RIEMANN'S EXISTENCE THEOREM

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Abstract. This paper provides a brief survey of Riemann's Existence Theorem from the perspective of its history, its connections to various areas of mathematics, its generalizations, and open problems.

1. Overview

Riemann's Existence Theorem is a foundational result that has connections to complex analysis, topology, algebraic geometry, and number theory. It arose as part of Riemann's groundbreaking work on what we now call Riemann surfaces. The theorem itself was for a while controversial, and decades passed before there was a precise statement or proof. But the process of making it precise itself led to developments of great consequence.

The term *Riemann's Existence Theorem* is in fact used in more than one way, to refer to several related but distinct assertions. From an analytic point of view, it concerns meromorphic functions on Riemann surfaces. In one form, it states that on any Riemann surface X (i.e. one-dimensional complex manifold) there exists a non-constant meromorphic function, and moreover meromorphic functions exist that separate any two given points. Such a function represents X as a branched cover of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, whose branch locus is a finite subset S of $\mathbb{P}^1_{\mathbb{C}}$. The existence of this covering map shows that the field of meromorphic functions on X is a finite algebraic extension of the function field $\mathbb{C}(x)$ of $\mathbb{P}^1_{\mathbb{C}}$; and therefore X can be viewed as a complex algebraic curve.

Modern proofs appear, for example, in [Völ96, Chapter 6] and in [Nar92] (Chapter 7, Theorems 3(b) and 4), using local-to-global methods in the complex metric topology. Another approach is to use Serre's result GAGA ([Ser55]), which makes it possible to pass between the complex analytic and algebraic settings in the case of compact (or projective) spaces. More specifically, that result gives an equivalence between sheaves in the Zariski and metric topologies that preserves cohomology, and therefore an equivalence between algebraic and analytic covers. For a more detailed discussion, see [Har03, Section 2].

A related assertion is that given a finite subset $S \subset \mathbb{P}^1_{\mathbb{C}}$, and given a permutation representation of the fundamental group of the complement of S, there is a Riemann surface X and a holomorphic map $X \to \mathbb{P}^1_{\mathbb{C}}$ which away from S is a covering space having this representation as its monodromy. This statement, together with the supplementary assertion that these representations classify the Riemann surfaces that arise as such covers, is also often referred to as Riemann's Existence Theorem. The theorem is also often stated in terms of branched covers of more general Riemann surfaces than $\mathbb{P}^1_{\mathbb{C}}$, and is sometimes stated in the special case of covers that are normal (i.e. Galois).

In this form, Riemann's Existence Theorem can be viewed as an assertion relating topological objects to algebraic ones. Given a non-empty finite subset $S = \{P_1, \ldots, P_n\}$ of $\mathbb{P}^1_{\mathbb{C}}$, every topological covering space of the complement U gives rise to a Riemann surface structure, which can be completed to a branched cover Y of $\mathbb{P}^1_{\mathbb{C}}$ with branch locus S; and this in turn has the structure of a projective algebraic curve. Since $\pi_1(U) = \langle \sigma_1, \dots, \sigma_n \mid \prod \sigma_i = 1 \rangle$, the algebraic curves that are finite and Galois over $\mathbb{P}^1_{\mathbb{C}}$ and branched only over S are in bijection with branch cycle data, i.e. equivalence classes of n-tuples $(g_1, \dots, g_n) \in G^n$ such that the g_i generate G and satisfy $\prod g_i = 1$. Here the equivalence relation is conjugation by an element of G, corresponding to a choice of base point on Y (e.g. see [Völ96, Theorem 4.32]). This bijection is compatible with taking intermediate covers.

The above bijection is not canonical, however, since it depends on a choice of homotopy basis $\{\sigma_i\}$ as above. But for any choice such that σ_i winds once, counterclockwise, around P_i and around no other P_j, the cover $Y \to \mathbb{P}^1_{\mathbb{C}}$ that corresponds to the equivalence class of $(g_1, \dots, g_n)$ has the property that g_i is the standard generator of inertia at some point Q_i of Y over P_i. That is, if x_i, y_i are uniformizers at P_i, Q_i such that $y_i^{e_i} = x_i$ where e_i is the ramification index over P_i, then $g_i(y_i) = \zeta_i y_i$ where $\zeta_i = e^{2\pi i/n}$. The assertion that such a bijection exists is purely algebraic, but no purely algebraic proof is known. See Section 3 below.

2. History

Riemann was led to the existence theorem by his consideration of multivalued functions on the Riemann sphere as single-valued functions defined on a Riemann surface, which he viewed as a union of complex domains glued together along boundary curves. Given a multivalued complex function $f(z)$, he constructed such a Riemann surface as a branched cover of $\mathbb{P}^1_{\mathbb{C}}$ whose branch points are the singular points of f (e.g. the points $0, \infty$ in the case of the multivalued function $z^{1/2}$). These ideas were developed in Riemann's thesis [Rie51] and in his paper [Rie57].

Riemann's contemporaries, especially Weierstrass, were skeptical of his arguments, because they relied on the unproven Dirichlet principle (which says that a solution to Poisson's equation is given by the minimizer of Dirichlet's energy functional). Later Hilbert gave a proof of Dirichlet's principle that sufficed for Riemann's approach. See [Bot03, Section 8.6] and [Mon99, Chapter 4] for a further discussion of this history.

Since the surfaces considered by Riemann arose as the domains of multi-valued functions on $\mathbb{C}$, they automatically had a non-constant meromorphic function that defined a branched covering map. Later Klein, motivated by a comment of Prym, originated the study of abstract Riemann surfaces, which was eventually developed into a rigorous theory by Weyl in [Wey13]. It is in this context that Riemann's Existence Theorem is now stated. See [Rem98, pp. 208-209, 218-219], [Sar55], and Part II of [Wey13] (especially Sections 12 and 19 in the 1955 edition).

A generalization of Riemann's Existence Theorem, showing that there are many holomorphic functions on abstract non-compact Riemann surfaces, was shown in [BS49] (see also [Rem98, §2.4]). Another generalization, in [GrRe58], extended Riemann's Existence Theorem to higher dimensions. It asserted that if W is a subvariety of a normal projective complex variety V, then any topological covering space $X^* \to V \smallsetminus W$ is the restriction of a morphism $X \to V$ of normal projective varieties. This provided motivation for Abhyankar's definition of $\pi(V \setminus W)$ over more general algebraically closed fields k, as the inverse system of Galois extensions of the function field of V that are unramified on V away from W. The point is that if $k = \mathbb{C}$, then this is the same as the inverse system of finite quotients of the topological fundamental group $\pi_1(V \setminus W)$. (See Abhyankar's Appendix 1 to Chapter VIII in the 1971 edition of [Zar35].)

These ideas motivated Grothendieck to define the *étale fundamental group* $\pi_1^{\text{et}}(X)$ of an algebraic variety X as the inverse limit of the Galois groups of the finite étale (flat and unramified) Galois covers $Y \to X$. If the ground field is $\mathbb{C}$, this group is the profinite completion of the topological fundamental group of X. For a more general algebraically closed field k, Grothendieck was able to obtain a version of Riemann's Existence Theorem for k-curves by relating covers of k-curves to covers of complex curves, and then citing the classical form of Riemann's Existence Theorem ([Gro71, XIII, Corollaire 2.12]).

For k of characteristic zero, Grothendieck's theorem says that the étale fundamental group of an n-punctured curve of genus g over an algebraically closed field k of characteristic zero is the same as over $\mathbb{C}$. For genus 0, this is the profinite completion of $\langle \sigma_1, \ldots, \sigma_n \mid \prod \sigma_i = 1 \rangle$, which is free of rank $n-1$ if $n \geq 1$. More generally, it is the profinite completion of

$$\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \sigma_1, \ldots, \sigma_n \mid \left(\prod [\alpha_j, \beta_j]\right)\left(\prod \sigma_i\right) = 1 \rangle,$$

which is free of rank $2g + n - 1$ if $n \geq 1$.

In non-zero characteristic p, Grothendieck's assertion is that the maximal prime-to-p quotient of the étale fundamental group is the same as over $\mathbb{C}$; and that the tame fundamental group π_1^{t} is a quotient of the analogous étale fundamental group over $\mathbb{C}$. (Here $\pi_1^{\text{t}}(U)$ is the inverse limit of the Galois groups of the Galois branched covers of the smooth completion of U that are unramified over U and are at most tamely ramified over the complement of U.) But the precise quotient of π_1^{et} is unspecified, and it remains unknown, though of course the maximal prime-to-p quotient of the étale fundamental group is a quotient of this group. The full étale fundamental group of an affine curve in characteristic p also remains unknown (even for the affine line). But in contrast to the situation in characteristic zero, π_1^{et} is known not to be free, since its maximal prime-to-p quotient has finite rank whereas its maximal p-quotient has infinite rank due to Artin-Schreier covers. (On the other hand, the maximal p-quotient of the étale fundamental group of a projective curve in characteristic p is a free pro-p-group of *finite* rank, by [Sha56].)

3. Algebraic proofs and constructions

Quite early there were attempts at an algebraic proof of Riemann's Existence Theorem, e.g. to show algebraically that there is a bijection between branch cycle data and algebraic branched covers of $\mathbb{P}^1_{\mathbb{C}}$ as discussed above. See for example [Sch03], [Sev15], [Sev21, Anhang F], [Sev22], and comments of Zariski ([Zar35, Chapter VIII, §4]). Especially since Grothendieck's work in SGA 1, there have been efforts to obtain an algebraic proof of Riemann's Existence Theorem in the following form:

Theorem 3.1. *Let $S = \{P_1, \ldots, P_n\}$ be a set of $n \geq 1$ distinct points on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, and let U be the complement of S. Let Π_n be the group on n generators $\sigma_1, \ldots, \sigma_n$*

subject to the single relation $\sigma_1 \cdots \sigma_n = 1$. Then there is an isomorphism i between the inverse system of Galois connected unramified covers of U and the inverse system of finite quotients of Π_n, such if $i(U' \to U) = (G; g_1, \ldots, g_n)$, then G is the Galois group of U' over U, and for each $i = 1, \ldots, n$, g_i is the standard generator of inertia of some point of Y over P_i, where Y is the smooth completion of U'.

Here, for a finite group G and a set of generators $g_1, \ldots, g_n \in G$ such that $g_1 \cdots g_n = 1$, we write $(G; g_1, \ldots, g_n)$ for the quotient of Π_n that takes σ_i to g_i.

Unlike the original form of Riemann's Existence Theorem that concerned the existence of a non-constant meromorphic function on a Riemann surface, the above assertion is purely algebraic, and therefore it seems reasonable to try to find a purely algebraic proof. Such a proof might also be generalizable to other situations, and thereby lead to results over general fields that go beyond those described in the previous section. In addition, such a proof could lead to more explicit forms of Riemann's Existence Theorem; see Section 5 below.

While Grothendieck's result in SGA 1 used algebraic methods to prove a generalization of Riemann's Existence Theorem over other algebraically closed fields, it ultimately relied on the complex case, where it had been proven complex analytically. Thus the problem of finding a completely algebraic proof remained.

For the forward direction, a difficulty in defining the correspondence i is that it depends on the choice of homotopy basis of U (a "bouquet of loops," in the terminology of Fried [Fri77]), with different choices leading to different correspondences. Since this choice is topological and non-canonical, it is unclear how to define i algebraically. For a discussion of this, see [Völ96], Remark 2.14b and the beginning of Chapter 10. For a further discussion, see [Fri77], especially p. 25, where a version appears for complex algebraic curves other than $\mathbb{P}^1_{\mathbb{C}}$, and for non-Galois covers.

For the reverse direction of the bijection in the above theorem, a first step is to construct, for each $(G; g_1, \ldots, g_n)$ as above, a cover U' of U with Galois group and standard generators of inertia g_i at some points over P_i. For certain *non-Galois* covers, this was carried out in [Ful69], using Grothendieck's machinery. This was done for the purpose of algebraizing the classical proof of Severi in [Sev21, Anhang F] that the moduli space of curves of genus g is irreducible, by realizing all curves of genus g as member of a family of branched covers of the projective line.

Fulton's paper helped motivate [Fri77], and together they helped motivate an algebraic construction of Galois branched covers in [Har80]. That construction relied on Grothendieck's Existence Theorem ([Gro61, III, Cor. 5.1.6]), an analog of GAGA for formal schemes; and it used that result in a way somewhat analogous to the use of GAGA in analytic proofs of Riemann's Existence Theorem. The construction in [Har80] realized covers with $2n$ branch points $P_1, \ldots, P_{2n}$, whose branch cycle description is of the form $(G, g_1, g_1^{-1}, \ldots, g_n, g_n^{-1})$ for some generators $g_1, \ldots, g_n$ of G. By deforming such covers by allowing the points $P_2, P_4, \ldots, P_{2n}$ to coalesce at some other point P_0, one can see topologically that one then obtains a branched cover with $n + 1$ branch points and with branch cycle description $(G, g_0, g_1, g_2, \ldots, g_n)$, as a boundary object of this family. This provides an algebraic construction of covers with prescribed descriptions, but it does not provide an algebraic *proof* that this construction works, since the fact that one of the boundary covers is connected relies on topology. It remains open whether one can prove algebraically that the above construction yields an inverse to a map i as in the theorem, and in that way obtain an algebraic proof of Riemann's Existence Theorem.

The above algebraic construction of covers having the special branch cycle description $(G, g_1, g_1^{-1}, \dots, g_n, g_n^{-1})$ carries over to more general situations, and provides (algebraic) proofs that every finite group is the Galois group of a branched cover of the projective line over the fraction field of any complete local domain that is not a field, and also of the projective line over any algebraically closed field; see [Har84] and [Har87]. This was later extended by F. Pop in [Pop96] to the more general class of fields that are called *large* (or "ample"). In addition, by using covers of the above type, in [Pop94] he proved a "Half Riemann Existence Theorem," giving a large quotient of $\mathbb{P}^1_k \smallsetminus S$, for appropriate S in the case that k is a Henselian valued field of arbitrary characteristic. (His proofs used rigid analytic methods, rather than formal schemes, though these are essentially equivalent by work of Raynaud.) But an additional difficulty in establishing a full "Riemann Existence Theorem" for k algebraically closed of characteristic p and S of order n (i.e. finding the structure of $\pi_1^{\text{et}}(\mathbb{P}^1_k \smallsetminus S)$) is that the group varies with the choice of S (e.g. see [Tam97]).

In the case of an algebraically closed field k of characteristic p, and a fixed branch locus $S \subset \mathbb{P}^1_k$, the structure of $\pi_1^{\text{et}}(\mathbb{P}^1_k \smallsetminus S)$ is not known, but its set of finite quotients (i.e. the Galois groups of finite étale covers of $\mathbb{P}^1_k \smallsetminus S$) is known. This was given by Abhyankar's Conjecture ([Abh57]), proven in [Ray94] and [Har94], by using rigid and formal methods.

4. Relationship to Galois theory

Riemann's Existence Theorem can be interpreted as a statement about the extension fields of $\mathbb{C}(z)$ and more generally of complex function fields in one variable. For $Y \to \mathbb{P}^1_{\mathbb{C}}$ branched only over a finite set S of $n > 0$ points of $\mathbb{P}^1_{\mathbb{C}}$, the function field $\mathbb{C}(Y)$ is a finite extension of the function field $\mathbb{C}(z)$ of $\mathbb{P}^1_{\mathbb{C}}$, unramified away from the places of $\mathbb{P}^1_{\mathbb{C}}$ corresponding to the points of S. This correspondence between topological covers and field extensions is a bijection that preserves the property of being Galois as well as preserving the Galois group (of deck transformations, on the topological side). Using the correspondence between branched covers and branch cycle data, it follows that the Galois groups of extensions of $\mathbb{C}(z)$ unramified over S are precisely the finite groups that can be generated by a set of $n-1$ or fewer elements (these being the finite quotients of $\pi_1(U)$); and similarly on $2g+n-1$ generators if the base is a complex curve of genus g. In particular, every finite group is a Galois group over $\mathbb{C}(z)$, and more generally over function fields of complex curves. Using Grothendieck's generalization of Riemann's Existence Theorem to other algebraically closed fields k of characteristic zero, the same conclusion holds for function fields over $k(z)$.

Building on Grothendieck's result (which had been announced in 1961), Douady showed in [Dou64] that the absolute Galois group of $\mathbb{C}(z)$ is isomorphic to the free profinite group on the elements of $\mathbb{C}$. This can be viewed as the inverse limit of the profinite completions of the groups $\pi_1(\mathbb{P}^1_{\mathbb{C}} \smallsetminus S)$, where S ranges over the finite subsets of $\mathbb{P}^1_{\mathbb{C}}$ that include the point ∞, each of which is free profinite on the elements of $S \smallsetminus \{\infty\}$ by Riemann's Existence Theorem. (See also [Sza09, §3.4].) Moreover, by reducing to the complex case, Douady also proved the analogous statement for any algebraically closed field k of characteristic zero.

Over an algebraically closed field k of characteristic $p > 0$, it similarly follows from Grothendieck's form of Riemann's Existence Theorem that every finite prime-to-p group is a Galois group over $k(x)$; and moreover that the Galois group of the maximal pro-prime-to-p extension of $k(x)$ is a free pro-prime-to-p group of rank equal to the cardinality of k. Although $\pi_1^{\mathrm{et}}(\mathbb{P}^1_k \smallsetminus S)$ is known not be to free for S a finite set of $n > 0$ elements, the analog of Douady's theorem holds, i.e. the absolute Galois group of $k(x)$ is the free profinite group of rank equal to the cardinality of k. This was shown in [Har95] by building on the methods of [Har80] (see Section 3); and it was simultaneously shown in [Pop95] via the same strategy using rigid analytic spaces (see also [HJ00]). While the generators of the free profinite Galois group in characteristic zero correspond to the elements of the field, it is unclear what the generators correspond to in characteristic p.

Riemann's Existence Theorem also arises in connection with realizing finite groups G as Galois groups over number fields k. By Hilbert's Irreducibility Theorem, it suffices to realize G as the Galois group of a branched cover of $\mathbb{P}^1_k$. By a result of Grothendieck (which was in fact known earlier), if the branch points of a branched cover $Y \to \mathbb{P}^1_{\mathbb{C}}$ lie at algebraic values of the parameter, then the cover is defined over $\bar{\mathbb{Q}}$, as is the Galois action in the Galois case. Each such cover is then defined over some number field k (a *field of definition* of the Galois cover). An explicit algebraic form of Riemann's Existence Theorem would produce such a field k if one is given the branch cycle description of the given cover Y. In general this is open. See Section 5 below.

5. Explicit forms

As noted above, if $P_1, \ldots, P_n$ are points on the projective line over $\bar{\mathbb{Q}}$, then every branched cover of $\mathbb{P}^1$ with that branch locus is in fact defined over $\bar{\mathbb{Q}}$, and not just over $\mathbb{C}$. Since a cover of curves is given by finitely many equations, it is then defined over some number field, i.e. a finite extension of $\mathbb{Q}$. Now a Galois branched cover is determined by its branch cycle description, once one chooses a homotopy basis for the complement of these points, as in Theorem 3.1. Thus, given a finite group G and generators $g_1, \ldots, g_n$, it is natural to ask for a number field k over which the cover of $U = \mathbb{P}^1 - \{P_1, \ldots, P_n\}$ is defined. In fact, it is natural to ask for the *minimum* field of definition of this Galois cover. Actually, there is not always such a minimum field, though often there is one (e.g. if the group has trivial center or is abelian). The intersection of the fields of definition, though, is a natural field k that can be associated to the cover as its "field of moduli," and $k(\zeta_n)$ is a field of definition for some n. (See [CH85].) So if the branch points P_i are algebraic, one can ask for an explicit form of Riemann's Existence Theorem that produces the field of moduli of the cover associated to a given branch cycle description.

Here one has group theoretic data as input, and the desired output is a number field. Thus it seems reasonable to expect that one should be able to pass from one to the other purely algebraically, possibly obtaining an algebraic proof of Riemann's Existence Theorem in the process. But in fact, the passage from group theory to number theory goes through a sequence of non-algebraic steps, making it hard to make this process explicit.

We can visualize these steps as follows:

Branch cycle description (group theory)
$\downarrow$
Topological cover
$\downarrow$
Non-constant map between Riemann surfaces
$\downarrow$
Non-constant morphism of complex algebraic curves
$\downarrow$
Non-constant morphism of curves over $\overline{\mathbb{Q}}$
$\downarrow$
Non-constant morphism of curves over some number field

The middle arrow corresponds to the classical form of Riemann's Existence Theorem, and the first three arrows correspond to the form in Theorem 3.1.

If one could carry out the above process in general, passing from a branch cycle description $(G; g_1, \ldots, g_n)$ to the field of moduli k, then one could use this to realize groups as Galois groups over those fields k (or at least, some $k(\zeta_n)$). Moreover, one could look for branch cycle descriptions such that the field of moduli is $\mathbb{Q}$, and in this way try to solve the inverse Galois problem.

Progress in this direction has been made in the special case that the branch cycle data $(g_1, \ldots, g_n)$ is *rigid*; i.e. has the property that any n-tuple $(h_1, \ldots, h_n)$ with $\prod h_i = 1$ that is entrywise conjugate to the given data is necessarily uniformly conjugate to it. The point is that in this case, there is no ambiguity in the bijection between covers and branch cycle data, and the action of the absolute Galois group of $\mathbb{Q}$ on branch cycle data is easy to understand. For early work in this direction see [Bel79], [Mat79], [Tho84], [Shi74], and [Fri77]; see also presentations in [MM99, Chapters I,II],[Ser92, Chapters 7,8], and [Völ96, Chapter 3].

But more generally, in order to pass from the group theory to the associated number theory for a given branch cycle description, one would like a formula for k in terms of $(G; g_1, \ldots, g_n)$. Such a formula was proposed by Richard Parker in the case of $n = 3$; see [Har87]. In that specific form, it was disproven by R. Daenzer; but a more symmetric form of the conjecture was then studied. According to that form, k is the field obtained from $\mathbb{Q}$ by adjoining the eigenvalues of the linear transformation on the group ring $\mathbb{Q}[G \times G]$ (viewed as a $\mathbb{Q}$-vector space) given by left multiplication by the element $\sum_{g \in G} (g^{-1} g_1 g, g^{-1} g_2 g) \in \mathbb{Q}[G \times G]$. Parker said that he was not committed to the specific form of this element; and that his aim was simply to obtain an extension of $\mathbb{Q}$ in a canonical way depending on the branch cycle description, such that the extension need not be abelian (since fields of moduli need not be abelian over $\mathbb{Q}$). In an unpublished manuscript [Sch06], Leila Schneps verified the conjecture for the case that G is either abelian of rank at most two, or is dihedral. In both of these cases, the field of moduli (which is equal to Parker's field) turns out to be abelian over $\mathbb{Q}$. An unpublished manuscript of Hoffman ([Hoff09]), though, asserts that in fact the field obtained using Parker's construction is *always* abelian over $\mathbb{Q}$, which would imply that it cannot always be the field of moduli k (though it could be the maximal abelian subgroup of k). This question deserves further study.

In some cases, one can explicitly compute not only the field of moduli, but even the equations of the branched cover, given the branch cycle description. This was done in the case of $n = 3$, where the top space has genus zero (but is not necessarily Galois), in parts III and IV of [Sch94], using Gröbner bases. While such computations do not appear to provide a general formula, they could be used to test conjectures about formulas for the field of moduli of non-Galois covers.

6. Open problems

We conclude with a summary of key open problems.

1. Find an algebraic proof of Riemann's Existence Theorem in the form Theorem 3.1.
2. Find an explicit form of Riemann Existence Theorem that would give at least the field of moduli of a cover with a given branch cycle description.
3. Describe the branch cycle descriptions that arise from tamely ramified covers over an algebraically closed field of characteristic p (i.e. the associated quotients of Π_n), at least in the case of covers of the projective line having branch locus $\{0, 1, \infty\}$.
4. Describe the étale fundamental group of an affine curve over an algebraically closed field of characteristic p, as a profinite group, at least in the case of the affine line over $\bar{\mathbb{F}}_p$.
5. Using explicit equations or fields of moduli for some special branched covers, realize their Galois groups over small number fields (especially $\mathbb{Q}$ itself).

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