Recall: If F is a global field, we have completions F_v of F at the absolute values of F. For F a global function field, these correspond to discrete valuations on F. We can then ask which F-varieties V satisfy a local-global principle (LGP):

$$V(F_v) \neq \emptyset \text{ for all } v \Rightarrow V(F) \neq \emptyset$$

Key case: V a G-torsor$/F$ for $G \subset GL_n$ a linear algebraic group over F.

Obstruction to LGP for all G-torsors $/F$:

$$Ш(F,G) = \operatorname{Ker}\left(H^1(F,G) \to \prod_v H^1(F_v,G)\right)$$

I.e. $Ш(F,G)$ is trivial $\Leftrightarrow$ LGP for all G-torsors$/F$.

For F a # fld, holds if G rat'l + connected.

LGP for G-torsors $\rightsquigarrow$ LGP for alg. structures (qf, csa, ...)

$\rightsquigarrow$ results on field invariants (u-inv, period)

We can also try to carry this over to <u>semi-global fields</u>, i.e. function fields F of curves over a complete discretely valued field K (cdvf)

e.g. $K = \mathbb{Q}_p,\ k((t))$

cdvf

So $F = K(x)$ or a finite extension.

$\quad$ = fn fld of a K-curve

Ex. $K(x)$ = function field of $\mathbb{P}^1_K$

$$\mathbb{P}^1_K \longrightarrow \bullet\ \operatorname{Spec} K$$

As before, we can take the completions F_v of F at discrete valuations v, & ask for LGP's for varieties$/F$, esp. G-torsors $/F$, with an obstruction $Ш(F,G)$ wrt F_v's.

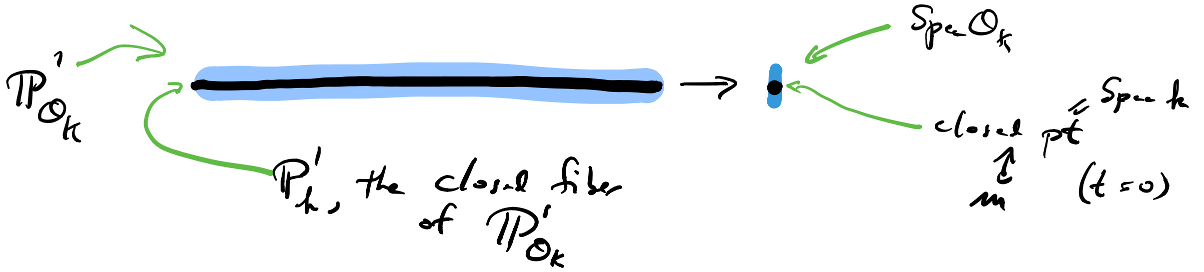
If LGP holds for G-torsors$/F$, get a LGP for alg. structures $/F$, & can then try to apply to field invariants.

For a cdvf K, have its valuation ring $\mathcal{O}_K$, a complete discrete valuation ring (cdvr).

Ex. $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$; $K = k((t))$, $\mathcal{O}_K = k[[t]]$

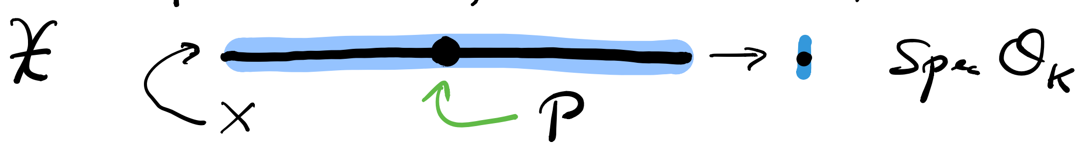
Can view F as function field of an $\mathcal{O}_K$-curve $\mathcal{X}$.

Key example: $K = k((t))$, $\mathcal{O}_K = k[[t]]$, $F = K(x)$.
$\quad$ cdvf $\quad$ $\mathfrak{m} = (t) \subset \mathcal{O}_K$ cdvr $\quad$ $\mathcal{O}_K/\mathfrak{m} = k$



See more information geometrically this way.
Also can get an alternative LGP:

Say have a regular projective curve $\mathfrak{X} \to \operatorname{Spec} \mathcal{O}_K$ with closed fiber $X \to \operatorname{Spec} k$. $\forall P \in X$, take the complete local ring $\widehat{\mathcal{O}}_{\mathfrak{X},P}$ of $\mathfrak{X}$ at P.
Let $F_P = \operatorname{frac} \widehat{\mathcal{O}}_{\mathfrak{X},P}$ $\quad$ Completion of the local ring $\mathcal{O}_{\mathfrak{X},P}$



Ex. $K = k((t))$, $\mathcal{O}_K = k[[t]]$, $F = K(x)$,
$\mathfrak{X} = \mathbb{P}^1_{k[[t]]}$, $X = \mathbb{P}^1_k$, $P : x = t = 0$. Then
$\widehat{\mathcal{O}}_{\mathfrak{X},P} = k[[x,t]]$, $F_P = \operatorname{frac} k[[x,t]] =: k((x,t))$

Can ask for a LGP wrt F_P's instead of F_v's

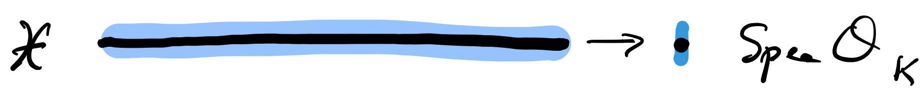
Obstruction: $Ш_{\mathfrak{X}}(F,G) = \ker\left(H^1(F,G) \to \prod_{P \in X} H^1(F_P,G)\right)$ $\quad$ easier to study + use.
for G-torsors $\quad$ $\cap$
$\quad$ $Ш(F,G)$.

In the above situation, we're taking a regular projective model $\mathcal{X}$ of a semi-global field F.

I.e. F is the function field of a curve over a cdvf K, or equivalently over the cdvr $\mathcal{O}_K$. $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ is reg. & proj.

As above, for $K = k((t))$, $\mathcal{O}_K = k[[t]]$, $F = K(x)$,

we can take $\mathcal{X} = \mathbb{P}^1_{\mathcal{O}_K}$;

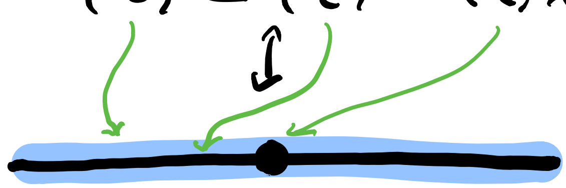


Here $\mathcal{X}$ is projective over $\mathcal{O}_K$, and is regular (no singularities; local rings are regular).

Since $\mathcal{O}_K$ is of dim $= 1$, $\mathcal{X}$ has dim $= 2$.

$\longleftrightarrow$ Krull dim of $k[[t]][x]$ is 2:

$$(0) \subset (t) \subset (t, x)$$

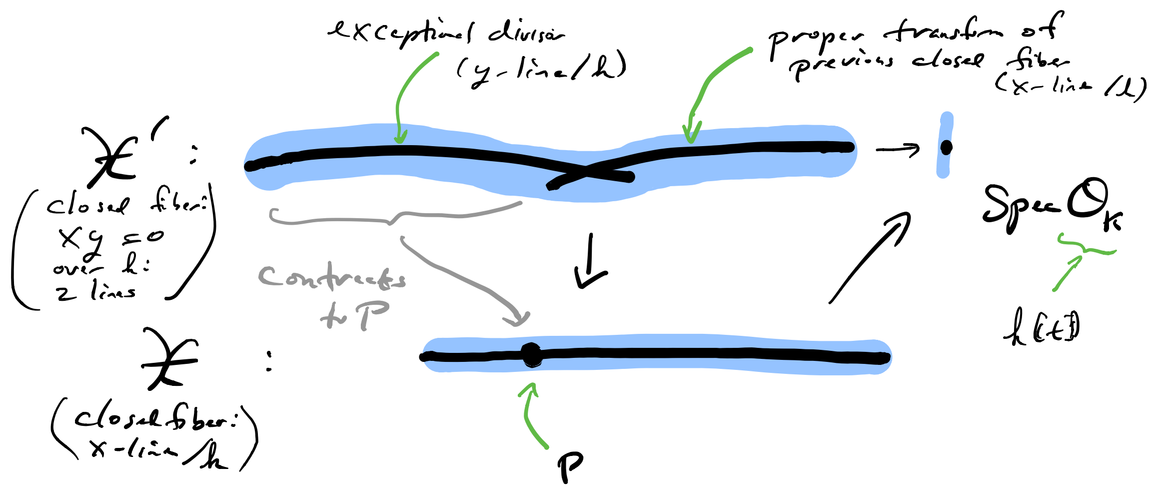


For curves over a field (dim = 1), there's a unique projective model for each function field.

But in higher dimensions: not unique; eg for surfaces / field, or curve / dvr.

E.g. Blow up a regular model $\mathcal{X}$ at a point: get a new model

Ex. Blow up $\mathcal{X} = \mathbb{P}^1_{k[[t]]}$ at $P: x = t = 0$, & get $\mathcal{X}'$ given by $xy = t$:

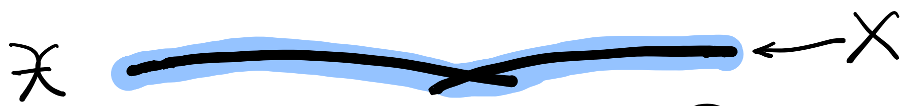


If we take a different model, we get a new $Ш_{\mathcal{X}}(F, G)$.

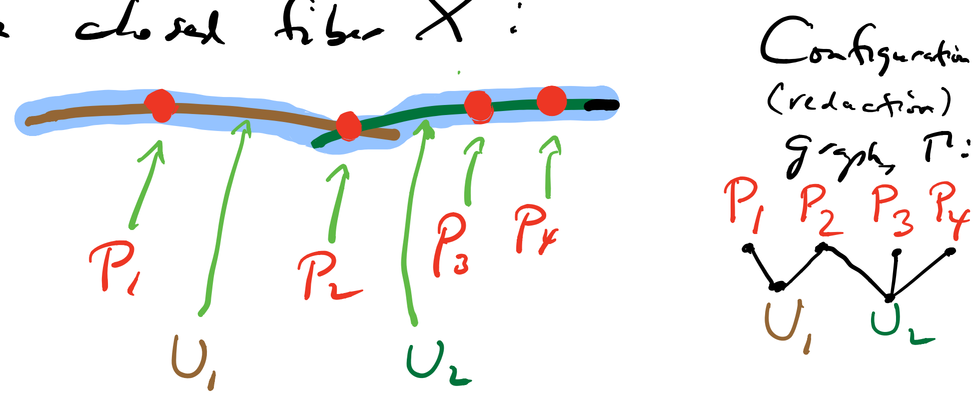
We can choose the model to help compute this.

Given a (regular projective) model $\mathcal{X}$ of a sgf F over $\mathcal{O}_K$, we can partition the closed fiber X (a curve over $k = \mathcal{O}_K/\mathfrak{m}$) into a finite collection of points & open sets.

Ex. For the above example



we could pick some points $P_i \in X$ and take the conn comps U_j of their complement in the closed fiber X:



The set $\mathcal{P}$ of these pts should be non-$\emptyset$ and include all the pts where irreducible components of X meet. The set $\mathcal{U}$ of the components of $X - \mathcal{P}$ will consist of affine open curves, one on each irreducible component of X.

In the above set-up, for each point $P \in \mathcal{P}$ we have an associated field $F_P \supset F$.

frac $\hat{\mathcal{O}}_{\mathcal{X},P}$

Ex. $x = t = 0$

$\mathcal{X} = \mathbb{P}^1_{k[[t]]}$, $P = P_0$, $F_P = k((x,t)) = \operatorname{frac} k[[x,t]]$

For each $U \in \mathcal{U}$, we also have an associated field $F_U \supset F$:

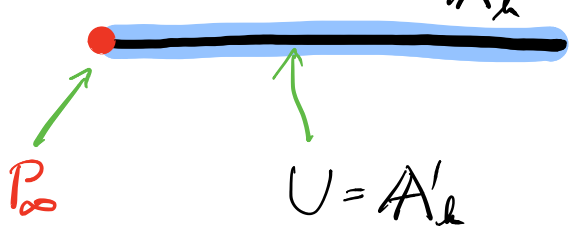
Ex. $K = k((t))$, $\mathcal{O}_K = k[[t]]$, $F = K(x)$,

$\mathcal{X} = \mathbb{P}^1_{\mathcal{O}_K}$; take $\mathcal{P} = \{P_\infty\}$,

pt at ∞ on $X = \mathbb{P}^1_k$

So $\mathcal{U} = \{U\}$

$\mathbb{A}^1_k$



$\mathcal{X} = \mathbb{P}^1_{\mathcal{O}_K}$

$F_P = k((x^{-1},t))$. Re F_U:

Take the subring of $F = \{\text{rat'l fns on } \mathcal{X}\}$ consisting of fns that are regular (no poles) on U. Complete wrt t-adic metric.

fns on U

In this example, we get $\hat{R}_U = k[x][[t]]$.

Define $F_U = \operatorname{frac} \hat{R}_U$. (See also PS#5)

In this example we have:

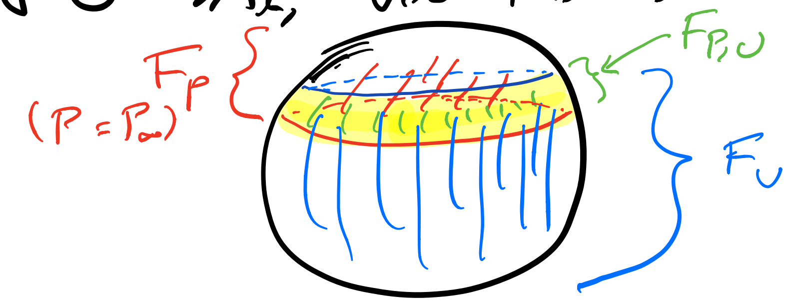
$$\begin{tikzcd}[column sep=small]
 & F_{P,U} = k((x^{-1}))((t)) \arrow[dl, dash] \arrow[dr, dash] & \\
k((x^{-1},t)) = F_P \arrow[dr, dash] & & F_U = \operatorname{frac} k[x][[t]] \arrow[dl, dash] \\
 & F = k((t))(x) &
\end{tikzcd}$$

and $F = F_P \cap F_U$.

More generally, for each pair P, U with P on the closure of U, have a field $F_{P,U}$ containing $F_P + F_U$, and the intersection of all these fields F_P, F_U, wrt these containments, is F.

As on PS 5, view F_P, F_U as the rational (or meromorphic) functions on some metric neighborhoods; + can view $F_{P,U}$ as the functions on the "overlap".

In the above example with $P_\infty \in X \subset \mathbb{P}^1_{k((t))}$, + $U = \mathbb{A}^1_k$, view this as analogous $/\mathbb{C}$ to



$$\begin{tikzcd}[column sep=tiny]
 & F_{P,U} \arrow[dl, dash] \arrow[dr, dash] & \\
F_P \arrow[dr, dash] & & F_U \arrow[dl, dash] \\
 & F = \mathbb{C}(x) &
\end{tikzcd}$$

Relationship of this to LGP wrt F and F_P's for all $P \in \mathcal{X}$:

In above example, both have F_{P_∞} as one of these fields.

All the other F_P's contain F_U. A'_k

Similarly in general.

If we take $K, F, \mathcal{X}, X, \mathcal{P}, \mathcal{U}$ in general, we can consider a LGP wrt F_P's & F_U's for $P \in \mathcal{P}$, $U \in \mathcal{U}$: a finite set of overfields.

(Analogous to simplicial cohomology vs. singular cohomology.)

Can study combinatorially, & get a LGP for G a rational connected linear algebraic group, asserting triviality of

$$Ш_{\mathcal{P}}(F,G) := \ker\left(H^1(F,G) \to \prod_{P \in \mathcal{P}} H^1(F_P,G) \times \prod_{U \in \mathcal{U}} H^1(F_U,G)\right).$$

Have $Ш_{\mathcal{P}}(F,G) \subseteq Ш_{\mathcal{X}}(F,G) \subseteq Ш(F,G)$

Triviality of $Ш_{\mathcal{P}}$ suffices for applications to u-invariant and period-index.

For use with a rational connected lin. alg. gp G, the key property: factorization.

In above example, with one F_P, one F_U, and one $F_{P,U}$, this asserts: Every element $g_0 \in G(F_{P,U})$ can be factored as

$$g_0 = g_P g_U$$

with $g_P \in G(F_P)$, $g_U \in G(F_U)$.

(In special case when $G = GL_n$: solve for coefficients inductively. For more general rational conn. gp G, use the birational iso of G with affine space to do the same.)

In more general examples, may have many F_P's, F_U's, $F_{P,U}$'s.

Say G is a rat'l conn lin. alg. group $/F$.

Given elts $g_{P,U} \in G(F_{P,U})$ for all pairs (P,U) with P on the closure of U, $\exists$ elts $g_P \in G(F_P)$, $g_U \in G(F_U)$ $(P \in \mathcal{P}, U \in \mathcal{U})$ st $g_{P,U} = g_P g_U \in G(F_{P,U})$ for all pairs (P,U).

(Simultaneous factorization holds)

This property gives a LGP for in particular, torsors homogeneous spaces under such a linear algebraic group:

Say G is a lin. alg. gp $/F$, & satisfies (simult.) fact'n.
Say V is a homogeneous space$/F$ under G,
in the sense that $\forall E/F$, $G(E)$ acts transitively on $V(E)$.

In this situation, we have:

Thm. (LCP) If V has a point over each F_P and each F_U $(P \in \mathcal{P},\ U \in \mathcal{U})$ then V has a point over F.

Pf. Let $\xi_P \in V(F_P)$, $\xi_U \in V(F_U)$.
So for each pair P, U, $\xi_P, \xi_U \in V(F_{P,U})$.

V homog$/G \Rightarrow \exists\, g_{P,U} \in G(F_{P,U})$ st
$$\xi_P \cdot g_{P,U} = \xi_U$$
Factorization $\Rightarrow \exists\, g_P \in G(F_P), g_U \in G(F_U)\ \forall P, U$
st $g_{P,U} = g_P g_U$.

So $\xi_U = \xi_P \cdot g_P g_U$, $\xi_U \cdot g_U^{-1} = \xi_P \cdot g_P$.
True $\forall P, U$. So:
the points $\xi_U \cdot g_U^{-1}$, $\xi_P \cdot g_P$ on V all agree,
& are defined over the intersection of all the fields F_P, F_U: i.e. have an F-pt. ✓

In particular, true for torsors. So if G satisfies (simult) factorization $/F$ wrt F_P's, F_U's, then $Ш_P(F, G)$ is trivial. semi-global field

<u>Key case</u>: G rat'l & conn.

<u>Ex</u>. For a regular q.f. q/F,

$SO(q)$ is a rat'l conn lin alg gp $/F$

So $Ш_P(F, SO(q))$ is trivial.

$$\begin{matrix} Ш_P(F, SO(q)) \\ \cap\!| \\ H^1(F, SO(q)) \end{matrix}$$

i.e. trivial over each F_P and F_U

i.e. every locally trivial $SO(q)$-torsor over a semi-global field F is trivial.

<u>Ex</u>. Let q be a q.f. / semi-global field F.

Let V be a connected homogeneous space over $O(q)$ (which is not a conn gp).

Since V is connected, & since $SO(q)$ is the connected component of the identity in $O(q)$, it follows that $SO(q)$ also acts transitively on V.

Hence if V has a pt / each F_P, F_U then V has an F-pt: LGP holds for $O(q)$.

rat'l & conn

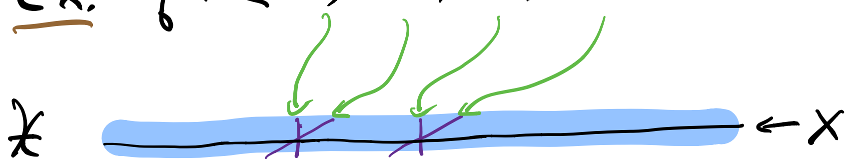
Application to u-invariant: $u(\mathbb{Q}_p(x)) = 8$.

(≥ 8 is easy b/c $\exists$ anisotropic form of dim 8; so STS ≤ 8)

Let q be a regular q.f. $/ \mathbb{Q}_p(x) = F$.

Take a model $\mathcal{X}$ of F, e.g. $\mathbb{P}^1_{\mathbb{Z}_p}$. WMA $q = \langle a_1, \ldots, a_n \rangle$.
After blowing up if necessary, WMA the locus where $a_1, \ldots, a_n$ aren't units has only normal crossings.

Ex. $q = \langle x-2, x-(2-p), x-3, x-(3-p) \rangle$



Let $\mathcal{P}$ contain the pts where this locus meets X.
By Springer's Thm on 1^{st} + 2^{D} residues, get: if $\dim q \geq 8$ then q is isotropic over F_P for $P \in \mathcal{P}$. Also get q isotropic over F_U for $U \in \mathcal{U}$. The hypersurface $Q : q = 0$ is a homog. sp. under $O(q)$ by Witt Extension. But since $\dim q \geq 2$, Q is connected, so a homogeneous space under $SO(q)$.
By the above, since Q has pts over each F_P, F_U, it has an F-pt.
I.e.: q is isotropic $/F$. This shows $u(F) = 8$.

Question: What about possibly disconnected homogeneous spaces under $O(q)$ (e.g. torsors)?

q.f. $/F$ of dim n

$O(q)$ is not conn, though each connected component is rat'l.

Turns out: $Ш_p(F, O(q)) = \mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z}/2)$

where Γ is the reduction graph of $\mathcal{X}$.

e.g. $\mathbb{P}^1_{O_K}$

$F = K(x)$

So $Ш_p(F, O(q))$ is trivial $\Leftrightarrow$ Γ is a tree.

kernel of global-to-local map on $H^1(F, O(q))$

classifies q.f. $/F$ of dim n

where Q may be disconnected

So get: Hasse-Minkowski$/F$ even in dim 2 $\Leftrightarrow$ Γ is a tree

e.g. $F = K(x)$

More generally, if G is a lin. alg. gp. that's not conn, but each conn comp is rat'l, $Ш_p(F, G) = \mathrm{Hom}(\pi_1(\Gamma), \bar{G})$,

always finite

where $\bar{G}$ is the group of connected components of G; i.e. $\bar{G} = G/G^\circ$;

identity component of G

(Eg. $\bar{G} = \mathbb{Z}/2$ if $G = O(q)$)

Application to LGP for csa's / semi-global field

- want analog of Albert-Brauer-Hasse-Noether

For this, want factorization to hold for PGL_n, since $H^1(F, PGL_n)$ classifies csa's $/F$ of degree n.

Recall: $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$

$\therefore \forall$ field E,

$$\begin{array}{ccccccc} & & & & GL_n(E) & \to & PGL_n(E) \\ & & & & \| & & \| \\ 1 \to H^0(E, \mathbb{G}_m) & \to & & & H^0(E, GL_n) & \to & H^0(E, PGL_n) \to \end{array}$$

$\to H^1(E, \mathbb{G}_m)$

$\quad \| $
$\quad 0$ by Hilbert 90

$\therefore \quad GL_n(E) \to PGL_n(E)$
is surjective.

To prove factorization for PGL_n, say we have a finite subset $\mathcal{P} \in X =$ closed fiber of $\mathcal{X}$, as before, $+ \ \mathcal{U} = \{\text{conn. comp's of } X - \mathcal{P}\}$. Say we're given elements $g_{P,U} \in PGL_n(F_{P,U})$ for all pairs P, U with P in the closure of U.

We want elts $g_P \in PGL_n(F_P)$, $g_U \in PGL_n(F_U)$
for all $P \in \mathcal{P}$, $U \in \mathcal{U}$, st
for all pairs (P, U) as above,

$$g_{P,U} = g_P g_U \in PGL_n(F_{P,U}).$$

By the above surjectivity,

$$\begin{array}{ccc} \exists\, \tilde{g}_{P,U} & \in & GL_n(F_{P,U}) \\ \downarrow & & \downarrow \\ g_{P,U} & \in & PGL_n(F_{P,U}) \end{array}$$

GL_n rat'l conn $\Rightarrow \exists\, \tilde{g}_P \in GL_n(F_P)$,
$\tilde{g}_U \in GL_n(F_U)$, st $\tilde{g}_{P,U} = \tilde{g}_P \tilde{g}_U$.

Let $g_P \in PGL_n(F_P)$, $g_U \in PGL_n(F_U)$
be the images of $\tilde{g}_P, \tilde{g}_U$.

Then $g_{P,U} = g_P g_U$, proving factorization
for PGL_n.

<u>Conclusion</u>: We have a LGP for PGL_n
wrt $\mathcal{P}, \mathcal{U}$.

Can then use this to obtain

$$\operatorname{ind}\alpha \mid (\operatorname{per}\alpha)^2 \quad \text{for } \alpha \in \operatorname{Br}(\mathbb{Q}_p(x)),$$

of per prime to p

similarly to the proof for $u = 8$:

Pick a model s.t. the bad locus for α (where α is not defined / $\mathcal{X}$) has only normal crossings.

Pick $\mathcal{P}$ to contain the points of $X \subset \mathcal{X}$ where the bad locus meets X.

Check locally on F_P, F_U that $\operatorname{ind} \mid \operatorname{per}^2$. Now use LGP to conclude it works globally.

Above results use LGP's wrt $\mathcal{P}$'s & $\mathcal{U}$'s to get numerical invariants. But in fact

$$Ш_{\mathcal{P}}(F,G) = Ш_{\mathcal{X}}(F,G)$$

if G is rat'l & conn & lin alg'l: $Ш_{\mathcal{X}}(F,G) = \varinjlim_{\mathcal{P}} Ш_{\mathcal{P}}(F,G)$.

So also get LGP w.r.t. pts on X.

Moreover, as CPS showed, for q.f.'s and csa's, we also have LGP's wrt the discrete valuations on F;

— closer analogs of H-M & A-B-H-N.

Some open questions:

- For a linear algebraic group G over a semi-global field F s.t. G isn't nec. rat'l, when is there a LGP? (& even for global F)
- When LGP fails, what is the obstruction $\text{Ш}(F,G)$, & when is it finite?
- Do $\text{Ш}(F,G)$ and $\text{Ш}_{\mathcal{X}}(F,G)$ always agree? (We can assume $\mathcal{X}$ is a "nice model", or even take $\varinjlim_{\mathcal{X}} \text{Ш}_{\mathcal{X}}(F,G)$.)
- What add'l applications can we obtain to LGP for algebraic structures, & to numerical field invariants?
- What happens in still higher dimensional generalizations of semi-global fields?
- Are there LGP's for "higher dimensional global fields", like $\mathbb{Q}(x)$? Can these be used to obtain results about u-invariant, period-index, etc?