We are proving: Hasse–Minkowski Theorem:

Say F a global field, q a q.f. $/F$.

Then: q isotropic $/F \iff q$ isotropic $/$every F_v.

For proof: WMA q regular. We saw:

For dim $q=1$: $q=\langle a\rangle$, anisotropic $/F, F_v$ ✓

For dim $q=2$: $q=\langle a,b\rangle$.

After mult by constant, STS for $\langle 1,-a\rangle$

$a\in F^\times$. This is isotropic $\iff a$ is a square.

So WTS: $a\in F_v^{\times 2}$ for all $v \Rightarrow a\in F^{\times 2}$.

Case of $F=\mathbb{Q}$:

Say $a\in \mathbb{Q}_p^{\times 2}$ $\forall p$, and $a\in \mathbb{R}^{\times 2}$.

i.e. $a>0$

So $v_p(a)$ is even

So prime factorization of a is $\prod_{i=1}^{r} p_i^{2n_i}$.

So a is a square in $\mathbb{Q}$, ✓

also a UFD

Similarly if $F=\mathbb{F}_p(x)=\mathrm{frac}\ \mathbb{F}_p[x]$.

In general case: a global field $F = \operatorname{frac} R$

Dedekind domain

Say $a \in F_v^{\times 2}$ $\forall v$; WTS $a \in F^{\times 2}$.

Contrapositive: Say $a \notin F^{\times 2}$.

WTS $a \notin F_v^{\times 2}$, some v.

Let $E = F[\sqrt{a}]$. Degree 2 field extension

(i.e., of R)

Then $\exists$ primes of F

that remain prime in S:

i.e.
$$\begin{tikzcd}
\exists ! P \arrow[d, dash] & \subset S \arrow[d, dash, "2"] & \subset E \arrow[d, dash, "2"] \\
\mathfrak{p} & \subset R & \subset F
\end{tikzcd}$$

(Tchebotarev Density Thm: half the primes split, half remain prime)

Take such a $\mathfrak{p} \longleftrightarrow v$

$$\begin{tikzcd}
E_P \arrow[d, dash, "2"'] = E \otimes_F F_{\mathfrak{p}} = & F_{\mathfrak{p}}[\sqrt{a}] \arrow[d, dash] \\
F_{\mathfrak{p}} = & F_{\mathfrak{p}}
\end{tikzcd}$$

So no $\sqrt{a}$ in $F_{\mathfrak{p}} = F_v$; $a \notin F_v^{\times 2}$.

For H-M in dim 3 & 4:

Two approaches:

- Lam: Use a special case of the local-global principle for splitting of csa's (Lam gives a proof of that LGP in case of $F=\mathbb{Q}$; the full thm is related to class field theory in number thy.)
- Serre, A Course in Arithmetic, Chap. IV, §3
  — gives a direct proof of H-M, but only over $\mathbb{Q}$.

Will follow Lam's approach to H-M

LGP for csa's: Theorem of

Albert — Brauer - Hasse - Noether

independently

Statement:

Let F be a global field.
Let A be a csa$/F$. Then:

A is split$/F$ $\Longleftrightarrow$

A is split$/F_v$ for all v.

Equivalently: the map

$$\mathrm{Br}(F) \to \prod_v \mathrm{Br}(F_v)$$

is injective.

For histories of these two local-global theorems, see:

1) R. Parimala, A Hasse Principle for Quadratic Forms over Function Fields.
Bull. AMS (81), April 2014, 447-461.

2) Peter Roquette, The Brauer-Hasse-Noether Theorem in Historical Perspective. Springer monograph, 2005

Both are available online.

We won't need the full strength of ABHN — just for quaternion algebras (rather than general csa's).

Can reformulate ABHN for quaternions, using that a quaternion algebra $A = \left(\frac{a,b}{F}\right)$ is classified by its norm form $q = \langle 1, -a, -b, ab \rangle$; & that A is split

$\iff q$ is hyperbolic

$\iff q$ is isotropic.

So we can rephrase ABHN in this situation as:

$$\left.\begin{array}{l} q = \langle 1, -a, -b, ab \rangle \text{ is hyperbolic } /F \\ \qquad\qquad\qquad\qquad \text{(or: isotropic)} \\ \iff q \text{ is hyperbolic } /F_v \text{ for all } v. \end{array}\right\} (*)$$

Of course $(*)$ is a special case of H-M. But it can also be used in the proof of H-M if shown separately.

Lam proves (*) for $F = \mathbb{Q}$

in Chap. VI, §4, as follows:

For each odd p, we have

$$W(\mathbb{Q}) \to W(\mathbb{Q}_p) \xrightarrow{\partial_2} W(\mathbb{F}_p)$$

second residue form

By abuse of notation, call this composition $\partial_{(p)}$.

Also, define $W(\mathbb{Q}) \xrightarrow{\partial_{(\infty)}} W(\mathbb{R}) = \mathbb{Z}$

by $q \longmapsto \operatorname{sign}(q)$ (signature)

And define $W(\mathbb{Q}) \xrightarrow{\partial_{(2)}} \mathbb{Z}/2$

by $q \longmapsto \nu_2(\det q) \mod 2$

Then have a map

$$\partial := (\partial_{(\infty)}, \partial_{(2)}, \partial_{(3)}, \partial_{(5)}, \dots) : W(\mathbb{Q}) \to \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \bigoplus_{p \neq 2} W(\mathbb{F}_p)$$

In Chap VI §4, Lam shows directly that

∂ is an <u>isomorphism</u>. This gives the structure

of $W(\mathbb{Q})$.

Using that ∂ is an iso, we can get the following:

<u>Prop.</u> If q.f.'s $q, q'/\mathbb{Q}$ become isometric $/$all $\mathbb{Q}_p$ + $/\mathbb{R}$, then they are isometric $/\mathbb{Q}$.

<u>Proof</u> q, q' isometric $/\mathbb{R} \Rightarrow$ same signature.
q, q' isometric $/\mathbb{Q}_2 \Rightarrow v_2(\det q) \equiv v_2(\det q') \mod 2$.

So $\partial_{(\infty)}(q) = \partial_{(\infty)}(q')$ and $\partial_{(2)}(q) = \partial_{(2)}(q')$.

Since isometric $/\mathbb{Q}_p$, have same 2^D f.i.l. form at p: $\partial_{(p)}(q) = \partial_{(p)}(q')$.

So $\partial(q) = \partial(q')$. ∂ iso $\Rightarrow$ q, q' in same class in $W(\mathbb{Q})$. But same dim. So $\underline{q \cong q'}$.

In particular, this applies to a q.f. q of dimension 4, and $q' = 2h = \langle 1, -1, 1, -1 \rangle$.

So: if q is hyperbolic at each completion of $\mathbb{Q}$, then q, q' are isometric over each completion; so above Prop $\Rightarrow$ $q \cong q'$. $\therefore$ q hyperbolic.

Apply to $q = \langle 1, -a, -b, ab \rangle$. Get the desired special case of ABHN for $F = \mathbb{Q}$:

q hyperbolic $/F \iff q$ hyperbolic $/$ all F_v

Equiv: q isotropic$/F \iff q$ isotropic $/$ all F_v

Equiv: $A = \left(\frac{a,b}{F}\right)$ split $/F \iff A$ splits $/$ all F_v.

To use this to prove H-M for the

Case dim $q = 3$:

Say dim $q = 3$, q regular$/F$, q isotropic $/$each F_v.
WTS q isotropic $/F$.

WMA $q = \langle 1, a, b \rangle$, $a, b \in F^\times$.

q isotropic $/F_v \Rightarrow \langle 1, a, b, ab \rangle$ isotropic $/ F_v$.

norm form of $\left(\frac{-a,-b}{F}\right)$

So $\left(\frac{-a,-b}{F}\right)$ splits $/ F_v$ for all v,
and $\langle 1, a, b, ab \rangle$ is hyperbolic$/F_v$.

By the above result, $\left(\frac{-a,-b}{F}\right)$ splits $/F$,

or equiv! $\langle 1, a, b, ab\rangle$ is hyperbolic $/F$.

i.e. $\langle 1, a, b, ab\rangle \cong 2h = \langle 1, -1, -ab, ab\rangle$

By Witt Cancellation,

$$q := \langle 1, a, b\rangle \cong \langle 1, -1, -ab\rangle$$

isotropic $/F$

$\therefore$ q isotropic $/F$.

Case: $\dim q = 4$.

First: another theorem of Springer:

Thm (Lam, Chap VII, Thm 2.7)
Say K/F is a field extension such that $n := [K:F]$ is odd. Let q be a q.f. $/F$ that is anisotropic $/F$. Then q is also anisotropic $/K$.

As a nice consequence of this thm:

Cor. If K/F has odd degree, then $u(K) \geq u(F)$.

Pf of above thm: By contradiction.

Suppose $\exists$ anisotropic q.f. q/F and K/F of odd degree n st q is isotropic $/K$. If so, take (K, q) st n is min'l. So $n > 1$.

Say $K = F(\alpha_1, \ldots, \alpha_m)$. By induction on m, we are reduced to the case $m = 1$ (since each step $\begin{array}{c} F(\alpha_1, \ldots, \alpha_{r+1}) \\ | \\ F(\alpha_1, \ldots, \alpha_r) \end{array}$ has odd degree)

So $K = F(\alpha)$. Let $p(t) =$ min'l poly of α / F.

So $\deg p(t) = n$.

q isotropic / $K \Rightarrow \exists\, \gamma_1, \dots, \gamma_d \in K$

s.t. $\quad q(\gamma_1, \dots, \gamma_d) = 0$, where $d = \dim q$.

$K = F[\alpha] \Rightarrow \gamma_i = g_i(\alpha)$, $\deg g_i < n = \deg p(t)$

We may choose $\gamma_1, \dots, \gamma_d$ s.t. $\max_i \deg g_i$ is <u>min'l</u>.

Then the poly's $g_1(t), \dots, g_d(t) \in F[t]$

are relatively prime (no common factor f)

— or else we'd have $g_i(t) = \tilde{g}_i(t) f(t)$

and then $q(\tilde{\gamma}_1, \dots, \tilde{\gamma}_d) = 0$ with $\tilde{\gamma}_i = \tilde{g}_i(\alpha)$,

contradicting minimality of $\max_i \deg g_i$.

So the ideal $(g_1(t), \dots, g_d(t)) = (1)$ in $F[t]$

& hence also in $\bar{F}[t]$.

Under $F[t] \xrightarrow{\varphi} K$, $\quad t \longmapsto \alpha$

$q(g_1(t), \dots, g_d(t)) \longmapsto q(\gamma_1, \dots, \gamma_d) = 0$

$\therefore$ in $\ker \varphi = (p(t)) \subset F[t]$.

So $\quad q(g_1(t), \dots, g_d(t)) = p(t)\, h(t) \qquad (*)$

some poly in $F[t]$.

$$\underset{\text{even}}{\deg}\; q(g_1(t), \dots, g_d(t)) \le 2 \max_i \deg g_i(t)$$
$$\le 2(n-1)$$

and $\deg p(t) = n$,

So $(*) \Rightarrow \deg h(t) \le n-2$.

LHS of $(*)$ has even degree,

& $p(t)$ has odd degree n,

So $\deg h(t)$ is odd.

$\deg h_1 \le \deg h \le n-2$

So $h(t) \in F[t]$ has an odd deg irred factor $h_1(t)$

Let $\beta \in \overline{F}$ be a root of $h_1(t)$. $\therefore [F(\beta):F] \le n-2$

So $h(\beta) = 0$. But we had $\deg h_1$

$$q(g_1(t), \dots, g_d(t)) = p(t)h(t). \quad (*)$$

So $q(g_1(\beta), \dots, g_d(\beta)) = p(\beta)h(\beta) = 0 \in F(\beta)$

So q is isotropic over an

odd degree extension$/F$ of $\deg \le n-2 < n$, $\quad \deg \le n-2 / \text{odd}$ F

Contradicting the minimality of n.

Above thm of Springer is for extensions of odd degree: can't gain new isotropy.

What about extensions of even degree?

Then a q.f. can become isotropic (e.g. $\langle 1,1 \rangle$, going from $\mathbb{R}$ to $\mathbb{C}$)

Q: When does this happen? Ans:

Thm (Lam, Ch VII, Thm 3.1.)

Let q be an anisotropic q.f. $/F$.

Let $a \in F^\times - F^{\times 2}$ and $K = F(\sqrt{a})$.

Then q becomes isotropic $/K$

$\iff$ $q \cong \langle -ab, b \rangle \perp q'$ for some $b \in F^\times$ and some q.f. q' over F.

Equiv: q remains anisotropic $/K$ unless q is of this form $/F$.

Pf of thm:

($\Leftarrow$) If q is of this form, then $/K$:

$q \cong \langle -ab, b\rangle \perp q'$

$\cong \langle -b, b\rangle \perp q'$ is isotropic $/K$.

$a \in K^{\times 2}$ $\quad h$

($\Rightarrow$) Say $q = \langle b_1, \ldots, b_n\rangle$, anisotropic $/F$, becomes isotropic $/K = F[\sqrt{a}]$.

WTS $q \cong \langle -ab, b\rangle \perp q'$.

q isotropic $/K \Rightarrow \exists\, z \neq 0 = (z_1, \ldots, z_n) \in K^n$ st $q(z) = 0$. Write $z_i = x_i + \sqrt{a}\, y_i$, $x_i, y_i \in F$. Write $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

So $x, y \in F^n$, not both 0, since $x + \sqrt{a}\, y = z \neq 0$.

$$0 = q(z) = \sum b_i z_i^2 = \sum b_i (x_i + y_i \sqrt{a})^2$$

$$= \alpha + \beta \sqrt{a}$$

where $\alpha = q(x) + a\, q(y) \in F$

and $\beta = 2B(x, y) \in F$, where $B \longleftrightarrow q$

$\alpha + \beta\sqrt{a} = 0$, $\alpha, \beta \in F \Rightarrow \alpha, \beta = 0$.

$\therefore B(x, y) = 0$, so $x \perp y$ wrt q.

x, y not both 0; q anisotropic

$\Rightarrow q(x), q(y)$ not both 0.

$0 = \alpha = q(x) + a\,q(y) \Rightarrow q(x) = -a\,q(y)$;
but $a \neq 0$. So neither of $q(x), q(y)$ is 0.

$\therefore x, y$ both non-0; and orthogonal.

So can extend $\{x, y\}$ to an orthogonal basis.

Wrt this basis,

$$q \cong \langle q(x), q(y), \ldots \rangle = \langle -a\,q(y), q(y), \ldots \rangle$$

q eval. at other basis elts

$$= \langle -a\,q(y), q(y) \rangle \perp q'$$

Set $b = q(y)$. So $q \cong \langle -ab, b \rangle \perp q'$.

Cor. Say q a qf $/F$, $\dim q = 4$, $\delta := \det q$. If q is anisotropic $/F$ then q is anisotropic $/K := F[\sqrt{\delta}]$.

Equiv: q isotropic $/K \Rightarrow q$ isotropic $/F$.

Pf. WMA $\delta \notin F^{\times 2}$ (otherwise trivial).

By the prev. thm,

q isotropic $/K = F[\sqrt{\delta}] \Rightarrow q \cong \langle -\delta b, b \rangle \perp q'$

for some $b \in F^\times$ and q', q.f. $/F$.

dim = 2

$\det(\langle -\delta b, b \rangle) = -\delta \in F^\times / F^{\times 2}$,

$\det q = \delta$, so $\det q' = -1$

$\therefore q' \cong \langle c, -c \rangle \cong h$, isotropic $/F$.

$\therefore q$ isotropic $/F$.

Using this, we can prove H-M for $\dim q = 4$:

Pf of Hasse-Minkowski for dim $q = 4$:

global fld

WMA q regular $/F$, & isotropic $/$each F_v.

WTS q isotropic $/F$. WMA $q \cong \langle 1, a, b, c \rangle$.

$\delta := \det q = abc$

global field

By above Cor, STS q isotropic $/K = F[\sqrt{\delta}]$

$$\begin{array}{ccc} K & \subset & K_w \\ | & & | \\ F & \subset & F_v \end{array}$$

abs val's on K restrict to abs val's on F

q isotropic $/$ each $F_v \Rightarrow q$ isotropic $/$each K_w

$\underset{abc}{\overset{\shortparallel}{\delta}} \in K^{\times^2} \Longrightarrow c, ab$ are in the same square class in $K^\times / K^{\times^2}$

So $q = \langle 1, a, b, c \rangle \cong \langle 1, a, b, ab \rangle \quad /K$

norm form of $\left(\frac{-a, -b}{F}\right)$

q isotropic$/$each K_w

$\Downarrow$ previous result (re ABHN)

q isotropic $/K$. So ✓.

To prove H-M for dim ≥ 5, by induction:

Will use the Weak Approximation Thm, an extension of Chinese Remainder Thm:

Weak Approx for a global field F:

Given distinct abs. values $v_1, \ldots, v_m$ on F, and $\varepsilon_1, \ldots, \varepsilon_m > 0$, and $a_i \in F_{v_i}$ for $i = 1, \ldots, m$, $\exists\, a \in F$ st $|a - a_i|_{v_i} < \varepsilon_i$ for all i.

Ex. $F = \mathbb{Q}$, $v_i \leftrightarrow p_i$ (primes), $\varepsilon_i = \frac{1}{p_i^{n_i}}$ for $i = 1, \ldots, m$, $a_i \in \mathbb{Z}_{p_i}$.

$\mathbb{Z}$ dense in $\mathbb{Z}_{p_i}$ wrt $|\cdot|_{p_i}$; take $b_i \in \mathbb{Z}$, $|b_i - a_i|_{p_i} \leq \frac{1}{p_i^{n_i}}$.

CRT $\Rightarrow \exists\, a \in \mathbb{Z}$ st. $a \equiv b_i \pmod{p_i^{n_i}}$ for all i; so $|a - a_i|_{p_i} \leq |a - b_i|_{p_i} \leq \frac{1}{p_i^{n_i}}$, ✓.

(By clearing denoms, OK even if $a_i \in \mathbb{Q}_{p_i}$; get $a \in \mathbb{Q}$.)

Weak approx also allows archimedean abs. val's — + so extends CRT.

It remains to prove Hasse-Minkowski for q.f.'s q of dim ≥ 5.

We will proceed by induction on dim q:

We'll take q of some dim ≥ 5, and assume it holds for all q.f.'s of lower dim. (We already know it holds in dim ≤ 4)

We'll assume q isotropic / all F_v.
WTS q isotropic / F.

<u>Strategy</u>: Write $q \cong \langle w \rangle \perp q'$

lower dim

st q' is isotropic / every completion F_v.

Once we do this, we can apply H-M to q'. (by inductive hypothesis). So get

q' isotropic / all $F_v \overset{HM}{\Rightarrow} q'$ isotropic / F

$\Rightarrow q$ isotropic / F.