_i')\cdot\gamma' = (\sum_{i=1}^n a_i(\alpha_i-\alpha))\cdot(\alpha\cdot\gamma) = (\sum_{i=1}^n a_i\alpha_i)\cdot\gamma,$$

using that $\sum_{i=1}^n a_i = 1$. So the sum is independent of the choice of lift.

(b) The first part of (b) follows from the definition of the action of M_A on M_P, and by the hypotheses on σ. The second assertion follows from this by the fact that the fibres of ν are the M_A-orbits on M_P. In particular, using this in the special case that $\gamma = \alpha' \in M_A(S)$ (corresponding to the case when $\overline{\gamma}$ is trivial), it follows that σ_* defines a homomorphism on $M_A(S)$.

For the final assertion, let $q = \mathrm{ord}(\sigma)$, and let $\tilde{S}$ be the disjoint union of q copies of S. Thus the cyclic group $\langle\sigma\rangle$ acts on $\tilde{S}$, by cyclically permuting the copies of S. Define $\tilde{\gamma} : \tilde{S} \to M_{\overline{P}}$ such that on the jth copy of S, $\tilde{\gamma}$ acts like $\sigma_*^j\circ\gamma$. Thus $\tilde{\gamma}\circ\sigma_* = \sigma_*\circ\tilde{\gamma} : \tilde{S} \to M_{\overline{P}}$. So replacing S by $\tilde{S}$ and $\overline{\gamma}$ by $\tilde{\gamma}$, we may assume that $\langle\sigma\rangle$ acts on S in a such a way that $\overline{\gamma}\circ\sigma_* = \sigma_*\circ\overline{\gamma} : S \to M_{\overline{P}}$.

Choose a base lift γ of $\overline{\gamma}$. The actions of σ_* on M_P and on S induce another lift γ' of $\overline{\gamma}$, given by $\gamma' = \sigma_*\circ\gamma\circ\sigma_*^{-1} : S \to M_{\overline{P}}$. Writing $\gamma_i = \alpha_i\cdot\gamma$, we have that $\sum_{i=1}^n a_i\gamma_i = \sum_{i=1}^n (a_i\alpha_i)\cdot\gamma$. So the left hand side of the final assertion of (b) is equal to

$$\sigma_*\big((\sum_{i=1}^n a_i\alpha_i)\cdot\gamma\big) = \sigma_*\big(\sum_{i=1}^n a_i\alpha_i\big)\cdot\sigma_*(\gamma) = \big(\sum_{i=1}^n a_i\sigma_*(\alpha_i)\big)\cdot(\gamma'\circ\sigma_*),$$

using that σ_* is a homomorphism on $M_A(S)$. Similarly, $\sigma_*(\gamma_i) = \sigma_*(\alpha_i)\cdot(\gamma'\circ\sigma_*)$. But each $\sigma_*(\gamma_i)$ is a lift of $\sigma_*(\overline{\gamma}) = \overline{\gamma}\circ\sigma_*$. So using (a), and taking $\gamma'\circ\sigma_*$ as the base lift of $\overline{\gamma}\circ\sigma_*$, we obtain that the right hand side of the final assertion in (b) is also equal to $(\sum_{i=1}^n a_i\sigma_*(\alpha_i))\cdot(\gamma'\circ\sigma_*)$. This proves the last part of (b). []

Lemma 3.4. *Let P be a finite p-group, let A be a central elementary abelian p-subgroup of P, let $\overline{P} = P/A$, and let $\nu : M_P \to M_{\overline{P}}$ be the induced morphism of moduli spaces. Let $\sigma \in \mathrm{Aut}(P)$ be an automorphism of order n prime to p, and assume that $\sigma(A) = A$. Let $\tau^{(0)},\dots,\tau^{(r)} \in \mathbf{A}^1 - \{0\}$, let $\eta^{(0)},\dots,\eta^{(r)} \in M_P$, and let $\overline{\eta}^{(e)} = \nu(\eta^{(e)})$ for all e. Suppose that for $1 \le e, e' \le r$, we have that $[\tau^{(e)}]^n = [\tau^{(e')}]^n$ if and only if $e = e'$. Let $\overline{F} : \mathbf{A}^1 \to M_{\overline{P}}$ be a morphism such that $\overline{F}(\tau^{(e)}) = \overline{\eta}^{(e)}$ for all e, and such that $\overline{F}(\zeta_n\tau) = \sigma_*(\overline{F}(\tau))$ for all $\tau \in \mathbf{A}^1$, where $\zeta_n \in k$ is a fixed nth root of unity. Then there is a morphism $F : \mathbf{A}^1 \to M_P$ such that $\nu\circ F = \overline{F}$, $F(\tau^{(e)}) = \eta^{(e)}$ for all e, and $F(\zeta_n\tau) = \sigma_*(F(\tau))$ for all $\tau \in \mathbf{A}^1$.*

Proof. By composing $\overline{F} : \mathbf{A}^1 \to M_{\overline{P}}$ with a section of $\nu : M_P \to M_{\overline{P}}$, we obtain a morphism $F_0 : \mathbf{A}^1 \to M_P$ such that $\nu\circ F_0 = \overline{F}$. For each $e = 0,\dots r$ and each $i = 0,\dots,n-1$, let $\eta_i^{(e)} = \sigma_*^i(\eta^{(e)})$ and $\tau_i^{(e)} = \zeta_n^i\tau^{(e)}$. In particular, $\eta_0^{(e)} = \eta^{(e)}$ and $\tau_0^{(e)} = \tau^{(e)}$, for all e.

For each e and i, $\nu(\eta_i^{(e)}) = \sigma_*^i(\nu(\eta^{(e)})) = \sigma_*^i(\overline{\eta}^{(e)}) = \sigma_*^i(\overline{F}(\tau^{(e)})) = \overline{F}(\tau_i^{(e)}) = \nu(F_0(\tau_i^{(e)}))$. So there is an element $\alpha_i^{(e)} \in M_A(k)$ such that $\alpha_i^{(e)}\cdot F_0(\tau_i^{(e)}) = \eta_i^{(e)}$, under the action of M_A on M_P. Since the set $\{\alpha_i^{(e)}\,|\,e = 0,\dots r; i = 0,\dots,n-1\}$ is finite, the points $\alpha_i^{(e)} \in M_A(k)$ all lie in a common affine space $\mathbf{A}^m$ in the direct limit defining M_A. For each e and i write $\alpha_i^{(e)} = (\alpha_{i,1}^{(e)}, \dots, \alpha_{i,m}^{(e)}) \in \mathbf{A}_k^m \subset M_A(k)$.

By hypothesis, the points $\tau_i^{(e)} \in \mathbf{A}^1$ $(e = 0, \dots r; i = 0, \dots, n-1)$ are distinct. Hence there are polynomials $\phi_1, \dots, \phi_m \in k[t]$ such that $\phi_j(\tau_i^{(e)}) = \alpha_{i,j}^{(e)}$ for all e, i, j. Let $\phi = (\phi_1, \dots, \phi_m) : \mathbf{A}^1 \to \mathbf{A}^m \subset M_A$. Thus $\phi(\tau_i^{(e)}) = \alpha_i^{(e)}$ for all e, i. Let $F_1 : \mathbf{A}^1 \to M_P$ be given by $F_1 = \phi \cdot F_0$. Then $\nu \circ F_1 = \nu \circ F_0 = \overline{F}$, and $F_1(\tau_i^{(e)}) = \alpha_i^{(e)} \cdot F_0(\tau_i^{(e)}) = \eta_i^{(e)}$ for all e, i. In particular, $F_1(\tau^{(e)}) = \eta^{(e)}$ for all e.

Finally, define $F : \mathbf{A}^1 \to M_P$ by $F(t) = \sum_{i=0}^{n-1} (1/n) \sigma_*^{-i} F_1(\zeta_n^i t)$. This is well defined by Proposition 3.3(a), since the projections $\nu(\sigma_*^{-i} F_1(\zeta_n^i t)) = \sigma_*^{-i} \overline{F}(\zeta_n^i t) = \overline{F}(t)$ are all equal. We wish to verify the three desired properties for F. First, the composition $\nu \circ F(t) = \sum_{i=0}^{n-1} (1/n) \sigma_*^{-i} \overline{F}(\zeta_n^i t) = \sum_{i=0}^{n-1} (1/n) \overline{F}(t) = \overline{F}(t)$. Second, we have $F(\tau^{(e)}) = \sum_{i=0}^{n-1} (1/n) \sigma_*^{-i} F_1(\tau_i^{(e)}) = \sum_{i=0}^{n-1} (1/n) \sigma_*^{-i}(\eta_i^{(e)}) = \sum_{i=0}^{n-1} (1/n) \eta^{(e)} = \eta^{(e)}$. And third, using Proposition 3.3(b) we obtain that $F(\zeta_n \tau) = \sum_{i=0}^{n-1} (1/n) \sigma_*^{-i} F_1(\zeta_n^{i+1} \tau) = \sigma_*(\sum_{i=0}^{n-1} (1/n) \sigma_*^{-(i+1)} F_1(\zeta_n^{i+1} \tau)) = \sigma_*(\sum_{i=0}^{n-1} (1/n) \sigma_*^{-i} F_1(\zeta_n^i \tau)) = \sigma_*(F(\tau))$. []

Proposition 3.5. *Let P be a finite p-group, and let C be a cyclic group of order n prime to p, having generator σ. Let C act on P, let σ_* be the induced automorphism of the moduli space M_P of pointed P-Galois étale covers of the pointed affine curve (U, u), and let $\xi_0 \in M_P$. Then there is a morphism $F : \mathbf{A}^1 \to M_P$ such that*

(1) $F(1) = \xi_0$,

(2) $F(\zeta_n \tau) = \sigma_*(F(\tau))$ for all $\tau \in \mathbf{A}^1$, where ζ_n is a fixed primitive nth root of unity in k, and

(3) For every proper normal subgroup N of P, the induced morphism $\mathbf{A}^1 \to M_{P/N}$ is non-constant.

Proof. The result is trivial if P is the trivial group. Proceeding by induction on the order of P, we assume that P is non-trivial, and that the result holds for all p-groups P' of order less than that of P. There are two cases:

Case 1: P is an elementary abelian p-group. Let $N_1, \dots, N_r$ be the subgroups of P having index p, let $P_e = P/N_e$, and let $\nu_e : M_P \to M_{P_e}$ be the induced morphism between moduli spaces. By the comments before Proposition 3.3 concerning lifting, ν_e is surjective on points. So for each $e = 1, \dots, r$, there is a k-point $\eta^{(e)} \in M_P$ such that $\nu_e(\eta^{(e)}) \neq \nu_e(\xi_0)$. Also, let $\eta^{(0)} = \xi_0$.

Let $\tau^{(0)} = 1$. Since k is infinite, we may choose non-zero elements $\tau^{(1)}, \dots, \tau^{(r)} \in k$ such that no $\tau^{(e)}/\tau^{(e')}$ is an nth root of unity, for $e \neq e'$, $0 \leq e, e' \leq r$. Thus the hypotheses of Lemma 3.4 are satisfied (with $A = P$, and with $\overline{P}$ and $\overline{F}$ trivial). Hence there is a morphism $F : \mathbf{A}^1 \to M_P$ such that $F(\tau^{(e)}) = \eta^{(e)}$ for all $e = 0, \dots, r$, and $F(\zeta_n \tau) = \sigma_*(F(\tau))$ for all $\tau \in \mathbf{A}^1$. So (1) and (2) hold. If remains to show (3).

It suffices to check (3) for maximal proper subgroups $N \subset P$. Thus we may assume $N = N_e$ for some e. Now $F(\tau^{(e)}) = \eta^{(e)}$, and so $\nu_e \circ F(\tau^{(e)}) = \nu_e(\eta_e)$. By construction, $\nu_e(\eta^{(e)}) \neq \nu_e(\xi_0)$, and $\nu_e(\xi_0) = \nu_e \circ F(1)$ by (1). Hence $\nu_e \circ F(\tau^{(e)}) \neq \nu_e \circ F(1)$. Thus the induced morphism $\nu_e \circ F : \mathbf{A}^1 \to M_{P/N}$ is non-constant, proving (3).

Case 2: Otherwise. Then the Frattini subgroup $Q \subset P$ is a non-trivial proper subgroup of P, and hence Q has a non-trivial center Z. Let $A \subset Z$ consist of the p-torsion elements. Thus A is a non-trivial characteristic subgroup of P (and so is σ-invariant), it is unequal to P, and it is an elementary abelian p-group. Let $\overline{P} = P/A$, let $\nu : M_P \to M_{\overline{P}}$ be the induced morphism of moduli spaces, and let $\overline{\xi}_0 = \nu(\xi_0)$. Since the order of $\overline{P}$ is strictly less than that of P, the inductive hypothesis implies that there is a morphism $\overline{F} : \mathbf{A}^1 \to M_{\overline{P}}$ such that $\overline{F}(1) = \overline{\xi}_0$; $\overline{F}(\zeta_n \tau) = \sigma_*(\overline{F}(\tau))$ for all $\tau \in \mathbf{A}^1$; and the induced morphism $\overline{F}' : \mathbf{A}^1 \to M_{P'}$ is non-constant for each non-trivial quotient P' of $\overline{P}$. Applying Lemma 3.4, we obtain a morphism $F : \mathbf{A}^1 \to M_P$ such that $\nu \circ F = \overline{F}$, $F(1) = \xi_0$, and $F(\zeta_n \tau) = \sigma_*(F(\tau))$ for all $\tau \in \mathbf{A}^1$. Thus (1) and (2) hold, and it remains to check (3).

As in case 1, it suffices to check (3) for maximal proper subgroups N of P. (Each such N is normal, since P is a p-group.) Since the Frattini subgroup Q is the intersection of the maximal proper subgroups of P, we have that N contains A. Since A is a characteristic subgroup of P, it is also a normal subgroup of N, and $(P/A)/(N/A)$ is isomorphic to $P' = P/N$. Let $\nu' : M_P \to M_{P'}$ and $\overline{\nu}' : M_{\overline{P}} \to M_{P'}$ be the induced morphisms of moduli spaces. Thus $\overline{\nu}' \circ \nu = \nu'$. By the inductive hypothesis, $\overline{F} : \mathbf{A}^1 \to M_{\overline{P}}$ induces a non-constant morphism $\overline{F}' = \overline{\nu}' \circ \overline{F} : \mathbf{A}^1 \to M_{P'}$. But $\nu \circ F = \overline{F}$ and $\overline{\nu}' \circ \nu = \nu'$. So the induced morphism $\nu' \circ F : \mathbf{A}^1 \to M_{P'}$ of (3) is equal to $\overline{F}'$, and so is non-constant, as desired. []

Section 4. Cyclic-by-p families.

Using the results of section 3, this section constructs and studies a family of cyclic-by-p unramified covers of the affine k-curve $\mathbf{P}^1 - \{0, \infty\}$ (where as before, k is algebraically closed of characteristic p). The family will be used in section 5, in proving Conjectures 1.1 and 1.2 in the case of $\mathbf{P}^1 - \{0, \infty\}$. Specifically, the results of this section will be used in section 5 to show that an extension G of a prime-to-p cyclic group C by a quasi-p-group Q must occur as a Galois group over the twice punctured projective line, with one branch point tame. Cf. Proposition 5.2 and Theorem 5.4.

Remark. At first glance, a much simpler strategy may suggest itself for constructing such a G-Galois cover of $\mathbf{P}^1 - \{0, \infty\}$. Namely, by [Ra], there is a Q-Galois cover $\pi : W \to \mathbf{P}^1$ of the projective t-line which is branched only over infinity. Let $\psi : \mathbf{P}^1 \to \mathbf{P}^1$ be the map from the t-line to the s-line given by $s = t^n$. Then the composition $\psi \circ \pi : W \to \mathbf{P}^1$ is a cover of the s-line branched only over $(s = 0)$ and $(s = \infty)$, and tamely ramified over $(s = 0)$. But this cover will not in general be Galois with group G, or even be Galois at all. So instead, Proposition 4.1 below is used to construct (in Lemma 5.1) a cyclic-by-p cover that can be patched (in 5.2) to the Q-cover W in a way that yields the desired G-cover. See also the remark after the proof of 5.2.

Observe that if P is a p-group, and $V \to \mathbf{P}^1$ is an irreducible P-Galois branched cover which is ramified only over one point, then the cover is totally ramified there. For otherwise, the inertia groups over the branch point are the conjugates of a proper p-subgroup I of P. Since P is a p-group, I and hence its conjugates lie in a proper normal subgroup N of P. Thus V/N is a nontrivial connected étale cover of $\mathbf{P}^1$, which is impossible.

Proposition 4.1. *Let H be a cyclic-by-p group with Sylow p-subgroup P and cyclic quotient C of order n (prime to p). Define $\theta : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ by $\theta(x,t) = (x,s)$ where $s = t^n$. Let $Y' \to \mathbf{A}^1$ be a connected P-Galois étale cover of the affine x-line over k. Then there is a connected normal k-surface Z and a P-Galois morphism $\pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ to (x,t)-space, such that*

(i) the restriction $\pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ of Z is finite étale;

(ii) the fibre of Z' over $\mathbf{A}^1 \times (t=1)$ is isomorphic to $Y' \to \mathbf{A}^1$;

(iii) the morphism π is totally ramified over $(x=\infty)$ and over $(t=\infty)$; and

(iv) the composition $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ is a H-Galois ramified cover, with branch locus $(s=0) \cup (s=\infty) \cup (x=\infty)$. It is tamely ramified over $(s=0)$ with inertia group C, it is totally ramified over $(s=\infty)$, and P is the inertia group over $(x=\infty)$.

Proof. Step 1: Construction of Z.

Since H is a semidirect product of P with C, C acts on P by conjugation. As in section 3, let σ be a generator of C, let ζ_n be a primitive nth root of unity in k, and let M_P be the fine moduli space of étale P-Galois covers of $\mathbf{A}^1$ pointed over $(x=0)$. Choose a base point on Y' over $(x=0)$, and let $\xi_0 \in M_P$ be the k-point corresponding to the pointed cover $Y' \to \mathbf{A}^1$. By Proposition 3.5, we obtain a morphism $F : \mathbf{A}^1 \to M_P$ satisfying (1) - (3) of that result.

The morphisms $F : \mathbf{A}^1 \to M_P$ and $\tilde{F} = \sigma_* \circ F : \mathbf{A}^1 \to M_P$, defined on the affine t-line, induce P-Galois étale covers $\pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1, \tilde{\pi}' : \tilde{Z}' \to \mathbf{A}^1 \times \mathbf{A}^1$ of (x,t)-space with sections over $(x=0) \times \mathbf{A}^1$. Let $\pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ be the normalization of $\mathbf{P}^1 \times \mathbf{P}^1$ in Z'. Thus (i) holds. By construction, the fibre Z'_α of Z' over $\mathbf{A}^1 \times (t=\alpha)$ is in the class of $F(\alpha)$, for all $\alpha \in \mathbf{A}^1$. In particular, by (1) of Proposition 3.5, the fibre over $\mathbf{A}^1 \times (t=1)$ in isomorphic to the P-Galois cover $Y' \to \mathbf{A}^1$, together with its base point over $(x=0)$. Thus (ii) holds.

Step 2: Showing that $\theta \circ \pi$ is H-Galois.

Let C act on the t-line $\mathbf{A}^1$ by $\sigma_* : (t=\alpha) \mapsto (t=\zeta_n\alpha)$. So $1 \times \sigma_*$ defines an action of C on (x,t)-space $\mathbf{A}^1 \times \mathbf{A}^1$. We will lift this to an action Φ of C on Z', in such a way that Φ and the given action of P on Z' generate a copy of H acting on Z'. This will then extend to a Galois action of H on Z over (x,s)-space $\mathbf{P}^1 \times \mathbf{P}^1$.

By (2) of Proposition 3.5, for each $\alpha \in \mathbf{A}^1$, the fibre $\tilde{Z}'_\alpha$ of $\tilde{Z}'$ over $\mathbf{A}^1 \times (t=\alpha)$ is in the class of $\sigma_*(F(\alpha)) = F(\zeta_n\alpha)$, and so is isomorphic to $Z'_{\zeta_n\alpha}$. Meanwhile, by definition of the action of C on M_P, this fibre is isomorphic to Z'_α as a pointed cover, but with a different P-action. Namely, for $g \in P$, $\sigma g \sigma^{-1}$ acts on $\tilde{Z}'_\alpha = Z'_{\zeta_n\alpha}$ in the same way that g acts on Z'_α.

Let $Z'^* \to \mathbf{A}^1 \times \mathbf{A}^1$ be the P-Galois cover with a section over $(x=0)$ whose underlying cover and section agree with those of $\tilde{Z}'$, but whose P-action is defined so that each $g \in P$ acts on Z'^* in the same way that $\sigma g \sigma^{-1}$ acts on $\tilde{Z}'$. Thus Z'^* is a family of P-Galois covers of $\mathbf{A}^1$ together with a section over $(x=0)$, parametrized by the t-line. Since M_P is a fine moduli space for such covers, it follows that $Z'^* \to \mathbf{A}^1 \times \mathbf{A}^1$ is induced by the morphism $F^* : \mathbf{A}^1 \to M_P$ taking each point $(t=\alpha)$ to the point of M_P corresponding to the class of Z'^*_α. But by construction, Z'^*_α is isomorphic to Z'_α, as a P-Galois pointed cover, for all α. So $F^* = F$, and hence Z' and Z'^* are isomorphic as P-Galois covers with sections. Equivalently, there is an isomorphism $\phi : Z' \to \tilde{Z}'$ of covers of (x,t)-space $\mathbf{A}^1 \times \mathbf{A}^1$ preserving the section over $(x = 0)$, and such that for each $g \in P$ and $z \in Z'$, $\phi((\sigma^{-1} g \sigma)(z)) = g(\phi(z))$. That is, $\tilde{\pi}' \circ \phi = \pi'$ and

(*) $\phi^{-1} g \phi = \sigma^{-1} g \sigma : Z' \to Z'$ for all $g \in P$.

With respect to the above action σ_* on the t-line, the fibre of the P-Galois cover $(1 \times \sigma_*^{-1}) \circ \pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ over $(t = \alpha)$ is the P-Galois cover $Z'_{\zeta_n \alpha} \to \mathbf{A}^1$. So $(1 \times \sigma_*^{-1}) \circ \pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ is induced by the map $F \circ \sigma_* : \mathbf{A}^1 \to M_P$. But $\sigma_* \circ F = F \circ \sigma_*$ by property (2) of Proposition 3.5, and $\tilde{\pi}' : \tilde{Z}' \to \mathbf{A}^1 \times \mathbf{A}^1$ is the P-Galois cover induced by the map $\sigma_* \circ F$. So there is an isomorphism between these two P-Galois covers, i.e. an isomorphism $\iota : \tilde{Z}' \to Z'$ such that $[(1 \times \sigma_*^{-1}) \circ \pi'] \circ \iota = \tilde{\pi}'$, and

(**) $\iota^{-1} g \iota = g : \tilde{Z}' \to \tilde{Z}'$ for all $g \in P$,

as well as preserving the section.

Let $\Phi = \iota \circ \phi : Z' \to Z'$. Then $(\theta \circ \pi') \circ \Phi = [\theta \circ (1 \times \sigma_*^{-1})] \circ \pi' \circ \iota \circ \phi = \theta \circ \tilde{\pi}' \circ \phi = \theta \circ \pi'$. So Φ is an automorphism of the cover $\theta \circ \pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$. Also, $(1 \times \sigma_*^{-1}) \circ \pi' \circ \iota = \tilde{\pi}' = \pi' \circ \phi^{-1}$, and so $\pi' \circ \Phi = (1 \times \sigma_*) \circ \pi'$. Thus $\Phi : Z' \to Z'$ lifts the action of $1 \times \sigma_*$ on the intermediate C-cover $\theta : \mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{A}^1$. As a consequence, Φ^n lifts the identity on (x,t)-space $\mathbf{A}^1 \times \mathbf{A}^1$, and so it is an automorphism of the P-Galois cover $\pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ of (x,t)-space which preserves the section. Hence $\Phi^n = 1$. Moreover $\Phi^{-1} g \Phi = \sigma^{-1} g \sigma$ for all $g \in P$, by (*) and (**). Thus Φ and P generate a group of automorphisms of $\theta \circ \pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ which is isomorphic to H, the semidirect product of P with C. Since the degree of $\theta \circ \pi'$ is equal to $(\#P)(\#C) = \#H$, it follows that $\theta \circ \pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ is Galois with group H, and hence so is $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$.

Step 3: Verification of the other properties.

It remains to show the rest of (iv), condition (iii), and the connectivity of Z. Now the branch locus of $\theta \circ \pi$ is the union of that of θ with the image of that of π, and the ramification indices multiply. So the branch locus of $\theta \circ \pi$ is $(s = 0) \cup (s = \infty) \cup (x = \infty)$. For all α, Φ takes the fibre Z'_α to $Z'_{\zeta_n \alpha}$, sending base point to base point. So Φ restricts to the identity on the fibre Z'_0. Thus the inertia group over $(s = 0)$ is $C \subset H$. Hence (iv) will follow from (iii), as will connectivity of Z.

So to complete the proof it suffices to show that (iii) holds. By [Ha1,1.12], there is a dense open subset $M_P^o \subset M_P$ corresponding to the connected étale P-Galois covers of $\mathbf{A}^1$. Since $Z'_\alpha \to \mathbf{A}^1$ is in the class of $F(\alpha)$ for all $\alpha \in \mathbf{A}^1$, and since $Z'_1 = Y'$ is connected, it follows that the point $F(1)$ lies in M_P^o. So for all α in a dense open subset of $\mathbf{A}^1$, $Z'_\alpha \to \mathbf{A}^1$ corresponds to a point in M_P^o, and hence $Z_\alpha \to \mathbf{P}^1$ is an irreducible P-Galois cover. Since P is a p-group, and since $Z_\alpha \to \mathbf{P}^1$ is ramified only over infinity, it follows by the comment just before the proposition that Z_α is totally ramified over infinity, for each α in the open set. Thus $\pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ is totally ramified over $(x = \infty)$. This shows half of condition (iii).

To conclude the proof, we need to show that π is totally ramified over $(t = \infty)$. So assume not, and let I be an inertia group over the generic point of $(t = \infty)$. By assumption, I is a proper subgroup of P. Since P is a p-group, I lies in a proper normal subgroup $N \subset P$. Let $\overline{P} = P/N$, let $\nu : M_P \to M_{\overline{P}}$ be the induced morphism between moduli spaces, and let $\overline{\pi} : \overline{Z} \to \mathbf{P}^1 \times \mathbf{P}^1$ be the quotient of Z by N. Let $\overline{Z}''$ be the inverse image of $\mathbf{A}^1 \times \mathbf{P}^1$ under $\overline{\pi}$. This is normal, and unramified in codimension 1; so by Purity of Branch Locus, it is étale. So $\overline{Z}'' \to \mathbf{A}^1 \times \mathbf{P}^1$ defines a family of pointed $\overline{P}$-Galois covers of the affine x-line, parametrized by the projective t-line. The restriction of this family to $\mathbf{A}^1 \times \mathbf{A}^1$ corresponds to the quotient Z'/N, and hence is induced by the morphism $\overline{F} = \nu \circ F : \mathbf{A}^1 \to M_{\overline{P}}$ on the affine t-line. Since $M_{\overline{P}}$ is a fine moduli space for $\overline{P}$-Galois covers of the affine line, the full family $\overline{Z}'' \to \mathbf{A}^1 \times \mathbf{P}^1$ is induced by a morphism $\psi : \mathbf{P}^1 \to M_{\overline{P}}$ which extends $\overline{F}$. Since $M_{\overline{P}}$ is a direct limit of affine spaces, it follows that ψ is constant, and hence so is $\overline{F}$. This contradicts (3) of Proposition 3.5, thereby completing the proof. []

Proposition 4.2. *In the situation of Proposition 4.1, let $\delta \in S = \mathbf{P}^1 \times \mathbf{P}^1$ be the point $(x = s = \infty)$ in (x, s)-space, and let $b : S' \to S$ be a blow-up at a finite set of distinct points including δ. Let $D \subset S'$ be the exceptional divisor over δ, and let Z' be the normalization of the pullback $Z \times_S S'$. Then the H-Galois cover $Z' \to S'$ is totally ramified over D.*

Proof. Let T be (x, t)-space $\mathbf{P}^1 \times \mathbf{P}^1$. In an affine neighborhood of the point $\delta \in S$, take local coordinates $\overline{x} = x^{-1}$ and $\overline{s} = s^{-1}$. Similarly, on an affine neighborhood of the point $\theta^{-1}(\delta) \in T$, take local coordinates $\overline{x} = x^{-1}$ and $\overline{t} = t^{-1}$. Thus in a neighborhood of δ, the morphism $\theta : T \to S$ is given by $\overline{s} = \overline{t}^n$. Also, over an affine neighborhood of δ that excludes the other blown-up points, S' is described in an affine patch S'_1 by $\overline{s} = w\overline{x}$ and in a complementary affine patch S'_2 by $\overline{x} = u\overline{s}$. The exceptional locus D is given in S'_1 by $(\overline{x} = 0)$, and it is given in S'_2 by $(\overline{s} = 0)$. Let $T' = T \times_S S'$. Thus T' is given over the patch S'_1 by $\overline{t}^n = w\overline{x}$, and it is given over S'_2 by $\overline{t}^n = \overline{s}$. So T' is normal, and $T' \to S'$ is totally ramified over D.

It remains to show that the P-Galois cover $Z' \to T'$ is totally ramified over the inverse image $L \subset T'$ of $D \subset S'$. Since $T' \to S'$ is totally ramified over D, it follows that L is irreducible. Let $Q \subset P$ be an inertia group of $Z' \to T'$ over the generic point of L. We wish to show that $Q = P$. If this does not hold, then Q is contained in a maximal subgroup M of the p-group P. Now M is normal in P of index p. Let $U = Z/M$ and $U' = Z'/M$. Thus $U \to T$ and $U' \to T'$ are $\mathbf{Z}/p$-Galois covers, and $U' \to T'$ is unramified over the generic point of L. Note that on the patch $T'_1 \subset T'$ over $S'_1 \subset S'$, L is given by $(\overline{x} = 0)$.

Now $U \to T$ is a $\mathbf{Z}/p$-Galois cover that ramified only over the loci $(x = \infty) \cup (t = \infty)$, and it is totally ramified there. So it restricts to a non-trivial $\mathbf{Z}/p$-Galois étale cover of the affine (x, t)-plane. Thus it is given over this affine plane by an equation of the form $z^p - z = f(x, t)$, for some $f(x, t) = \sum_{(i,j) \in I} a_{ij} x^i t^j \in k[x, t]$ (where I is a finite index set and each a_{ij} is non-zero). This cover of the affine plane is unchanged if f is altered by adding a polynomial of the form $g^p - g$, where $g \in k[x, t]$. Since k is algebraically closed, we may thus assume that f has no constant term. Similarly, we may replace any term of f by its p^mth power, for any positive integer m; and so we may assume that there is a positive integer N such that for all $(i, j) \in I$, we have $p^N \leq ni + j < p^{N+1}$. Moreover the index set I remains non-empty, since the cover is non-trivial.

Generically on T' (and using the variables on the patch T'_1), the p-cover $U' \to T'$ is given by

$$z^p - z = \sum_{(i,j)\in I} a_{ij} w^i \bar{t}^{-ni-j},$$

since $\bar{x} = \bar{t}^n/w$. Completing at the point $(w = \bar{t} = \bar{x} = 0)$ on T', we obtain a normalized p-extension of $k[[w,\bar{x}]][\bar{t}]/(\bar{t}^n - w\bar{x})$ that is unramified over the generic point of $(\bar{x} = 0)$. Passing to the overring $k((w))[[\bar{t}]]$ (where the inclusion of $k[[w,\bar{x}]][\bar{t}]/(\bar{t}^n - w\bar{x})$ takes $\bar{x}$ to $\bar{t}^n/w$), the induced normalized p-extension given by the same equation is thus unramified over $(\bar{t} = 0)$. Hence the same is true for the induced normalized p-extension of $K[[\bar{t}]]$, where K is the algebraic closure of $k((w))$. But $K[[\bar{t}]]$ has no non-trivial unramified extensions. So the above extension of $K[[\bar{t}]]$ is trivial, and hence $\sum_{(i,j)\in I}(a_{ij}w^i)\bar{t}^{-ni-j}$ is of the form $g^p - g$ for some polynomial $g \in K[\bar{t}^{-1}] = K[t]$. Since $p^N \le ni+j < p^{N+1}$ for each $(i,j) \in I$, it follows that $g = 0$, and hence $f = 0$. Viewing $f = \sum_I a_{ij} w^i t^{ni+j}$ as a polynomial in t with coefficients in $k[w]$, we have for every non-negative integer d that the coefficient f_d of t^d must equal 0. Here

$$f_d = \sum_{(i,j)\in I_d} a_{ij} w^i, \qquad (*)$$

where $I_d = \{(i,j) \in I \mid ni + j = d\}$. But no terms in the sum $(*)$ can cancel, and each $a_{ij} \neq 0$; hence I_d is empty. This holds for all d, so I is empty: a contradiction. □

Section 5. The twice punctured line.

The purpose of this section is to prove Abhyankar's Conjecture 1.1 in the case of the curve $\mathbf{P}^1 - \{0,\infty\}$, over an algebraically closed field k of characteristic p. Moreover, we also prove Conjecture 1.2 in this case (cf. Theorem 5.4).

For the affine curve $C = \mathbf{P}^1_k - \{0,\infty\}$, Abhyankar's Conjecture asserts that the groups G that occur as Galois groups over C are precisely those such that $G/p(G)$ occurs over the analogous complex curve, $\mathbf{P}^1_{\mathbf{C}} - \{0,\infty\}$; i.e. such that $G/p(G)$ is cyclic of order prime to p. Since it is known that any group which does occur over $\mathbf{P}^1_k - \{0,\infty\}$ must be of this form, it suffices to show conversely that each group of this form must indeed occur as the Galois group of a Galois étale cover of $\mathbf{P}^1_k - \{0,\infty\}$.

We first prove a version of the result in a special case in which G is a semi-direct product (Proposition 5.2). The proof uses Raynaud's result [Ra] for $\mathbf{A}^1$; Propositions 4.1 and 4.2 above; and the patching and deformation results of section 2. See also the remarks at the beginning of section 4 and after the proof of 5.2, concerning the strategy. The proof begins with Lemma 5.1 below, which constructs a certain cyclic-by-p cover $B^* \to T^*$, in which the general fibre of T^* is a conic and the special fibre consists of two lines meeting at a point τ. (In the statement of 5.1, an auxiliary space L^* appears for reasons that may not be essential; cf. the remark after Proposition 2.1.)

Lemma 5.1. *Let H be a semi-direct product of a p-group P and a cyclic group C of order prime to p. Let $Y' \to \mathbf{A}^1$ be an irreducible P-Galois étale cover of the affine line. Let L be the projective u-line, let $L^* = L \times_k \mathrm{Spec}(k[[v]])$, and let $K = k((v))$. Then there* *is an irreducible normal cover $\phi^* : T^* \to L^*$ and an irreducible normal H-Galois cover $\chi^* : B^* \to T^*$ such that:*

(i) The closed fibre X of ϕ^* consists of two irreducible components X_1, X_2, each isomorphic to the projective line, and meeting at a single point $\tau \in X$;

(ii) The open fibre $T^{*o} = T^* \times_{\mathrm{Spec}(k[[v]])} \mathrm{Spec}(K)$ of ϕ^* is isomorphic to the projective s-line $\mathbf{P}^1_K$, and the closures of $(s = 0)$ and $(s = \infty)$ in T^* meet $X_2' = X_2 - \{\tau\}$;

(iii) The fibre of χ^* over $X_1' = X_1 - \{\tau\}$ is isomorphic to the H-Galois étale cover $\mathrm{Ind}_P^H Y' \to \mathbf{A}^1$, and P is the inertia group of χ^* at a point $\beta \in B^*$ lying over τ.

(iv) The open fibre $B^{*o} \to T^{*o}$ of χ^* is étale away from $(s = 0) \cup (s = \infty)$, and B^{*o} is K-smooth away from $(s = \infty)$. Moreover, C is an inertia group over $(s = 0)$ and H is the inertia group over $(s = \infty)$.

Proof. Let $Y \to \mathbf{P}^1$ be the normalization of $\mathbf{P}^1$ in Y'. By the observation just prior to Proposition 4.1, this is a smooth P-Galois cover which is totally ramified at a point η over the point $(x = \infty)$, and is elsewhere unramified.

By Proposition 4.1, we obtain a connected normal P-Galois branched cover $\pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ of (x, t)-space. Its restriction $\pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ is finite étale; the fibre of Z' over $\mathbf{A}^1 \times (t = 1)$ is isomorphic to $Y' \to \mathbf{A}^1$; π is totally ramified over $(x = \infty)$ and $(t = \infty)$; and the composition $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ is an H-Galois ramified cover of (x, s)-space (where $\theta : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ is given by $\theta(x, t) = (x, s)$, $s = t^n$). Moreover, this H-Galois cover is tamely ramified over $(s = 0)$ with inertia group C, it is totally ramified over $(s = \infty)$, and the inertia group over $(x = \infty)$ is P.

Write $\pi_\alpha : Z_\alpha \to \mathbf{P}^1$ for the fibre of $\pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ over $\mathbf{P}^1 \times (t = \alpha)$, and $\pi'_\alpha : Z'_\alpha \to \mathbf{A}^1$ for its restriction to $\mathbf{A}^1 \times (t = \alpha)$. Then for $\alpha \neq \infty$, Z_α is smooth except possibly over $(x = \infty)$, and the normalization of Z_1 is isomorphic (as a P-Galois cover of $\mathbf{P}^1 \times (t = 1)$) to Y. Letting ζ be the unique ramified point of Z_1, we have isomorphisms $\hat{\mathcal{K}}_{Z_1,\zeta} \approx \hat{\mathcal{K}}_{Y,\eta}$ of P-Galois $k((x^{-1}))$-algebras. Note that by Proposition 4.1(iv), P is the inertia group of $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ at ζ.

Now define a rational map $\phi_0 : \mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ by $(x, s) \mapsto (u, v)$, where $u = (s - 1 + x^{-1})^{-1}$ and $v = (s - 1)/x$. Thus ϕ_0 is defined except at the two points $\delta_1 : (x = 0, s = 1)$ and $\delta_2 : (x = s = \infty)$. Letting $\mathrm{pr}_2 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ be the second projection from (u, v)-space to the v-line, we obtain the composition $\mathrm{pr}_2 \circ \phi_0 : \mathbf{P}^1 \times \mathbf{P}^1 \dashrightarrow \mathbf{P}^1$ from (x, s)-space to the v-line, given by $v = (s - 1)/x$.

Let the surface T be the blow-up of (x, s)-space $\mathbf{P}^1 \times \mathbf{P}^1$ at δ_1 and δ_2, and let D be the exceptional locus over δ_2. We obtain a blow-up map $b : T \to \mathbf{P}^1 \times \mathbf{P}^1$ to (x, s)-space, and a morphism $\phi : T \to \mathbf{P}^1 \times \mathbf{P}^1$ to (u, v)-space, such that $\phi = \phi_0 \circ b$ as a rational map. Here ϕ is a branched covering map of degree 2, and the morphism $\mathrm{pr}_2 \circ \phi : T \to \mathbf{P}^1$ to the v-line is smooth except at the points $(x = \infty, s = 1)$ and $(x = 0, s = \infty)$. The H-Galois ramified cover $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ of (x, s)-space pulls back via the blow-up morphism b to a morphism $\chi_0 : \Sigma \to T$. Since $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ is smooth away from $(x = \infty) \cup (s = 0) \cup (s = \infty)$, so is the pullback $\chi_0 : \Sigma \to T$. Hence the composition $\mathrm{pr}_2 \circ \phi \circ \chi_0 : \Sigma \to \mathbf{P}^1$ is also smooth away from the locus of $(x = \infty) \cup (s = 0) \cup (s = \infty)$.

But $\mathrm{pr}_2 \circ \phi_0 \circ \theta : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ restricts to a well-defined smooth morphism over $(s = 0)$ (i.e. over $(t = 0)$), given by $v = -x^{-1}$. Since $\pi' : Z' \to \mathbf{A}^1 \times \mathbf{A}^1$ is étale over $(t = 0)$

and since the pullback $b' : \Sigma \to Z$ of $b : T \to \mathbf{P}^1 \times \mathbf{P}^1$ is the identity over $(s = 0)$, it follows that the composition $(\mathrm{pr}_2 \circ \phi_0 \circ \theta) \circ \pi \circ b' = \mathrm{pr}_2 \circ \phi \circ \chi_0 : \Sigma \to \mathbf{P}^1$ is smooth along $(s = 0)$, away from $(x = \infty)$. Hence $\Sigma \to \mathbf{P}^1$ is actually smooth away from $(x = \infty) \cup (s = \infty)$.

In T, the locus of $(x = \infty)$ is the union of D with the proper transform of $(x = \infty) \subset \mathbf{P}^1 \times \mathbf{P}^1$ under $b : T \to \mathbf{P}^1 \times \mathbf{P}^1$. The same statement holds with x replaced by s. Now the proper transform of $(x = \infty)$ lies over the locus of $(v = 0)$. Hence on the open set $(v \neq 0)$, $\mathrm{pr}_2 \circ \phi \circ \chi_0 : \Sigma \to \mathbf{P}^1$ is smooth away from (the total transform of) $(s = \infty)$, and $\Sigma \to T$ is étale away from $(s = 0) \cup (s = \infty)$ on this open set.

Let $K = \hat{\mathcal{K}}_{\mathbf{P}^1,(v=0)} = k((v))$, the fraction field of the complete local ring of the projective v-line $\mathbf{P}^1$ at the point $(v = 0)$. Let $T^{*o} \to \mathrm{Spec}(K)$ be the pullback of $\mathrm{pr}_2 \circ \phi : T \to \mathbf{P}^1$ under $\mathrm{Spec}(K) \to \mathbf{P}^1$, and let $\Sigma^{*o} \to \mathrm{Spec}(K)$ be the pullback of $\mathrm{pr}_2 \circ \phi \circ \chi_0 : \Sigma \to \mathbf{P}^1$ under $\mathrm{Spec}(K) \to \mathbf{P}^1$. By the previous paragraph, it follows that the pullback morphism $\Sigma^{*o} \to \mathrm{Spec}(K)$ is smooth away from $(s = \infty)$. Also, C is an inertia group for the cover $\Sigma^{*o} \to T^{*o}$ over $(s = 0)$, since C is an inertia group for $\theta \circ \pi : Z \to \mathbf{P}^1 \times \mathbf{P}^1$ there.

Let B be the normalization of Σ, let $\nu : B \to \Sigma$ be the canonical morphism, let $\chi = \chi_0 \circ \nu : B \to T$, and let $\beta \in B$ be the unique point lying over $\zeta \in Z$. Thus P is the inertia group of $\chi : B \to T$ at β. Applying Proposition 4.2 (with B and T playing the roles of Z' and S' there), we find that $B \to T$ is totally ramified over D, and thus the inertia group there is H. Let B^{*o} be the pullback of B to $\mathrm{Spec}(K)$. Thus H is the inertia group of $B^{*o} \to T^{*o}$ over $(s = \infty)$, and B^{*o} agrees with Σ^{*o} away from $(s = \infty)$. In particular, B^{*o} is K-smooth away from $(s = \infty)$, and C is an inertia group over $(s = 0)$ for the cover $B^{*o} \to T^{*o}$. Moreover, $B^{*o} \to T^{*o}$ is étale away from $(s = 0) \cup (s = \infty)$. This shows (iv).

Identify the projective u-line L with the u-axis $(v = 0)$ in (u, v)-space. Its inverse image $\phi^{-1}(L) \subset T$ is the union of the proper transforms X_1, X_2 of the lines $(s = 1)$ and $(x = \infty)$ under the blow-up map b. Here X_1 and X_2 meet at a point τ lying over the point $\lambda : (u = \infty, v = 0)$ on L. Thus (i) holds.

We may regard L^* as the formal completion of (u, v)-space $\mathbf{P}^1 \times \mathbf{P}^1$ along the projective line L. Thus there is an "inclusion" morphism $L^* \to \mathbf{P}^1 \times \mathbf{P}^1$. The pullback of $\phi : T \to \mathbf{P}^1 \times \mathbf{P}^1$ under $L^* \to \mathbf{P}^1 \times \mathbf{P}^1$ is a degree 2 branched covering morphism $\phi^* : T^* \to L^*$, where T^* is the formal completion of T at $X = \phi^{-1}(L)$. Let $\chi^* : B^* \to T^*$ be the pullback of $\chi : B \to T$. Thus the second assertion in (iii) holds, since it holds for χ.

Now let L^{*o} be the generic fibre of L^*, i.e. $L^{*o} = L \times_k \mathrm{Spec}(K) = L^* - (v = 0)$. Note that T^{*o} is the inverse image of L^{*o} under $\phi^* : T^* \to L^*$. Thus T^{*o} is the projective curve over $K = k((v))$ given as a branched cover of the u-line by the equation $Y^2 - u^{-1}Y + v = 0$ (whose roots are $s - 1$ and x^{-1}). Identifying Y with $s - 1$, the function field of T^{*o} is $K(Y) = K(s)$, with $u^{-1} = Y + v/Y = (s - 1) + v/(s - 1)$. So this curve is rational, and indeed is isomorphic to the projective s-line over K, showing the first part of (ii). Since $T^* \to \mathrm{Spec}(k[[v]])$ is proper, the closures of the K-points $(s = 0)$ and $(s = \infty)$ of T^{*o} must meet the closed fibre $X \subset T^*$. But X_1 lies in the locus of $(s = 1)$. So the loci of $(s = 0)$ and $(s = \infty)$ in T^* cannot meet X_1, and so must meet X_2'. This proves the second part of (ii).

Letting ζ_n be a primitive nth root of unity in k, and identifying the affine x-line with $X_1' \subset T$, the fibre of χ^* over X_1' becomes identified with the disjoint union of the P-Galois étale covers $\pi'_{\zeta_n^i} : Z'_{\zeta_n^i} \to \mathbf{A}^1$ of the x-line, as i ranges over $i = 0, \ldots, n-1$ (since the ζ_n^i are the values of t at which $s = 1$). This union is in turn isomorphic to the étale cover $\mathrm{Ind}_P^H Y' \to \mathbf{A}^1$, since the cover Y' is isomorphic to Z'_1. This proves the first assertion in (iii), and concludes the proof. []

Proposition 5.2. *Let Q be a quasi-p group, let C be a cyclic group of order prime to p, and let G be a semi-direct product of Q with C. Suppose that C is contained in the normalizer of some Sylow p-subgroup $P \subset G$. Then there is a smooth connected G-Galois cover of $\mathbf{P}^1$, branched only over the points $(s = 0)$ and $(s = \infty)$, such that the inertia groups over $(s = 0)$ are the conjugates of C.*

Proof. Since Q is a quasi-p-group, by [Ra] there is a smooth connected Q-Galois cover $W \to \mathbf{P}^1$ of the projective x-line, branched only at the point $(x = \infty)$, and whose inertia groups are the Sylow p-subgroups of Q. Let ω be a ramified point of W with inertia group P. Now the complete local ring of $\mathbf{P}^1$ at $(x = \infty)$ is $k[[x^{-1}]]$, with fraction field $k((x^{-1}))$, and $\hat{\mathcal{K}}_{W,\omega}$ is a P-Galois field extension of $k((x^{-1}))$. So by [Ha1,Cor.2.4], there is a unique P-Galois étale cover $Y' \to \mathbf{A}^1$ of the affine x-line which induces $\hat{\mathcal{K}}_{W,\omega}$ over $k((x^{-1}))$. Let $Y \to \mathbf{P}^1$ be the normalization of $\mathbf{P}^1$ in Y'. By the observation just before Proposition 4.1, this is a smooth P-Galois cover having a unique ramified point η, lying over the point $(x = \infty)$. By construction, we have an isomorphism $\hat{\mathcal{K}}_{Y,\eta} \approx \hat{\mathcal{K}}_{W,\omega}$ of P-Galois $k((x^{-1}))$-algebras.

Let $H \subset G$ be the subgroup generated by P and C. Since $C \subset N_G(P)$ by hypothesis, H is a semi-direct product of P with C. Taking L to be the projective u-line and $L^* = L \times_k \mathrm{Spec}(k[[v]])$, Lemma 5.1 yields an irreducible normal cover $\phi^* : T^* \to L^*$ and an irreducible normal H-Galois cover $\chi^* : B^* \to T^*$ satisfying (i) - (iv) of that result. In the notation of Lemma 5.1 (i) and (iii), we may write $X'_j = \mathrm{Spec}(R_j)$, where $R_1 = k[x]$ and $R_2 = k[y]$. For $j = 1, 2$ let X'^*_j and $\hat{X}'^*_j$ be the formal completions of T^* along X'_j and at $\mathrm{Spec}(\hat{\mathcal{K}}_{X_j,\tau})$, respectively.

For $j = 1, 2$, we obtain pullbacks $\chi_j : B'^*_j \to X'^*_j$, which are H-Galois ramified covers. Here B'^*_2 is normal since B^* is. Now by 5.1(ii) and (iv), the H-Galois cover $\chi_2 : B'^*_2 \to X'^*_2$ it totally ramified over the point at which $(s = \infty)$ meets X'_2. So since X'^*_2 is irreducible, it follows that B'^*_2 is also. Meanwhile, pulling back B^* to $\hat{T}^* = \mathrm{Spec}(\hat{\mathcal{O}}_{T^*,\tau})$ yields the H-Galois cover $\mathrm{Ind}_P^H \hat{B}^* \to \hat{T}^*$, where $\hat{B}^* = \mathrm{Spec}(\hat{\mathcal{O}}_{B^*,\beta})$ is a P-Galois cover of $\hat{T}^*$. Let $\hat{\chi}_j : \hat{B}^*_j \to \hat{X}'^*_j$ be the pullback of $\hat{B}^*$ to $\hat{X}'^*_j$. Thus we have an isomorphism $B'^*_j \times_{X'^*_j} \hat{X}'^*_j \approx \mathrm{Ind}_P^H \hat{B}^*_j$ of H-Galois covers of $\hat{X}'^*_j$. By (iii) of Lemma 5.1, the closed fibre of χ_1 is isomorphic to the étale cover $\mathrm{Ind}_P^H Y' \to \mathbf{A}^1$. So χ_1 and hence $\hat{\chi}_1$ are also étale, and thus are trivial deformations of their closed fibres. In particular, the P-Galois cover $\hat{B}^*_1$ is a trivial deformation of $\mathrm{Spec}(\hat{\mathcal{K}}_{Y,\eta})$.

Meanwhile, the Q-Galois ramified cover $W \to \mathbf{P}^1$ of the projective x-line restricts to a Q-Galois étale cover $W' \to \mathbf{A}^1 = X'_1$, where we identify the affine x-line with $X'_1 \subset T^*$. Letting $\hat{W}' = \mathrm{Spec}(\hat{\mathcal{K}}_{W,\omega})$, which is a P-Galois cover of $\hat{X}'_1 = \mathrm{Spec}(k((x^{-1})))$, we have an isomorphism $W' \times_{\mathbf{A}^1} \mathrm{Spec}(k((x^{-1}))) \approx \mathrm{Ind}_P^Q \hat{W}' = \mathrm{Spec}(\mathrm{Ind}_P^Q \hat{\mathcal{K}}_{W,\omega}) \approx \mathrm{Spec}(\mathrm{Ind}_P^Q \hat{\mathcal{K}}_{Y,\eta})$. Pulling back W' and $\hat{W}'$ by the morphisms $X'^*_1 \to X'_1$ and $\hat{X}'^*_1 \to \hat{X}'_1$ corresponding to the inclusions of rings, we obtain a Q-Galois étale cover $W'^* \to X_1'^*$ and a P-Galois étale cover $\hat{W}'^* \to \hat{X}_1'^*$; these are trivial deformations of $W' \to \mathbf{A}^1$ and $\hat{W}' \to \mathrm{Spec}(k((x^{-1})))$, respectively. Now $\hat{B}_1'^* \to \hat{X}_1'^*$ is a trivial deformation of $\mathrm{Spec}(\hat{\mathcal{K}}_{Y,\eta}) \to \mathrm{Spec}(k((x^{-1})))$, and we have an isomorphism $\hat{W}' = \mathrm{Spec}(\hat{\mathcal{K}}_{W,\omega}) \approx \mathrm{Spec}(\hat{\mathcal{K}}_{Y,\eta})$. This yields an isomorphism from $\hat{W}'^*$ to $\hat{B}_1^*$, as P-Galois étale covers of $\hat{X}_1'^*$. Since $\hat{B}^* \times_{\hat{T}^*} \hat{X}_j'^* \approx \hat{B}_j^*$, we obtain an isomorphism $\hat{B}^* \times_{\hat{T}^*} \hat{X}_1'^* \approx \hat{W}'^*$. Note that W'^* is regular since W' is, and that $B_2'^*$ and $\hat{B}^*$ are normal since B^* is.

We thus can apply Proposition 2.3, with $G_1 = Q$, $G_2 = H$, $I = I_1 = I_2 = P$, $W_1'^* = W'^*$, $W_2'^* = B_2'^*$, and $\hat{N}^* = \hat{B}^*$. This yields an irreducible normal G-Galois cover $V^* \to T^*$ such that $V^* \times_{T^*} X_1'^* \approx \mathrm{Ind}_Q^G W'^*$ and $V^* \times_{T^*} X_2'^* \approx \mathrm{Ind}_H^G B_2'^*$ as G-Galois covers of $X_1'^*$ and $X_2'^*$, and $V^* \times_{T^*} \hat{T}^* \approx \mathrm{Ind}_P^G \hat{B}^*$ as G-Galois covers of $\hat{T}^*$. Since its branching agrees with W'^*, $B_2'^*$, and $\hat{B}^*$ respectively over $X_1'^*$, $X_2'^*$, and $\hat{T}^*$, it follows by Lemma 5.1(iv) that its generic fibre is branched only at $(s = 0) \cup (s = \infty)$. Moreover C is an inertia group at $(s = 0)$, and H is an inertia group over $(s = \infty)$, by Lemma 5.1 (ii) and (iv). And since B^{*o} is K-smooth away from $(s = \infty)$ (where $K = k((v))$), so is the generic fibre V^{*o} of V^*.

Now V^{o*} is a dense open subset of V^* having dimension one; hence it is irreducible and regular. The G-Galois branched cover $V^{*o} \to T^{*o} = \mathbf{P}^1_K$ is branched at the points $(s = 0)$ and $(s = \infty)$, with C and H respectively occuring as inertia groups, and V^{*o} is smooth over $(s = 0)$. The conjugates of H contain the conjugates of P, and the conjugates of P (being the Sylow p-subgroups of G) generate Q. Since Q and C generate G, it follows that the conjugates of H and C generate G. So the result follows from Corollary 2.7, under case (i) of Proposition 2.6. []

Remark. The strategy in Proposition 5.2 began with a quasi-p cover of the affine line (from [Ra]), and found (via 4.1 and 5.1) a cyclic-by-p cover of $\mathbf{P}^1_{k((x^{-1}))}$ branched at $\{0, \infty\}$ and tame over $(s = 0)$, whose behavior at $(s = 1)$ allowed it to be patched to the quasi-p cover at its branch point. This patching used formal geometry. M. Raynaud has observed to the author that this strategy can also be carried out with rigid geometry, using ideas from [Ra]. This requires a cyclic-by-p cover of the $k((v))$-line with branching as above, agreeing with a given p-cover over an annulus centered at $(s = 1)$. The existence of such a cover follows from a variant of [Ra, Cor. 4.2.6]. There, from a cover $V \to U$ and a lifting $\mathcal{V}' \to \mathcal{U}$ of a corresponding rigid cover $\mathcal{V} \to \mathcal{U}$, one obtains a lift $V' \to U$ of V having enlarged group. In the variant of [Ra, Cor. 4.2.6], $V \to U$ is allowed tame ramification over a given finite set S, and $\mathcal{V}'$ is required to be unramified only over $\mathcal{V}$ (not $\mathcal{U}$); the conclusion is that $V' \to U$ exists and V' is unramified over V (rather than over U). This applies to the present situation by taking $X = \mathbf{P}^1_K$; $U = X - \{1, \infty\}$; $S = \{0\}$; $V \to U$ the n-cyclic cover branched at $\{0, \infty\}$; $\mathcal{U}$ the boundary of a disc centered at $(s = 1)$; and $\mathcal{V}' \to \mathcal{U}$ the disconnected cyclic-by-p cover induced by the p-cover. The above variant of [Ra, Cor. 4.2.6] itself follows from a variant of [Ra, Prop. 4.2.5] with π_1 replaced by a suitable tame fundamental group.

Before stating the main result of this section, we prove a group-theoretic lemma concerning the quasi-p part $p(G)$ of a finite group G:

Lemma 5.3. *Let G be a finite group, let $Q = p(G)$, and let $\pi : G \to G/Q$ be the natural quotient map. Let P be a Sylow p-subgroup of G, and let $G' = N_G(P)$. Then G' contains a subgroup F having order prime to p, such that $\pi(F) = G/Q$.*

Proof. Let $C = G/Q$. We claim that $\pi(G') = C$. To see this, let $c \in C$; we will show that $\pi(g') = c$ for some $g' \in G'$. By definition of C, we know that there is a $g \in G$ such that $\pi(g) = c$. Since gPg^{-1} is a Sylow p-subgroup of G, it follows that $gPg^{-1} \subset Q = p(G)$, and hence that gPg^{-1} is a Sylow p-subgroup of Q. Since P (being a Sylow p-subgroup of G) is also a Sylow p-subgroup of $Q = p(G)$, there must be an element $q \in Q \subset G$ such that $q(gPg^{-1})q^{-1} = P$. Let $g' = qg$. Thus $g'Pg'^{-1} = P$, and so $g' \in N_G(P) = G'$. Also, $\pi(g') = \pi(q)\pi(g) = c$. So g' is as desired, proving the claim.

Let $Q' = N_Q(P) = G' \cap Q$. Paralleling the proof of [Se1,I,Prop.45], we have

$$1 \to Q' \to G' \to C \to 1, \tag{1}$$

$$1 \to P \to G' \to G'/P \to 1, \tag{2}$$

$$1 \to Q'/P \to G'/P \to C \to 1. \tag{3}$$

Here (2) splits since $\#P$ is a power of p and $\#(G'/P)$ is prime to p. So G' contains a subgroup F of order prime to p which maps isomorphically to G'/P in (2). Since the map $G'/P \to C$ in (3) is surjective, it follows that F maps onto C in (1); i.e. $\pi(F) = C$. []

Theorem 5.4. *Conjectures 1.1 and 1.2 hold in the case of $g = 0, r = 1$, i.e. for étale covers of $\mathbf{P}^1 - \{0, \infty\}$.*

Proof. In this situation, Conjecture 1.1 asserts that if G is a finite group, then there is a smooth connected G-Galois cover of $\mathbf{P}^1$, branched only at the points $(s = 0)$ and $(s = \infty)$, if and only if $G/p(G)$ is cyclic. Conjecture 1.2 asserts that for such groups G, the cover may be chosen so as to be tamely ramified over one of these points (which we may take to be the point $(s = 0)$). One direction of 1.1, viz. that if there is a cover then $G/p(G)$ is cyclic, is a consequence of [Gr, XIII, Cor.2.12]. The other direction is subsumed by 1.2, so it remains to show that if $G/p(G)$ is cyclic then there is a cover with the properties asserted in 1.2.

Let $Q = p(G)$, and let $C = G/p(G)$. Thus Q is a quasi-p-group. Let P be a Sylow p-subgroup of G (and hence of Q), and let c be a generator of C. By Lemma 5.3, $N_G(P)$ contains a subgroup F of order prime to p, mapping surjectively to C under $G \to C$. Let $c' \in F$ be an element lying over $c \in C$, and let C' be the cyclic subgroup of G generated by c'. Since Q is normal in G, the subgroup C' acts on Q by conjugation, and we may form the corresponding semidirect product G', of Q with C'. Since $C' \subset N_G(P)$, C' acts on P by conjugation, and the subgroup H of G' generated by P and C' is the semidirect product of these two groups. Since the action of C' on Q in G' is the same as the action of C' on Q in G, we have that $C' \subset N_{G'}(P)$. So by Proposition 5.2, there is a smooth connected G'-Galois cover $Y' \to \mathbf{P}^1$, branched only over the points $(s = 0)$ and $(s = \infty)$, such that the inertia groups over $(s = 0)$ are the conjugates of C'.

Now there is a surjective homomorphism $G' \to G$ defined by mapping Q to itself by the identity map, and mapping $C' \to C$ by taking $c' \mapsto c$. Let $N \subset G'$ be the kernel. Thus G is isomorphic to G'/N, and the cover $Y = Y'/N$ of $\mathbf{P}^1$ is as desired. []

The results of this section will be used to prove the general cases of the conjectures, in section 6.

Section 6. The general case.

This section proves the general case of Abhyankar's Conjecture on covers of affine curves over an algebraically closed field k of characteristic p (Conjecture 1.1). Namely, a group G is a Galois group over a smooth k-curve of genus g with $r + 1$ points deleted if and only if $G/p(G)$ is a quotient of the group $\Gamma_{g,r}$ generated by elements $a_1, \ldots, a_g, b_1, \ldots, b_g$, $c_0, \ldots, c_r$ subject to the relation $\prod_{j=1}^{g}[a_j, b_j] \prod_{i=0}^{r} c_i = 1$. In fact, we prove the stronger version of Abhyankar's Conjecture, viz. Conjecture 1.2.

As in section 5, we begin with a version in the semi-direct case:

Proposition 6.1. *Let Q be a quasi-p group, let F be a finite quotient of $\Gamma_{g,r}$ having order prime to p, and let G be a semi-direct product of Q with F. For $i = 0, \ldots, r$ let $\bar{c}_i \in F$ be the image of $c_i \in \Gamma_{g,r}$, and let $C_i \subset F$ be the cyclic subgroup generated by $\bar{c}_i$. Suppose that F is contained in the normalizer of some Sylow p-subgroup $P \subset G$. If X is a smooth connected projective k-curve of genus g and $\xi_0, \ldots, \xi_r \in X$ are distinct, then there is a smooth connected G-Galois cover of X, branched only over the points ξ_i and tamely ramified except over ξ_0, such that the inertia groups over ξ_i are the conjugates of C_i for $i > 0$, and such that the maximal prime-to-p quotients of the inertia groups over ξ_0 are the conjugates of C_0 in $F = G/Q$.*

Proof. Since F is a finite quotient of $\Gamma_{g,r}$ of order prime to p, it follows by [Gr, XIII, Cor. 2.12] that there is a smooth connected F-Galois branched cover $U \to X$ which is branched only over the points ξ_i, and with the inertia group at a point μ_i over ξ_i generated by $\bar{c}_i$ $(i = 0, \ldots, r)$. Write $c = \bar{c}_0$, $C = C_0$, and $\mu = \mu_0$. Let $x \in \mathcal{O}_{X,\xi_0}$ be a local uniformizer of X at ξ_0. Then there is a local uniformizer $u \in \hat{\mathcal{O}}_{U,\mu}$ of U at μ such that $u^n = x$. So $c(u) = \zeta_n u$ for some primitive nth root of unity $\zeta_n \in k$.

Let E be the subgroup of G generated by Q and C; this is the semidirect product of these two subgroups. By Proposition 5.2, there is a smooth connected E-Galois branched cover $W \to Y$ of the projective y-line Y over k, branched only at $(y = 0)$ and $(y = \infty)$, whose ramification over $(y = 0)$ is tame. Moreover, by that result, we may assume that the inertia group of $W \to Y$ at some point $\omega \in W$ over $(y = 0)$ is equal to C. Thus $W/Q \to Y$ is a C-Galois cover, branched only at the points $(y = 0)$ and $(y = \infty)$, and hence totally ramified there. So if H is the inertia group of $W \to Y$ at a point over $(y = \infty)$, then H is a cyclic-by-p group and the natural map $H/p(H) \to E/Q$ is an isomorphism to $C = E/Q \subset F = G/Q$.

In the complete local ring of W at ω, there is a local uniformizer w such that $w^n = y$. Thus $c(w) = \zeta_n^{-j} w$ for some integer j, which may be chosen such that $1 \le j < n$. Pulling back the cover $W \to Y$ by $y'^j = y$ and then normalizing, we are reduced to the case that $j = 1$. So we may assume that $c(w) = \zeta_n^{-1} w$.

Now let S be the blow-up of $X \times \mathbf{P}^1$ at the point $(\xi_0, (s = 0))$ (where the second factor is the s-line over k), and let $\sigma \in S$ be the point at which the exceptional divisor meets the proper transform of $(s = 0) \subset X \times \mathbf{P}^1$. Then S is a regular subvariety of $X \times \mathbf{P}^1 \times Y$, where as before Y is the projective y-line over k, and where S is given in a neighborhood of σ by $s = xy$. Thus $\sigma \in X \times \mathbf{P}^1 \times Y$ is the point $(\xi_0, (s = 0), \eta)$, where η is the point $(y = 0)$ on Y. Next, define $\theta : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ from (x, t)-space to (x, s)-space, by $\theta(x, t) = (x, s)$ where $s = t^n$. Let $T \to \mathbf{P}^1 \times \mathbf{P}^1$ be the pullback of $S \to \mathbf{P}^1 \times \mathbf{P}^1$ with respect to the morphism θ, and let $\tau \in T$ be the inverse image of σ under $T \to S$. Thus $T \subset X \times \mathbf{P}^1 \times Y$ is given locally near τ by $t^n = xy$ (where the second factor is the projective t-line), and is elsewhere regular. So T is a normal variety.

The blow-up S contains a copy of X, viz. the proper transform of $(s = 0) \subset X \times \mathbf{P}^1$. Also, S contains a copy of Y, viz. the exceptional locus of the blow-up map $S \to X \times \mathbf{P}^1$. The pullbacks of these copies under $T \to S$ are copies of X and Y in T, and we will identify these with X and Y, respectively. Note that these curves in T intersect only at τ, where $\tau \in T$ is identified with $\xi_0 \in X$ and with $\eta \in Y$.

Since $T \subset X \times \mathbf{P}^1 \times Y$, the rational functions x, t, y on T define morphisms $T \to \mathbf{P}^1$. Let $z = x + y$. Then we obtain a morphism $\Phi = (z, t) : T \to \mathbf{P}^1 \times \mathbf{P}^1$ to (z, t)-space. This morphism is finite and generically separable, and the fibre over $(t = 0)$ is the union of X and Y in T (under the identifications made in the previous paragraph), meeting at τ.

Observe that the complete local ring of T at τ has a degree n extension which is ramified precisely over the closed point. Namely, $\hat{\mathcal{O}}_{T,\tau} = k[[x, y]][t]/(t^n - xy)$, and this is a subring of $k[[u, w]]$ via the inclusion $x \mapsto u^n$, $y \mapsto w^n$, $t \mapsto uw$. The extension $\hat{\mathcal{O}}_{T,\tau} \subset k[[u, w]]$ is C-Galois with respect to the action given by $c(u) = \zeta_n u$ and $c(w) = \zeta_n^{-1} w$. It corresponds to the C-Galois cover $\hat{N}^* \to \hat{T}^*$, where $\hat{N}^* = \operatorname{Spec}(k[[u, w]])$ and $\hat{T}^* = \operatorname{Spec}(\hat{\mathcal{O}}_{T,\tau})$. Here $\hat{N}^*$ is the completion of the $k[t]$-scheme $N^* = \operatorname{Spec}(k[t, u, w]/(uw - t))$ at the closed point $(t = u = w = 0)$.

Recall that the C-Galois extension $\hat{\mathcal{O}}_{X,\xi_0} \subset \hat{\mathcal{O}}_{U,\mu}$ is given by $u^n = x$ and $c(u) = \zeta_n u$, and that the C-Galois extension $\hat{\mathcal{O}}_{Y,\eta} \subset \hat{\mathcal{O}}_{W,\omega}$ is given by $w^n = y$ and $c(w) = \zeta_n^{-1} w$. So viewing $\hat{\mathcal{K}}_{X,\xi_0}[[t]] = k((x))[[t]] = k((x))[[y]][t]/(t^n - xy)$ and $\hat{\mathcal{K}}_{Y,\eta}[[t]] = k((y))[[t]] = k((y))[[x]][t]/(t^n - xy)$ as algebras over $\hat{\mathcal{O}}_{T,\tau}$, we obtain C-Galois algebra isomorphisms

$$k[[u, w]] \otimes_{\hat{\mathcal{O}}_{T,\tau}} \hat{\mathcal{K}}_{X,\xi_0}[[t]] \tilde{\to} \hat{\mathcal{K}}_{U,\mu}[[t]] \tag{1}$$

and

$$k[[u, w]] \otimes_{\hat{\mathcal{O}}_{T,\tau}} \hat{\mathcal{K}}_{Y,\eta}[[t]] \tilde{\to} \hat{\mathcal{K}}_{W,\omega}[[t]] \tag{2}$$

over $\hat{\mathcal{K}}_{X,\xi_0}[[t]]$ and $\hat{\mathcal{K}}_{Y,\eta}[[t]]$ respectively.

Let L be the projective z-line over k, let $\lambda \in L$ be the point $(z = 0)$, let L^* be the projective z-line over $k[[t]]$, let $L' = L - \{\lambda\} = \operatorname{Spec}(k[z^{-1}])$, and let $L'^* = \operatorname{Spec}(k[z^{-1}][[t]])$. Let $\phi : T^* \to L^*$ be the pullback of Φ under $L^* \to L \times \mathbf{P}^1$. Thus T^* is the completion of

T along the locus of $(t = 0)$. Also, $\hat{T}^* = \mathrm{Spec}(\hat{\mathcal{O}}_{T^*,\tau})$. By construction, the closed fibre of ϕ is the union of X and Y, which meet only at τ. Let $K = k((t))$, let $X^o = X \times_k K$, and let T^o be the generic fibre of $T^* \to \mathrm{Spec}(k[[t]])$.

Let $X' = X - \{\xi_0\}$; let R_1 the ring of functions on X'; let $X'^* = \mathrm{Spec}(R_1[[t]])$; and let $\hat{X}'^* = \mathrm{Spec}(\hat{\mathcal{K}}_{X,\xi_0}[[t]])$. Also, let U' be the inverse image of X' in U; $U'^* = U' \times_{X'} X'^*$; and $\hat{U}'^* = \mathrm{Spec}(\hat{\mathcal{K}}_{U,\mu}[[t]])$. Similarly, let $Y' = Y - \{\eta\} = \mathrm{Spec}(R_2)$, where $R_2 = k[y^{-1}]$; let $Y'^* = \mathrm{Spec}(R_2[[t]])$; and $\hat{Y}'^* = \mathrm{Spec}(\hat{\mathcal{K}}_{Y,\eta}[[t]])$. Also, let W' be the inverse image of Y' in W; $W'^* = W' \times_{Y'} Y'^*$; and $\hat{W}'^* = \mathrm{Spec}(\hat{\mathcal{K}}_{W,\omega}[[t]])$.

By (1) and (2), we can apply Proposition 2.3, with $G_1 = F$, $G_2 = E$, $I = I_1 = I_2 = C$, $v = t$, $X_1 = X$, $X_2 = Y$, $W_1'^* = U'^*$, $W_2'^* = W'^*$, and $\hat{N}^* = \mathrm{Spec}(k[[u, w]])$. This yields an irreducible normal G-Galois cover $\psi : V^* \to T^*$ such that $V^* \times_{T^*} X'^* \approx \mathrm{Ind}_F^G U'^*$, $V^* \times_{T^*} Y'^* \approx \mathrm{Ind}_E^G W'^*$, and $V^* \times_{T^*} \hat{T}^* \approx \mathrm{Ind}_C^G \hat{N}^*$ as G-Galois covers of $X'^*, Y'^*, \hat{T}^*$ respectively.

The branching of $\psi : V^* \to T^*$ is determined by that of its patches. Hence it is tamely ramified over the loci of $\xi_i^* = \xi_i \times_k \mathrm{Spec}(k[[t]])$ for $i > 0$, and C_i is an inertia group over ξ_i^*. It is also ramified over the locus of $(y = \infty)$, which is the pullback under $T^* \to T \to S$ of the proper transform of $\xi_0^* = \xi_0 \times_k \mathrm{Spec}(k[[t]])$ under the blow-up map $S \to X \times \mathbf{P}^1$; and one of the inertia groups there is H. In addition, it is ramified over the singular point $\tau \in T$. Otherwise, $V^* \to T^*$ is unramified. Since the closed fibre of T^* is generically smooth, it follows that the closed fibre of V^* is also generically smooth.

Now by construction of T, the generic fibre T^{*o} of $T^* \to \mathrm{Spec}(K)$ is isomorphic to X^o. Identifying T^{*o} with X^o, we find that the restriction $\psi^o : V^{*o} \to X^o$ of the morphism ψ to the generic fibre is ramified precisely over the points $\xi_i^o = \xi_i \times_k \mathrm{Spec}(K)$ of X^o. Here the inertia groups over ξ_i^o are the conjugates of C_i, for all $i > 0$; and the inertia groups over ξ_0^o are the conjugates of the subgroup H. Also, since $U \times_k \mathrm{Spec}(K)$, $W \times_k \mathrm{Spec}(K)$, and $N^* \times_{k[t]} \mathrm{Spec}(K)$ are smooth over K, it follows that V^{*o} is also smooth over K. Thus V^{*o} is regular. The result now follows from Corollary 2.7, under case (ii) of Proposition 2.6. []

Theorem 6.2. *Conjectures 1.1 and 1.2 hold in general.*

Proof. Half of 1.1 (viz. the fact that $G/p(G)$ is a quotient of $\Gamma_{g,r}$) follows from [Gr, XIII, Cor.2.12], and the other half is subsumed by 1.2. So it suffices to prove 1.2.

Let $Q = p(G)$, let $\pi : G \to G/Q$ be the quotient map, and let P be a Sylow p-subgroup of G. By Lemma 5.3, there is a prime-to-p subgroup F of $N_G(P)$ such that $\pi(F) = G/Q$. Since $\Gamma_{g,r}$ is free, the surjective homomorphism $\Gamma_{g,r} \to G/Q$ lifts to a homomorphism $\Gamma_{g,r} \to F$. After shrinking F, we may assume that $\Gamma_{g,r} \to F$ is surjective.

Now G is generated by F and Q, since $\pi(F) = G/Q$. Since Q is normal in G, F acts on Q by conjugation. Let G' be the semidirect product of Q and F with respect to this action. Thus (as in the proof of 5.4), G is a quotient of G', say $G = G'/N$, and $F \subset N_{G'}(P)$. So by Proposition 6.1, there is a smooth connected G'-Galois cover $Y' \to X$, branched only over the points ξ_i, and tamely ramified except over ξ_0. Thus the cover $Y = Y'/N$ of X is as desired. []

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