# Recall: Galois cohomology

- defined in terms of group coho.

Given an action of a profinite group $\Gamma$ on an abelian group $A$, we can define coho gps $H^i(\Gamma, A)$ for $i = 0, 1, 2, \ldots$. Here $H^0(\Gamma, A) = A^\Gamma$.
Given a s.e.s. $0 \to A \to B \to C \to 0$ of abelian gps w $\Gamma$-actions, we get a l.e.s. of coho gps involving $H^0, H^1, H^2, \ldots$

If $\Gamma$ acts on a non-abelian group $G$, we can still define $H^i(\Gamma, G)$ for $i = 0, 1$.
$H^0(\Gamma, G) = G^\Gamma$. $H^1(\Gamma, G)$ is just a ptd set.
A s.e.s. $1 \to N \to G \to H \to 1$ gives a l.e.s. of 6 terms, involving $H^0, H^1$, ending with $H^1(\Gamma, H)$. (If $N \subset Z(G)$, get a $7^{th}$ term, $H^2(\Gamma, N)$.)

If $S$ is just a pointed set and we have an action of $\Gamma$ on $S$, we can still define

$H^0(\Gamma, S) := S^\Gamma$. But no $H^1$ ...

Suppose $D \subseteq G$, groups, $D$ not nec. normal, with compatible $\Gamma$-actions.

Get a s.e.s. of pointed sets with $\Gamma$-actions:

$$1 \to D \to G \to G/D \to 1$$

ptd set of left cosets

Then get a 5-term coho exact seq:

$$1 \to H^0(\Gamma, D) \to H^0(\Gamma, G) \to H^0(\Gamma, G/D) \to H^1(\Gamma, D) \to H^1(\Gamma, G)$$

(Serre, Gal. Coho. Chap I, §5.4)

Galois coho in terms of group cohomology:

$$H^i(F,G) := H^i(\mathrm{Gal}(F), G(F^{sep})),$$

$$H^i(E/F, G) := H^i(\mathrm{Gal}(E/F), G(E))$$

for $G$ a linear algebraic group $/F$, i.e. iso to a Zariski closed subgp of $GL_n$; eg $GL_n, SL_n, O_n, \mathbb{G}_m, \mathbb{G}_a$

Recall:

"Hilbert's Thm 90": $H^1(E/F, GL_n) = 1$.

e.g. $\mathbb{G}_m$

Cor $H^1(E/F, SL_n) = 1$

Pf. $1 \to SL_n \to GL_n \xrightarrow{\det} \mathbb{G}_m \to 1$

$$\begin{array}{ccccccc} 1 \to & SL_n(F) & \to & GL_n(F) & \xrightarrow{\det} & F^\times & \\ & \| & & \| & & \| & \\ \rightsquigarrow 1 \to & H^0(E/F, SL_n) & \to & H^0(E/F, GL_n) & \xrightarrow{\det} & H^0(E/F, \mathbb{G}_m) & \end{array}$$

$\to H^1(E/F, SL_n) \to H^1(E/F, GL_n)$ trivial

So:

$$GL_n(F) \xrightarrow{\det} F^\times \longrightarrow H^1(E/F, SL_n) \to 1$$

surj $\Rightarrow$ trivial

$\therefore$ trivial.

Recall, in Galois thy there is also an additive form of Hilbert 90.

Corresponding coho result:

$H^1(E/F, \mathbb{G}_a) = 0.$

More generally: $H^i(E/F, \mathbb{G}_a) = 0$ for $i > 0$.
(Serre, Local Fields, Chap X, §1)

The analog for $\mathbb{G}_m$ is false.

In fact, $H^2(F, \mathbb{G}_m) = Br(F)$ !

(See Serre, Local Fields, Chap X, §4-5)

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Another approach to $H^1$: torsors.

(= principal homogeneous spaces)

Recall: In topology, if $G$ acts simply transitively on a space $X$, say $X$ is a $G$-torsor (or $G$-PHS).

Ex. $A \subset \mathbb{R}^3$ a 2-dim'l vector subspace,

a group.

Let $X \subset \mathbb{R}^3$ be a plane parallel to $A$
(not nec. through origin)

Then $A$ acts simply transitively on $X$
by translation: $a \cdot x := a + x$.

If $G$ acts on a space $X$, have

$$\begin{array}{ccc} G \times X & \xrightarrow{\sim} & X \times X \\ (g, x) & \longmapsto & (g \cdot x, x). \end{array}$$

(homeo if torsor)

Can use this as the <u>def'n</u> of a torsor.

In above example with a plane $A$, + parallel $X$,
if pick a pt $x_0 \in X$, have

$$\begin{array}{ccc} A & \xrightarrow{\sim} & X \\ a & \longmapsto & a \cdot x_0 \end{array}$$

(homeo depends on $x_0$)

Can also consider torsors in algebraic geometry –
but now a variety $/F$ might not have a <u>point</u> $/F$.

Ex. Let $F=\mathbb{R}$, $G=SO_2$ $/\mathbb{R}$,

$X$ : curve in $\mathbb{A}^2_{\mathbb{R}}$ given by

$x^2+y^2=c$ (for some choice of $c \in \mathbb{R}^\times$),

$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \, a,b \in \mathbb{R},\ a^2+b^2=1 \right\}$

acts on $X(\mathbb{R})$ by rotation,

and similarly $SO_2(\mathbb{C}) \cong \mathbb{C}^\times$

acts on $X(\mathbb{C})$ by rotation:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If $c>0$, then $X(\mathbb{R}) \neq \emptyset$;

& then can pick an $\mathbb{R}$-pt $P \in X$,

& get $SO_2 \xrightarrow{\sim} X$

$$A \longmapsto A \cdot P.$$

But if $c<0$, there are no $\mathbb{R}$-pts on $X$, but there are $\mathbb{R}$ pts on $SO_2$, so not isomorphic varieties $/\mathbb{R}$.

But even if have $c<0$,
we still have $G \times X \xrightarrow{\sim} X \times X$
as before. (Neither side has an $\mathbb{R}$-pt)

In general, we say that a $G$-torsor $X$
over $F$ is <u>trivial</u> if $G \xrightarrow{\sim} X$, over $F$.
Equiv: $X$ has an $F$-point.

So in above example
$$X : x^2+y^2=c \quad /\mathbb{R} \quad (c\in\mathbb{R}^\times)$$
is a trivial torsor under $G = SO_{2,\mathbb{R}}$
iff $c>0$.

Note: In the above example,
with $G = SO_{2,\mathbb{R}}$, the group $H := \mathbb{G}_{m,\mathbb{R}}$ is

also a linear algebraic group $/\mathbb{R}$, and

$$H(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times = G(\mathbb{C})$$

But $H(\mathbb{R}) \not\cong G(\mathbb{R})$, $H \not\cong G$.

We say $G, H$ are (twisted) forms $/\mathbb{R}$ of the same group $\mathbb{G}_m$ over $\mathbb{C}$ — i.e. these groups $/\mathbb{R}$ become isomorphic $/\mathbb{C}$.

In the situation of the above example, we have $K = \mathbb{R}$, $K^{sep} = \mathbb{C}$,

$$\Gamma = \mathrm{Gal}(\mathbb{R}) = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong C_2.$$

$\Gamma$ acts on $G(\mathbb{C}) \cong \mathbb{C}^\times$

and $G(\mathbb{C})^\Gamma = G(\mathbb{R}) = SO_2(\mathbb{R})$.

$$\text{So } H^1(\mathbb{R}, G) = H^1(C_2, \mathbb{C}^\times)$$

where the action of $C_2$ on $\mathbb{C}^\times$ is not the obvious one, but rather the one from viewing $\mathbb{C}^\times \cong SO_2(\mathbb{C})$

Complex special orthogonal gp $SO(\langle 1, 1, \ldots, 1 \rangle)$, not special unitary gp

This suggests a relationship between Galois cohomology $H^1(F, G)$ and $G$-torsors $/F$.

In fact: have natural bijection

$$H^1(F,G) \longleftrightarrow \left\{\begin{array}{l}\text{iso. classes}\\ \text{of } G\text{-torsors}\\ \text{over } F\end{array}\right\}$$

Moreover, for $E/F$ Galois,

$$\begin{array}{ccc} H^1(E/F,G) & \longleftrightarrow & \left\{\begin{array}{l}\text{iso. classes of}\\ G\text{-torsors}/F \text{ that}\\ \text{split } /E\end{array}\right\} \\ \cap & & \\ H^1(F,G) & & \end{array}$$

i.e. become trivial.

inclusion induced by surjection

$$\begin{tikzcd} \mathrm{Gal}(F) \arrow[r, two heads] \arrow[dr, dashed, "\in Z^1(F,G)"'] & \mathrm{Gal}(E/F) \arrow[d, "\in Z^1(E/F,G)"] \\ & G(E) \end{tikzcd}$$

What is the correspondence

$$H^1(F,G) \longleftrightarrow \left\{ \begin{array}{l} \text{iso. classes} \\ \text{of } G\text{-torsors} \\ \text{over } F \end{array} \right\} ?$$

First, a notational issue:

For a $G$-torsor $X$ over $F$,

$G$ acts on $X(F^{sep})$, by the torsor action.

Also $\Gamma = Gal(F)$ acts on $X(F^{sep})$.

These actions don't commute.

So need to have them act on opposite sides of $X$.

Convention: $\Gamma$ acts on $X(F^{sep})$ on the left, and $G$ acts on $X(F^{sep})$ on the right.

So the $G$ action is of the form (rt action)

$$X \times G \longrightarrow X \qquad (x,g) \mapsto x \cdot g$$

and the torsor condition becomes:

$$X \times G \xrightarrow{\sim} X \times X \qquad (x,g) \mapsto (x, x \cdot g)$$

Now, to describe the correspondence

$$H^1(F, G) \leftrightarrow \left\{ \begin{array}{l} \text{iso. classes} \\ \text{of } G\text{-torsors} \\ \text{over } F \end{array} \right\}:$$

Given a $G$-torsor $X$ over $F$, with an action of $\Gamma = \mathrm{Gal}(F)$ on $X(F^{sep})$:

<u>pick</u> $x_0 \in X(F^{sep})$.

left action of Galois gp on $X(F^{sep})$

For $\gamma \in \Gamma$, $\gamma \cdot x_0 \in X(F^{sep})$,

so $\exists!$ elt. $f(\gamma) \in G(F^{sep})$ st

$$\underset{\Gamma}{\gamma} \cdot \underset{X(F^{sep})}{x_0} = \underset{X(F^{sep})}{x_0 \cdot f(\gamma)}$$

right action of lin. alg. gp. $G$

<u>Easy to check</u>: $f : \Gamma \to G(F^{sep})$ is a 1-cocycle; i.e. $f \in Z^1(\Gamma, G(F^{sep}))$.

Changing the above choice of $x_0 \in X(F^{sep})$ gives a new $f$, cohomologous to the above $f$.

So we get a well defined element of

$$H^1(\Gamma, G(F^{sep})) = H^1(F, G)$$

(Similarly, for a $G$-torsor $X$ over $F$ that splits $/E$, we have an action of $\mathrm{Gal}(E/F)$ on $X(E)$, and we get an element of $H^1(E/F, G)$)

For the reverse direction, given an elt of $H^1(F, G)$, pick a representing cocycle $f \in Z^1(F, G)$.

So $f : \underset{\mathrm{Gal}(F^{sep}/F)}{\underset{\parallel}{\Gamma}} \longrightarrow G(F^{sep})$ st $df = 0$.

Let $X$ be $G$ (over $F^{sep}$) viewed as a set.

Put a right $G$-action on $X$: if $\underset{X}{\underset{\cap}{x}} \leftrightarrow \underset{G}{\underset{\cap}{g}}$

then for $h \in G$ define $\underset{X}{\underset{\cap}{x}} \cdot h \leftrightarrow \underset{G}{\underset{\cap}{gh}}$

Also put a left $\Gamma$-action on $X$:

For $\gamma \in \Gamma$, $x \in X$, $\gamma \cdot x \longleftrightarrow f(\gamma)\,\gamma(g)$

$x \in X \leftrightarrow g \in G$, $\gamma \cdot x \in X$, $f(\gamma)\,\gamma(g) \in G$

This defines a $G$-torsor with a $\Gamma$-action, so a $G$-torsor$/F$. Can check this direction is inverse to the other direction.

Ex. For any field $F$, Hilbert 90 says $H^1(F, GL_n) = 1$. So every $GL_n$-torsor$/F$ is trivial. Similarly for $SL_n$-torsors.

But for other linear algebraic groups $G$, often $\exists$ non-trivial torsors.

— as we saw with $SO_{2,\mathbb{R}}$. Also:

Ex.: $$H^1(F, O_n) \xleftrightarrow{\text{bij}} \left\{\begin{array}{l}\text{Isometry classes of}\\ \text{regular quadratic forms}\\ \text{over } F \text{ of dim } n\end{array}\right\}$$

The existence of non-iso. q.f.'s$/F$ $\Leftrightarrow$ $\exists$ non-trivial $O_n$-torsors$/F$.

The above comes from a general principle:

Given an algebraic object $\Delta$ over a field $F$, let $G = \mathrm{Aut}(\Delta)$, the gp of aut's of $\Delta$.

For reasonable objects $\Delta$, this is an alg. gp $/F$.

(& in key cases, a linear alg gp, i.e. $\subseteq GL_{n,F}$)

& for each $E/F$, have $G(E) = \mathrm{Aut}(\Delta(E))$.

Then:

$$\begin{array}{ccc} H^1(F, G) & \longleftrightarrow & \left\{ \begin{array}{l} \text{iso classes of objects } /F \\ \text{that become iso to } \Delta \text{ over } F^{sep} \end{array} \right\} \\ \cup\!\!\!\!\mid & & \cup\!\!\!\!\mid \\ \text{distinguished element} & \longleftrightarrow & [\Delta] \end{array}$$

forms of $\Delta$

More gen'lly,

$$H^1(E/F, G) \longleftrightarrow \left\{ \begin{array}{l} \text{iso classes of objects } /F \\ \text{that become iso to } \Delta \text{ over } E \end{array} \right\}$$

Back to Example:

regular quadratic forms of dim $n$ /F $\leftarrow$ char $\neq 2$

Take $q = \langle 1, 1, \ldots, 1 \rangle$. $G = \mathrm{Aut}(q) = O(q) = O_{n,F}$.

$H^1(E/F, O_n) \longleftrightarrow$ reg. q.f./F that become isometric to $q$ over E.

Case $E = F^{sep}$:

All reg qf/F become iso to $q$ / $F^{sep}$.

So:

$H^1(F, O_n) \longleftrightarrow \{$iso. cl. of reg qf of dim $n$ over F.$\}$

$\longleftrightarrow \{$iso cl. of $O_n$-torsors /F$\}$

as claimed.

To prove the above general principle, that

$H^1(E/F, G)$ classifies the forms $\Delta'$ of $\Delta$ /F that become iso to $\Delta$ over E: ($G = \mathrm{Aut}(\Delta)$)

Use: the structure consists of a v.s. together with add'l data, & aut's of the structure form a subgp of $GL_n$, functorially in the field.

Ex. For qf's of dim $n$: a quadratic space $(V,q)$, VS of dim $n$

$G = \mathrm{Aut}(q) = O(q) \subset GL_n$.

Ex. $A$ csa of deg $n$: a $n^2$-dim'l vs, with add'l structure

$G = \mathrm{Aut}(A) \subset GL_{n^2}$.

In general, say we have an alg. object $\Delta/F$, (e.g. qf, csa, ...), viewed as a structure on $F^n$.

So $G := \mathrm{Aut}\,\Delta \subset GL_n$. Take extension $E/F$.

Let $X = \{\text{objects}/E \text{ iso to } \Delta \text{ over } E\}$. viewed as structures on $E^n$

A pointed set, with distinguished elt $\Delta$

$X$ is acted on by $\Gamma := \mathrm{Gal}(E/F)$; a $\Gamma$-set.

$H^0(\Gamma, X)$

$X^\Gamma = \{\text{objects}/F \text{ iso to } \Delta \text{ over } E\}$

$= \{\text{forms of } \Delta_{/F}, \text{ iso to } \Delta \text{ over } E\}$.

Two such forms are iso $/F$ if in the same orbit of

$$GL_n(F) = GL_n(E)^\Gamma$$
$$= H^0(E/F, GL_n)$$

So the iso classes of forms of $\Delta$ over $F$ that become iso to $\Delta$ over $E$ are in bijection with the orbits of $GL_n(F)$ on $X^\Gamma = H^0(\Gamma, X)$.

The set $X$ is acted on transitively by $GL_n(E)$, and the stabilizer of $\Delta \in X$ is $\mathrm{Aut}(\Delta)(E) = G(E)$.

So $X \xleftrightarrow[\substack{\text{preserving} \\ \Gamma\text{-action}}]{\text{bij}} GL_n(E) / G(E)$ (left cosets)

(iso of ptd sets)

The inclusion $G = \mathrm{Aut}(\Delta) \hookrightarrow GL_n$ then gives a 5-term ; cohomology exact sequence, as discussed. This will then show the general principle.