DIFFERENTIAL GALOIS GROUPS OVER LAURENT SERIES FIELDS

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Abstract. In this manuscript, we apply patching methods to give a positive answer to the inverse differential Galois problem over function fields over Laurent series fields of characteristic zero. More precisely, we show that any linear algebraic group (i.e. affine group scheme of finite type) over such a Laurent series field does occur as the differential Galois group of a linear differential equation with coefficients in any such function field (of one or several variables).

Introduction

Differential Galois theory studies linear homogeneous differential equations by means of their symmetry groups, the differential Galois groups. Such a group acts on the solution space of the equation under consideration, and this furnishes it with the structure of a linear algebraic group over the field of constants of the differential field. In analogy to the inverse problem in ordinary Galois theory, an answer to the question of which linear algebraic groups occur as differential Galois groups over a given differential field provides information about the field and its extensions.

Classically, differential Galois theory was mostly concerned with one variable function fields over the complex numbers. In that case, the inverse problem is related to the Riemann-Hilbert problem (Hilbert's 21st problem) about monodromy groups of differential equations. In fact, Tretkoff and Tretkoff ([TT79]) showed that every linear algebraic group over $\mathbb{C}$ occurs as the differential Galois group of some equation over $\mathbb{C}(x)$, as a consequence of Plemelj's solution to the (modified) Riemann-Hilbert problem ([Ple08]; see also [AB94]). The solution to the inverse differential Galois problem over $C(x)$ for an arbitrary algebraically closed field C of characteristic zero was given in [Har05], building on decades of work by other authors, e.g. [Kov69], [Kov71], [Sin93], [MS96], [MS02]. It is worth noting that contrary to ordinary Galois theory, there seems to be no direct way of deducing this from the complex case for general groups (see [Sin93]).

More recently, there have been results concerning differential Galois theory over non-algebraically closed fields of constants; see, e.g. [AM05], [And01], [Dyc08], [CHvdP13]. (There was actually an earlier attempt, in [Eps55a] and [Eps55b], but the approach did not yield a full Galois correspondence and seems to have been dropped.) The differential Galois groups in this more general setting are still linear algebraic groups, but in the sense

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of group schemes rather than of point groups. However, only very limited results are known about the corresponding inverse problem (in particular, see [Dyc08]).

In this manuscript (Theorem 4.14), we solve the inverse problem for function fields over complete discretely valued fields of equal characteristic zero, which are Laurent series fields. While such fields are never algebraically closed, they share with the complex numbers the property of being complete with respect to a metric topology, therefore allowing local calculations involving series. This relates our approach to the classical approach in [TT79], although we use different techniques.

As a general result that may also be useful in other contexts, we show that in order to solve the inverse problem over finitely generated differential fields over a field of constants K, it suffices to solve it over $K(x)$ with derivation d/dx (Corollary 4.13). This type of approach has previously been used for groups over algebraically closed fields (e.g. for connected groups in [MS96]) and is sometimes called the *Kovacic trick*.

Our approach to solving the inverse problem over rational function fields over Laurent series fields is based on a recent version of patching methods ([HH10]) and inspired by the use of patching methods in ordinary Galois theory (see [Har03] for an overview). In ordinary Galois theory, the patching machinery reduces the realization of a given finite group to the (local) realization of generating (e.g. cyclic) subgroups. In the differential setup, there are two additional complications. The first is that in differential Galois theory, it is not possible to patch the local Picard-Vessiot rings (which are the analogs of Galois extensions), since these are not finite as algebras. Instead, we apply patching directly to the differential equations via a factorization property related to patching; this is equivalent to patching the corresponding differential modules. The second complication is that while any finite group is generated by its cyclic subgroups, an analogous statement for linear algebraic groups is true only over algebraically closed fields: Every linear algebraic group over an algebraically closed field is generated by (a finite number of) finite cyclic groups and copies of the multiplicative and additive group. To overcome this issue, we apply the patching method after base change to a finite extension of the field of constants in such a way that the result descends to the original field.

The advantage of our method is that the building blocks (i.e., the local differential equations for finite cyclic, multiplicative, and additive groups) are very easy to find and verify. Because we apply the patching machinery only in a very simple case in which the process can be described explicitly, we do not require the reader to be familiar with [HH10]. We note that a related but somewhat different strategy, to handle a special case of the problem, was sketched by the second author in [Har07].

We expect that the results of this manuscript can be applied to solve the inverse differential Galois problem for function fields over constant fields other than Laurent series, for example p-adic fields (this is work in progress).

Organization of the manuscript: Section 1 lists some basic facts about Picard-Vessiot theory over non-algebraically closed fields of constants. It also contains auxiliary results about the linearization of the differential Galois group, a statement and consequences of the Galois correspondence, and descent results for Picard-Vessiot rings. Section 2 describes the patching setup for differential equations, proves the main patching result and gives some examples. Section 3 is concerned with the generation of linear algebraic groups by "simple" subgroups and with finding differential equations which have those groups as differential Galois groups, the so-called *building blocks* for patching. Finally, Section 4 solves the inverse problem over rational function fields by combining the results of the previous two sections. It also contains a result stating that solving the inverse problem over rational function fields implies a solution over arbitrary finitely generated differential fields (over the same field of constants). Combining this with the result obtained for rational function fields gives the main Theorem 4.14.

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1. Picard-Vessiot theory

Throughout this manuscript, all fields are assumed to be of characteristic zero. If F is a field, we write $\bar{F}$ for its algebraic closure. If R is an integral domain, we write $\mathrm{Frac}(R)$ for its field of fractions. If R is a differential ring, we write C_R for its ring of constants. If F is a differential field, then C_F is a field which is algebraically closed in F.

We first record some facts from the Galois theory of differential fields with arbitrary fields of constants. We refer to [Dyc08] for details.

Let (F, ∂) be a differential field with field of constants $C_F = K$. Given a matrix $A \in F^{n\times n}$ and a differential ring extension R/F, a **fundamental solution matrix for** A is a matrix $Y \in \mathrm{GL}_n(R)$ which satisfies the differential equation $\partial(Y) = AY$. A differential ring extension R/F is called a **Picard-Vessiot ring** for $A \in F^{n\times n}$ if it satisfies the following conditions: The ring of constants of R is $C_R = K$, there exists a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$, R is generated by the entries of Y and Y^{-1} (we write $R = F[Y, Y^{-1}]$), and R is differentially simple (i.e., has no nontrivial ideals which are stable under ∂). Differential simplicity implies that a Picard-Vessiot ring is an integral domain. Its fraction field is called a **Picard-Vessiot extension**. Note that since K is not assumed to be algebraically closed, Picard-Vessiot rings need not exist (e.g. [Sei56]) and might not be unique. The **torsor theorem** states that if R is a Picard-Vessiot ring, there is an R-linear isomorphism of differential rings $R \otimes_F R \cong R \otimes_K C_{R\otimes_F R}$.

The **differential Galois group** of a Picard-Vessiot ring R/F is defined as the group functor $\underline{\mathrm{Aut}}^\partial(R/F)$ from the category of K-algebras to the category of groups that sends a K-algebra S to the group $\mathrm{Aut}^\partial(R \otimes_K S/F \otimes_K S)$ of differential automorphism of $R \otimes_K S$ that are trivial on $F \otimes_K S$. This functor is represented by the K-Hopf algebra $C_{R\otimes_F R} = K[Y \otimes Y^{-1}, Y^{-1} \otimes Y]$. We conclude that the differential Galois group of R/F is an affine group scheme of finite type over K which is necessarily (geometrically) reduced since the characteristic is zero ([Oor66]; see also [Car62]); i.e. it is a **linear algebraic group over** K with coordinate ring K-isomorphic to $C_{R\otimes_F R}$. Moreover, the torsor theorem asserts that the affine variety $\mathrm{Spec}(R)$ defined by a Picard-Vessiot ring R with Galois group G is a G-torsor and $R \otimes_F \bar{F} \cong C_{R\otimes_F R} \otimes_K \bar{F}$, which implies $\mathrm{trdeg}(\mathrm{Frac}(R)/F) = \dim(G)$. It is immediate from the definitions that if K'/K is an algebraic **extension of constants**, then $R \otimes_K K'$ is a Picard-Vessiot ring over $F \otimes_K K'$ whose differential Galois group is the base change of the differential Galois group of R/F from K to K'. We remark that the definition of the differential Galois group requires the use of a Picard-Vessiot ring rather than a Picard-Vessiot extension; for this reason most statements in this manuscript are phrased in terms of a Picard-Vessiot ring.

When constructing Picard-Vessiot rings, we will frequently use the following well-known criterion ([Dyc08, Corollary 2.7]).

Proposition 1.1. *Let L/F be an extension of differential fields with constants $C_L = C_F$ and let $A \in F^{n\times n}$. Assume that there exists a fundamental solution matrix $Y \in \mathrm{GL}_n(L)$ for A, i.e., $\partial(Y) = AY$. Then $R = F[Y, Y^{-1}] \subseteq L$ is a Picard-Vessiot ring for A.*

The next proposition gives a criterion to determine which elements of a Picard-Vessiot extension are contained in a Picard-Vessiot ring. Elements satisfying the condition in the proposition are also called *differentially finite*.

Proposition 1.2. *Let R/F be a Picard-Vessiot ring for some matrix A. Then an element $a \in \mathrm{Frac}(R)$ lies in R if and only if $a, \partial(a), \partial^2(a), \ldots$ span a finite dimensional F-vector space.*

Proof. In the case that C_F is algebraically closed, this is a well-known statement (see [vdPS03, Cor. 1.38]), which can be applied to $R \otimes_K \bar{K}$ over the field $L := F \otimes_K \bar{K}$. Here $R \otimes_K \bar{K}$ is a Picard-Vessiot ring since $\bar{K}/K$ is an algebraic extension of constants. Let $a \in R \subseteq R \otimes_K \bar{K}$. Thus $a, \partial(a), \partial^2(a), \ldots$ span a finite dimensional vector space over L, so there is an $r \in \mathbb{N}$ such that $a, \partial(a), \partial^2(a), \ldots, \partial^r(a)$ are linearly dependent over L. Let V be the F-vector space spanned by $a, \partial(a), \ldots, \partial^r(a)$. Then

$$\dim_F(V) = \dim_L(V \otimes_F L) = \dim_L(V \otimes_K \bar{K}) \leq r;$$

and we conclude that $a, \partial(a), \ldots, \partial^r(a)$ are linearly dependent over F, giving the forward direction. The proof of the converse direction is the same as in the case C_F is algebraically closed; see [vdPS03, Cor. 1.38, proof of (3)⇒(1)]. □

In particular, any two Picard-Vessiot rings R, R' with the same field of fractions (but possibly for different matrices) are necessarily equal.

1.1. **Linearization of the differential Galois group.** The differential Galois group of a Picard-Vessiot ring was defined above as an abstract linear algebraic group. A choice of fundamental solution matrix determines an embedding into a general linear group as follows.

Proposition 1.3. *Let $R = F[Y, Y^{-1}]$ be a Picard-Vessiot ring over F with field of constants $C_F = K$. Then there is a closed embedding of linear algebraic groups*

$$\Psi_Y = \Psi_{Y,K} \colon \underline{\mathrm{Aut}}^\partial(R/F) \hookrightarrow \mathrm{GL}_{n,K}$$

such that for all K-algebras S:

$$\Psi_Y(S) \colon \mathrm{Aut}^\partial(R \otimes_K S/F \otimes_K S) \hookrightarrow \mathrm{GL}_n(S),\ \sigma \mapsto (Y \otimes 1)^{-1}\sigma(Y \otimes 1).$$

Proof. Since σ is a differential automorphism, $\partial\left((Y\otimes 1)^{-1}\sigma(Y\otimes 1)\right)=0$, i.e., $(Y\otimes 1)^{-1}\sigma(Y\otimes 1)$ has entries in $C_{R\otimes_K S}=K\otimes_K S\cong S$. Therefore, $\Psi_Y(S)$ is a well-defined map. Let $K[Z,Z^{-1}]$ denote the coordinate ring of GL_n over K. Let $\rho\colon \underline{\mathrm{Aut}}^\partial(R/F)\hookrightarrow \mathrm{GL}_{n,K}$ be the closed embedding induced by the surjection on the coordinate rings $K[Z,Z^{-1}]\to K[Y\otimes Y^{-1},Y^{-1}\otimes Y]$, $Z\mapsto Y\otimes Y^{-1}$. It is then easy to deduce that $\rho(S)=\Psi_Y(S)$ for every K-algebra S by examining the isomorphism $\mathrm{Aut}^\partial(R\otimes_K S/F\otimes_K S)\to \mathrm{Hom}_K(K[Y\otimes Y^{-1},Y^{-1}\otimes Y],S)$. Hence $\Psi_Y=\rho$ is in fact a closed embedding of linear algebraic groups. □

We define $\underline{\mathrm{Gal}}^\partial_Y(R/F)\leq \mathrm{GL}_n$ as the image of the differential Galois group $\underline{\mathrm{Aut}}^\partial(R/F)$ under the linearization Ψ_Y defined in Proposition 1.3 and we also call it **the differential Galois group of** R/F **(with respect to** Y**)**. A different choice of fundamental solution matrix leads to a differential Galois group which is conjugate by an element of $\mathrm{GL}_n(K)$. We will write expressions like $\mathcal{G}\leq\mathrm{GL}_n$ to denote a linear algebraic group with an embedding into a fixed copy of GL_n.

The following example illustrates the above and will be used in our building blocks in Section 3.2.

Example 1.4. Let F be a differential field with field of constants K and let $a\in F$. We consider the differential equation $\partial(y)=Ay$ with $A=\begin{pmatrix}0&a\\0&0\end{pmatrix}$. If there exists a differential field extension L/F with $C_L=K$ and an element $y\in L$ with $\partial(y)=a$, then $R=F[y]$ is a Picard-Vessiot ring for A over F by Proposition 1.1. Indeed, $Y=\begin{pmatrix}1&y\\0&1\end{pmatrix}\in\mathrm{GL}_2(L)$ is a fundamental solution matrix for A and $R=F[Y,Y^{-1}]$. Let S be a K-algebra and $\sigma\in\mathrm{Aut}^\partial(R\otimes_K S/F\otimes_K S)$. As

$$\partial(\sigma(y\otimes 1))=\sigma(\partial(y\otimes 1))=\sigma(a\otimes 1)=a\otimes 1=\partial(y\otimes 1),$$

there exists an $\alpha_\sigma\in C_{R\otimes_K S}=K\otimes_K S\simeq S$ with $\sigma(y\otimes 1)=y\otimes 1+\alpha_\sigma$. Moreover, $Y^{-1}\sigma(Y)=\begin{pmatrix}1&\alpha_\sigma\\0&1\end{pmatrix}$, hence $\underline{\mathrm{Gal}}^\partial_Y(R/F)(S)$ is a subgroup of $\mathbb{G}_a(S)$ in its two-dimensional representation and $\underline{\mathrm{Gal}}^\partial_Y(R/F)$ is a subgroup scheme of $\mathbb{G}_a$. (Note that since $\mathbb{G}_a$ has no proper closed subgroups in characteristic zero, it must in fact be either the whole group or trivial.)

Remark 1.5. One advantage of working within a fixed GL_n is the following. Two linear algebraic groups $\mathcal{G}\leq\mathrm{GL}_n$ and $\mathcal{H}\leq\mathrm{GL}_n$ defined over K are equal (as subgroups of GL_n) if and only if $\mathcal{G}(\bar K)=\mathcal{H}(\bar K)$ inside $\mathrm{GL}_n(\bar K)$, since the $\bar K$-rational points of a linear algebraic group are dense. In particular, $\mathcal{G}$ and $\mathcal{H}$ are isomorphic over K if $\mathcal{G}(\bar K)=\mathcal{H}(\bar K)$; whereas in general, $\mathcal{G}(\bar K)\cong\mathcal{H}(\bar K)$ for abstract linear algebraic groups would imply only that $\mathcal{G}$ and $\mathcal{H}$ are isomorphic over $\bar K$.

The following lemma is immediate from the definitions (in the language of differential modules, the transformation corresponds to a change of basis).

Lemma 1.6. *Let* R/F *be a Picard-Vessiot ring for a matrix* A *with fundamental solution matrix* $Y\in\mathrm{GL}_n(R)$*. Then for each* $B\in\mathrm{GL}_n(F)$*,* R/F *is also a Picard-Vessiot ring for*

$\partial(B)B^{-1} + BAB^{-1}$ *with fundamental solution matrix* $BY \in \mathrm{GL}_n(R)$, *and* $\underline{\mathrm{Gal}}^\partial_Y(R/F) = \underline{\mathrm{Gal}}^\partial_{BY}(R/F)$ *inside* GL_{n,C_F}.

Lemma 1.7. *Let* R/F *be a Picard-Vessiot ring for a matrix* A *with fundamental solution matrix* $Y \in \mathrm{GL}_n(R)$. *Let* L/F *be a differential field extension such that* R *and* L *are contained in a common differential field extension with constants* C_F. *Then the compositum* $LR = L[Y, Y^{-1}]$ *is a Picard-Vessiot ring for* A *over* L, *and* $\underline{\mathrm{Gal}}^\partial_Y(LR/L) \le \underline{\mathrm{Gal}}^\partial_Y(R/F)$ *as subgroups of* GL_{n,C_F}.

Proof. Abbreviate $K = C_F$. By Proposition 1.1, LR is a Picard-Vessiot ring for A over L. For the second assertion, let S be any K-algebra, and let $\sigma \in \underline{\mathrm{Aut}}^\partial(LR/L)(S) = \mathrm{Aut}^\partial(LR \otimes_K S/L \otimes_K S)$. Then σ acts on $LR \otimes_K S$ and hence on $\mathrm{GL}_n(LR \otimes_K S)$, taking $Y \otimes 1$ to $(Y \otimes 1)Z$ where $Z = \Psi_Y(S)(\sigma) \in \underline{\mathrm{Gal}}^\partial_Y(LR/L)(S) \le \mathrm{GL}_n(S)$ with Ψ_Y as in Proposition 1.3. Since $R = F[Y, Y^{-1}]$, σ restricts to an automorphism of $R \otimes_K S$. The corresponding monomorphism $\underline{\mathrm{Gal}}^\partial_Y(LR/L)(S) \to \underline{\mathrm{Gal}}^\partial_Y(R/F)(S)$ maps Z to Z. This defines an inclusion $\underline{\mathrm{Gal}}^\partial_Y(LR/L) \le \underline{\mathrm{Gal}}^\partial_Y(R/F)$ of subgroups of $\mathrm{GL}_{n,K}$. □

If moreover the tensor product of L and R is a compositum without new constants, we even obtain equality:

Proposition 1.8. *Let* R/F *be a Picard-Vessiot ring for a matrix* A *with fundamental solution matrix* $Y \in \mathrm{GL}_n(R)$. *Let* L/F *be a differential field extension and assume that* $L \otimes_F R$ *is an integral domain such that its field of fractions has constants* C_F. *Then* $L \otimes_F R$ *is a Picard-Vessiot ring for* A *over* L *with fundamental solution matrix* $1 \otimes Y$. *Moreover, we have an equality of differential Galois groups inside* GL_{n,C_F}*:*

$$\underline{\mathrm{Gal}}^\partial_{1\otimes Y}(L \otimes_F R/L) = \underline{\mathrm{Gal}}^\partial_Y(R/F)$$

Proof. Abbreviate $K = C_F$. View $L \otimes_F R$ as the compositum of L and R inside $\mathrm{Frac}(L \otimes_F R)$. Then Lemma 1.7 implies that $L \otimes_F R = F[1 \otimes Y, (1 \otimes Y)^{-1}]$ is a Picard-Vessiot ring over L and $\underline{\mathrm{Gal}}^\partial_{1\otimes Y}(L \otimes_F R/L) \le \underline{\mathrm{Gal}}^\partial_Y(R/F)$ as subgroups of $\mathrm{GL}_{n,K}$. To obtain equality, it suffices to show that for any K-algebra S and any $\sigma_0 \in \mathrm{Aut}^\partial(R \otimes_K S/F \otimes_K S)$ there exists an element $\sigma \in \mathrm{Aut}^\partial((L \otimes_F R) \otimes_K S/L \otimes_K S) = \mathrm{Aut}^\partial(L \otimes_F (R \otimes_K S)/L \otimes_F (F \otimes_K S))$ that restricts to σ_0. But $\sigma = \mathrm{id}_L \otimes_F \sigma_0$ has that property, which concludes the proof. □

In the special case when R/F is a finite Galois extension and L/F is a finite extension, the condition that $L \otimes_F R$ is an integral domain is equivalent to saying that L and R are linearly disjoint over F and the condition that the field $L \otimes_F R$ does not have new constants is equivalent to saying that $L \otimes_F R$ is regular over K. (Recall that a field extension A/K is *regular* if A is linearly disjoint from $\bar{K}$ over K.)

1.2. **The Galois correspondence.** Let R/F be a Picard-Vessiot ring together with a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$. Set $G := \underline{\mathrm{Aut}}^\partial(R/F)$, $\mathcal{G} := \underline{\mathrm{Gal}}^\partial_Y(R/F) \le \mathrm{GL}_n$ and $E := \mathrm{Frac}(R)$, $K := C_F$. Then for a closed subgroup $\mathcal{H} \le \mathcal{G}$ (defined over K), consider the inverse image $H \le G$ of $\mathcal{H}$ under the isomorphism $\Psi_Y : G \to \mathcal{G}$. The **functorial invariants** E^H in E are defined as the elements $a/b \in \mathrm{Frac}(R)$ satisfying

$$\sigma(a \otimes 1)(b \otimes 1) = (a \otimes 1)\sigma(b \otimes 1)$$

for all K-algebras S and all $\sigma \in H(S) \subseteq \mathrm{Aut}^\partial(R \otimes_K S/F \otimes_K S)$. As $H(\bar{K})$ is dense in H, it even suffices to check the above condition for $S = \bar{K}$. The **Galois correspondence** ([Dyc08, Thm. 4.4]) then states that there is an inclusion-reversing bijection between closed subgroups of G (defined over K) and differential subfields $F \subseteq L \subseteq E$, given by

$$H \mapsto E^H \text{ and } L \mapsto \underline{\mathrm{Aut}}^\partial(R/L).$$

We illustrate with an example why we need to consider functorial invariants rather than invariants on K-points.

Example 1.9. Consider $F = \mathbb{Q}(x)$ with derivation d/dx and $R = F[\sqrt[3]{x}]$. Then $R = \mathrm{Frac}(R)$ is a Picard-Vessiot ring over F for the differential equation $\partial(y) = \frac{1}{3x}y$ with fundamental solution matrix $y = \sqrt[3]{x}$ and differential Galois group $G \cong \mu_3$, the subgroup of GL_3 defined by the equation $X^3 - 1 = 0$. Note that $\mu_3(\mathbb{Q}) = \{1\}$; hence $\mathrm{Aut}^\partial(R/F) = \mathrm{Aut}(R/F) = \{\mathrm{id}\}$, and $R^{\mathrm{Aut}(R/F)}$ strictly contains F. But $R^G = F$.

It is known ([Dyc08, Proposition 4.3]) that if $H \leq G$ is a closed normal subgroup, then E^H is the fraction field of a Picard-Vessiot ring R_H over F with $\underline{\mathrm{Aut}}^\partial(R_H/F) \cong G/H$. For H equal to the identity component of G, we obtain the following.

Lemma 1.10. *Let F be a differential field and let R/F be a Picard-Vessiot ring with differential Galois group G. Write G^0 for the identity component of G and $E = \mathrm{Frac}(R)$. Then E^{G^0} is the algebraic closure of F in E, and it is a finite field extension of degree $[E^{G^0} : F] = |G(\bar{K})/G^0(\bar{K})|$.*

Proof. Since $\mathrm{trdeg}(E/E^{G^0}) = \dim(G^0) = \dim(G) = \mathrm{trdeg}(E/F)$ and E is finitely generated over F, the extension E^{G^0}/F is finite. On the other hand, every algebraic subextension $F \subseteq L \subseteq E$ is a differential extension, and thus $H := \underline{\mathrm{Aut}}^\partial(R/L)$ is a closed subgroup of G of the same dimension. Therefore, $G^0 \subseteq H$ and $L = E^H \subseteq E^{G^0}$. Hence E^{G^0} is the algebraic closure of F in E. While the finite field extension E^{G^0}/F might not be Galois, the compositum $E^{G^0}\bar{K} \cong E^{G^0} \otimes_K \bar{K}$ is a finite extension of $F\bar{K} \cong F \otimes_K \bar{K}$ of the same degree as E^{G^0}/F. Moreover, $\mathrm{Aut}(E^{G^0}\bar{K}/F\bar{K}) \cong (G/G^0)(\bar{K}) \cong G(\bar{K})/G^0(\bar{K})$ and $(E^{G^0}\bar{K})^{\mathrm{Aut}(E^{G^0}\bar{K}/F\bar{K})} = F\bar{K}$ (because $\bar{K}$ is algebraically closed). Hence $E^{G^0}\bar{K}/F\bar{K}$ is a finite Galois extension of degree $|G(\bar{K})/G^0(\bar{K})|$ and the claim follows. ☐

1.3. **Galois descent for Picard-Vessiot rings.** Let F_0 be a differential field with field of constants K_0. Given a linear algebraic group $\mathcal{G} \leq \mathrm{GL}_n$ over K_0, it might be easier to realize $\mathcal{G}_K$ as a differential Galois group over F_0K for some finite extension of constants K/K_0. We assume that K/K_0 is a finite Galois extension with Galois group Γ. As K_0 is algebraically closed in F_0, $F_0K \cong F_0 \otimes_{K_0} K$ is Galois over F_0 with group Γ (and the action of Γ commutes with the derivation). In our applications, we construct a Picard-Vessiot ring over F_0K such that the fundamental solution matrix is Γ-invariant. The following lemma explains that this Picard-Vessiot ring then descends to a Picard-Vessiot ring over F_0 with differential Galois group $\mathcal{G}$.

Lemma 1.11. *Let K/K_0 be a finite Galois extension with Galois group Γ. Let F_0 be a differential field with $C_{F_0} = K_0$ and set $F = F_0K$. Further, let L/F be an extension of* *differential fields with $C_L = C_F = K$ and such that the action of Γ on F extends to an action on L via differential automorphisms. If $R = F[Y, Y^{-1}] \subseteq L$ is a Picard-Vessiot ring over F such that $Y \in \mathrm{GL}_n(L)$ is invariant under the action of Γ, then $R_0 := F_0[Y, Y^{-1}]$ is a Picard-Vessiot ring over F_0 with $\underline{\mathrm{Gal}}^\partial_Y(R_0/F_0)_K = \underline{\mathrm{Gal}}^\partial_Y(R/F)$ as subgroups of GL_n. In particular, if $\underline{\mathrm{Gal}}^\partial_Y(R/F) = \mathcal{G}_K$ for a linear algebraic group $\mathcal{G} \leq \mathrm{GL}_{n,K_0}$, then $\underline{\mathrm{Gal}}^\partial_Y(R_0/F_0) = \mathcal{G}$.*

Proof. Note that $A := \partial(Y)Y^{-1} \in F^{n\times n}$ is Γ-invariant, hence $A \in F_0^{n\times n}$. The field L^Γ of Γ-invariants in L is a differential field extension of F_0 with $C_{L^\Gamma} = K_0 = C_{F_0}$, since $C_L = K$. As Y is contained in $\mathrm{GL}_n(L^\Gamma)$, Proposition 1.1 implies that R_0 is a Picard-Vessiot ring for A over F_0. Let $\mathcal{H} \leq \mathrm{GL}_n$ be the linear algebraic group $\underline{\mathrm{Gal}}^\partial_Y(R_0/F_0)$ defined over K_0. Since K_0 is algebraically closed in $\mathrm{Frac}(R_0)$, the natural map $R_0 \otimes_{K_0} K \to R$ is an isomorphism, and the induced map on matrices sends $Y \otimes_{K_0} 1$ to Y. Hence the natural map $\iota : \mathrm{Aut}^\partial(R \otimes_K \bar{K}/F \otimes_K \bar{K}) \to \mathrm{Aut}^\partial(R_0 \otimes_{K_0} \bar{K}/F_0 \otimes_{K_0} \bar{K})$ is an isomorphism. It is given by $\sigma \mapsto \phi^{-1}\sigma\phi$, where the differential $\bar{K}$-isomorphism ϕ is the composition

$$R_0 \otimes_{K_0} \bar{K} \to R_0 \otimes_{K_0} K \otimes_K \bar{K} \to R \otimes_K \bar{K},$$

whose induced map on matrices sends $Y \otimes_{K_0} 1$ to $Y \otimes_K 1$.

We claim that the isomorphism ι yields the equality

$$\mathcal{H}(\bar{K}) = \underline{\mathrm{Gal}}^\partial_Y(R_0/F_0)(\bar{K}) = \underline{\mathrm{Gal}}^\partial_Y(R/F)(\bar{K})$$

as subsets of $\mathrm{GL}_n(\bar{K})$. Namely, for any $\sigma \in \mathrm{Aut}^\partial(R \otimes_K \bar{K}/F \otimes_K \bar{K})$,

$$\begin{aligned}
\Psi_{Y,K}(\bar{K})(\sigma) &= (Y \otimes_K 1)^{-1}\sigma(Y \otimes_K 1)\\
&= \phi((Y \otimes_{K_0} 1)^{-1}\iota(\sigma)(Y \otimes_{K_0} 1))\\
&= (Y \otimes_{K_0} 1)^{-1}\iota(\sigma)(Y \otimes_{K_0} 1)\\
&= \Psi_{Y,K_0}(\bar{K})(\iota(\sigma)).
\end{aligned}$$

(The third equality holds since all entries of the matrix are contained in $\bar{K}$.) The claim follows.

We conclude that $\mathcal{H}_K = \underline{\mathrm{Gal}}^\partial_Y(R/F)$ as subgroups of GL_n as asserted. In particular, if $\underline{\mathrm{Gal}}^\partial_Y(R/F) = \mathcal{G}_K$ for some $\mathcal{G} \leq \mathrm{GL}_{n,K_0}$, then $\mathcal{H}(\bar{K}) = \mathcal{G}_K(\bar{K}) = \mathcal{G}(\bar{K})$ as subgroups of $\mathrm{GL}_n(\bar{K})$, and hence $\mathcal{H} = \mathcal{G}$. □

2. Application of patching to differential equations

Throughout this section, let $K = k((t))$ for some field k of characteristic zero; let $F = K(x)$; and consider a derivation ∂ on F with $C_F = K$. Note that $\partial = \partial(x) \cdot \partial/\partial x$. Moreover, F is the function field of the projective x-line $\mathbb{P}^1_{k[[t]]}$ over the discrete valuation ring $k[[t]]$. In [HH10], a collection of field extensions F_P, $F_\wp$ and F_U were considered. Whereas the definitions in loc. cit. are geometric, we only need a special case, in which the description is very explicit:

If $P \in \mathbb{A}^1_k \subset \mathbb{P}^1_k$ is a rational point defined by $x = b$ for some $b \in k$, then we consider the fields

$$\begin{aligned}
F_P &= k((x-b, t)) \quad \text{and}\\
F_{\wp(P)} &= k((x-b))((t)),
\end{aligned}$$

where $k((x-b,t))$ denotes the fraction field of the power series ring $k[[x-b,t]]$ in two variables. Given a non-empty finite set $\mathcal{P}$ of k-rational points, we have an index set $\mathcal{B} = \{\wp(P) | \; P \in \mathcal{P}\}$ in bijection with $\mathcal{P}$. If $\mathcal{P}$ consists of points $z = b_i$ for some $b_i \in k$ and $1 \le i \le m$, then we let U be the complement of $\mathcal{P}$ in $\mathbb{P}^1_k$, and we write

$$F_U = \mathrm{Frac}(k[(x-b_1)^{-1}, \ldots, (x-b_m)^{-1}][[t]]).$$

As explained in [HH10], there are inclusions $F \subseteq F_U, F_P \subseteq F_{\wp(P)}$ for all $P \in \mathcal{P}$. These are in fact inclusions of differential fields, where we equip F_U, F_P, and $F_{\wp(P)}$ with the derivation $\partial(x) \cdot \partial/\partial x$; moreover $C_{F_U} = C_{F_P} = C_{F_{\wp(P)}} = C_F = K$ for all $P \in \mathcal{P}$. In particular, if $A \in F^{n\times n}$ is such that there exists a fundamental solution matrix $Y \in \mathrm{GL}_n(F_U)$, then $F[Y, Y^{-1}]$ is a Picard-Vessiot ring for A over F (by Proposition 1.1).

The method of patching over the fields $(F, F_U, F_P, F_\wp)$ relies on the following two properties.

Theorem 2.1.

(a) Simultaneous factorization property: Let $n \in \mathbb{N}$. If $(Y_\wp)_{\wp\in\mathcal{B}}$ is a collection of matrices $Y_\wp \in \mathrm{GL}_n(F_\wp)$ then there exist matrices $B_P \in \mathrm{GL}_n(F_P)$ for each $P \in \mathcal{P}$ and one matrix $Y \in \mathrm{GL}_n(F_U)$ such that for each $P \in \mathcal{P}$, $Y_{\wp(P)} = B_P^{-1} Y$ in $\mathrm{GL}_n(F_{\wp(P)})$.

(b) Intersection property: If $x \in F_U$ is such that for each $P \in \mathcal{P}$, x is contained in F_P when considered as an element of $F_{\wp(P)}$, then $x \in F$.

For a proof of the simultaneous factorization property, see [HH10, Thm 5.10] and [HHK11, Prop. 2.2]. The intersection property is stated in [HH10, Prop. 6.3].

Definition 2.2. In the above setup, an *action* of a finite group Γ on the *differential patching data* $(\mathcal{P}, \mathcal{B}, U)$ consists of the following:

(1) a left action of Γ on F via differential automorphisms.
(2) a left action of Γ on F_U via differential automorphisms, extending the action of Γ on F.
(3) a right action of Γ on the finite set $\mathcal{P}$.
(4) for each $\sigma \in \Gamma$ and each $P \in \mathcal{P}$, an isomorphism $\sigma\colon F_{P^\sigma} \to F_P$ of differential fields extending $\sigma\colon F \to F$ such that for all $\sigma, \tau \in \Gamma$, $\sigma\tau\colon F_{P^{\sigma\tau}} \to F_P$ is the composition $\sigma \circ \tau\colon F_{P^{\sigma\tau}} \to F_{P^\sigma} \to F_P$.
(5) for each $\sigma \in \Gamma$ and each $P \in \mathcal{P}$, an isomorphism of differential fields $\sigma\colon F_{\wp(P^\sigma)} \to F_{\wp(P)}$ extending both $\sigma\colon F_U \to F_U$ and $\sigma\colon F_{P^\sigma} \to F_P$ such that for all $\sigma, \tau \in \Gamma$, $\sigma\tau\colon F_{\wp(P^{\sigma\tau})} \to F_{\wp(P)}$ is the composition $\sigma \circ \tau\colon F_{\wp(P^{\sigma\tau})} \to F_{\wp(P^\sigma)} \to F_{\wp(P)}$.

Example 2.3. (a) Let $k_0 \le k$ such that k/k_0 is a finite Galois extension, let $e \ge 1$ be a natural number such that k contains a primitive e-th root of unity, and set $t_0 = t^e$. Then $K = k((t))$ is a finite Galois extension of $K_0 = k_0((t_0))$, and $\Gamma := \mathrm{Gal}(K/K_0)$ is the semi-direct product of the cyclic group of order e and the group $\mathrm{Gal}(k/k_0)$. In particular, Γ surjects onto $\mathrm{Gal}(k/k_0)$. Note that $F = K(x)$ is a finite Galois extension of $F_0 = K_0(x)$ whose Galois group is isomorphic to Γ and acts on F as a differential field. The action of Γ on F over F_0 (from the left) induces an action of Γ on the x-line $\mathbb{P}^1_{k[[t]]}$ over $\mathbb{P}^1_{k_0[[t_0]]}$ from the right. In particular, there is an induced action of Γ on $\mathbb{P}^1_k$ over $\mathbb{P}^1_{k_0}$ from the right. If $P \in \mathbb{P}^1_k$ is a point of the form $x = b$ for some $b \in k$, then P^σ is defined by $x = \sigma^{-1}(b)$. The induced isomorphism

$$\sigma\colon F_{P^\sigma} = k((x-\sigma^{-1}(b),t)) \to F_P = k((x-b,t))$$

is a differential isomorphism, as is the induced isomorphism

$$\sigma\colon F_{\wp(P^\sigma)} = k((x-\sigma^{-1}(b)))((t)) \to F_{\wp(P)} = k((x-b))((t)).$$

Notice that if $\mathcal{P} \subseteq \mathbb{P}^1_k$ is a finite set of closed points invariant under the action of Γ, then the action of Γ on $\mathbb{P}^1_{k[[t]]}$ also induces an action of Γ on F_U as a differential field. Therefore, Γ acts on the differential patching data $(\mathcal{P},\mathcal{B},U)$.

(b) Observe that the Γ-orbit of a closed point $x = b$ as above consists of at most $|\mathrm{Gal}(k/k_0)| \le |\Gamma|$ elements. To obtain an action with orbits of full length $|\Gamma|$ for use in Section 4, we will use the following construction. Suppose again that k contains a primitive e-th root of unity ζ. Let $z = x/t$. We will work with the z-line $\mathbb{P}^1_{k[[t]]}$ instead of the x-line. (In other words, we perform a blow-up at the origin, and then blow down the original component.) Let $\sigma \in \Gamma$. As t is an e-th root of $t_0 \in K_0$, $\sigma(t) = \zeta^{n_\sigma} t$ for some $n_\sigma \in \mathbb{N}$, and hence $\sigma(z) = \zeta^{-n_\sigma} z$. Note that the induced action of Γ on the z-line $\mathbb{P}^1_k$ maps a point P of the form $z = b$ to the point P^σ defined by $z = \zeta^{n_{\sigma^{-1}}}\sigma^{-1}(b)$; and the induced isomorphisms $\sigma\colon F_{P^\sigma} = k((z-\zeta^{n_{\sigma^{-1}}}\sigma^{-1}(b),t)) \to F_P = k((z-b,t))$ and $F_{\wp(P^\sigma)} = k((z-\zeta^{n_{\sigma^{-1}}}\sigma^{-1}(b)))((t)) \to F_{\wp(P)} = k((z-b))((t))$ are again isomorphisms of differential fields (they map $z-\zeta^{n_{\sigma^{-1}}}\sigma^{-1}(b)$ to $\zeta^{-n_\sigma}(z-b)$). The Γ-orbit of such a point $z = b$ then consists of all points of the form $z = \zeta^{n_\sigma}\sigma(b)$ for $\sigma \in \Gamma$. We will show later that there exist elements $b \in k$ such that this orbit consists of $|\Gamma|$ points.

Theorem 2.4. *Let $n \in \mathbb{N}$.*

a) Assume that for each $P \in \mathcal{P}$, a matrix $A_P \in F_P^{n\times n}$ is given together with a fundamental solution matrix $Y_P \in \mathrm{GL}_n(F_{\wp(P)})$. Let $\mathcal{G}_P = \underline{\mathrm{Gal}}^\partial_{Y_P}(R_P/F_P) \le \mathrm{GL}_{n,K}$ be the differential Galois group of the Picard-Vessiot ring $R_P = F_P[Y_P, Y_P^{-1}]$ for A_P (see Proposition 1.1). Then there exists a matrix $A \in F^{n\times n}$ and a fundamental solution matrix $Y \in \mathrm{GL}_n(F_U)$ for A such that the Picard-Vessiot ring $R = F[Y,Y^{-1}]$ over F has the following property: Its differential Galois group equals the Zariski closure of the group generated by the various subgroups $\mathcal{G}_P$ of $\mathrm{GL}_{n,K}$; i.e. $\underline{\mathrm{Gal}}^\partial_Y(R/F) = \overline{<\mathcal{G}_P \mid P \in \mathcal{P}>}$.

b) Assume that moreover a finite group Γ acts on the differential patching data $(\mathcal{P},\mathcal{B},U)$ and assume that $\sigma(Y_{P^\sigma}) = Y_P$ in $\mathrm{GL}_n(F_{\wp(P)})$ for each $P \in \mathcal{P}$ and each $\sigma \in \Gamma$. Then the fundamental solution matrix $Y \in \mathrm{GL}_n(F_U)$ can be chosen such that its entries are Γ-invariant.

Proof. Simultaneous factorization (Theorem 2.1.a) implies that there exists a matrix $Y \in \mathrm{GL}_n(F_U)$ and matrices $B_P \in \mathrm{GL}_n(F_P)$ for each $P \in \mathcal{P}$ such that $Y_P = B_P^{-1}\cdot Y$ when viewed inside $\mathrm{GL}_n(F_{\wp(P)})$ for each $P \in \mathcal{P}$. We set $A = \partial(Y)Y^{-1} \in F_U^{n\times n}$. For each $P \in \mathcal{P}$, we compute inside $F_{\wp(P)}^{n\times n}$, viewing F_U and F_P as subfields of $F_{\wp(P)}$:

$$\begin{aligned} A &= \partial(Y)Y^{-1} \\ &= \partial(B_P Y_P)Y_P^{-1}B_P^{-1} \\ &= \partial(B_P)B_P^{-1} + B_P A_P B_P^{-1} \in F_P^{n\times n}, \end{aligned}$$

using that Y_P is a fundamental solution matrix for A_P. The intersection property (Theorem 2.1.b) implies that all entries of A are contained in F. By Proposition 1.1, $R = F[Y, Y^{-1}] \subseteq F_U$ is a Picard-Vessiot ring for $A \in F^{n\times n}$ over F. Set $\mathcal{G} = \underline{\mathrm{Gal}}^{\partial}_Y(R/F) \leq \mathrm{GL}_{n,K}$.

We first show that $\mathcal{G}_P \leq \mathcal{G}$ for all $P \in \mathcal{P}$. Note that $R_P = F_P[Y_P, Y_P^{-1}] = F_P[Y, Y^{-1}] = F_P R$ (the compositum is taken inside $F_{\wp(P)}$) for all $P \in \mathcal{P}$ since $Y_P = B_P^{-1} \cdot Y$. Hence

$$\mathcal{G}_P = \underline{\mathrm{Gal}}^{\partial}_{Y_P}(R_P/F_P) = \underline{\mathrm{Gal}}^{\partial}_Y(R_P/F_P) \leq \underline{\mathrm{Gal}}^{\partial}_Y(R/F) = \mathcal{G}$$

by Lemma 1.6 and Lemma 1.7. We now let $\mathcal{H} \leq \mathcal{G}$ be the Zariski closure of $< \mathcal{G}_P \mid P \in \mathcal{P} >$ in GL_n. We claim that $\mathcal{G} = \mathcal{H}$. Let $H \leq \underline{\mathrm{Aut}}^{\partial}(R/F)$ be the inverse image of $\mathcal{H} \leq \mathcal{G}$ under the isomorphism $\Psi_Y : \underline{\mathrm{Aut}}^{\partial}(R/F) \to \mathcal{G}$ (Proposition 1.3). By the Galois correspondence, it suffices to show that $E^H = F$, where $E = \mathrm{Frac}(R)$. Suppose there exists an element $a/b \in E^H \setminus F$ for some $a, b \in R$. Note that $a/b \in E \subseteq F_U \subseteq F_\wp$ for all $\wp \in \mathcal{B}$. The intersection property implies that there exists a $P \in \mathcal{P}$ with $a/b \in F_{\wp(P)} \setminus F_P$. On the other hand, $\mathcal{G}_P \leq \mathcal{H}$, hence $\underline{\mathrm{Aut}}^{\partial}(R_P/F_P) \leq H$. Indeed, $\underline{\mathrm{Aut}}^{\partial}(R_P/F_P)$ can be identified with a subgroup scheme of $\underline{\mathrm{Aut}}^{\partial}(R/F)$ via restriction and $\Psi_Y : \underline{\mathrm{Aut}}^{\partial}(R/F) \to \mathcal{G}$ maps this subgroup scheme to $\underline{\mathrm{Gal}}^{\partial}_Y(R_P/F_P) = \mathcal{G}_P \leq \mathcal{H} = \Psi_Y(H)$. We conclude that a/b is an element of $\mathrm{Frac}(R) \subseteq \mathrm{Frac}(R_P)$ that is invariant under $\underline{\mathrm{Aut}}^{\partial}(R_P/F_P)$. The Galois correspondence applied to the Picard-Vessiot ring R_P/F_P implies that a/b is contained in F_P, a contradiction. This proves part (a).

To prove part (b), it suffices to show that there exists a $B \in \mathrm{GL}_n(F)$ such that $B^{-1}Y \in \mathrm{GL}_n(F_U^\Gamma)$, since then $\underline{\mathrm{Gal}}^{\partial}_Y(R/F) = \underline{\mathrm{Gal}}^{\partial}_{B^{-1}Y}(R/F)$ by Lemma 1.6.

We first claim that for each $\sigma \in \Gamma$, $Y\sigma(Y)^{-1} \in \mathrm{GL}_n(F_U)$ has entries in F. Let $\sigma \in \Gamma$. By the intersection property, it suffices to show that $Y\sigma(Y)^{-1} \in \mathrm{GL}_n(F_P)$ when viewed as an element in $\mathrm{GL}_n(F_{\wp(P)})$ for each $P \in \mathcal{P}$. Let $P \in \mathcal{P}$ and set $Q = P^\sigma \in \mathcal{P}$. By assumption, there is a differential isomorphism $\sigma\colon F_{\wp(Q)} \to F_{\wp(P)}$ restricting to $\sigma\colon F_Q \to F_P$ and restricting to $\sigma\colon F_U \to F_U$. In $\mathrm{GL}_n(F_{\wp(Q)})$, we have $Y = B_Q Y_Q$, with notation as in the proof of part (a); hence $\sigma(Y) = \sigma(B_Q)\sigma(Y_Q) = \sigma(B_Q)\sigma(Y_{P^\sigma}) = \sigma(B_Q)Y_P$ in $\mathrm{GL}_n(F_{\wp(P)})$. On the other hand, $Y = B_P Y_P$ and we compute inside $\mathrm{GL}_n(F_{\wp(P)})$: $Y\sigma(Y)^{-1} = B_P Y_P(\sigma(B_Q)Y_P)^{-1} = B_P\sigma(B_Q)^{-1} \in \mathrm{GL}_n(F_P)$, proving the claim.

Therefore, we obtain a 1-cocycle $\chi\colon \Gamma \to \mathrm{GL}_n(F), \sigma \mapsto Y\sigma(Y)^{-1}$. By Hilbert's Theorem 90, $H^1(\Gamma, \mathrm{GL}_n(F))$ is trivial, hence there exists a $B \in \mathrm{GL}_n(F)$ such that for all $\sigma \in \Gamma$: $Y\sigma(Y)^{-1} = B\sigma(B)^{-1}$. This implies that $B^{-1}Y \in \mathrm{GL}_n(F_U)$ is Γ-invariant as we wanted to show. □

Note that part (b) of the above theorem enables us to use Lemma 1.11 and hence to obtain Galois descent.

Example 2.5.
(1) In this example, we apply Theorem 2.4 to show that SL_2 is a differential Galois group over $F = \mathbb{R}((t))(x)$ (with derivation $\partial = \partial/\partial x$). Let $\mathcal{G}_1, \mathcal{G}_2 \leq \mathrm{SL}_2$ be the subgroups of upper and lower unitary triangular matrices in SL_2. Let P_1, P_2 be the closed points $x = 1$ and $x = 2$ on $\mathbb{P}^1_{\mathbb{R}}$ and consider the patching data $(\mathcal{P}, \mathcal{B}, U)$ induced by $\mathcal{P} = \{P_1, P_2\}$. Then for $j = 1, 2$, $F_{P_j} = \mathbb{R}((x - j, t))$ and $F_{\wp(P_j)} = \mathbb{R}((x - j))((t))$; and $F_U = \mathrm{Frac}(\mathbb{R}[(x - 1)^{-1}, (x - 2)^{-1}][[t]])$.

Set $a_j = -t(x-j)^{-1}(x-j-t)^{-1} \in F \subseteq F_{P_j}$ for $j = 1,2$. Also write $A_{P_1} = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}$, $A_{P_2} = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}$ and $Y_{P_1} = \begin{pmatrix} 1 & y_1 \\ 0 & 1 \end{pmatrix}$, $Y_{P_2} = \begin{pmatrix} 1 & 0 \\ y_2 & 1 \end{pmatrix}$, with $y_j = \sum_{r=1}^{\infty} \frac{1}{r(x-j)^r} t^r \in F_U \subseteq F_{\wp(P_j)}$ for $j = 1,2$. Then $\partial(Y_P) = A_P Y_P$ for both points $P \in \mathcal{P}$, and $R_P = F_P[Y_P, Y_P^{-1}]$ is a Picard-Vessiot ring for A_P over F_P. It is easy to see that y_j is transcendental over F_{P_j} and to deduce that $\underline{\mathrm{Gal}}^{\partial}_{Y_{P_j}}(R_{P_j}/F_{P_j}) = \mathcal{G}_j$ in GL_2 for $j = 1,2$ (use Example 1.4). Theorem 2.4(a) now implies that there exists an $A \in F^{2\times 2}$ and a $Y \in \mathrm{GL}_2(F_U)$ such that the Picard-Vessiot ring $R = F[Y, Y^{-1}]$ for A over F has differential Galois group $\underline{\mathrm{Gal}}^{\partial}_Y(R/F) = \langle \mathcal{G}_1, \mathcal{G}_2 \rangle = \mathrm{SL}_2$.

(2) As another example, let us consider SO_2 in its two-dimensional representation

$$\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$$

over $\mathbb{R}$. We realize SO_2 as a differential Galois group over $\mathbb{R}((t))(x)$ by using patching over $F = \mathbb{C}((t))(x)$ (with derivation $\partial = \partial/\partial x$ on both fields). Consider the isomorphism over $\mathbb{C}$

$$\psi\colon \mathbb{G}_m \to \mathrm{SO}_2,\ \lambda \mapsto \frac{1}{2}\begin{pmatrix} \lambda + \lambda^{-1} & i(-\lambda + \lambda^{-1}) \\ i(\lambda - \lambda^{-1}) & \lambda + \lambda^{-1} \end{pmatrix}.$$

Let P_1, P_2 be the closed points $x = i$ and $x = -i$ on $\mathbb{P}^1_{\mathbb{C}}$ and consider the patching data $(\mathcal{P}, \mathcal{B}, U)$ induced by $\mathcal{P} = \{P_1, P_2\}$. Then $F_P = \mathbb{C}((x \pm i, t))$ and $F_{\wp(P)} = \mathbb{C}((x \pm i))((t))$ for $P = P_1, P_2$, respectively, and $F_U = \mathrm{Frac}(\mathbb{C}[(x-i)^{-1}, (x+i)^{-1}][[t]])$. Note that $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on F and on F_U via differential automorphisms and the non-trivial element $\sigma \in \Gamma$ induces ∂-isomorphisms $F_{P_1} \to F_{P_2}$ and $F_{\wp(P_1)} \to F_{\wp(P_2)}$. Therefore, Γ acts on the differential patching data $(\mathcal{P}, \mathcal{B}, U)$. Set $y = e^{\frac{t}{x-i}} \in F_U$ and $Y_{P_1} = \psi(y) \in \mathrm{GL}_2(F_U) \leq \mathrm{GL}_2(F_{\wp(P_1)})$. Note that $R_{P_1} = F_{P_1}[Y_{P_1}, Y_{P_1}^{-1}] = F_{P_1}[y, y^{-1}]$ is a Picard-Vessiot ring over F_{P_1} for the one-dimensional equation $\partial(y) = \frac{-t}{(x-i)^2} y$. It can be shown that $A_{P_1} := \partial(Y_{P_1})Y_{P_1}^{-1} \in R_{P_1}^{2\times 2}$ is contained in $F_{P_1}^{2\times 2}$. Thus $R_{P_1} \subseteq F_{\wp(P_1)}$ is also a Picard-Vessiot ring for A_{P_1} over F_{P_1}. Set $Y_{P_2} = \sigma(Y_{P_1})$. Then $A_{P_2} := \partial(Y_{P_2})Y_{P_2}^{-1} = \sigma(A_{P_1}) \in F_{P_2}^{2\times 2}$, and $R_{P_2} = F_{P_2}[Y_{P_2}, Y_{P_2}^{-1}]$ is a Picard-Vessiot ring for A_{P_2} over F_{P_2}. It is easy to see that y is transcendental over F_{P_1} hence $\underline{\mathrm{Gal}}^{\partial}_y(R_{P_1}/F_{P_1}) = \mathbb{G}_{m,\mathbb{C}}$ and thus $\underline{\mathrm{Gal}}^{\partial}_{Y_{P_1}}(R_{P_1}/F_{P_1}) = \psi(\mathbb{G}_{m,\mathbb{C}}) = (\mathrm{SO}_2)_{\mathbb{C}}$. Similarly, $\underline{\mathrm{Gal}}^{\partial}_{Y_{P_2}}(R_{P_2}/F_{P_2}) = (\mathrm{SO}_2)_{\mathbb{C}}$. Theorem 2.4(a) implies that there exists an $A \in F^{2\times 2}$ with fundamental solution matrix $Y \in \mathrm{GL}_2(F_U)$ and Picard-Vessiot ring $R = F[Y, Y^{-1}]$ such that $\underline{\mathrm{Gal}}^{\partial}_Y(R/\mathbb{C}((t))(x)) = (\mathrm{SO}_2)_{\mathbb{C}}$. By Theorem 2.4(b), we can moreover assume that Y is Γ-invariant. Lemma 1.11 then implies that $R_0 := \mathbb{R}((t))(x)[Y, Y^{-1}]$ is a Picard-Vessiot ring over $\mathbb{R}((t))(x)$ with $\underline{\mathrm{Gal}}^{\partial}_Y(R_0/\mathbb{R}((t))(x)) = \mathrm{SO}_2$.

(3) Let $T \leq \mathrm{GL}_n$ be a one-dimensional torus defined over $\mathbb{Q}((t))$ that splits over the finite extension $\mathbb{Q}((t))(\sqrt{t}) = \mathbb{Q}((s))$ with $s := \sqrt{t}$, i.e., there is an isomorphism $\psi\colon \mathbb{G}_m \to T$ defined over $\mathbb{Q}((s))$. Set $K = \mathbb{Q}((s))$ and $F = K(x)$ with derivation $\partial = \partial/\partial x$. Then $K/\mathbb{Q}((t))$ is Galois with Galois group $\Gamma \cong C_2$ and $F/\mathbb{Q}((t))(x)$ is also Galois with Galois group Γ. Let $\sigma \in \Gamma$ be the non-trivial automorphism. Consider the change of variables $z = x/s \in F$. Then $F = K(z)$ with $\partial(z) = 1/s$ and $\sigma(z) = -z$. Let P_1, P_2 be the

closed points $z = 1$ and $z = -1$ on the z-line $\mathbb{P}^1_{\mathbb{Q}}$ and consider the patching data $(\mathcal{P}, \mathcal{B}, U)$ induced by $\mathcal{P} = \{P_1, P_2\}$. Then $F_P = \mathbb{Q}((z \pm 1, s))$ and $F_{\wp(P)} = \mathbb{Q}((z \pm 1))((s))$ for $P = P_1, P_2$, respectively, and $F_U = \mathrm{Frac}(\mathbb{Q}[(z-1)^{-1}, (z+1)^{-1}][[s]])$. As explained in Example 2.3(b), Γ acts on F and on F_U via differential automorphisms and σ induces ∂-isomorphisms $F_{P_1} \to F_{P_2}$ and $F_{\wp(P_1)} \to F_{\wp(P_2)}$. Therefore, Γ acts on the differential patching data $(\mathcal{P}, \mathcal{B}, U)$ permuting P_1 and P_2. We set $y = e^{\frac{s}{z-1}} \in F_{\wp(P_1)}$ which is transcendental over F_{P_1}. Note that $\partial(y)y^{-1} = \frac{1}{s} \cdot \frac{-s}{(z-1)^2} \in F_{P_1}$, since $\partial(z) = 1/s$. We define $Y_{P_1} = \psi(y) \in T(F_{\wp(P_1)})$ and $Y_{P_2} = \sigma(Y_{P_1}) \in T(F_{\wp(P_2)})$ and proceed as in the previous example. Eventually, we obtain an $A \in F^{n\times n}$ with fundamental solution matrix $Y \in \mathrm{GL}_n(F_U)$ and Picard-Vessiot ring $R = F[Y, Y^{-1}]$ such that $\underline{\mathrm{Gal}}^{\partial}_Y(R/\mathbb{Q}((s))(x)) = T_K$ and moreover $R_0 = \mathbb{Q}((t))(x)[Y, Y^{-1}]$ is a Picard-Vessiot ring over $\mathbb{Q}((t))(x)$ with $\underline{\mathrm{Gal}}^{\partial}_Y(R_0/\mathbb{Q}((t))(x)) = T$.

3. Constructing extensions

3.1. **Linearizations.** In this subsection, K denotes a field of characteristic zero. We begin by showing that every linear algebraic group admits a finite set of "simple" generating subgroups, after passing to a finite field extension.

Proposition 3.1. *Let $\mathcal{G}$ be a linear algebraic group defined over K. Then there exists a finite extension L/K and finitely many closed subgroups $\mathcal{G}_1, \dots, \mathcal{G}_r \leq \mathcal{G}_L$ such that $\mathcal{G}_L$ is generated by $\mathcal{G}_1, \dots, \mathcal{G}_m$ and such that each $\mathcal{G}_i$ is isomorphic (over L) to either $\mathbb{G}_a$ or $\mathbb{G}_m$ or a finite (constant) cyclic group.*

Proof. It suffices to show this for L replaced by an algebraic closure $\bar{K}$ of K. By the theorem of Borel-Serre [BS64, Lemme 5.11], $\mathcal{G}_{\bar{K}}$ is generated by its identity component together with some finite group $H \leq \mathcal{G}_{\bar{K}}$. Clearly, H is generated by finitely many finite constant cyclic subgroups (defined over $\bar{K}$). We may thus assume that $\mathcal{G}_{\bar{K}}$ is connected.

Theorem 6.4.5 in [Spr09] implies that $\mathcal{G}_{\bar{K}}$ is generated by the centralizers of finitely many maximal tori $T \leq \mathcal{G}_{\bar{K}}$. Such a centralizer $C(T)$ is a connected closed subgroup of $\mathcal{G}_{\bar{K}}$ and it is nilpotent with maximal torus T ([Spr09, Prop. 6.4.2]). Let C_u denote the set of unipotent elements in $C(T)$ (which is a closed, connected subgroup). Then $C = TC_u$. Now T is isomorphic to a direct product of copies of $\mathbb{G}_m$ (over $\bar{K}$) and is thus generated by finitely many subgroups that are isomorphic to $\mathbb{G}_m$.

It remains to show that C_u is generated by finitely many subgroups that are isomorphic to $\mathbb{G}_a$. For $x \in C_u(\bar{K})$, let $G(x) \leq C_u$ denote the Zariski closure of the subgroup generated by x. For each $x \neq 1$, $G(x)$ is an infinite, closed, abelian, unipotent and thus also connected subgroup of C_u (defined over $\bar{K}$). Since C_u is finite dimensional, there exist finitely many elements $x \in C_u(\bar{K})$ such that the finitely many subgroups $G(x)$ generate C_u. Now each of the groups $G(x)$ is isomorphic to $\mathbb{G}_a^m$ by [Spr09, Lemma 3.4.7.c] for some $m \in \mathbb{N}$. Actually, $m = 1$ since $\mathbb{G}_a^m$ does not contain a dense cyclic subgroup for $m \geq 2$. □

In order to construct Picard-Vessiot rings whose differential Galois groups are given subgroups of GL_n that are isomorphic to $\mathbb{G}_a$, $\mathbb{G}_m$, or a finite cyclic group, we use the following statement that allows us to modify the representation of the differential Galois group. It is based on standard Tannakian arguments.

Proposition 3.2. *Let R be a Picard-Vessiot ring over a differential field F with field of constants K, and let G be its differential Galois group. Suppose that $\rho : G \to \mathrm{GL}_{n,K}$ is any linear representation. Then there exists a Picard-Vessiot ring $R' \subseteq R$ over F and a fundamental solution matrix $Y' \in \mathrm{GL}_n(R')$ such that $\underline{\mathrm{Gal}}^{\partial}_{Y'}(R'/F) = \rho(G)$. If moreover ρ is faithful, then $R' = R$.*

Proof. Consider the differential equation associated to R and let M be the corresponding differential module over F ([vdPS03], discussion preceding Lemma 1.7). Let $\{\{M\}\}$ be the full subcategory of the category of differential modules over F generated by M (i.e., $\{\{M\}\}$ is the smallest full subcategory that contains M and is closed under subquotients, finite direct sums, tensor products, and duals). Further, let Repr_G denote the category of finite dimensional K-representations of G. The Picard-Vessiot extension R determines an equivalence of symmetric tensor categories $S : \{\{M\}\} \to \mathrm{Repr}_G$ by [AM05], Theorem 4.10. Here, for an object N in $\{\{M\}\}$, $S(N)$ is the solution space $\ker(\partial, R \otimes_F N)$ with G-action induced by the action of G on R, and trivial on N. Since R is a Picard-Vessiot ring for M, $\dim_K(S(N)) = \dim_F(N)$ for all N; hence there exists a fundamental solution matrix Y_N over R generating a Picard-Vessiot ring $R_N \leq R$, and the induced morphism $G \to \underline{\mathrm{Gal}}^{\partial}_{Y_N}(R_N/F)$ is equivalent to $S(N)$. Applying this to an object M' in $\{\{M\}\}$ such that $S(M')$ is equivalent to the given representation ρ yields the first claim. If ρ is faithful then the fraction fields of R' and R are equal by the Galois correspondence. Since a Picard-Vessiot ring is characterized as the set of differentially finite elements in a Picard-Vessiot extension (see Proposition 1.2), this implies $R' = R$. □

3.2. **Building blocks.** As before, we fix a field k of characteristic zero. Next, we construct explicit extensions with differential Galois group $\mathbb{G}_a$, $\mathbb{G}_m$ and cyclic group C_r of order r, via the following lemmas. For the $\mathbb{G}_a$ case, we have:

Lemma 3.3. *Let $c \in k((t))^\times$ and consider the derivation $\partial = c\frac{\partial}{\partial x}$ on the fields $k((x,t)) \subseteq k((x))((t))$. Set $y = -\log(1 - t/x) := \sum_{r=1}^{\infty} x^{-r}t^r/r \in k((x))((t))$. Then $R = k((x,t))[y]$ is a Picard-Vessiot ring over $k((x,t))$ with fundamental solution matrix $Y = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, and $\underline{\mathrm{Gal}}^{\partial}_{Y}(R/k((x,t)))$ is the image of $\mathbb{G}_a$ in its two-dimensional representation $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.*

Proof. Note that y satisfies the differential equation $\partial(y) = \frac{c}{x} \cdot \frac{1}{1-x/t} \in k((x,t))$, hence $R \subseteq k((x))((t))$ is a Picard-Vessiot ring over $k((x,t))$ with $\underline{\mathrm{Gal}}^{\partial}_{Y}(R/k((x,t))) \leq \mathbb{G}_a$ in its two-dimensional representation (see Example 1.4). To see that this containment is an equality, it suffices to show that the differential Galois group is nontrivial. By the Galois correspondence, this is equivalent to showing that y does not lie in $k((x,t))$. We will show by contradiction that y does not even lie in the overfield $K((x))$ of $k((x,t))$ (where $K = k((t))$ as above).

If $y = \sum_{i=m}^{\infty} a_i x^i$ with $a_i \in K$, then

$$\partial(y) = \sum_{i=m}^{\infty} i a_i x^{i-1}.$$

Here there is no term of degree -1. But the term of lowest degree in

$$\frac{c}{x} \cdot \frac{1}{1-x/t} = cx^{-1}\big(1 + x/t + x^2/t^2 + \cdots\big)$$

has degree -1. Thus $\partial(y)$ cannot equal $\frac{c}{x} \cdot \frac{1}{1-x/t}$, and this is a contradiction. $\square$

To treat the $\mathbb{G}_m$ case, we first show a generalization of Theorem 2.4 in [Völ96] (where k was assumed algebraically closed).

Lemma 3.4. *Let κ be a field of characteristic zero, let $K = \kappa((\pi))$, and let L be a finite field extension of K. Then there is a finite field extension κ' of κ and a positive integer e such that L is contained in the field extension $\kappa'((\pi))(\pi^{1/e}) = \kappa'((\pi^{1/e}))$ of K.*

Proof. After replacing L by its Galois closure over K, we may assume that L/K is Galois. Let $\bar{K}$ be an algebraic closure of K that contains L.

Let $R = \kappa[[\pi]]$, let S be the integral closure of R in L, and let e be the ramification index of S over the prime πR of R. The residue field of $R' = R[\pi^{1/e}]$ is again κ. Let S' be the integral closure of the compositum of R' and S inside $\bar{K}$. Thus R' and S' are each complete discrete valuation rings, and S' is finite over R'. Let κ' be the residue field of S'.

By Abhyankar's Lemma ([Gro71], Lemme X.3.6), S' is unramified over the prime $\pi^{1/e}R'$ of R'. Thus the degree of S' over R' is equal to the degree d of the residue field extension κ'/κ. Hence the inclusion $\kappa'[[\pi^{1/e}]] \hookrightarrow S'$ is an isomorphism, each ring being a degree d integrally closed extension of R'. It follows that S is contained in $\kappa'[[\pi^{1/e}]]$ and thus that L is contained in $\kappa'((\pi))(\pi^{1/e})$ as asserted. $\square$

Lemma 3.5. *Let $c \in k((t))^\times$ and consider the derivation $\partial = c\frac{\partial}{\partial x}$ on the fields $k((x,t)) \subseteq k((x))((t))$. Let $y = exp(t/x) := \sum_{r=0}^{\infty} x^{-r}t^r/r! \in k((x))((t))$. Then $R = k((x,t))[y, y^{-1}]$ is a Picard-Vessiot ring over $k((x,t))$ with $\underline{\mathrm{Gal}}_y^\partial(R/k((x,t))) = \mathbb{G}_m$.*

Proof. Note that y satisfies the differential equation $\partial(y) = -ctx^{-2}y$ over $k((x,t))$, hence $R \subseteq k((x))((t))$ is a Picard-Vessiot ring for the one-by-one matrix $A = -ctx^{-2}$ by Proposition 1.1 and $\underline{\mathrm{Gal}}_y^\partial(R/k((x,t))) \leq \mathrm{GL}_1 = \mathbb{G}_m$. To see that this containment is an equality, it suffices by the Galois correspondence to show that y is transcendental over $k((x,t))$ (since the proper closed subgroups of the multiplicative group are finite). We will show by contradiction that no finite extension of the differential field extension $k((t))((x))$ of $k((x,t))$ contains an element y that satisfies $\partial(y) = -ctx^{-2}y$. (This is sufficient since derivations extend uniquely to finite extensions.)

Suppose there is a finite extension $L/k((t))((x))$ containing such a y. Applying Lemma 3.4 with $\kappa = k((t))$, we obtain a finite field extension κ' of κ and a positive integer e such that L is contained in $\kappa'((z))$, where $z^e = x$.

We consider $k((t))((x))$ as a differential field with respect to $\partial = c\,\partial/\partial x$. This derivation has a unique extension to a derivation on $\kappa'((z))$ with constant field κ', given by $\partial(z) = cz^{1-e}/e$.

Since $y \in \kappa'((z))$, we may write $y = \sum_{i=m}^{\infty} a_i z^i$, where each $a_i \in \kappa'$ and $a_m \neq 0$. Now $\partial(z^i) = iz^{i-1}\partial(z) = \frac{ci}{e} z^{i-e}$, and so

$$\partial(y) = \sum_{i=m}^{\infty} \frac{ia_i c}{e} z^{i-e}.$$

The coefficient of z^j in this expression vanishes for $j < m-e$, and in particular for $j = m-2e$. Meanwhile, the coefficient of z^{m-2e} in $-ctx^{-2}y = -ctz^{-2e}y$ is $-cta_m$, which is non-zero. Thus $\partial(y)$ cannot equal $-ctx^{-2}y$. This is a contradiction. □

Lemma 3.6. *Let r be a positive integer, and assume that k contains a primitive r-th root of unity. Then $y := x\big(1-x^{-1}t\big)^{1/r}$ is contained in $k((x))((t))$ and $R = k((x,t))[y] \subseteq k((x))((t))$ is a Picard-Vessiot ring over $k((x,t))$ with $\underline{\mathrm{Gal}}^{\partial}_{y}(R/k((x,t))) = C_r$ in its one-dimensional representation.*

Proof. First note that the element $y/x = \big(1 - x^{-1}t\big)^{1/r}$ lies in the subring $k[x^{-1}][[t]]$ of $k((x))((t))$ because it is a power series in $x^{-1}t$, hence $y \in k((x))((t))$.

Now y^r lies in $k[[x,t]]$. We claim that r is minimal for this property. To show this, let m be minimal such that $y^m \in k[[x,t]]$. Then m divides r; let $d = r/m$. Thus $y^r = x^r - x^{r-1}t$ is a d-th power of some element f in $k[[x,t]]$. Since $f^d = x^r - x^{r-1}t$, it follows that f has no term of total degree less than m; and among its terms of total degree m there is a non-vanishing x^m term as well as some $x^i t^j$ term with $i + j = m$ and $j > 0$. Let j be maximal for this property. Then $x^r - x^{r-1}t = f^d$ has a non-vanishing $x^{id}t^{jd}$ term. Hence $i = r - 1$ and $j = d = 1$. Thus $r = m$, as claimed.

Since k contains a primitive r-th root of unity, it then follows from Kummer theory that $k((x,t))(y)$ is a cyclic extension of $k((x,t))$ of degree r. The derivation ∂ extends uniquely to $R = k((x,t))(y) = k((x,t))[y]$, and y satisfies the differential equation $\partial(y) = ay$ for $a := \partial(y)/y = \partial(y^r)/(ry^r) \in k((x,t))$. Hence R is a Picard-Vessiot ring over $k((x,t))$ by Proposition 1.1, and $\underline{\mathrm{Gal}}^{\partial}_{y}(R/k((x,t))) = C_r$. □

Proposition 3.7. *Let k be a field of characteristic zero. Let P be a k-point of the affine x-line $\mathbb{A}^1_k \subset \mathbb{P}^1_k$, and let F_P and $F_{\wp(P)}$ be as defined in Section 2. Endow these fields with the derivation $c\frac{\partial}{\partial x}$ for some $c \in k((t))^\times$. Suppose that $\mathcal{G} \subseteq \mathrm{GL}_n$ is a linear algebraic group over $K = k((t))$ that is K-isomorphic to $\mathbb{G}_m$, $\mathbb{G}_a$, or C_r where r is such that k contains a primitive r-th root of unity. Then there exist a matrix $A \in F_P^{n\times n}$, a Picard-Vessiot ring R/F_P for A that is contained in $F_{\wp(P)}$, and a fundamental solution matrix $Y \in \mathrm{GL}_n(R)$ for A, such that $\underline{\mathrm{Gal}}^{\partial}_{Y}(R/F_P) = \mathcal{G}$.*

Proof. After a change of variables, we may assume that P is the point of $\mathbb{P}^1_k$ where $x = 0$. We then have inclusions $F_P = k((x,t)) \subset F_{\wp(P)} = k((x))((t))$. By Lemma 3.3, 3.5, and 3.6, there exists a Picard-Vessiot ring over F_P contained in $F_{\wp(P)}$ with differential Galois group $\mathbb{G}_m$, $\mathbb{G}_a$, or C_r as in the statement, respectively. The result then follows from Proposition 3.2. □

4. The differential inverse Galois problem

In this section, we solve the inverse problem over function fields over $k((t))$, where as before k is any field of characteristic zero. We begin by making the following

Definition 4.1. Let F be a differential field with field of constants K. We say that a linear algebraic group G defined over K **is a differential Galois group** over F if there exists a Picard-Vessiot ring R/F with Galois group K-isomorphic to G.

Note that if G is a differential Galois group over F, this also implies that every faithful representation $\mathcal{G} \leq \mathrm{GL}_n$ of G arises as a differential Galois group over F by Proposition 3.2; namely, there exists a $Y \in \mathrm{GL}_n(R)$ such that $\underline{\mathrm{Gal}}_Y^{\partial}(R/F) = \mathcal{G}$.

We next note that in order to solve the inverse problem, we may modify the derivation.

Lemma 4.2. *Let F be a field, and let ∂ and ∂' be derivations on F such that there is an $a \in F^{\times}$ with $\partial' = a\partial$. Let $\mathcal{G} \leq \mathrm{GL}_n$ be a linear algebraic group over C_F, and suppose that $\mathcal{G}$ is a differential Galois group over (F, ∂). Then $\mathcal{G}$ is also a differential Galois group over (F, ∂').*

Proof. By assumption, there exists a matrix $A \in F^{n\times n}$ with Picard-Vessiot ring R over (F, ∂) and fundamental solution matrix $Y \in \mathrm{GL}_n(R)$ such that $\underline{\mathrm{Gal}}_Y^{\partial}(R/F) = \mathcal{G}$ in the given representation $\mathcal{G} \leq \mathrm{GL}_n$. Then $\partial' = a\partial$ extends to a derivation on R with $\partial'(Y) = aAY$. By Proposition 1.1, R is a Picard-Vessiot ring over (F, ∂') for the matrix aA with fundamental solution matrix Y. Note that $\underline{\mathrm{Aut}}^{\partial'}(R/F) = \underline{\mathrm{Aut}}^{\partial}(R/F)$ and thus $\underline{\mathrm{Gal}}_Y^{\partial'}(R/F) = \underline{\mathrm{Gal}}_Y^{\partial}(R/F) = \mathcal{G}$. $\square$

4.1. **The inverse problem over $k((t))(x)$.** As before, k denotes a field of characteristic zero. In this section, we show that every linear algebraic group over $k((t))$ is a differential Galois group over the rational function field $k((t))(x)$, equipped with any nontrivial derivation with field of constants $k((t))$.

Lemma 4.3. *Let $k_0 \leq k$ such that k/k_0 is a finite Galois extension of degree d. Assume further that k contains a primitive e-th root of unity ζ for some $e \in \mathbb{N}$. Then there exist infinitely many elements $a \in k$ such that there are $d\cdot e$ distinct elements among the* $\mathrm{Gal}(k/k_0)$*-conjugates of $a, \zeta a, \ldots, \zeta^{e-1}a$.*

Proof. We may assume that $d > 1$. Since k/k_0 is separable, there exists some primitive element $b \in k$. Thus there are exactly d conjugates of b. For any $c \in k_0$, consider the $\mathrm{Gal}(k/k_0)$-conjugates of $\zeta^i(c + b)$ for $0 \leq i < e$. If the number of conjugates is less than $d \cdot e$, then there exists a non-trivial $\sigma \in \mathrm{Gal}(k/k_0)$ such that $\zeta^i(c + b) = \sigma(\zeta^j(c + b))$ for some $0 \leq i \leq j < e$. Note that $\zeta^i \neq \sigma(\zeta^j)$, since $\sigma(c) = c$ and $\sigma(b) \neq b$. Hence there exists $0 < l < e$ with $\frac{\zeta^i}{\sigma(\zeta^j)} = \zeta^l$. We obtain $c(\zeta^l - 1) = \sigma(b) - \zeta^l \cdot b$, hence c is contained in the finite set $S = \{\frac{\sigma(b)-\zeta^l\cdot b}{\zeta^l-1} \mid 0 < l < e, 1 \neq \sigma \in \mathrm{Gal}(k/k_0)\} \subseteq k$. Therefore, almost all choices of c yield an element $a = c + b$ with the desired property. $\square$

Suppose that k/k_0 is a finite Galois extension of degree d and that k contains a primitive e-th root of unity, for a positive integer e. Consider the Laurent series field extension $k((t))/k_0((t_0))$ with $t_0 := t^e$. Define $\Gamma = \mathrm{Gal}(k((t))/k_0((t_0)))$. Recall from Example 2.3(a) that Γ has order $d \cdot e$, and it is a semi-direct product of the cyclic group $\mathrm{Gal}\big(k((t))/k((t_0))\big)$ of order e with the group $\mathrm{Gal}(k/k_0)$. The action of Γ on $K = k((t))$ over $K_0 = k_0((t_0))$ extends to an action of $F = K(x)$ over $K_0(x)$ by taking x to x. Consider the variable change $z = x/t$.

Lemma 4.4. *In the above setup, consider the induced action of Γ on the z-line $\mathbb{P}^1_k$ (as explained in Example 2.3(b)). Then for any positive integer r, there exist r k-points $P_1, \ldots, P_r \in \mathbb{A}^1_k \subset \mathbb{P}^1_k$ whose orbits $P_1^\Gamma, \ldots P_r^\Gamma$ are disjoint and are each of order $|\Gamma|$.*

Proof. For each $\sigma \in \Gamma$, there exists an integer $0 \le n_\sigma \le e-1$ with $\sigma(t) = \zeta^{n_\sigma} t$ and thus $\sigma(z) = \zeta^{-n_\sigma} z$. Recall from Example 2.3(b) that if $\sigma \in \Gamma$ and $P \in \mathcal{P}$ is the k-point $z = b$, then $P^{\sigma^{-1}}$ is the point $z = \zeta^{n_\sigma}\sigma(b)$.

We apply Lemma 4.3 and obtain an $a_1 \in k$ such that there are $d \cdot e$ distinct elements among the $\mathrm{Gal}(k/k_0)$-conjugates of $\zeta^i a_1$ $(0 \le i \le e-1)$. Let P_1 be the point $z = a_1$. Then P_1^Γ consists of the points $z = \zeta^{n_\sigma}\sigma(a_1)$. We claim that these $|\Gamma| = de$ points are distinct. It suffices to check that

$$\{\zeta^{n_\sigma}\sigma(a_1) \mid \sigma \in \Gamma\} = \{\tau(\zeta^i a_1) \mid 0 \le i < e, \tau \in \mathrm{Gal}(k/k_0)\}.$$

Given an element $\tau(\zeta^i a_1)$ in the right hand side, let ψ be the element of $\mathrm{Gal}\big(k((t))/k_0((t))\big) \le \Gamma$ that lifts $\tau \in \mathrm{Gal}(k/k_0)$. Thus $n_\psi = 0$. Since $\tau(\zeta^i)$ is an e-th root of unity, there is an element π in the cyclic group $\mathrm{Gal}\big(k((t))/k((t_0))\big)$ with $\zeta^{n_\pi} = \tau(\zeta^i)$. Define $\sigma = \pi \circ \psi \in \Gamma$. As $n_\psi = 0$, $\zeta^{n_\sigma} = \zeta^{n_\pi} = \tau(\zeta^i)$. Therefore $\zeta^{n_\sigma}\sigma(a_1) = \tau(\zeta^i a_1)$, and the claim follows.

To find a second point P_2, we apply Lemma 4.3 to obtain an element $a_2 \in k$ with the same property and such that a_2 is not contained in the finite set of $\mathrm{Gal}(k/k_0)$-conjugates of $\zeta^i a_1$. Therefore, $P_2 \notin P_1^\Gamma$. Thus P_1^Γ and P_2^Γ are disjoint orbits of order $|\Gamma|$. By induction, we get points $P_1, \ldots, P_r$ as asserted. $\square$

Theorem 4.5. *Let k be a field of characteristic zero, let $K = k((t))$, and let F be a rational function field of transcendence degree one over K. Let ∂ be a non-trivial derivation on F with field of constants K. Then every linear algebraic group defined over K is a differential Galois group over F.*

Proof. In order to make the notation in the proof consistent with the notation used in Section 2, we rename our base field as follows: We consider the rational function field $F_0 = K_0(x)$ over $K_0 = k_0((t_0))$, the field of formal Laurent series over a field k_0 of characteristic zero. By Lemma 4.2 we may assume without loss of generality that the derivation ∂ on F_0 satisfies $\partial(x) = 1$, with $C_{F_0} = K_0$ (hence $\partial = \partial/\partial x$).

Let $\mathcal{G} \le \mathrm{GL}_n$ be a linear algebraic group defined over K_0. We will show that there exists a differential equation $A \in F_0^{n\times n}$ with a Picard-Vessiot ring R and fundamental solution matrix Y such that $\underline{\mathrm{Gal}}^\partial_Y(R/F_0) = \mathcal{G}$.

By Proposition 3.1, there exists a finite extension K/K_0 and finitely many K-subgroups $\mathcal{G}_1, \ldots, \mathcal{G}_r \le \mathcal{G}_K$ that generate $\mathcal{G}_K$ and such that each $\mathcal{G}_i$ is either a finite cyclic group or is K-isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$. After enlarging K, we may assume it contains a primitive m-th root of unity for each m that is the order of one of the finite cyclic groups among $\mathcal{G}_1, \ldots, \mathcal{G}_r$. We may further assume that $K = k((t))$ for a finite Galois extension k/k_0 and an e-th root t of t_0 for some $e \in \mathbb{N}$, by Lemma 3.4. Without loss of generality, we may assume that k contains a primitive e-th root of unity ζ. Then K/K_0 is a finite Galois extension (see Example 2.3) and we set $\Gamma = \mathrm{Gal}(K/K_0)$.

We first construct a Picard-Vessiot ring over $F = K(x)$ with Galois group $\mathcal{G}_K$, using patching. Let $z = x/t$. By Lemma 4.4, we can fix k-points $P_1, \ldots, P_r$ on the affine z-line $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ such that $|P_i^\Gamma| = |\Gamma|$ for all i and such that these orbits are disjoint. We set

$\mathcal{P} = P_1^\Gamma \cup \cdots \cup P_r^\Gamma$. Let $(\mathcal{P}, \mathcal{B}, U)$ be the corresponding differential patching data as explained in the beginning of Section 2. As explained in Example 2.3(b), Γ acts on the differential patching data $(\mathcal{P}, \mathcal{B}, U)$ (see also Definition 2.2).

By Proposition 3.7 (with $c = \partial(z) = t^{-1}$), we can now fix Picard-Vessiot rings R_{P_j}/F_{P_j} for each $j = 1, \ldots, r$ with fundamental solution matrices $Y_{P_j} \in \mathrm{GL}_n(F_{\wp(P_j)})$ such that $\underline{\mathrm{Gal}}^\partial_{Y_{P_j}}(R_{P_j}/F_{P_j}) = \mathcal{G}_j \leq \mathcal{G}_K \leq \mathrm{GL}_{n,K}$. Now let $P \in \mathcal{P}$ be arbitrary. Then there exists a unique $\sigma \in \Gamma$ and a unique integer $1 \leq j \leq r$ such that $P^\sigma = P_j$. Recall that there is a differential isomorphism $\sigma\colon F_{\wp(P_j)} \to F_{\wp(P)}$ restricting to $\sigma\colon F_{P_j} \to F_P$ (see Definition 2.2 and Example 2.3). Set $Y_P = \sigma(Y_{P_j}) \in \mathrm{GL}_n(F_{\wp(P)})$ and $R_P = F_P[Y_P, Y_P^{-1}] = \sigma(R_{P_j}) \subseteq F_{\wp(P)}$. We define $A_P = \partial(Y_P)Y_P^{-1}$. Note that $A_P = \sigma(A_{P_j}) \in \mathrm{GL}_n(F_P)$. Hence R_P is a Picard-Vessiot ring over F_P for A_P with fundamental solution matrix Y_P. Fix an extension of σ from K to $\bar{K}$. Then σ induces a differential isomorphism $\sigma\colon R_{P_j} \otimes_K \bar{K} \to R_P \otimes_K \bar{K}$. Hence

$$\underline{\mathrm{Aut}}^\partial(R_P/F_P)(\bar{K}) = \sigma \underline{\mathrm{Aut}}^\partial(R_{P_j}/F_{P_j})(\bar{K})\sigma^{-1}$$

and thus

$$\begin{aligned}\underline{\mathrm{Gal}}^\partial_{Y_P}(R_P/F_P)(\bar{K}) &= \underline{\mathrm{Gal}}^\partial_{\sigma(Y_{P_j})}(\sigma(R_{P_j})/\sigma(F_{P_j}))(\bar{K}) \\ &= \sigma(\underline{\mathrm{Gal}}^\partial_{Y_{P_j}}(R_{P_j}/F_{P_j})(\bar{K})) \\ &= \sigma(\mathcal{G}_j(\bar{K})).\end{aligned}$$

We conclude by Remark 1.5 that $\underline{\mathrm{Gal}}^\partial_{Y_P}(R_P/F_P) = \mathcal{G}_j^\sigma$, the linear algebraic group obtained from $\mathcal{G}_j$ by applying σ to the defining equations.

Theorem 2.4 then yields a matrix $A \in F^{n\times n}$ and a Picard-Vessiot ring $R = F[Y, Y^{-1}]$ for A over F ($Y \in \mathrm{GL}_n(F_U)$ a fundamental solution matrix) such that

$$\underline{\mathrm{Gal}}^\partial_Y(R/F) = \overline{< \mathcal{G}_j^\sigma \mid 1 \leq j \leq r, \sigma \in \Gamma >} = \overline{< \mathcal{G}_K^\sigma \mid \sigma \in \Gamma >} = \mathcal{G}_K.$$

Moreover, we constructed the fundamental solution matrices $(Y_P)_{P\in\mathcal{P}}$ such that $\sigma(Y_{P^\sigma}) = Y_P$ for all $P \in \mathcal{P}$, $\sigma \in \Gamma$. Therefore, Theorem 2.4(b) asserts that we can assume that Y is Γ-stable, which in turn implies that all entries of A are contained in $F^\Gamma = F_0$. Therefore, $R_0 := F_0[Y, Y^{-1}]$ is a Picard-Vessiot ring for A over F_0 with differential Galois group $\underline{\mathrm{Gal}}^\partial_Y(R_0/F_0) = \mathcal{G} \leq \mathrm{GL}_{n,K_0}$ (via Lemma 1.11 with $L = F_U$). □

Proposition 4.6. *The Picard-Vessiot ring R in Theorem 4.5 can be constructed such that $R \subseteq K((x))$.*

Proof. We switch to the notation as in the proof of Theorem 4.5, so we need to show that $R_0 \subseteq K_0((x)) = k_0((t_0))((x))$. We refine the choice of $\mathcal{P}$ such that the point $z = 0$ is not contained in $\mathcal{P}$. Set $m = |\Gamma| \cdot r$, where r is as in the proof of Theorem 4.5, and let $\alpha_1, \ldots, \alpha_m \in k^\times$ such that $\mathcal{P}$ is the set of all points $z = \alpha_i$. Now $R_0 = F_0[Y, Y^{-1}]$ and $Y \in \mathrm{GL}_n(F_U)$ is Γ-invariant. As all α_i are non-zero,

$$F_U = \mathrm{Frac}([(z-\alpha_1)^{-1}, \ldots, (z-\alpha_m)^{-1}][[t]]) \subseteq \mathrm{Frac}(k[[z,t]]).$$

Moreover $k[[z,t]]$ is contained in $k((t))[[x]]$, as can be seen by replacing z by x/t in any power series in $k[[z,t]]$. Thus $\mathrm{Frac}(k[[z,t]]) \subseteq \mathrm{Frac}(k((t))[[x]]) = k((t))((x))$. Hence F_U can be regarded as a differential subfield of $k((t))((x))$; and it is easy to check that the embedding $F_U \subseteq k((t))((x))$ is Γ-equivariant. Therefore, all entries of Y lie in $F_U^\Gamma \subseteq k((t))((x))^\Gamma = k_0((t_0))((x))$ and the assertion follows. □

Remark 4.7. Theorem 4.5, in conjunction with [Hru02], can be used to obtain a new proof of the fact that every linear algebraic group G over $\bar{\mathbb{Q}}$ is a differential Galois group over $\bar{\mathbb{Q}}(x)$. (In [Har05], the second author gave an earlier proof of this, which also handled the case of $C(x)$ for C an arbitrary algebraically closed field of characteristic zero). Namely, we may regard G as defined over $\bar{\mathbb{Q}}((t))$; and then by Theorem 4.5, there is a Picard-Vessiot extension R of $\bar{\mathbb{Q}}((t))(x)$ with differential Galois group $G_{\bar{\mathbb{Q}}((t))}$, for some matrix $A \in \bar{\mathbb{Q}}((t))(x)^{n\times n}$. This data descends to some finitely generated $\bar{\mathbb{Q}}$-subalgebra D of $\bar{\mathbb{Q}}((t))$; and by [Hru02, Section V.1], infinitely many specializations to closed points of $\mathrm{Spec}(D)$ yield the same differential Galois group G. Since $\bar{\mathbb{Q}}$ is algebraically closed, these closed points are all $\bar{\mathbb{Q}}$-rational; and so G is a differential Galois group over $\bar{\mathbb{Q}}(x)$.

Remark 4.8. Theorem 4.5 in particular asserts that every finite étale group scheme G over a characteristic zero Laurent series field $K = k((t))$ is a differential Galois group over $K(x)$. Equivalently, for each such G, there is a finite morphism of smooth connected projective K-curves $C \to \mathbb{P}^1_K$, with a faithful action of G on C, such that the generic point of C is a G-torsor over $K(x)$. In the case that G is a finite *constant* group, this says that G is a Galois group over $K(x)$; and that was proven (in fact in arbitrary characteristic) in [Har87, Theorem 2.3], using formal patching methods. In the case of finite étale group schemes G that need not be constant, this was shown (again in arbitrary characteristic) in [MB01]. That paper in fact showed more, viz. that this holds if K is more generally a large field in the sense of Pop ([Pop96]); and also that one fiber of the morphism $C \to \mathbb{P}^1_K$ can be given in advance; this extended results of Pop ([Pop96, Main Theorem A]) and Colliot-Thélène ([CT00, Theorem 1]) for finite constant groups.

4.2. **Passage to finitely generated extensions.** In this subsection, we prove our main result (Theorem 4.14). This is a consequence of Theorem 4.5, together with the following result (see Theorem 4.12): If F is any differential field of characteristic zero with the property that every linear algebraic group defined over C_F is a differential Galois group over F, then every finitely generated extension L/F of differential fields with $C_L = C_F$ also has this property. To prove the latter, we first require some preparation.

Lemma 4.9. *Let L/F be an extension of differential fields. If $x_1, \dots, x_n \in C_L$ are algebraically independent over C_F, then they are algebraically independent over F.*

Proof. We prove the contrapositive. Assume that $x_1, \dots, x_n \in C_L$ are algebraically dependent over F. Let r be the smallest positive integer such that there exists a polynomial expression over F, vanishing on $(x_1, \dots, x_n)$, with exactly r monomial terms. Let p be such a polynomial. After dividing by an element in $F^\times$, we can assume that some coefficient equals one. Differentiating the coefficients of p yields a polynomial with less than r monomial terms that also vanishes on $(x_1, \dots, x_n)$ and must be the zero polynomial by minimality of p. Hence all coefficients of p are contained in C_F, and thus $x_1, \dots, x_n$ are algebraically dependent over C_F. □

Lemma 4.10. *Let F be a differential field, let R/F be a Picard-Vessiot ring and set $G = \underline{\mathrm{Aut}}^\partial(R/F)$ and $E = \mathrm{Frac}(R)$. Let $H_1, \dots, H_r$ be closed subgroups of G (defined over C_F) and set $H = \bigcap_{i=1}^{r} H_i$. Then*

$$E^{H_1} \cdots E^{H_r} = E^H \subseteq E.$$

Proof. The compositum of $E^{H_1}, \dots, E^{H_r}$ is a differential subfield of E, so it equals $E^{\tilde{H}}$ for some closed subgroup $\tilde{H}$ of G, by the Galois correspondence. Then $E^{H_i} \subseteq E^{\tilde{H}}$, so $\tilde{H} \subseteq H_i$ for all i, and thus $\tilde{H} \subseteq H$. On the other hand, every element in the compositum is invariant under H, hence $\tilde{H} \supseteq H$. □

Lemma 4.11. *Let $F_1, \dots, F_r$ be finite field extensions of a field F. Embed F into $F_1 \otimes_F \cdots \otimes_F F_r$ via $1 \mapsto 1 \otimes \cdots \otimes 1$. Let $V \subseteq F_1 \otimes_F \cdots \otimes_F F_r$ be an F-subspace of dimension $d < r$ with $F \subseteq V$. If $1 \leq s \leq r - d + 1$, then after relabeling the indices,*

$$(F_1 \otimes_F \cdots \otimes_F F_s) \cap V = F,$$

where the intersection is taken inside $F_1 \otimes_F \cdots \otimes_F F_r$.

Proof. Since F is contained in each F_i, the right hand side is contained in the left hand side. For the other containment, it suffices to prove the assertion for $s = r - d + 1$.

For each $i \leq r$, fix an F-basis $\{v_{i,j} \mid 1 \leq j \leq [F_i : F]\}$ of F_i with $v_{i,1} = 1$. We use induction on r. If $r = 2$, then $d = 1$ and $V = F$, and the claim follows. Let $r > 2$. If $(F_1 \otimes_F \cdots \otimes_F F_r) \cap V = F$, there is nothing to prove. Otherwise, fix an $x \in (F_1 \otimes_F \cdots \otimes_F F_r) \cap V$ that is not contained in F. Then we can write x as an F-linear combination of the elements $v_{1,j_1} \otimes \cdots \otimes v_{r,j_r}$ in a unique way. Since $x \notin F$, this sum contains a term $v_{1,j_1} \otimes \cdots \otimes v_{r,j_r}$ with $j_i \neq 1$ for some i. After relabeling the indices we may assume $i = r$, which implies $x \notin F_1 \otimes_F \cdots \otimes_F F_{r-1}$. Hence $\dim_F((F_1 \otimes_F \cdots \otimes_F F_{r-1}) \cap V) \leq d - 1$ and the claim follows by induction. □

As before, let F be a differential field, and let G be a linear algebraic group defined over $K = C_F$. We now prove the above mentioned general result, using a strategy sometimes called the *Kovacic trick*: Assume there is a Picard-Vessiot extension R over F with differential Galois group G^r. The idea is to show that $R \otimes_F L$ must contain a Picard-Vessiot ring over L with differential Galois group G if r is sufficiently large.

Theorem 4.12. *Let F be a differential field of characteristic zero and write $K = C_F$. Let L/F be a differential field extension that is finitely generated over F with $C_L = K$. Let G be a linear algebraic group defined over K with the property that for every $r \in \mathbb{N}$, G^r is a differential Galois group over F. Then G is a differential Galois group over L.*

Proof. Let L' be the algebraic closure of F in L and let L'' be the normal closure of L' in $\bar{F}$. Set $d = [L'' : F]$ and $m = \mathrm{trdeg}(L/F) + 1$. Note that $1 \leq d, m < \infty$.

First step: We show that there is a Picard-Vessiot ring R/F with differential Galois group $\underline{\mathrm{Aut}}^\partial(R/F)$ isomorphic to G^{2m}, and with the following property: If F' denotes the algebraic closure of F in $\mathrm{Frac}(R)$, then (1) $F' \otimes_F L'$ is a field in which (2) K is algebraically closed.

By assumption, there is a Picard-Vessiot ring R/F with differential Galois group G^{2m+2d}. We show that we can achieve (1) and (2) by replacing R with a suitable subring.

Set $E = \mathrm{Frac}(R)$ and let E_i be the fixed field $E^{G\times\cdots\times G\times 1\times G\times\cdots\times G}$ (omitting the i-th factor) for $1 \leq i \leq 2m+2d$. Thus $C_E = C_{E_i} = K$. Since $G\times\cdots\times G\times 1\times G\times\cdots\times G$ is a normal subgroup of G^{2m+2d}, the Galois correspondence implies that $E_i = \mathrm{Frac}(R_i)$ for a Picard-Vessiot ring R_i/F with differential Galois group $\underline{\mathrm{Aut}}^\partial(R_i/F) \cong G$ for each i.

We let $F' \subseteq E$ be the algebraic closure of F in E and similarly let $F'_i \subseteq E_i$ the algebraic closure of F in E_i. Then Lemma 1.10 implies $F' = E^{G^0\times\cdots\times G^0}$ and $F'_i = E^{G\times\cdots\times G\times G^0\times G\times\cdots\times G}$. By Lemma 4.10, the compositum $F'_1\cdots F'_{2m+2d}$ equals $E^{G^0\times\cdots\times G^0} = F'$. Again by Lemma 1.10, $[F' : F] = |G(\bar{K})/G^0(\bar{K})|^{2m+2d}$ and $[F'_i : F] = |G(\bar{K})/G^0(\bar{K})|$ for each $1 \leq i \leq 2m+2d$. A dimension count yields

$$F' \cong F'_1 \otimes_F \cdots \otimes_F F'_{2m+2d}.$$

Let $V = F' \cap L''$. Thus $\dim(V) \leq \dim(L'') = d$. An application of Lemma 4.11 yields $(F'_1 \otimes_F \cdots \otimes_F F'_{2m+d}) \cap L'' = F$, possibly after renumbering the indices.

We now change notation: we replace E by $E^{1\times\cdots\times 1\times G\times\cdots\times G}$ (with d copies of G on the right); we replace R with the unique Picard-Vessiot ring in $E^{1\times\cdots\times 1\times G\times\cdots\times G}$ (hence $\underline{\mathrm{Aut}}^\partial(R/F) \cong G^{2m+d}$); and we replace F' by the algebraic closure $F'_1\otimes_F\cdots\otimes_F F'_{2m+d}$ of F in $E^{1\times\cdots 1\times G\times\cdots\times G}$. (Note that this tensor product is indeed equal to the relative algebraic closure, since both have the same degree over F by Lemma 1.10 and since the former field extension is contained in the latter.) We similarly replace E_i and R_i. In this new situation, $F' \cap L'' = F$. As L''/F is a finite Galois extension, $F' \cap L'' = F$ implies that $F' \otimes_F L''$ is a field. In particular, $F' \otimes_F L'$ is a field which proves (1) for this choice of R.

To establish (2), let K' be the algebraic closure of K in the field $F' \otimes_F L'$. Thus $K' = C_{F'\otimes_F L'}$. Moreover, K' and F' are linearly disjoint over K, since K'/K is algebraic whereas K is algebraically closed in F' (using $C_{F'} \subseteq C_E = K$). Hence a K-basis of K' is linearly independent over F'. Therefore, $[K' : K] = [F'K' : F'] \leq [F' \otimes_F L' : F'] = [L' : F] \leq d$ and thus $[K'L' : L'] \leq d$, where all composita are taken inside $F' \otimes_F L'$. Recall that $F' \cong F'_1 \otimes_F \cdots \otimes_F F'_{2m+d}$, hence

$$F' \otimes_F L' \cong (F'_1 \otimes_F L') \otimes_{L'} \cdots \otimes_{L'} (F'_{2m+d} \otimes_F L').$$

Since the L'-vector space $V = L'K'$ has dimension at most d, by Lemma 4.11 we can relabel the indices such that

$$(F'_1 \otimes_F L') \otimes_{L'} \cdots \otimes_{L'} (F'_{2m} \otimes_F L') \cap L'K' = L'.$$

In particular $(F'_1 \otimes_F L') \otimes_{L'} \cdots \otimes_{L'} (F'_{2m} \otimes_F L') \cap K' = L' \cap K' = K$, where the last equality uses that $C_L = K$.

We again change notation, replacing E by $E^{1\times\cdots\times 1\times G\times\cdots\times G}$ (with d copies of G on the right) and replacing R, F', E_i, and R_i correspondingly as above. In this new notation, we obtain that $F' \otimes_F L'$ is a field in which K is algebraically closed. Note that $\underline{\mathrm{Aut}}^\partial(R/F) \cong G^{2m}$ for this new R.

Second step: We claim that if R is as constructed in the first step and $E = \mathrm{Frac}(R)$, then $E \otimes_F L$ is an integral domain and K is algebraically closed in $\tilde{E} = \mathrm{Frac}(E \otimes_F L)$.

As E/F' and L/L' are regular, both $E \otimes_{F'} (F' \otimes_F L')$ and $(F' \otimes_F L') \otimes_{L'} L$ are regular over $F' \otimes_F L'$. Therefore, their tensor product over $F' \otimes_F L'$ is regular over $F' \otimes_F L'$ ([Bou90, Proposition V.17.3b]). But this tensor product is isomorphic to $E \otimes_F L$, and we conclude that $E \otimes_F L$ is a regular $(F' \otimes_F L')$-algebra. In particular, $E \otimes_F L$ is an integral domain and its fraction field $\tilde{E}$ is a regular extension of the field $F' \otimes_F L'$ ([Bou90, Proposition V.17.4]).

Recall that K is algebraically closed in the field $F' \otimes_F L'$ (by the first step). But $F' \otimes_F L'$ is algebraically closed in $\tilde{E}$, as $\tilde{E}$ is regular over $F' \otimes_F L'$. We conclude that K is algebraically closed in $\tilde{E}$ which completes the second step.

Third step: With notation as above, for $1 \leq i \leq 2m$ we define $\tilde{R}_i = R_i \otimes_F L$; this is an integral domain, being contained in $\tilde{E}$. We also let $\tilde{E}_i = \mathrm{Frac}(\tilde{R}_i)$. Thus $\tilde{E}_i$ is the compositum of E_i and L inside $\tilde{E}$. We claim that $C_{\tilde{E}_i} = K$ for some i.

Suppose to the contrary that $C_{\tilde{E}_i}$ strictly contains K for all $1 \leq i \leq 2m$. Then for each i, $\tilde{E}_i$ contains a constant that is transcendental over K (since K is algebraically closed in $\tilde{E}_i \subseteq \tilde{E}$ by the second step) and thus transcendental over L (by Lemma 4.9, using $C_L = K$). Therefore, $\mathrm{trdeg}(C_{\tilde{E}_i}L/L) \geq 1$ for all i and thus

$$\mathrm{trdeg}(\tilde{E}_i/C_{\tilde{E}_i}L) \leq \mathrm{trdeg}(\tilde{E}_i/L) - 1 \leq \mathrm{trdeg}(E_i/F) - 1 = \dim(G) - 1,$$

where the last equality follows from the fact that R_i/F is a Picard-Vessiot ring with differential Galois group G. Base change from $C_{\tilde{E}_i}L$ to $C_{\tilde{E}}L$ then yields $\mathrm{trdeg}(\tilde{E}_i C_{\tilde{E}}/C_{\tilde{E}}L) \leq \dim(G) - 1$ for all $1 \leq i \leq 2m$. By Lemma 4.10, the compositum $E_1 \cdots E_{2m}$ equals E. Therefore, $\tilde{E}$ equals the compositum of $\tilde{E}_1, \ldots, \tilde{E}_{2m}$ inside $\tilde{E}$ and we conclude that

$$\mathrm{trdeg}(\tilde{E}/C_{\tilde{E}}L) \leq 2m(\dim(G) - 1). \tag{1}$$

On the other hand, any subset of $C_{\tilde{E}}$ that is algebraically independent over K remains algebraically independent over E by Lemma 4.9, since $C_E = K$. Therefore, $\mathrm{trdeg}(C_{\tilde{E}}/K) = \mathrm{trdeg}(EC_{\tilde{E}}/E) \leq \mathrm{trdeg}(\tilde{E}/E)$ and we obtain

$$\mathrm{trdeg}(C_{\tilde{E}}L/L) \leq \mathrm{trdeg}(C_{\tilde{E}}/K) \leq \mathrm{trdeg}(\tilde{E}/E) \leq \mathrm{trdeg}(L/F) = m - 1, \tag{2}$$

where we used that $\tilde{E}$ is a compositum of E and L to obtain the last inequality. Equations (1) and (2) together yield $\mathrm{trdeg}(\tilde{E}/L) \leq 2m\dim(G) - m - 1$, and therefore

$$\mathrm{trdeg}(E/F) \leq \mathrm{trdeg}(\tilde{E}/F) \leq 2m\dim(G) - 2.$$

But R/F is a Picard-Vessiot ring with differential Galois group G^{2m}, so $\mathrm{trdeg}(E/F) = 2m\dim(G)$, a contradiction, which completes the third step.

Conclusion of the proof: We conclude that for some $1 \leq i \leq 2m$, $\tilde{R}_i = R_i \otimes_F L$ is an integral domain and $\tilde{E}_i = \mathrm{Frac}(\tilde{R}_i)$ satisfies $C_{\tilde{E}_i} = K$. By Proposition 1.8, $\tilde{R}_i$ is a Picard-Vessiot ring over L with $\underline{\mathrm{Aut}}^\partial(\tilde{R}_i/L) \cong \underline{\mathrm{Aut}}^\partial(R_i/F) \cong G$. □

A field of characteristic zero with a nontrivial derivation is always a transcendental extension of its field of constants (since derivations extend uniquely to separable algebraic extensions); hence we may find a copy of a rational function field within the given field. This gives the following

Corollary 4.13. *Let L be a differential field that is finitely generated over its field of constants $K \neq L$. Let G be a linear algebraic group defined over K with the property that for any $r \in \mathbb{N}$, G^r is a differential Galois group over $K(x)$; here $K(x)$ denotes a rational function field with derivation d/dx. Then G is a differential Galois group over L.*

Proof. Let $z \in L$ be transcendental over K. By Lemma 4.2, we may assume that $\partial(z) = 1$ by replacing ∂ with $\partial(z)^{-1} \cdot \partial$. Then $F = K(z)$ is a differential subfield of L differentially isomorphic to $K(x)$ and the claim follows from Theorem 4.12. □

Combining Theorem 4.5 and Corollary 4.13, we obtain our main result:

Theorem 4.14. *Let $K = k((t))$ be a field of Laurent series over a field k of characteristic zero. Let L/K be a finitely generated field extension with a non-trivial derivation ∂ on L such that $C_L = K$. Then every linear algebraic group defined over K is a differential Galois group over L.*

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