Recall: q a q.f. $\rightsquigarrow$ Clifford alg $A = C(V, q)$
$= T(V)/I(q)$; has $\mathbb{Z}/2$-grading.
$F, V \subseteq A$, $v^2 = q(v)$ for $v \in V$.

Ex. $V = F^2$, basis $\{i, j\}$, $q = \langle a, b\rangle$, dim 2.

$C(V, q) = \left(\frac{a, b}{F}\right)$. dim $= 4 = 2^2$. Grading:

$C_0 = F \oplus kF$, $C_1 = iF \oplus jF$.

As a graded alg, write $\left\langle \frac{a, b}{F} \right\rangle$.

Ex. $V = F$, basis 1, $q = \langle a \rangle$ (dim 1). As gr. alg., write $F\langle\sqrt{a}\rangle$,

$C(V, q) = F[t]/(t^2 - a)$; $C_0 = F \cdot 1$, $C_1 = F \cdot t$

So $\dim C(V, q) = 2 = 2^1$ here.

Can use this grading to show that
$\dim C(V, q) = 2^n$, where $n = \dim V$. (As in above ex's)
Had $\leq$, with spanning set $x_1^{e_1} \cdots x_n^{e_n}$, $e_i \in \{0, 1\}$. Want $\geq$.
Use <u>graded $\otimes$</u> of $\mathbb{Z}/2$-graded alg's A, B:
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $A_0 \oplus A_1$, $B_0 \oplus B_1$

$A \hat{\otimes} B$: as v.s. same as $A \otimes B$

Mult on homog elts:

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\partial b \, \partial a'} aa' \otimes bb'$$

$a, a' \in A$, $b, b' \in B$ homog. elements

interchanging b, a'

Can check: have natural hom of gr. algs:

$$C((V,q)\perp(V',q')) \longrightarrow C(V,q)\,\hat{\otimes}\,C(V',q')$$

& this is a surj (Lam, Chap V, Lu 1.7)

So if diagonalize $q = \langle a_1, \dots, a_n\rangle$

$$= \langle a_1\rangle \perp \cdots \perp \langle a_n\rangle$$

& apply surj, & use $\dim C(F, \langle a\rangle) = 2$,

get $\dim C(V,q) \geq \prod \dim C(F, \langle a_i\rangle) = 2^n$. ✓

So the elts $x_1^{e_1} \cdots x_n^{e_n}$ are a basis of $C(V,q)$

Basis of $C_0(V,q)$: those with $\sum e_i$ even.

" " $C_1(V,q)$: " " " " odd.

Ex. Had $C(h) = \hat{M}_2(F)$.

For $m \geq 1$, $C(mh) = \hat{M}_2(F)^{\otimes m}$

$$=: \hat{M}_{2^m}(F), \text{ graded};$$

with basis elts E_{ij}; $\partial E_{ij} = \begin{cases} 0 & i+j \text{ even} \\ 1 & i+j \text{ odd} \end{cases}$

("checkerboard grading")

$(V,q) \rightsquigarrow C(V,q)$, graded alg.

If $q = \langle a,b\rangle$, $C(V,q) = \left\langle \frac{a,b}{F} \right\rangle$ as gr. alg;

as an alg, quat. alg $\left(\frac{a,b}{F}\right)$ (csa)

If $q = \langle a \rangle$, $C(V, q) = F[t]/(t^2 - a)$ as graded alg; commutative, so not central.

To remedy this: notion of

central simple graded algebra (csga)

[= "super csa", scsa]

For this, define graded centralizer $\hat{C}_A(S)$, gen by homog elts c st $cs = (-1)^{\partial c \partial s} sc$ for all homog $s \in S$.

Get graded center $\hat{Z}(A) := \hat{C}_A(A)$.

A is (graded) central if $\hat{Z}(A) = F$.

Graded ideal: ideal that's a graded subspace (i.e., $\oplus$ of homog parts)

If a graded alg A has no proper graded ideals: simple graded alg.

If gr. alg. A is central & simple: "central simple graded alg."

Get analogs of results on csa's with essentially the same proofs:

Thm (Lam, Chap IV, Thm 2.3)

1) A, B gr F-alg, $A' \subseteq A$, $B' \subseteq B$ graded sub-alg's

$$\Rightarrow \hat{C}_{A \hat{\otimes} B}(A' \hat{\otimes} B') = \hat{C}_A(A') \hat{\otimes} \hat{C}_B(B')$$

2) A is csga $/F$, B is sga $/F$
$\Rightarrow A \hat{\otimes} B$ is sga $/F$.

3) A, B csga $/F \Rightarrow$ so is $A \hat{\otimes} B$.

Also get:

Thm (Lam, Chap V, Thm 2.1)

$C(V, q)$ is a csga $/F$.

even though not nec csa

Pf by induction on $n = \dim V$.

$n=1$: $C(V, q) = F\langle \sqrt{a} \rangle$. Check directly
$q = \langle a \rangle$

Inductive step: use
$C(q \perp q') = C(q) \hat{\otimes} C(q')$.

Can form "graded Brauer group" from equiv classes of csga's:

A, A' equiv if $A \hat{\otimes} \widehat{\mathrm{End}}(V) \cong A' \hat{\otimes} \widehat{\mathrm{End}}(V')$.

Equiv classes $\leadsto$ group: inverse of class of A is class of A^*, graded opposite alg:

$$a^* \cdot b^* = (-1)^{\partial a \partial b} (ba)^*.$$

[This is a cga (resp csga) if A is.]

Brauer-Wall group, $BW(F)$.

(C.T.C. Wall)

Natural inclusion $Br(F) \xrightarrow{i} BW(F)$ by trivial grading on csa A: $A_0 = A$, $A_1 = 0$.

In fact, have s.e.s. (Lam, ch. IV, Th 4.4):

$$0 \longrightarrow Br(F) \longrightarrow BW(F) \rightarrow Q(F) \rightarrow 0$$

Same group as before, from q.f.'s!

Recall: we had

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^\times/F^{\times 2} & \rightarrow & Q(F) & \rightarrow & \mathbb{Z}/2 & \rightarrow & 0 \\ & & \cong & & \cong & & \cong & & \\ 0 & \longrightarrow & I(F)/I^2(F) & \rightarrow & W(F)/I^2(F) & \rightarrow & W(F)/I(F) & \rightarrow & 0 \end{array}$$

Above, in fact we have a comm. diag. with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & I^2(F) \arrow[r] \arrow[d, "\gamma"] & W(F) \arrow[r] \arrow[d, "\Gamma"] & W(F)/I^2(F) \arrow[r] \arrow[d, "s"] & 0 \\
0 \arrow[r] & Br(F) \arrow[r] & BW(F) \arrow[r] & Q(F) \arrow[r] & 0
\end{tikzcd}$$

where Γ is induced by taking the class of the Clifford invariant, & $\gamma = \Gamma|_{I^2(F)}$. Also,

$$\ker \Gamma = \ker \gamma = I^3(F).$$

To explain this:

1st describe $BW(F) \to Q(F)$.

Elts of $Q(F)$: (e,d) where $e \in \mathbb{Z}/2 = W(F)/I(F)$ and $d \in F^\times/F^{\times 2} \cong I(F)/I^2(F)$.

So given a csga A over F, representing a class in $BW(F)$, want to give a pair (e,d).

Here $e \in \mathbb{Z}/2$ is called the <u>type</u> of A: either 0 (even) or 1 (odd).

To define the type:

Say A is of <u>even type</u> $(e=0)$ if A is a csa (as an ungraded alg)

Ex. $q=\langle a,b\rangle$, $C(q)=\left(\frac{a,b}{F}\right)$; csa, even type

Otherwise: <u>odd type</u> $(e=1)$.

Ex. $q=\langle a\rangle$, $C(q)=F\langle\sqrt{a}\rangle$, csga but not csa; odd type.

More gen'ly: Say q is a q.f. of dim n. View $C(q)$ as a csga. Then turns out:

$C(q)$ is of even type $\Leftrightarrow$ n is even

(as in above ex's). Will see this.

First, to understand the type better:

Let $A=A_0\oplus A_1$ be a csga, with center Z. Then Z is a csga, with grading $Z=Z_0\oplus Z_1$, where $Z_i=Z\cap A_i$.

Here $Z_0=F$ since A is a csga.

So $Z=F\oplus Z_1$. Suppose $A_1\neq 0$, i.e. $A\neq A_0$.

Can show!

as ungraded alg

1) $Z_1 = 0 \iff A$ is a csa (easy)

2) $Z_1 \neq 0 \iff A_0$ is a csa

(See Lam, Ch IV, Th 3.4)

So A of even type (i.e. csa)
$\Updownarrow$ by (1)
$Z_1 = 0$

So A of odd type $\iff Z_1 \neq 0$
$\iff$ by (2) A_0 is a csa

So either A or A_0 is a csa — as ungraded alg

but not both; corresp to even & odd cases.

For $BW(F) \longrightarrow Q(F)$

$\langle A \rangle \longmapsto (e, d)$, $\in F^\times / F^{\times 2}$

above gives e, the type. Re d:

Take a csga $A = A_0 \oplus A_1$.

If A is of even type and $A_1 = 0$

(i.e. $A = A_0$, a csa viewed as csga in deg 0)

take $d = 1$, trivial square class.

Otherwise: (Lam, Ch IV, Th 3.8, 3.6)

centralizer $C_A(A_0) = F \oplus Fz$

for some $z \in A$ st $z^2 \in F^\times$, where

i) if A of even type (& $A_1 \neq 0$), can take $z \in Z(A_0)$;

ii) if A of odd type, can take $z \in Z_1$.

In each case, z is unique up to mult by $F^\times$.

(well def.)

In either case, take $\underset{=:\delta(A)}{d} = z^2 \in F^\times/F^{\times 2}$

So $\forall A$, have an elt $d \in F^\times/F^{\times 2}$.

Now define $BW(F) \longrightarrow Q(F)$

$\langle A \rangle \longmapsto (e, d)$.

Can check: This is a well def. group hom (Lam, Ch IV, Th 4.3).

Moreover, it's surjective.

Also: the restriction to $Br(F) \subseteq BW(F)$ is trivial (immediate); in fact, $Br(F)$ is the full kernel. So get ses.

(Lam Ch IV Th 4.4) $0 \to Br(F) \to BW(F) \to Q(F) \to 0$.

Coming back to Clifford alg's $C(q)$ assoc. to q.f.'s q:

Say $\dim q = n$; diagonalize, take orthog. basis $e_1, \ldots, e_n$. Let $z = e_1 \cdots e_n \in C(q)$. (a basis element)

i) If n odd, $\deg z = 1 \in \mathbb{Z}/2$, and z commutes with each e_i; so $z \in Z_1$ $\Rightarrow C(q)$ of odd type. (use $e_ie_j = -e_je_i$ for $i \neq j$)

$0 \neq \ldots \neq 0$

ii) If n even, $\deg z = 0 \in \mathbb{Z}/2$, and z anti commutes with all e_i, & so commutes with all e_ie_j. So $z \in Z(C_0(q))$

$F \not\ni$

So $C_0(q)$ is not a cse$/F$; so $C(q)$ not odd type; so even type.

So indeed, $C(q)$ is $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ if $\dim q$ is $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$, as asserted. (Lam, Ch V, Th 2.2)

This describes the type $e \overset{\mathbb{Z}/2}{\in}$ of $C(q)$.

What about $d = \delta(A)$ of $\overset{A}{\hat{C}}(q)$, in terms of q?

$\in F^\times/F^{\times 2}$ (Introduced earlier)

Ans: It's $\det_{\pm} q := (-1)^{\frac{n(n-1)}{2}} \det q$.

Easy computation using $z = e_1 \cdots e_n$ above.

See Lam, Chap V, Th 2.3.

Recall: We're constructing:

$$\begin{tikzcd} 0 \arrow[r] & I^2(F) \arrow[r] \arrow[d, "\gamma"] & W(F) \arrow[r] \arrow[d, "\Gamma"] & W(F)/I^2(F) \arrow[r] \arrow[d, "S"] & 0 \\ 0 \arrow[r] & Br(F) \arrow[r] & BW(F) \arrow[r] & Q(F) \arrow[r] & 0 \end{tikzcd}$$

We have the rows; taking Clifford alg's defines a map $\hat{W}(F) \to BW(F)$.

Hyperbolic forms are in the kernel, because $C(h) = \hat{M}_2(F)$, trivial in $BW(F)$.

So get $\Gamma : W(F) \to BW(F)$.

This is a group hom, because

$$C(q \perp q') \cong C(q) \hat{\otimes} C(q').$$

Can check this diagram commutes:

$$\begin{tikzcd} W(F) \arrow[r, two heads] \arrow[d, "\Gamma"] & W(F)/I^2(F) \arrow[d, "f", "S"'] \\ BW(F) \arrow[r, two heads] & Q(F) \end{tikzcd}$$

the iso from earlier

So under Γ, $I^2(F) = \ker\left(W(F) \to W(F)/I^2(F)\right)$ maps to $Br(F) = \ker(BW(F) \to Q(F))$

So get comm. diag. with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & I^2(F) \arrow[r] \arrow[d, "\gamma"] & W(F) \arrow[r] \arrow[d, "\Gamma"] & W(F)/I^2(F) \arrow[r] \arrow[d, "f", "\cong"'] & 0 \\ 0 \arrow[r] & Br(F) \arrow[r] & BW(F) \arrow[r] & Q(F) \arrow[r] & 0 \end{tikzcd}$$

as asserted

In particular: If q is a q.f. in $I^2(F)$ then $C(q)$ is concentrated in degree 0, & is a csa.

Ex. Let $q = \langle 1, -a, -b, ab \rangle$
= norm form of $\left(\frac{a,b}{F}\right)$.

What is $\Gamma(q)$?
(i.e. class of $C(q)$ in $BW(F)$)

Ans: $A = \left(\frac{a,b}{F}\right)$, csa, concentrated in deg 0.

Why? Reason:
$q = \langle 1, -a \rangle \otimes \langle 1, -b \rangle \in I^2(F)$
So $\Gamma(q) = \gamma(q) \in Br(F) \subseteq BW(F)$.

(Lam, Ch V, La 3.2 (by computation)) Why is it $\left(\frac{a,b}{F}\right)$? Follows from:

La If $q = \langle a, b, c, d \rangle$ and $\det q = 1$ then
$$\Gamma(q) = \gamma(q) = \left(\frac{-ab, -ac}{F}\right) \in Br(F)$$

Note here:

$\Gamma(\langle a, b\rangle) = \left\langle \frac{a,b}{F} \right\rangle$, with non-trivial grading

$\Gamma(\langle 1, -a, -b, ab\rangle) = \left(\frac{a,b}{F}\right)$, with trivial grading.

Recall: As a group, $I(F)$ is gen by forms $\langle 1, -a\rangle$. So $I^2(F)$ is " " " $\langle 1, -a\rangle \otimes \langle 1, -b\rangle = \langle 1, -a, -b, ab\rangle$.

So: $\gamma(I^2(F))$ is gen by quaternion algs. Recall: these are of order 2 in $Br(F)$ (or trivial)

So: $im\,\gamma \subseteq Br(F)[2]$ ← 2-torsion

In fact, Merkurjev showed: the quaternion alg's generate $Br(F)[2]$.

So: $image\,\gamma = Br(F)[2]$

What about $\ker\gamma$? Answer: $I^3(F)$.

Easy inclusion:

Prop $I^3(F) \subseteq \ker\gamma$.

I.e. If $q \in I^3(F)$, $C(q)$ is trivial in $Br(F)$.

Proof $I^3(F)$ is gen by elts of form

$$\langle 1,-a\rangle \otimes \langle 1,-b\rangle \otimes \langle 1,-c\rangle$$

$$= \langle 1,-a,-b,-c,ab,ac,bc,-abc\rangle$$

$$= \langle 1,-a,-b,ab\rangle - \langle c,-ca,-cb,cab\rangle$$

in $Br(F)$

$$\begin{array}{ccc} \langle 1,-a,-b,ab\rangle & - & \langle c,-ca,-cb,cab\rangle \\ \Big\downarrow \gamma & & \Big\downarrow \gamma \quad \text{(by above formula)} \\ \left(\dfrac{a,b}{F}\right) & & \left(\dfrac{c^2a,\, c^2b}{F}\right) = \left(\dfrac{a,b}{F}\right) \end{array}$$

$= 0 \in Br(F)$.

So $I^3(F) \subseteq \ker\gamma$.

Merkurjev showed $\supseteq$, so: $=$.

So: $\gamma : I^2(F) \to Br(F)$ induces an iso $I^2(F)/I^3(F) \xrightarrow{\sim} Br(F)[2]$.

So have:

$$W(F)/I(F) \cong \mathbb{Z}/2$$

$$I(F)/I^2(F) \cong F^\times/F^{\times 2}$$

$$I^2(F)/I^3(F) \cong Br(F)[2]$$

General pattern?