

DIFFERENTIAL EMBEDDING PROBLEMS OVER COMPLEX FUNCTION FIELDS

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ABSTRACT. We introduce the notion of differential torsors, which allows the adaptation of constructions from algebraic geometry to differential Galois theory. Using these differential torsors, we set up a general framework for applying patching techniques in differential Galois theory over fields of characteristic zero. We show that patching holds over function fields over the complex numbers. As the main application, we prove the solvability of all differential embedding problems over complex function fields, thereby providing new insight on the structure of the absolute differential Galois group, i.e., the fundamental group of the underlying Tannakian category.

INTRODUCTION

This paper concerns embedding problems in differential Galois theory. Our main result generalizes two classical results over one-variable complex function fields:

- the solution of the inverse problem in differential Galois theory, and
- the solvability of all embedding problems in ordinary Galois theory.

Inverse problems in Galois theory ask for the existence of Galois extensions of a given field with prescribed Galois group. Embedding problems generalize this and are used to study how Galois extensions fit together in towers. In other words, the inverse problem asks which groups are epimorphic images of the absolute Galois groups, whereas solutions to embedding problems yield epimorphisms that in addition factor over a given epimorphism of groups. Therefore, solvability of embedding problems provides additional information about the structure of the absolute Galois group. This applies to both classical and differential Galois theory, where the absolute differential Galois group of a differential field F is the fundamental group of the Tannakian category of all differential modules over F .

In the arithmetic context, the study of embedding problems has led to realizing all solvable groups as Galois groups over \mathbb{Q} ([Sha54]), and determining the structure of the maximal prosolvable extension of \mathbb{Q}^{ab} ([Iwa53]). Solving embedding problems led to the proof of freeness of the absolute Galois group of a function field over an algebraically closed base field ([Hrb95], [Pop95]), and contributed to the proof of Abhyankar's conjecture on fundamental

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groups in characteristic p (see [Ser90], [Ray94], [Hrb94]). See [NSW08, Chapter IX] and [Hrb03, Section 5] for more about the arithmetic and geometric cases, respectively.

Differential Galois theory is an analog of Galois theory for (linear homogeneous) differential equations, over fields of characteristic zero. The symmetry groups that occur are no longer finite (or profinite), but rather are linear algebraic groups over the field of constants of the differential field. The corresponding inverse problem has been studied by a number of researchers. It was first solved for function fields of complex curves by Tretkoff and Tretkoff ([TT79]), as a relatively straightforward consequence of Plemelj’s solution of the Riemann-Hilbert problem ([Ple08]). More generally, it has been solved when the base field is the field of functions of a curve over an algebraically closed field. See [Hrt05] for the case of rational function fields; the general case, which appeared in [Obe03], is based on the rational case and on Kovacic’s trick (see also Proposition 3.1). This solution built on prior work by Kovacic, Mitschi, and Singer ([Kov69], [Kov71], [Sin93], [MS96], [MS02]). The differential inverse problem has also been solved for function fields over certain non-algebraically closed fields, including over the real numbers ([Dyc08]) and over fields of Laurent series ([BHH16]).

Embedding problems in differential Galois theory have been considered by several researchers ([MvdP03], [Hrt05], [Obe03], [Ern14]). In fact, they were already used by Kovacic in his seminal work ([Kov69], [Kov71]) on the inverse problem and played a crucial role in the solution of the inverse problem over algebraically closed constant fields. In his thesis, Oberlies solved some types of differential embedding problems over function fields of curves over algebraically closed fields, including all differential embedding problems with connected cokernel ([Obe03]). However, the general solvability of embedding problems remained open, even in the classical case where the base field is $\mathbb{C}(x)$, although the inverse problem there had long been solved. In this paper, we close this gap and prove the following (see Theorem 3.7):

Theorem. *Every differential embedding problem over a one-variable complex function field (equipped with any nontrivial derivation), has a proper solution.*

To prove the theorem, we first introduce the notion of *differential torsors*. Differential torsors generalize Picard-Vessiot rings (the differential analogues of finite Galois extensions). We use a criterion given in [AMT09] to characterize those differential torsors that are Picard-Vessiot rings. This can be viewed as a converse to a well-known theorem of Kolchin. One advantage of working with differential torsors is that we can adapt constructions from algebraic geometry, such as passing to quotients, inducing differential torsors from subgroups, and transporting differential structures along morphisms.

The other main advantage of working with differential torsors is that this allows us to apply *patching*. We first deduce a patching result for differential torsors from the corresponding assertion for torsors without differential structure in [HHK15]. Building on this, we prove a patching result for Picard-Vessiot rings (Theorem 2.5) and another patching result designed for solving embedding problems (Theorem 2.14). These results are stated in a very general framework amenable to further applications.

We then show that our patching results apply over finite extensions of $\mathbb{C}(x)$ by proving a factorization statement for matrices whose entries are meromorphic functions on connected metric open subsets of a compact Riemann surface (Lemma 3.4). Similar factorization results were an important ingredient in the solution of the Riemann-Hilbert problem.

The strategy of using torsors for the purpose of Galois realizations has previously been employed by other researchers, e.g. [Jua07], [JL07], [JL08], usually to produce generic extensions for specific groups (see also the references there). We expect that the finer notion of differential torsors may be a new tool in finding further generic differential Galois extensions. It is already applied in an upcoming preprint on differential Galois theory over large fields of constants ([BHHP17]). Moreover, our patching results (Theorem 2.5, 2.14) are used in a second upcoming preprint on that topic ([BHH17]). The explicit framework for applying patching to differential Galois theory over $\mathbb{C}(x)$ that we develop in Section 3 is used in another project ([BW17]).

The organization of the paper is as follows. In Section 1, we define the notion of differential torsors over a differential field of characteristic zero. We show that differential structures on a torsor correspond to derivations that are invariant under translation. It is then shown that a differential structure can be detected locally, by relating invariant derivations to certain point derivations. Using this, differential structures can be transported along morphisms of torsors, under certain additional conditions. Finally, we show that simple differential torsors with no additional constants correspond to Picard-Vessiot rings. Section 2 extends the patching result for torsors from [HHK15] to differential torsors, and deduces a patching result for Picard-Vessiot rings. This is applied to give a result on split differential embedding problems, under the hypothesis that kernel and cokernel have suitable realizations as differential Galois groups (Theorem 2.14). In Section 3, we describe a patching setup where the base field F is the function field of a complex curve, and use Theorem 2.14 to show that every split differential embedding problem over F has a proper solution. Finally, using the results from Section 1 on transporting differential structures along morphisms, we show that the solution of arbitrary differential embedding problems can be reduced to the split case. The appendix collects definitions and results about group actions, torsors, quotients, and induction of torsors from a subgroup, for affine group schemes of finite type over an arbitrary field.

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1. DIFFERENTIAL TORSORS

This section introduces the notion of differential torsors. We show that their differential structure is determined locally, and that it is possible to transport a differential structure along a morphism of torsors, under certain conditions. This will be used in Section 3 to reduce the solution of all differential embedding problems (over complex function fields) to that of all split differential embedding problems. We also prove that simple differential G_F -torsors correspond to Picard-Vessiot rings over F with differential Galois group G . For notation and basic facts about torsors, see the the appendix to this manuscript.

Recall that a *differential ring* is a commutative ring R equipped with a derivation $\partial: R \rightarrow R$. A *differential homomorphism* is a ring homomorphism between differential rings that commutes with the derivations. The ring of *constants* of R is defined to be $C_R := \{f \in R \mid \partial(f) = 0\}$. If R is a field then so is C_R . We call R *simple* if it has no non-trivial differential ideals (i.e., ideals closed under ∂).

Throughout this section, (F, ∂) is a differential field of characteristic zero, and we let K denote its field of constants C_F , which is algebraically closed in F . The letter G denotes an affine group scheme of finite type over K . Since K has characteristic zero, G is smooth and we will refer to G as a *linear algebraic group*.

We will often consider the base change G_F from K to F and we will view $F[G_F] = F \otimes_K K[G]$ as a differential ring extension of F by considering all elements in $K[G]$ as constants. We write $F[G] := F[G_F]$.

1.1. Differential torsors and invariant derivations.

Definition 1.1. A *differential G_F -space* is an affine G_F -space X equipped with an extension of the given derivation ∂ from F to $F[X]$, such that the co-action $\rho: F[X] \rightarrow F[X] \otimes_F F[G]$ corresponding to the action α of G_F on X is a differential homomorphism (a definition of the terms *G_F -space* and *co-action* is given in A.1). We also call such an extension a *differential structure* on X . A *morphism of differential G_F -spaces* $\phi: X \rightarrow Y$ is a morphism of affine varieties that is G_F -equivariant and such that the corresponding homomorphism $F[Y] \rightarrow F[X]$ is a differential homomorphism. A differential G_F -space X is *simple* if $F[X]$ is a simple differential ring. Note that a G_F -space X is a *differential G_F -torsor* (i.e., a differential G_F -space that is a G_F -torsor) if and only if the left $F[X]$ -linear extension $F[X] \otimes_F F[X] \rightarrow F[X] \otimes_F F[G]$ of ρ is a differential isomorphism, or equivalently $(\alpha, \text{pr}_1): X \times G \rightarrow X \times X$ is a differential isomorphism.

The next lemma gives a criterion for when a derivation on a G_F -torsor defines a differential torsor.

Let \bar{F} denote an algebraic closure of F . Note that $\partial: F \rightarrow F$ uniquely extends to a derivation $\partial: \bar{F} \rightarrow \bar{F}$. Let \bar{K} denote the algebraic closure of $K = C_F$ in \bar{F} and note that $C_{\bar{F}} = \bar{K}$. In particular, if K is algebraically closed, $C_{\bar{F}} = K$. Let Γ denote the Galois group of \bar{F} over F .

Lemma 1.2. *Let X be a G_F -torsor, and let $\partial: F[X] \rightarrow F[X]$ be a derivation extending $\partial: F \rightarrow F$. Then X is a differential G_F -torsor if and only if $g \circ \partial = \partial \circ g$ on $F[X] \otimes_K \bar{K}$ for all $g \in G(\bar{K})$. Here $g \in G(\bar{K})$ is viewed as the automorphism $g: F[X] \otimes_K \bar{K} \rightarrow F[X] \otimes_K \bar{K}$ corresponding to $X \rightarrow X$, $x \mapsto x.g$, and $\partial: F[X] \otimes_K \bar{K} \rightarrow F[X] \otimes_K \bar{K}$ is the unique extension of ∂ .*

Proof. Assume that X is a differential G_F -torsor. Let g be an arbitrary element of $G(\bar{K})$. Then the left square in

$$\begin{array}{ccccc} F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & \xrightarrow{\text{id}_{F[X]} \otimes g} & F[X] \otimes_K \bar{K} \\ \partial \downarrow & & \downarrow \partial \otimes \text{id}_{K[G]} & & \downarrow \partial \otimes \text{id}_{\bar{K}} \\ F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & \xrightarrow{\text{id}_{F[X]} \otimes g} & F[X] \otimes_K \bar{K} \end{array}$$

commutes. Since the right square commutes, the outer rectangle also commutes; i.e., $g \circ \partial = \partial \circ g$ for all $g \in G(\bar{K})$.

Conversely, if $g \circ \partial = \partial \circ g$ for all $g \in G(\bar{K})$, then the outer rectangle commutes. So $\rho(\partial(f))$ and $\partial(\rho(f)) \in F[X] \otimes_K K[G]$ have the same image in $F[X] \otimes_K \bar{K}$ for all $g \in G(\bar{K})$ and $f \in F[X]$. Since G is reduced this implies that $\rho(\partial(f)) = \partial(\rho(f))$; i.e., X is a differential torsor. \square

If K is algebraically closed, $G(\bar{K}) = G(K) \subset G(F)$, and Lemma 1.2 simplifies to

Corollary 1.3. *Assume $K = \bar{K}$, let X be a G_F -torsor and let $\partial: F[X] \rightarrow F[X]$ be a derivation extending $\partial: F \rightarrow F$. Then X is a differential G_F -torsor if and only if $g \circ \partial = \partial \circ g$ on $F[X]$ for all $g \in G(K)$.*

Thus a derivation $\partial: F[X] \rightarrow F[X]$ that extends $\partial: F \rightarrow F$ is G -invariant (i.e., $g \circ \partial = \partial \circ g$ for all $g \in G(\bar{K})$) if and only if it turns the G_F -torsor X into a differential G_F -torsor.

1.2. Point derivations. Our next goal is to show that a differential structure on a torsor can be detected locally, i.e., at a point. For a linear algebraic group G , the space of G -invariant derivations on the coordinate ring of G is isomorphic to the tangent space at the identity ([Hum75, Theorem 9.1]). A similar construction applies in our setting, as we now discuss.

Let R be a ring and let M be an R -module. A *derivation* $\partial: R \rightarrow M$ is an additive map that satisfies the Leibniz rule $\partial(r_1 r_2) = r_1 \partial(r_2) + r_2 \partial(r_1)$. Notice that it is necessary to specify an R -module structure on M for the Leibniz rule to make sense.

Let L be any differential field (e.g., F or \bar{F}). Given an affine variety X over L , together with an L -algebra A and an A -point $x \in X(A)$, the map $L[X] \rightarrow A$ given by $f \mapsto f(x)$ defines an $L[X]$ -module structure on A . A *point derivation* $\tilde{\partial}: L[X] \rightarrow A$ at x is a derivation (with respect to this $L[X]$ -module structure) that extends $\partial: L \rightarrow L$.

Let G be a linear algebraic group over K and let X be a G_F -torsor. Consider an element $x \in X(\bar{F})$ and a point derivation $\tilde{\partial}: F[X] \rightarrow \bar{F}$ at x . If R is an \bar{F} -algebra and $y \in X(R)$, we may define a point derivation $\tilde{\partial}^y: F[X] \rightarrow R$ at y as follows. Let $g \in G(R) = G_{\bar{F}}(R) = \text{Hom}_{\bar{F}}(\bar{F} \otimes_K K[G], R)$ be the unique element such that $y = x.g$. Now define $\tilde{\partial}^y$ as the composition

$$\tilde{\partial}^y: F[X] \xrightarrow{\rho} F[X] \otimes_F F[G] = F[X] \otimes_K K[G] \xrightarrow{\tilde{\partial} \otimes \text{id}_{K[G]}} \bar{F} \otimes_K K[G] \xrightarrow{g} R.$$

Note that $\tilde{\partial} \otimes \text{id}_{K[G]}$ is a derivation, where $\bar{F} \otimes_K K[G]$ is considered as a $F[X] \otimes_K K[G]$ -module via $F[X] \otimes_K K[G] \xrightarrow{x \otimes \text{id}_{K[G]}} \bar{F} \otimes_K K[G]$. Therefore $\tilde{\partial}^y$ is a derivation with respect to the $F[X]$ -module structure on R given by $g \circ (x \otimes \text{id}_{K[G]}) \circ \rho = x.g = y \in G(R) = \text{Hom}_F(F[X], R)$.

For $\tau \in \Gamma := \text{Gal}(\bar{F}/F)$, the map $\tau(\tilde{\partial}) := \tau \circ \tilde{\partial}: F[X] \rightarrow \bar{F}$ is a point derivation at $\tau(x)$. We call $\tilde{\partial}$ *Galois-equivariant* if $(\tau(\tilde{\partial}))^x = \tilde{\partial}$ for all $\tau \in \Gamma$.

The next lemma lists some basic properties of point derivations.

Lemma 1.4. *With notation as above, suppose $\tilde{\partial}: F[X] \rightarrow \bar{F}$ is a point derivation at $x \in X(\bar{F})$. Then the following hold:*

- (a) $\tilde{\partial}^x = \tilde{\partial}$.
- (b) If $y, z \in X(\bar{F})$ then $(\tilde{\partial}^y)^z = \tilde{\partial}^z$.
- (c) If $\tilde{\partial}$ is Galois-equivariant and $y \in X(\bar{F})$, then $\tau(\tilde{\partial})^y = \tilde{\partial}^y$ for all $\tau \in \Gamma$.
- (d) For $y \in X(\bar{F})$ and $\tau \in \Gamma$ we have $\tau(\tilde{\partial})^{\tau(y)} = \tau(\tilde{\partial}^y)$. In particular, $\tau(\tilde{\partial}^{\tau^{-1}(y)}) = \tau(\tilde{\partial})^{\tau(\tau^{-1}(y))} = \tau(\tilde{\partial})^y$.

Proof. To prove (a) note that $x = x.1$. The assertion now follows from the fact that the two inner diagrams in

$$\begin{array}{ccccc}
F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & \xrightarrow{\tilde{\partial} \otimes \text{id}_{K[G]}} & \bar{F} \otimes_K K[G] \\
& \searrow \text{id}_{F[X]} & \downarrow \text{id}_{F[X]} \cdot 1 & & \downarrow \text{id}_{\bar{F}} \cdot 1 \\
& & F[X] & \xrightarrow{\tilde{\partial}} & \bar{F}
\end{array}$$

commute, where 1 denotes evaluation at $1 \in G(K)$.

To prove (b), first note that the two inner diagrams in

$$\begin{array}{ccccc}
F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & \xrightarrow{\tilde{\partial} \otimes \text{id}_{K[G]}} & \bar{F} \otimes_K K[G] \\
\rho \downarrow & & \downarrow \text{id}_{F[X]} \otimes \Delta & & \downarrow \text{id}_{\bar{F}} \otimes \Delta \\
F[X] \otimes_K K[G] & \xrightarrow{\rho \otimes \text{id}_{K[G]}} & F[X] \otimes_K K[G] \otimes_K K[G] & \xrightarrow{\tilde{\partial} \otimes \text{id}_{K[G] \otimes K[G]}} & \bar{F} \otimes_K K[G] \otimes_K K[G]
\end{array}$$

commute. Therefore the outer rectangle also commutes. Let $g \in G(\bar{F})$ such that $y = x.g$ and let $h \in G(\bar{F})$ such that $z = y.h$. Then $z = x.gh$; and we see that $\tilde{\partial}^z$ is the upper right path from $F[X]$ to $\bar{F} \otimes_K K[G] \otimes_K K[G]$ composed with $g \circ (h \otimes \text{id}): \bar{F} \otimes_K K[G] \otimes_K K[G] \rightarrow \bar{F}$, whereas $(\tilde{\partial}_y)^z$ is the lower left path composed with $g \circ (h \otimes \text{id})$. This proves (b).

To prove (c), note that by Galois-equivariance and (a) we have $\tau(\tilde{\partial})^x = \tilde{\partial}^x$. Therefore $\tau(\tilde{\partial})^y = (\tau(\tilde{\partial})^x)^y = (\tilde{\partial}^x)^y = \tilde{\partial}^y$ by (b).

Finally for (d), note that if $y = x.g$, then $\tau(y) = \tau(x).\tau(g)$. Since the diagram

$$\begin{array}{ccc}
F[X] \otimes_K K[G] & \xrightarrow{\tilde{\partial} \otimes \text{id}_{K[G]}} & \bar{F} \otimes_K K[G] \\
\tau(\tilde{\partial}) \otimes \text{id}_{K[G]} \downarrow & & \downarrow g \\
\bar{F} \otimes_K K[G] & & \bar{F} \\
& \searrow \tau(g) & \downarrow \tau \\
& & \bar{F}
\end{array}$$

commutes, the claim follows. \square

If X is a G_F -torsor as above and $\partial : F[X] \rightarrow F[X]$ is a derivation that extends the given derivation $\partial : F \rightarrow F$, then there is an *induced* point derivation ∂_x at $x \in X(\bar{F})$ obtained by evaluation at x ; i.e., $\partial_x : F[X] \xrightarrow{\partial} F[X] \xrightarrow{x} \bar{F}$.

The following explains the connection between differential structures on a torsor and point derivations.

Proposition 1.5. *Let $x \in X(\bar{F})$ be a fixed element. The assignment $\partial \mapsto \partial_x$ induces a bijection between the derivations $\partial : F[X] \rightarrow F[X]$ that endow X with the structure of a differential G_F -torsor and the point derivations $\tilde{\partial} : F[X] \rightarrow \bar{F}$ at x that are Galois-equivariant. In particular, $(\partial_x)^y = \partial_y$ for all $y \in Y(\bar{F})$.*

Proof. Given a derivation ∂ on $F[X]$ which endows X with the structure of a differential G_F -torsor, let $\tilde{\partial} = \partial_x$. This is a point derivation at x by definition. To see that $\tilde{\partial}$ is

Galois-equivariant, let $\tau \in \Gamma$ and let $g \in G(\bar{F})$ be such that $x = \tau(x).g$. The square in

$$\begin{array}{ccccc} F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & & \\ \partial \downarrow & & \downarrow \partial \otimes \text{id}_{K[G]} & & \\ F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & \xrightarrow{\tau(x)} & \bar{F} \otimes_K K[G] \xrightarrow{g} \bar{F} \end{array}$$

commutes. The upper path all the way to \bar{F} is $\tau(\tilde{\partial})^x$ and the lower path is $\tilde{\partial}$ because $x = \tau(x).g$. Thus $\tilde{\partial}$ is Galois-equivariant.

Conversely, given a point derivation $\tilde{\partial}$ that is Galois-equivariant, we want to associate to $\tilde{\partial}$ a derivation on $F[X]$ that extends ∂ on F . Let $j : \bar{F}[X] \rightarrow \bar{F}[G]$ correspond to the scheme isomorphism $G_{\bar{F}} \rightarrow X_{\bar{F}}$ defined by $g \mapsto x.g$; and let $\text{id} \in \text{Hom}_{\bar{F}}(\bar{F}[X], \bar{F}[X]) = X_{\bar{F}}(\bar{F}[X])$ be the identity map on $\bar{F}[X]$, corresponding to an $\bar{F}[X]$ -point y_0 on X . Consider the point derivation $\tilde{\partial}^{y_0} : F[X] \rightarrow \bar{F}[X]$. If $f \in F[X]$, then $\tilde{\partial}^{y_0}(f) \in \bar{F}[X]$ defines a morphism $\partial(f) : X_{\bar{F}} \rightarrow \mathbb{A}_{\bar{F}}^1$. There is a unique $g_0 \in G(\bar{F}[X])$ satisfying $x.g_0 = y_0$; i.e., such that the composition $\bar{F}[X] \xrightarrow{j} \bar{F}[G] \xrightarrow{g_0} \bar{F}[X]$ is equal to $\text{id} : \bar{F}[X] \rightarrow \bar{F}[X]$. Thus $j = g_0^{-1}$.

We claim that if R is any \bar{F} -algebra and if $y \in X_{\bar{F}}(R)$, then $\partial(f)(y) = \tilde{\partial}^y(f)$. By the definition of $\tilde{\partial}^y$, it suffices to show that the composition $\bar{F}[G] \xrightarrow{g_0} \bar{F}[X] \xrightarrow{y} R$ is equal to $g : \bar{F}[G] \rightarrow R$, where $g \in G(R)$ is the element satisfying $x.g = y$. This last equality says that the composition $\bar{F}[X] \xrightarrow{j} \bar{F}[G] \xrightarrow{g} R$ is equal to $y : \bar{F}[X] \rightarrow R$. Hence $g = y \circ g_0 : \bar{F}[G] \rightarrow R$, proving the claim.

Using the Galois-equivariance of $\tilde{\partial}$, we will check that $\partial(f)$ is in fact in $F[X]$. For this, it suffices to show that $\partial(f) \in F[X] \otimes_F \bar{F}$ is fixed by the Γ -action. For $y \in X(\bar{F})$ and $\tau \in \Gamma$ we find

$$\tau(\partial(f))(y) = \tau(\partial(f)(\tau^{-1}(y))) = \tau(\tilde{\partial}^{\tau^{-1}(y)}(f)) = \tau(\tilde{\partial}^{\tau^{-1}(y)})(f) = \tilde{\partial}^y(f) = \partial(f)(y)$$

using Lemma 1.4(c) and (d). Thus $\partial(f) \in F[X]$ and we have constructed a well-defined map

$$\partial : F[X] \rightarrow F[X],$$

as desired. It is now straightforward to check that ∂ is a derivation and extends $\partial : F \rightarrow F$.

Our next goal is to show that $\partial : F[X] \rightarrow F[X]$ endows X with the structure of a differential G -torsor using Lemma 1.2. For $g \in G(\bar{K})$ and $y \in X(\bar{F})$ the square in

$$\begin{array}{ccccc} F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] & \xrightarrow{\text{id}_{F[X]} \otimes g} & F[X] \otimes_K \bar{K} \\ & & \downarrow \tilde{\partial}_y \otimes \text{id}_{K[G]} & & \downarrow \tilde{\partial}_y \otimes \text{id}_{\bar{K}} \\ & & \bar{F} \otimes_K K[G] & \xrightarrow{\text{id}_{\bar{F}} \otimes g} & \bar{F} \otimes_K \bar{K} \longrightarrow \bar{F} \end{array}$$

commutes. Tracking an element $f \in F[X]$ from the upper left to the lower right along both paths, we find that

$$\tilde{\partial}^{y.g}(f) = \tilde{\partial}^y(g(f)),$$

where we extend $\tilde{\partial}^y : F[X] \rightarrow \bar{F}$ to $\tilde{\partial}^y : F[X] \otimes_K \bar{K} \rightarrow \bar{F}$ by \bar{K} -linearity. Now

$$g(\partial(f))(y) = \partial(f)(y.g) = \tilde{\partial}^{y.g}(f) = \tilde{\partial}^y(g(f)) = \partial(g(f))(y).$$

Hence by Lemma 1.2, X is a differential G_F -torsor with respect to ∂ .

It remains to see that the constructed maps are inverse to each other. If we start with $\tilde{\partial}: F[X] \rightarrow \bar{F}$ and construct $\partial: F[X] \rightarrow F[X]$ as above, then $\partial(f)(x) = \tilde{\partial}^x(f) = \tilde{\partial}(f)$ for $f \in F[X]$ by Lemma 1.4(a). So ∂_x is the given $\tilde{\partial}$.

Conversely, if we start with $\partial: F[X] \rightarrow F[X]$ and define $\tilde{\partial} := \partial_x$, it remains to be seen that $\tilde{\partial}^y(f) = \partial(f)(y)$ for all $f \in F[X]$ and $y \in X(\bar{F})$. That is, we want to check that $(\partial_x)^y = \partial_y$. Let $g \in G(\bar{F})$ be such that $y = x.g$. Since the two inner diagrams in

$$\begin{array}{ccc}
F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] \\
\partial \downarrow & & \downarrow \partial \otimes \text{id}_{K[G]} \\
F[X] & \xrightarrow{\rho} & F[X] \otimes_K K[G] \\
& \searrow y & \downarrow x \otimes \text{id}_{K[G]} \\
& & \bar{F} \otimes_K K[G] \\
& & \downarrow g \\
& & \bar{F}
\end{array}$$

commute, the outer diagram also commutes. This shows that $(\partial_x)^y = \partial_y$. \square

1.3. Transport of differential structures. In this subsection, we study how differential structures behave under morphisms of torsors.

Definition 1.6. Let $\phi: G \rightarrow G'$ be a morphism of linear algebraic groups over K , let X be a differential G_F -torsor, and let X' be a differential G'_F -torsor. A G_F -equivariant morphism $\psi: X \rightarrow X'$ is called *differential* if the corresponding dual morphism $\psi^*: F[X'] \rightarrow F[X]$ is differential (i.e., commutes with the derivation).

The following lemma gives a local criterion for a morphism to be differential.

Lemma 1.7. *Consider the situation of Definition 1.6, and fix $x \in X(\bar{F})$. The morphism ψ is differential if and only if*

$$(1) \quad \begin{array}{ccc}
F[X'] & \xrightarrow{\psi^*} & F[X] \\
& \searrow \partial_{\psi(x)} & \swarrow \partial_x \\
& & \bar{F}
\end{array}$$

commutes (for this fixed x).

Proof. Unraveling the definitions shows that ψ is differential if and only if the diagram (1) commutes for all $x \in X(\bar{F})$. It remains to show that the commutativity for one fixed x is sufficient. For $\tilde{x} \in X(\bar{F})$, there exists $g \in G(\bar{F})$ such that $\tilde{x} = x.g$; and then $\psi(\tilde{x}) = x'.\phi(g)$,

where $x' = \psi(x)$. Since the three inner diagrams in

$$\begin{array}{ccc}
F[X'] & \xrightarrow{\psi^*} & F[X] \\
\downarrow & & \downarrow \\
F[X'] \otimes_K K[G'] & \xrightarrow{\psi^* \otimes \phi^*} & F[X] \otimes_K K[G] \\
\downarrow \partial_{x'} \otimes \text{id}_{K[G']} & & \downarrow \partial_x \otimes \text{id}_{K[G]} \\
\bar{F} \otimes_K K[G'] & \xrightarrow{\text{id}_{\bar{F}} \otimes \phi^*} & \bar{F} \otimes_K K[G] \\
\searrow \phi(g) & & \swarrow g \\
& \bar{F} &
\end{array}$$

commute, the outer diagram also commutes. That is,

$$\begin{array}{ccc}
F[X'] & \xrightarrow{\psi^*} & F[X] \\
\searrow (\partial_{x'})^{\psi(\tilde{x})} & & \swarrow (\partial_x)^{\tilde{x}} \\
& \bar{F} &
\end{array}$$

commutes. By Proposition 1.5, $(\partial_{x'})^{\psi(\tilde{x})} = \partial_{\psi(\tilde{x})}$ and $(\partial_x)^{\tilde{x}} = \partial_{\tilde{x}}$, hence both paths from the upper left to the lower right in

$$\begin{array}{ccc}
F[X'] & \xrightarrow{\psi^*} & F[X] \\
\partial \downarrow & & \downarrow \partial \\
F[X'] & \xrightarrow{\psi^*} & F[X] \\
& & \downarrow \tilde{x} \\
& & \bar{F}
\end{array}$$

yield the same result; i.e., $\psi^*(\partial(f))(\tilde{x}) = \partial(\psi^*(f))(\tilde{x})$ for all $f \in F[X']$ and $\tilde{x} \in X(\bar{F})$. Thus ψ is differential. \square

Proposition 1.8. *Let $\phi: G \rightarrow G'$ be a morphism of linear algebraic groups over K and let $\psi: X \rightarrow X'$ be a G_F -equivariant morphism from a G_F -torsor X to a G'_F -torsor X' (where G acts on X' via ϕ). Then:*

- (a) *If X is differential, then there exists a unique differential structure on X' such that ψ is differential.*
- (b) *If X' is differential, K is algebraically closed, and ϕ is surjective, then there exists a differential structure on X such that ψ is differential.*

Proof. For part (a), let $x \in X(\bar{F})$ and set $x' = \psi(x) \in X'(\bar{F})$. Define $\partial_{x'} := \partial_x \circ \psi^* : F[X'] \rightarrow \bar{F}$. We next show that $\partial_{x'}$ is Galois-equivariant. If $\tau \in \Gamma$ and $g \in G(\bar{F})$ such that

$x = \tau(x).g$, then $x' = \tau(x').\phi(g)$. Since the three inner diagrams in

$$\begin{array}{ccc}
F[X'] & \xrightarrow{\psi^*} & F[X] \\
\downarrow & & \downarrow \\
F[X'] \otimes_K K[G'] & \xrightarrow{\psi^* \otimes \phi^*} & F[X] \otimes_K K[G] \\
\downarrow \tau(\partial_{x'}) \otimes \text{id}_{K[G']} & & \downarrow \tau(\partial_x) \otimes \text{id}_{K[G]} \\
\bar{F} \otimes_K K[G'] & \xrightarrow{\text{id}_{\bar{F}} \otimes \phi^*} & \bar{F} \otimes_K K[G] \\
\searrow \phi(g) & & \swarrow g \\
& \bar{F} &
\end{array}$$

commute, the outer diagram also commutes. So $\tau(\partial_{x'})^{x'} = \tau(\partial_x)^x \circ \psi^* = \partial_x \circ \psi^* = \partial_{x'}$, using that ∂_x is Galois-equivariant by Proposition 1.5. Thus $\partial_{x'}$ is Galois-equivariant, as asserted. Hence by Proposition 1.5, $\partial_{x'}$ defines a differential structure on X' . By Lemma 1.7 the morphism ψ is differential. The uniqueness is clear from Proposition 1.5 and Lemma 1.7.

To prove (b), we first assume that $F = \bar{F}$ is algebraically closed. Let $x \in X(F)$. Since F is algebraically closed, every point derivation at x is Galois-equivariant. Hence Proposition 1.5 implies that the differential structures on X are in bijection with the set $\text{Der}_{\partial,x}(F[X], F)$ of point derivations at x . Let $M_x \subseteq F[X]$ be the maximal ideal corresponding to x and let $\mathcal{O}_{X,x} = F[X]_{M_x}$, the local ring at x . Recall that the (Zariski) tangent space $T_x X$ at x is the dual F -vector space of $\mathfrak{m}_x/\mathfrak{m}_x^2$ where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. We then have a chain of bijections:

$$(2) \quad \text{Der}_{\partial,x}(F[X], F) \cong \text{Der}_{\partial,x}(\mathcal{O}_{X,x}, F) \cong T_x X.$$

(Note that while $T_x X$ is an F -vector space, $\text{Der}_{\partial,x}(F[X], F)$ is not closed under addition of derivations, since the sum of two derivations extending $\partial: F \rightarrow F$ extends 2∂ rather than ∂ .) Although $\tilde{\partial} \in \text{Der}_{\partial,x}(\mathcal{O}_{X,x}, F)$ need not be F -linear, the induced map $\mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow F$ given by $f \bmod \mathfrak{m}_x^2 \mapsto \tilde{\partial}(f)$ is well defined and F -linear by the Leibniz rule; and this gives the forward direction of the second bijection in (2). For the reverse direction, a tangent vector $v: \mathfrak{m}_x/\mathfrak{m}_x^2 \rightarrow F$ defines a derivation $\tilde{\partial}: \mathcal{O}_{X,x} = \mathfrak{m}_x \oplus F \rightarrow F$ in $\text{Der}_{\partial,x}(\mathcal{O}_{X,x}, F)$ by $f \mapsto v((f - f(x)) \bmod \mathfrak{m}_x^2) + \partial(f(x))$.

Let $x' = \psi(x) \in X'(F)$ and define $\partial_{x'}: F[X'] \xrightarrow{\partial} F[X'] \xrightarrow{x'} F$. Since $\phi: G \rightarrow G'$ is surjective and the groups G, G' are smooth, it follows from [DG70, Chapter II, §5, Prop. 5.3] that the induced map on the Lie algebras is surjective. Since F is assumed to be algebraically closed for now, X and X' are trivial torsors and it follows that the tangent map $T_x X \rightarrow T_{x'} X'$ is surjective. Therefore the image of $\partial_{x'}$ in $T_{x'} X'$ lifts to a tangent vector in $T_x X$ which corresponds to a derivation $\partial_x: F[X] \rightarrow F$. Chasing through the bijections in (2) we see that

$$\begin{array}{ccc}
F[X'] & \xrightarrow{\psi^*} & F[X] \\
\searrow \partial_{x'} & & \swarrow \partial_x \\
& F &
\end{array}$$

10

commutes. Thus, if we define a differential structure on X via ∂_x by virtue of Proposition 1.5, it follows from Lemma 1.7 that ψ is differential. This proves (b) in case that F is algebraically closed.

Now let F be arbitrary again. Since $\phi: G \rightarrow G'$ is surjective we can identify $F[X']$ with a subring of $F[X]$. As $X_{\bar{F}}$ is a differential $G_{\bar{F}}$ -torsor, it follows from what we proved previously that there exists a G -invariant derivation $\partial_{\bar{F}[X]}: \bar{F}[X] \rightarrow \bar{F}[X]$ that extends the given derivation $\partial_{F[X']}: F[X'] \rightarrow F[X']$.

We next show that $\partial_{\bar{F}[X]}: \bar{F}[X] \rightarrow \bar{F}[X]$ restricts to a G -invariant derivation $\partial_{E[X]}: E[X] \rightarrow E[X]$, where E is a suitable finite Galois extension of F . Namely, suppose that $f_1, \dots, f_n \in F[X]$ generate $F[X]$ as an F -algebra. Then there exists a finite Galois extension E of F such that $\partial_{\bar{F}[X]}(f_1), \dots, \partial_{\bar{F}[X]}(f_n) \in E[X]$ and thus $\partial_{\bar{F}[X]}$ restricts to a derivation $\partial_{E[X]}$ on $E[X]$.

For $\tau \in \text{Gal}(E/F)$, the restriction of $\tau(\partial_{E[X]}) = \tau \circ \partial_{E[X]}$ to $F[X]$ is a derivation from $F[X]$ into $E[X]$, where $E[X]$ is considered as an $F[X]$ -module via the inclusion $F[X] \subseteq E[X]$. Moreover, $\tau(\partial_{E[X]})$ agrees with $\partial_{F[X']}$ on $F[X']$, and $\tau(\partial_{E[X]})$ is G -invariant, since the $G(K)$ -action commutes with the $\text{Gal}(E/F)$ -action (using here that K is algebraically closed).

This implies that

$$\mathcal{R}(\partial_{E[X]}) := \frac{1}{|\text{Gal}(E/F)|} \sum_{\tau \in \text{Gal}(E/F)} \tau(\partial_{E[X]})$$

defines a derivation from $F[X]$ into $E[X]$ that agrees with $\partial_{F[X']}$ on $F[X']$. By definition, an element in the image of $\mathcal{R}(\partial_{E[X]})$ is fixed by the $\text{Gal}(E/F)$ -action and thus lies in $F[X]$. Therefore $\mathcal{R}(\partial_{E[X]})$ actually defines a derivation $\mathcal{R}(\partial_{E[X]}): F[X] \rightarrow F[X]$. As $\tau(\partial_{E[X]})$ is G -invariant for every $\tau \in \text{Gal}(E/F)$, the derivation $\mathcal{R}(\partial_{E[X]})$ is also G -invariant and thus defines the desired differential structure on X by Corollary 1.3. Here ψ is a differential morphism over F because it is the restriction of a differential morphism over \bar{F} (cf. the algebraically closed case). \square

Example 1.9. Let $X = G_F$ be a trivial G_F -torsor. Then the derivations $\partial: F[X] \rightarrow F[X]$ that turn X into a differential G_F -torsor are in bijection with the Lie algebra of G_F .

To see this, let $x = 1 \in X(F) \subseteq X(\bar{F})$ be the identity element and let $\tilde{\partial}: F[X] \rightarrow \bar{F}$ be a derivation extending $\partial: F \rightarrow F$. Since $\tau(x) = x$ for $\tau \in \Gamma$ we see that $\tau(\tilde{\partial}): F[X] \rightarrow \bar{F}$ is a derivation with respect to the $F[X]$ -module structure on \bar{F} given by $f \mapsto f(x)$. From Lemma 1.4(a) it follows that $\tau(\tilde{\partial})^x = \tau(\tilde{\partial})$. Thus $\tilde{\partial}$ is Galois-equivariant if and only if $\tau(\partial) = \partial$ for all $\tau \in \Gamma$; i.e., $\partial(F[X]) \subseteq F$. So by Proposition 1.5 the derivations $\partial: F[X] \rightarrow F[X]$ that turn X into a differential G_F -torsor are in bijection with the point derivations $\tilde{\partial}: F[X] \rightarrow F$ at x . As in (2) above, the latter set is in bijection with $T_x X$.

Example 1.10. In particular, consider the group GL_n and let $X = \text{GL}_{n,F}$ be the trivial $\text{GL}_{n,F}$ -torsor, with coordinate ring $F[X] = F[T, \det(T)^{-1}] := F[t_{ij}, \det(T)^{-1} \mid 1 \leq i, j \leq n]$; here $T = (t_{ij})$ is an $n \times n$ -matrix of indeterminates t_{ij} over F . We can turn X into a differential $\text{GL}_{n,F}$ -torsor by defining a derivation $\partial_A: F[X] \rightarrow F[X]$ extending $\partial: F \rightarrow F$ by $\partial(T) = AT$ for a fixed $A \in F^{n \times n}$, the Lie algebra of $\text{GL}_{n,F}$. Conversely, if X is a differential $\text{GL}_{n,F}$ -torsor, then X is a trivial torsor since $H^1(F, \text{GL}_{n,F})$, which classifies $\text{GL}_{n,F}$ -torsors, is trivial by Hilbert's Theorem 90 (e.g., see [Ser97, Chapter III, Sect. 1.1, Lemma 1]). It follows from Example 1.9 that the derivation on $F[X]$ is of the above form ∂_A . Thus differential

$\mathrm{GL}_{n,F}$ -torsors correspond to universal solution rings for linear differential equations $\partial(z) = Az$ with $A \in F^{n \times n}$ (and where z is an n -tuple of indeterminates).

Distinct choices of A can lead to non-isomorphic differential torsors. For example, if $n = 1$, $F = \mathbb{C}(x)$ with derivation d/dx , then $A = (0)$ defines the trivial differential \mathbb{G}_m -torsor (i.e., its constants are $\mathbb{C}[\mathbb{G}_m]$), whereas $A = (1)$ defines a simple differential torsor, with constants \mathbb{C} . In particular, a differential torsor can be trivial as a torsor but non-trivial as a differential torsor.

Remark 1.11. Let H be a closed subgroup of G and let Y be a differential H_F -torsor. Recall that there is an induced G_F -torsor $\mathrm{Ind}_{H_F}^{G_F}(Y)$ (see the appendix, Subsection A.3). Since $Y \rightarrow \mathrm{Ind}_{H_F}^{G_F}(Y)$ is H_F -equivariant, it follows from Proposition 1.8(a) that there exists a unique differential structure on $\mathrm{Ind}_{H_F}^{G_F}(Y)$ such that $Y \rightarrow \mathrm{Ind}_{H_F}^{G_F}(Y)$ is differential. In the sequel we will always consider $\mathrm{Ind}_{H_F}^{G_F}(Y)$ as a differential G_F -torsor by virtue of this differential structure. In view of Proposition A.8, this structure can be made explicit: $\mathrm{Ind}_{H_F}^{G_F}(Y) = (Y \times G)/H = \mathrm{Spec}((F[Y] \otimes_K K[G])^H)$, and the derivation on $(F[Y] \otimes_K K[G])^H$ is the restriction of the one on $F[Y] \otimes_K K[G]$ induced from $F[Y]$ by declaring the elements of $K[G]$ to be constant.

1.4. Simple differential torsors and Picard-Vessiot rings. As before, (F, ∂) is a differential field of characteristic zero with field of constants $K = C_F$, and G is a linear algebraic group over K .

Recall that a *Picard-Vessiot ring* over F is a differential ring extension R/F of the form $R = F[Z, \det(Z)^{-1}] := F[z_{ij}, \det(Z)^{-1} \mid 1 \leq i, j \leq n]$ for some matrix $Z = (z_{ij}) \in \mathrm{GL}_n(R)$ with $\partial(Z)Z^{-1} \in F^{n \times n}$, such that R is a simple differential ring and $C_R = K$. (The elements z_{ij} need not be algebraically independent over F .) Equivalently, a Picard-Vessiot ring over F is a differential ring without zero divisors of the form $R = F[Z, \det(Z)^{-1}]$ with $\partial(Z)Z^{-1} \in F^{n \times n}$ and such that $C_{\mathrm{Frac}(R)} = K$. In this situation, we also say that R is a *Picard-Vessiot ring for the linear differential equation* $\partial(z) = Az$, where $A = \partial(Z)Z^{-1} \in F^{n \times n}$ and z is an n -tuple of indeterminates. A *Picard-Vessiot extension* of F is the fraction field of a Picard-Vessiot ring over F . If E/F is an extension of differential fields that is finitely generated as a field extension, then E is a Picard-Vessiot extension of F if and only if $C_E = C_F$, $E = \mathrm{Frac}(R)$ for some differential ring extension R of F , and the left R -module $R \otimes_F R$ is generated by its constants (see Definition 1.8 and Theorem 3.11 of [AMT09]). In this situation, R is the associated Picard-Vessiot ring.

The *differential Galois group* of a Picard-Vessiot ring R is defined as the group functor $\mathrm{Aut}^\partial(R/F)$. It is an affine group scheme of finite type over K represented by the K -algebra $C_{R \otimes_F R} = K[Z^{-1} \otimes Z, \det(Z^{-1} \otimes Z)^{-1}]$, where $Z^{-1} \otimes Z$ is a short-hand notation for the matrix product $(Z^{-1} \otimes 1) \cdot (1 \otimes Z) \in \mathrm{GL}_n(R \otimes_F R)$. For more details about differential Galois theory, see [vdPS03] for the case that the constant field K is algebraically closed; and see [Dyc08] and [AMT09] for the general case. In particular, there is a differential analog of the usual Galois correspondence; see [Dyc08, Theorem 4.4] and [AMT09, Theorem 2.11]. (In the former reference, one must be more careful in defining the invariant subfield E^H of a Picard-Vessiot extension, because $E \otimes_K A$ is not necessarily H -stable in the total ring of fractions of $R \otimes_K A$, for A a K -algebra. Instead, one can use the definition given at the beginning of Section 3 of [BHH16].) See also the observation after the proof of Proposition 1.17 below.

In the above situation, there is a linearization $\underline{\text{Aut}}^\partial(R/F) \hookrightarrow \text{GL}_n$ that depends on the choice of a fundamental solution matrix $Z \in \text{GL}_n(R)$. Details can be found in [BHH16] and [Dyc08]. The following proposition explains the relation between Picard-Vessiot rings and differential torsors.

Proposition 1.12. *Let F, K, G be as above.*

- (a) *Let R/F be a Picard-Vessiot ring with differential Galois group G . Then $\text{Spec}(R)$ is a simple differential G_F -torsor.*
- (b) *Let $X = \text{Spec}(R)$ be a differential G_F -torsor such that R is an integral domain and assume that $C_{\text{Frac}(R)} = K$. Then R/F is a Picard-Vessiot ring with differential Galois group G .*

Proof. (a) By definition, R is a simple differential ring. By Kolchin’s Theorem, $X = \text{Spec}(R)$ is a G_F -torsor. (See [vdPS03, Theorem 1.30] in the case that K is algebraically closed, or [AMT09, Proposition 2.13] for the general case.) Therefore we have isomorphisms $R \otimes_F R \cong R \otimes_K K[G]$ and $C_{R \otimes_F R} \cong K[G]$ (using $C_R = K$). The homomorphism corresponding to the G_F -action $X \times G_F \rightarrow X$ is

$$\rho : R \rightarrow R \otimes_F R \xrightarrow{\cong} R \otimes_K K[G] \xrightarrow{\cong} R \otimes_F F[G]$$

where the first map is the inclusion into the second factor. (Cf. [AMT09, Lemma 1.9].) Clearly this is a differential morphism, and so X is a differential torsor.

(b) Let $\rho : R \rightarrow R \otimes_F F[G] \xrightarrow{\cong} R \otimes_K K[G]$ be the differential homomorphism corresponding to the G_F -action on X . Since X is a G_F -torsor, the left R -linear extension $R \otimes_F R \rightarrow R \otimes_K K[G]$ is a differential isomorphism; and as $K[G]$ is constant, we see that $R \otimes_F R$ is generated by constants as a left R -module. Since $C_{\text{Frac}(R)} = K$ and since $\text{Frac}(R)/F$ is a finitely generated field extension, it follows that $\text{Frac}(R)/F$ is a Picard-Vessiot extension with Picard-Vessiot ring R (using the equivalent criterion given in [AMT09, Definition 1.8]). Moreover, the differential Galois group of R/F is G . Indeed, the isomorphism $R \otimes_F R \rightarrow R \otimes_K K[G]$ identifies $C_{R \otimes_F R}$ with $K[G]$, and it is easy to check that this is an isomorphism of Hopf algebras. \square

Remark 1.13. The close relationship between Picard-Vessiot rings and differential torsors explained in Prop. 1.12 has a parallel in Kolchin’s approach to differential Galois theory in [Kol73]. There, the Galois groups are not necessarily linear, and the corresponding field extensions (which are Picard-Vessiot extensions in the case of linear groups) are called *strongly normal*. Roughly speaking, Theorem 9 of [Kol73, VI.10] says that there is a bijection between the strongly normal extensions E of F with differential Galois group G , and the principal homogeneous G -spaces X over F such that E is generated as a differential field over F by an “ X -primitive”. We note that Kolchin’s framework uses universal domains rather than scheme theory, and so his principal homogeneous spaces (see [Kol73, V.3]) are not the same as torsors in our sense.

Corollary 1.14. *Let F, K, G be as above, and let R/F be a differential ring extension.*

- (a) *Suppose that $C_R = K$. Then R/F is a Picard-Vessiot ring with differential Galois group G if and only if $\text{Spec}(R)$ is a simple differential G_F -torsor.*
- (b) *Suppose that R is an integral domain and that $C_{\text{Frac}(R)} = K$. Then R/F is a Picard-Vessiot ring with differential Galois group G if and only if $\text{Spec}(R)$ is a differential G_F -torsor.*

Proof. The forward direction of (a) was given in Proposition 1.12(a), while the reverse direction follows from (b) because a simple differential K -algebra R is an integral domain that satisfies $C_R = C_{\text{Frac}(R)}$. The forward direction of (b) follows from Proposition 1.12(a), while the reverse direction is immediate from Proposition 1.12(b). \square

One advantage of working with differential torsors as compared to Picard-Vessiot rings is that for subgroups H of G , we can induce differential H_F -torsors to differential G_F -torsors, which will allow us to apply patching techniques. See Remark 1.11.

Proposition 1.15. *Let X be a differential G_F -torsor. If K is algebraically closed, then there exists a closed subgroup H of G and a simple differential H_F -torsor Y with $C_{\text{Frac}(F[Y])} = K$ such that $X \cong \text{Ind}_{H_F}^{G_F}(Y)$ (as differential torsors).*

Proof. Let $I \triangleleft F[X]$ be maximal among the proper differential ideals in $F[X]$ and let $Y \subseteq X$ be defined by I . By [vdPS03, Lemma 1.17]), the ideal I is prime, and in particular radical. So $F[Y] \cong F[X]/I$, and this is an integral domain. Moreover, also by [vdPS03, Lemma 1.17], there are no new constants in the field of fractions E of $F[Y]$.

Let $B \subseteq F[Y] \otimes_F F[Y]$ denote the image of $K[G]$ in $F[Y] \otimes_F F[Y]$ under

$$(3) \quad F[X] \otimes_K K[G] \cong F[X] \otimes_F F[X] \rightarrow F[Y] \otimes_F F[Y].$$

Since $F[X] \rightarrow F[Y]$ is a surjective differential homomorphism so is the map in (3). Therefore the elements of B are constants, and $F[Y] \otimes_F F[Y]$ is generated by B as a left $F[Y]$ -module. Hence $E \otimes_F F[Y]$ is generated by B as an E -module. It is a general fact that the constants of a differential E -algebra are linearly disjoint from E over C_E ([Kol73, Chapter II, Section 1, Corollary 1, p. 87]). Thus $E \otimes_F F[Y] \cong E \otimes_K B$. It follows that $F[Y] \otimes_F F[Y] \cong F[Y] \otimes_K B$ and that the constants of $F[Y] \otimes_F F[Y]$ equal B .

We have seen that $E = \text{Frac}(F[Y])$ satisfies $C_E = K$ and that $F[Y] \otimes_F F[Y]$ is generated by its constants as a left $F[Y]$ -module. Thus E is a Picard-Vessiot extension of F by the criterion in [AMT09, Definition 1.8], with Picard-Vessiot ring $F[Y]$. By [AMT09, Lemma 1.9] and the discussion on pages 137-138 there, the associated differential Galois group is $H := \text{Spec}(B)$. Hence Y is an H_F -torsor, by Corollary 1.14(b). Since $K[G] \rightarrow B = K[H]$ is surjective, H is a closed subgroup of G .

We have a commutative diagram:

$$(4) \quad \begin{array}{ccc} F[X] \otimes_F F[X] & \xrightarrow{\cong} & F[X] \otimes_K K[G] \\ \downarrow & & \downarrow \\ F[Y] \otimes_F F[Y] & \xrightarrow{\cong} & F[Y] \otimes_K K[H] \end{array}$$

The morphisms dual to the action of G_F and H_F on X and Y , respectively, are obtained from the horizontal isomorphisms in (4) by precomposing with the inclusions into the second factor. Thus the commutativity of (4) shows that the inclusion morphism $Y \rightarrow X$ is H_F -equivariant. Thus X and $\text{Ind}_{H_F}^{G_F}(Y)$ are isomorphic as G_F -torsors by Remark A.9(a), and as differential torsors by Remark 1.11. \square

Lemma 1.16. *Let H be a closed subgroup of the linear algebraic group G , and let Y be a differential H_F -torsor. If $\text{Ind}_{H_F}^{G_F}(Y)$ is a simple differential G_F -torsor, then $H = G$.*

Proof. Since $Y \rightarrow X := \text{Ind}_{H_F}^{G_F}(Y)$ is differential, the morphism $F[X] \rightarrow F[Y]$ is differential. It is surjective by Lemma A.10, and injective because $F[X]$ is simple. Thus $F[X] \cong F[Y]$. This implies $F[X] \otimes_K K[G] \cong F[Y] \otimes_K K[H]$ and therefore $H = G$. \square

The following proposition concerns the passage from a linear algebraic group G to its quotient by a normal subgroup N . See Subsection A.2 for a discussion of quotients of torsors.

Proposition 1.17. *Let $N \trianglelefteq G$ be a normal closed subgroup. If X is a differential G_F -torsor, then $Y := X/N$ is a differential $(G/N)_F$ -torsor. Moreover, if X is a simple differential G_F -torsor, then Y is a simple differential $(G/N)_F$ -torsor.*

Proof. First recall that X/N exists as an affine variety and is a $(G/N)_F$ -torsor by Proposition A.6(b); viz. $X/N = \widetilde{X//N_F} = \text{Spec}(F[X]^{N_F})$. Applying Proposition 1.8(a) to the quotient morphism $\psi : X \rightarrow Y = X/N$ yields a unique differential structure on Y that is compatible with that of X .

Next, assume that X is a simple differential G_F -torsor; i.e., that $F[X]$ has no non-trivial differential ideals. Let $I \trianglelefteq F[X]^{N_F}$ be a non-zero differential ideal. Then $J = I \cdot F[X]$ is a non-zero differential ideal of $F[X]$, hence $J = F[X]$. By Proposition A.6(b), $F[X]$ is faithfully flat over $F[X]^{N_F}$ and thus $I = J \cap F[X]^{N_F}$ (see [Mat89, Thm. 7.5.(ii)]). Hence $I = F[X]^{N_F}$. Thus $F[X]^{N_F}$ is differentially simple, and so Y is a simple differential $(G/N)_F$ -torsor. \square

Observe that Corollary 1.14 and Proposition 1.17 provide a partial version of the Galois correspondence in differential Galois theory, via torsors. Namely, if R is a Picard-Vessiot ring over F with differential Galois group G , and if N is a closed normal subgroup of G , then R^N is a Picard-Vessiot ring over F with differential Galois group G/N , by considering the corresponding torsors $X = \text{Spec}(R)$ and $X/N = \text{Spec}(R^N)$. Also, if $X = \text{Spec}(R)$ is as above and H is any closed subgroup of G such that $F(X/H) = F$, then $H = G$. To see this, first recall that X/H is an integral quasi-projective F -variety (by Proposition A.6(a) and the integrality of X); so if its function field is F then it is isomorphic to a single F -point $\text{Spec}(F)$. Thus $\text{Spec}(\bar{F}) = (X/H)(\bar{F}) = X(\bar{F})/H(\bar{F}) \cong G(\bar{F})/H(\bar{F})$; i.e. $H(\bar{F}) = G(\bar{F})$, and hence $H = G$ since \bar{F} -points are dense. The above facts will be useful later.

Lemma 1.18. *Let H be a closed subgroup of the linear algebraic group G , let Y be a differential H_F -torsor, and write $X = \text{Ind}_{H_F}^{G_F}(Y)$. Then*

$$\psi : Y \times_{G_F} \rightarrow Y \times X, (y, g) \mapsto (y, y.g)$$

is an isomorphism. If L is a differential field containing $R := F[Y]$ as a differential subring, the pullback of ψ via $\text{Spec}(L) \rightarrow Y$ yields an isomorphism of differential G_L -torsors $G_L \rightarrow X_L$. With $S := F[X] = \text{Ind}_{H_F}^{G_F}(R)$, the dual isomorphism on coordinate rings

$$\Theta : L[X] = L \otimes_F S = L \otimes_F (R \otimes_F F[G])^H \xrightarrow{\sim} L[G]$$

is given by $\Theta(\sum_i a_i \otimes r_i \otimes f_i) = \sum_i a_i \cdot r_i \cdot f_i$, for all elements $a_i \in L \subseteq L[G]$, $r_i \in R \subseteq L \subseteq L[G]$, $f_i \in F[G] \subseteq L[G]$ such that $\sum_i a_i \otimes r_i \otimes f_i \in L \otimes_F (R \otimes_F F[G])^H$.

Proof. Since $X = \text{Ind}_{H_F}^{G_F}(Y)$ is a G_F -torsor, the morphism $X \times_{G_F} \rightarrow X \times X$ given by $(x, g) \mapsto (x, x.g)$ is an isomorphism. Restricting to $Y \times_{G_F}$ yields the isomorphism ψ . As ψ is G_F -equivariant (where G_F is acting on the right factors), $G_L \rightarrow X_L$ is an isomorphism of

G_L -torsors. The inclusion $(F[Y] \otimes_F F[G])^H \subseteq F[Y] \otimes_F F[G]$ corresponds to the morphism $Y \times G_F \rightarrow \text{Ind}_{H_F}^{G_F}(Y)$, $(y, g) \mapsto y.g$, and so the dual map of ψ is given by

$$F[Y] \otimes_F (F[Y] \otimes_F F[G])^H \longrightarrow F[Y] \otimes_F F[G], \quad \sum_i a_i \otimes r_i \otimes f_i \mapsto \sum_i a_i \cdot r_i \otimes f_i$$

for all $a_i \in F[Y], r_i \in F[Y], f_i \in F[G]$ such that $\sum_i a_i \otimes r_i \otimes f_i \in F[Y] \otimes_F (F[Y] \otimes_F F[G])^H$. Tensoring over $F[Y]$ with L yields the last assertion. \square

2. DIFFERENTIAL PATCHING AND EMBEDDING PROBLEMS

In this section, we construct differential torsors over a given field F by using differential torsors over larger fields, and patching them. This builds on the method of patching over fields (see [HH10]), and in particular on a result in [HHK15] on patching torsors. The new aspect in our situation is the differential structure on the torsors. Using this approach and the correspondence between simple differential torsors and Picard-Vessiot extensions from the previous section, we can construct Picard-Vessiot extensions with desired properties. This is useful both in the inverse differential Galois problem and for solving embedding problems in differential Galois theory.

2.1. Patching differential torsors. The basic situation is the following: We have a quadruple of fields (F, F_1, F_2, F_0) together with inclusions $F \hookrightarrow F_i \hookrightarrow F_0$ for $i = 1, 2$, such that the diagram

$$\begin{array}{ccc} & F_0 & \\ & \swarrow & \searrow \\ F_1 & & F_2 \\ & \swarrow & \searrow \\ & F & \end{array}$$

commutes and such that F is the intersection of F_1 and F_2 taken inside F_0 . (Thus F is the inverse limit of the finite inverse system consisting of the fields F_i .) We refer to such a quadruple as a *diamond*. We say that a diamond has the *factorization property* if for every $n > 0$, every element $A \in \text{GL}_n(F_0)$ can be written as $A_2^{-1}A_1$ with $A_i \in \text{GL}_n(F_i)$.

It has been useful in applications (e.g. in Section 9 of [HHK15]) to consider more general collections of fields and inclusions; and the applications in that more general situation do not seem to follow easily from the case of diamonds of fields. More precisely, as in [HHK15], a *factorization inverse system* over a field F is a finite inverse system $\{F_i\}_{i \in \mathcal{I}}$ of fields whose inverse limit (in the category of rings) is F , whose index set \mathcal{I} has a partition $\mathcal{I} = \mathcal{I}_v \cup \mathcal{I}_e$ into a disjoint union such that for each index $k \in \mathcal{I}_e$ there are exactly two elements $i, j \in \mathcal{I}_v$ for which the inverse system contains maps $F_i, F_j \rightarrow F_k$, and such that there are no other maps given in the inverse system. If there is a map $F_i \rightarrow F_k$ in the inverse system, we write $i \succ k$; this defines a partial ordering on \mathcal{I} .

A factorization inverse system determines a (multi-)graph Γ whose vertices are the elements of \mathcal{I}_v and whose edges are the elements of \mathcal{I}_e . The vertices of an edge $k \in \mathcal{I}_e$ correspond to the elements $i, j \in \mathcal{I}_v$ such that $i, j \succ k$. Note that the graph Γ is connected (otherwise, the inverse limit F would have zero divisors). For every $k \in \mathcal{I}_e$, we fix a labeling $l = l_k$ and $r = r_k$ of its vertices l and r (i.e., we assign each edge a left vertex and a right vertex). Note

that an element $i \in \mathcal{I}_v$ can be a left vertex of an edge and a right vertex of another edge. When working with factorization inverse systems, we always assume that such an orientation of the edges has been fixed. A factorization inverse system $\{F_i\}_{i \in \mathcal{I}}$ has the *simultaneous factorization property* if for any collection of matrices $A_k \in \mathrm{GL}_n(F_k)$, for $k \in \mathcal{I}_e$, there exist matrices $A_i \in \mathrm{GL}_n(F_i)$ for all $i \in \mathcal{I}_v$ such that $A_k = A_{r_k}^{-1} \cdot A_{l_k}$ for all $k \in \mathcal{I}_e$, where we view A_{r_k} and A_{l_k} as matrices with entries in F_k via the inclusions $F_{r_k}, F_{l_k} \hookrightarrow F_k$. In the case that \mathcal{I}_e has just one element 0, and \mathcal{I}_v has two elements $l_k = 1, r_k = 2$, we recover the notions of a diamond and of a diamond with the factorization property.

Turning now to our situation, a *differential factorization inverse system* over a field F is a factorization inverse system over F , such that all fields F_i , $i \in \mathcal{I}$, are differential fields of characteristic zero and such that the inclusions $F_{l_k}, F_{r_k} \hookrightarrow F_k$ are differential homomorphisms for all $k \in \mathcal{I}_e$. Note that then F inherits a structure as a differential field (of characteristic zero) and the embeddings $F \hookrightarrow F_i$ given by mapping to the i -th component in the inverse limit are differential homomorphisms for all $i \in \mathcal{I}$. In the case of a diamond, we call the quadruple a *differential diamond*. A differential factorization inverse system (and in particular a differential diamond) may have the factorization property, as defined above.

- Example 2.1.** (a) Let $F = \mathbb{C}(x)$, the field of rational functions over \mathbb{C} , or equivalently the field of meromorphic functions on the Riemann sphere $\mathbb{P}_{\mathbb{C}}^1$. Choose open discs U_1 centered at the point $x = 0$ and U_2 centered at $x = \infty$, such that $U_1 \cup U_2 = \mathbb{P}_{\mathbb{C}}^1$. Let U_0 be the annulus $U_1 \cap U_2$, and write F_i for the field of meromorphic functions on U_i for $0, 1, 2$. Then (F, F_1, F_2, F_0) is a differential diamond with the factorization property with respect to the derivation d/dx . See Lemma 3.4 for a more general statement concerning open subsets of Riemann surfaces. This will be used in solving differential embedding problems over complex function fields.
- (b) There are also examples of factorization inverse systems with the simultaneous factorization property over function fields over a complete discretely valued field K ; see [HHK15, Corollary 3.4, Proposition 2.2]. In the case of the function field $K(x)$ together with the derivation d/dx in characteristic zero, we obtain differential factorization inverse systems. Such examples can be viewed as rigid analytic analogs of Example 2.1(a).

Following [HHK15, Section 2], for a linear algebraic group G over F and a factorization inverse system of fields $\{F_i\}_{i \in \mathcal{I}}$ with inverse limit F , we define a *G -torsor patching problem* to be a system of G_{F_i} -torsors $X_i = \mathrm{Spec}(S_i)$ together with F_k -isomorphisms of G_{F_k} -torsors $\nu_{ik} : F_k \times_{F_i} X_i \rightarrow X_k$ for all pairs of distinct indices with $i \succ k$ (i.e., such that $F_i \subseteq F_k$). Here ν_{ik} corresponds to an isomorphism of coordinate rings $\nu_{ik}^* : S_k \rightarrow F_k \otimes_{F_i} S_i$ that respects the G -actions. A *solution* to the patching problem is a torsor over F that induces the torsors X_i compatibly via base change. That is, a solution is given by a G -torsor $X = \mathrm{Spec}(S)$ over F together with F_i -isomorphisms of G_{F_i} -torsors $\gamma_i : F_i \times_F X \rightarrow X_i$ for all $i \in \mathcal{I}_v$ such that the two maps $\nu_{ik} \circ (\mathrm{id}_{F_k} \otimes_{F_i} \gamma_i) : F_k \times_F X \rightarrow X_k$ agree, for $i = l_k, r_k$. Patching problems and solutions can also be described on the level of coordinate rings. If we write $\Theta_{ki} = (\nu_{ik}^*)^{-1} : F_k \otimes_{F_i} S_i \rightarrow S_k$ and $\Phi_i = (\gamma_i^{-1})^* : F_i \otimes_F S \rightarrow S_i$, the compatibility condition is that the two isomorphisms $\Theta_{klk} \circ (\mathrm{id}_{F_k} \otimes_{F_{l_k}} \Phi_{l_k}), \Theta_{krk} \circ (\mathrm{id}_{F_k} \otimes_{F_{r_k}} \Phi_{r_k}) : F_k \otimes_F S \rightarrow S_k$ coincide.

It was shown at [HHK15, Proposition 2.2, Theorem 2.3] that if the factorization inverse system above has the simultaneous factorization property, then up to isomorphism there is

a unique solution $X = \text{Spec}(S)$, given on the level of coordinate rings by the inverse limit. That is, $S = \lim_{\leftarrow} S_i$, where the limit is over all $i \in \mathcal{I}$; and Φ_i is induced by the canonical map $S \rightarrow S_i$ for $i \in \mathcal{I}_v$. Thus for $j \in \mathcal{I}_v$, the map Φ_j sends $a \otimes (x_i)_{i \in \mathcal{I}} \in F_j \otimes_F S$ to ax_j in S_j . Also, since each γ_i is an isomorphism of torsors, the co-action homomorphism $\rho : S \rightarrow S \otimes_F F[G]$ corresponding to the G -action on X is the restriction of the product map $(\prod \rho_i) : \prod S_i \rightarrow (\prod S_i) \otimes_F F[G]$, where $\rho_i : S_i \rightarrow S_i \otimes_{F_i} F_i[G] \cong S_i \otimes_F F[G]$ is the co-action map corresponding to the G_{F_i} -action on $\text{Spec}(S_i)$ for $i \in \mathcal{I}$.

The notions of patching problems and solutions carry over to the differential situation. Namely, consider a differential factorization inverse system $\{F_i\}_{i \in \mathcal{I}}$ over a differential field F of characteristic zero, and let G be a linear algebraic group over $K := C_F$. A system of differential G_{F_i} -torsors X_i such that the maps ν_{ik} as above are differential isomorphisms will be called a *patching problem of differential G_F -torsors*. (Recall that in our setup, G is defined over the constants K of F . Also, recall that the derivation on $F_i[G]$ is defined to extend the given derivation on F_i and to be constant on $K[G]$.) Similarly, if X is a differential G_F -torsor and each γ_i as above is an isomorphism of differential torsors, we have a *solution* to this differential patching problem. The result from [HHK15] cited above then carries over to this situation, since the solution given by [HHK15] inherits a unique compatible derivation on the coordinate ring by restriction:

Theorem 2.2. *Let $\{F_i\}_{i \in \mathcal{I}}$ be a differential factorization inverse system over F with the simultaneous factorization property and let $K = C_F$ be the field of constants of F . Let G be a linear algebraic group over K and let $(\{S_i\}_{i \in \mathcal{I}}, \{\Theta_{kl_k}, \Theta_{kr_k}\}_{k \in \mathcal{I}_e})$ define a patching problem of differential G_F -torsors. Then up to differential isomorphism, there exists a unique solution $(S, \{\Phi_i\}_{i \in \mathcal{I}_v})$, given by $S = \lim_{\leftarrow} S_i$, with Φ_i induced by the natural map $S \rightarrow S_i$, and with the G -action and derivation on S given by restriction from those on the rings S_i .*

In particular, this holds in the case of differential diamonds (F, F_1, F_2, F_0) with the factorization property. There, we may identify S with the intersection $\Theta_{01}(S_1) \cap \Theta_{02}(S_2) \subseteq S_0$. With this identification, the isomorphism $\Phi_i : F_i \otimes_F S \rightarrow S_i$ sends $a \otimes s$ to $a \cdot \Theta_{0i}^{-1}(s)$ for $i = 1, 2$, for $a \in F_i$ and $s \in S$.

2.2. Patching Picard-Vessiot rings. Using the above theorem, we prove a result about patching Picard-Vessiot rings, by relying on the relationship between Picard-Vessiot rings and differential torsors. This enables us to construct Picard-Vessiot extensions of F with a given group G , by using Picard-Vessiot extensions of appropriate overfields F_i with differential Galois groups G_i that generate G .

Lemma 2.3. *Let R be a simple differential ring with field of constants K and let A be a K -algebra. We consider A as a constant differential ring. Then there is a bijection between the differential ideals in $R \otimes_K A$ and the ideals in A , given by $I \mapsto I \cap A$ for differential ideals $I \trianglelefteq R \otimes_K A$, with inverse $J \mapsto R \otimes_K J$ for ideals $J \trianglelefteq A$.*

Proof. This is a well-known statement in differential algebra and it follows as in [Kov03, Prop. 5.6]. See also [Mau10, Lemma 10.7]. \square

In the following, we use the notation $\text{Ind}_H^G(R)$ for the coordinate ring of $\text{Ind}_H^G(\text{Spec}(R))$, when H is a subgroup of G and $\text{Spec}(R)$ is an H -torsor. By the comment before Remark A.12, $\text{Ind}_H^G(R)$ is the ring of invariants $(R \otimes_K K[G])^H$ with respect to a certain H -action on

$\text{Spec}(R) \times G$. A definition of the ring of invariants can be found in the second paragraph of Section A.1. For a G -space $\text{Spec}(R)$ (and in particular for G itself) we also use the notion of H -stable ideals in R for subgroups H of G (see Lemma A.1 and the subsequent paragraph).

Lemma 2.4. *Let $F \subseteq F_1 \subseteq F_0$ be extensions of differential fields, and let G be a linear algebraic group over $K := C_F$. Let S be a differential ring containing F , and for $i = 0, 1$ write $S_i = F_i \otimes_F S$. Let H_1 be a closed subgroup of G , and suppose that $S_1 = \text{Ind}_{H_1}^G(R_1)$ for a Picard-Vessiot ring $R_1 \subseteq F_0$ over F_1 with differential Galois group $(H_1)_{C_{F_1}}$. Let I be a differential ideal of S , and for $i = 0, 1$ write $I_i := F_i \otimes_F I \subseteq S_i$. View I_0 as an ideal in $F_0[G]$ via the isomorphism $\Theta : S_0 = F_0 \otimes_{F_1} \text{Ind}_{H_1}^G(R_1) \rightarrow F_0[G]$ given in Lemma 1.18. For $i = 0, 1$, consider the right action of G_{F_i} on itself given by $(g', g) \mapsto g^{-1}g'$. Then*

- (a) *For every closed subgroup $H \subseteq G$, the ideal $I_0 \subseteq F_0[G]$ is H -stable if and only if $I_0 \cap F_1[G] \subseteq F_1[G]$ is H -stable.*
- (b) *The ideals $I_0 \subseteq F_0[G]$ and $I_0 \cap F_1[G] \subseteq F_1[G]$ are H_1 -stable.*

Proof. The forward direction of part (a) is clear. For the converse direction, it suffices to show that $(I_0 \cap F_1[G])F_0[G] = I_0$.

Let $\vartheta : R_1 \otimes_{F_1} F_1[G] \rightarrow F_0[G]$ be the differential homomorphism induced by the inclusions of R_1 and F_1 into F_0 . By Lemma 1.18, $\Theta|_{S_1} = \vartheta|_{S_1}$. Since $\text{Ind}_{H_1}^G(R_1) = (R_1 \otimes_{F_1} F_1[G])^{H_1}$ by Proposition A.8, we have the following commutative diagram of differential F -algebras, where $\Phi_i : S \rightarrow S_i$ is the natural map for $i = 0, 1$:

$$\begin{array}{ccc}
 F_0[G] & \xleftarrow{\quad} & F_1[G] \\
 \uparrow \Theta & \swarrow \vartheta & \downarrow \iota \\
 S_0 & & R_1 \otimes_{F_1} F_1[G] \\
 \uparrow \Phi_0 & \searrow & \uparrow \\
 S & \xrightarrow{\Phi_1} & S_1 = (R_1 \otimes_{F_1} F_1[G])^{H_1}
 \end{array}$$

Note that I_i is the ideal of S_i generated by $\Phi_i(I)$, for $i = 0, 1$. Write \tilde{J}_1 for the ideal of $R_1 \otimes_{F_1} F_1[G]$ generated by I_1 . If we identify S_0 with $F_0[G]$ via Θ , then I_0 is identified with the ideal of $F_0[G]$ generated by $\Theta(I_1)$, or equivalently the ideal generated by $\vartheta(\tilde{J}_1)$.

Let $K_1 = C_{F_1}$. As R_1 is differentially simple, every ideal in $R_1 \otimes_{F_1} F_1[G] \cong R_1 \otimes_{K_1} K_1[G]$ is generated by its intersection with $K_1[G]$ by Lemma 2.3; and in particular it is generated by its intersection with $F_1[G]$. Thus $J_1 := \tilde{J}_1 \cap F_1[G]$ generates \tilde{J}_1 as an ideal of $R_1 \otimes_{F_1} F_1[G]$. But $\vartheta(\tilde{J}_1)$ generates I_0 as an ideal of $F_0[G]$; hence $I_0 = J_1 F_0[G]$. Also, $J_1 = I_0 \cap F_1[G]$ since $F_0[G] \cong F_0 \otimes_{F_1} F_1[G]$ is faithfully flat over $F_1[G]$ (see [Mat89, Theorem 7.5(b)]). Thus $(I_0 \cap F_1[G])F_0[G] = I_0$, proving part (a).

The ideal $\tilde{J}_1 \subseteq R_1 \otimes_{F_1} F_1[G]$ is H_1 -stable (with respect to the action given in Proposition A.8) since it is the extension of the ideal $I_1 \subseteq S_1 = (R_1 \otimes_{F_1} F_1[G])^{H_1}$. Since the projection morphism $\iota^* : (\text{Spec } R_1) \times_{F_1} G_{F_1} \rightarrow G_{F_1}$ is H_1 -equivariant, and since \tilde{J}_1 is the extension of $J_1 \subseteq F_1[G]$ with respect to ι , it follows that J_1 is H_1 -stable (by stability condition (b) of Lemma A.1). The H_1 -stability of $J_1 = I_0 \cap F_1[G] \subseteq F_1[G]$ implies the H_1 -stability of $I_0 \subseteq F_0[G]$, by part (a). This proves part (b). \square

Theorem 2.5. *In the context of Theorem 2.2, suppose that G is generated by a set of closed K -subgroups H_i , for $i \in \mathcal{I}_v$; and that for each $i \in \mathcal{I}_v$ there is a Picard-Vessiot ring R_i/F_i with differential Galois group $(H_i)_{K_i}$ such that $S_i = \text{Ind}_{H_i}^G(R_i)$, where $K_i = C_{F_i}$. Suppose also that $S_k = F_k[G]$ for all $k \in \mathcal{I}_e$; and suppose that for each $i \succ k$ there is an embedding $R_i \hookrightarrow F_k$ of differential rings. Suppose moreover that the map $\Theta_{ki}: F_k \otimes_{F_i} S_i \rightarrow S_k$ is the isomorphism of differential G_{F_k} -torsors given in Lemma 1.18 with respect to that embedding (for all i). Then S/F is a Picard-Vessiot ring with differential Galois group G .*

Proof. By Theorem 2.2, $\text{Spec}(S)$ is a differential G_F -torsor and S is the inverse limit of $\{S_i\}_{i \in \mathcal{I}}$ with respect to the maps Θ_{ki} . As in Theorem 2.2, we write $\Phi_i: S \rightarrow S_i$ for the projection onto the i -th component. We identify S with $F \otimes_F S \subseteq F_i \otimes_F S \subseteq F_k \otimes_F S$, and similarly identify S_i with $F_i \otimes_{F_i} S_i \subseteq F_k \otimes_{F_i} S_i$. By Proposition A.8, $S_i = \text{Ind}_{H_i}^G(R_i)$ consists of the invariants of $R_i \otimes_{F_i} F_i[G]$ under the $(H_i)_{F_i}$ -action on $\text{Spec}(R_i) \times G_{F_i}$ given by the formula $(x, g).h = (x.h, h^{-1}g)$. This action restricts to an H_i -action on G given by $g.h = h^{-1}g$, which is also the restriction of the (right) G -action on G given by $g.g' = g'^{-1}g$.

To prove the theorem, it suffices to show that $C_S = K$ and that S is differentially simple, by Corollary 1.14(a).

First step: We show $C_S = K$.

Let $x \in S$ be constant, and write $x = (x_i)_{i \in \mathcal{I}}$ with $x_i \in S_i$ constant. We wish to show that $x \in K$. For $i \in \mathcal{I}_v$, the constants of $R_i \otimes_{F_i} F_i[G] = R_i \otimes_{K_i} K_i[G]$ equal $K_i[G]$, since $C_{R_i} = K_i$; and thus $C_{S_i} = K_i[G]^{(H_i)_{K_i}} \subseteq F_i[G]^{(H_i)_{F_i}}$. Hence $x_i \in F_i[G]^{(H_i)_{F_i}} \subseteq F_i[G]$. Suppose that $i \succ k$. Then x_k is the image of x_i under the natural inclusion $F_i[G] \rightarrow F_k[G]$ (see Lemma 1.18). The co-action on $F_i[G]$ is the restriction of the co-action on $F_k[G]$; so x_i is invariant under a given subgroup $H \subseteq G$ if and only if x_k is invariant under H . Since the graph Γ associated to $\{F_i\}_{i \in \mathcal{I}}$ is connected, it follows that all the x_i (for $i \in \mathcal{I}$) are invariant under the same subgroups of G . As a consequence, each of these elements is invariant under H_j for every $j \in \mathcal{I}_v$, since x_j is.

Thus $x_i \in \bigcap_{j \in \mathcal{I}_v} F_i[G]^{(H_j)_{F_i}}$ for all $i \in \mathcal{I}_v$, and this intersection equals F_i by Lemma A.2(a). Hence $x = (x_i)_{i \in \mathcal{I}} \in \varprojlim_{i \in \mathcal{I}} F_i = F$ and thus $x \in C_F = K$.

Second step: We show that S is differentially simple. Let I be a proper differential ideal of S . It suffices to show that $I = (0)$.

For an edge $k \in \mathcal{I}_e$ and a vertex $i \in \mathcal{I}_v$ of k , let I_i be the ideal of S_i generated by $\Phi_i(I)$, and let I_k be the ideal of $S_k = F_k[G]$ generated by $\Theta_{ki}(I_i)$ (which is independent of the choice of vertex i of k). Let $J_i = I_k \cap F_i[G]$. We may now apply Lemma 2.4, with F_i, F_k playing the roles of F_1, F_0 , and where we consider the right action of G on itself given by $(g', g) \mapsto g^{-1}g'$. By part (a) of the lemma, for any subgroup H of G , J_i is H -stable under this action if and only if I_k is. Since this holds for all such pairs (k, i) , and since the graph is connected, it follows that all the ideals $J_i \trianglelefteq F_i[G]$ (for $i \in \mathcal{I}_v$) and $I_k \trianglelefteq F_k[G]$ (for $k \in \mathcal{I}_e$) are stabilized by the same subgroups of G . But by part (b) of the lemma, J_i and I_k are H_i -stable, with respect to the above action. Thus for every $k' \in \mathcal{I}_e$, the ideal $I_{k'} \trianglelefteq F_{k_0}[G]$ is stable under H_i for all $i \in \mathcal{I}_v$. But the subgroups H_i generate G . So $I_{k'}$ is $G_{F_{k'}}$ -stable by Lemma A.2(b) and thus $I_{k'} = (0)$ by Lemma A.2(c). Hence $I \subseteq I_{k'}$ is also the zero ideal, as asserted. \square

As an example application, we show how Theorem 2.5 can be applied to show that SL_2 is a differential Galois group over F under the assumption that there exists a differential

diamond (F, F_1, F_2, F_0) with the factorization property such that F_0 contains “logarithmic elements” over F_1 and F_2 .

Example 2.6. Let (F, F_1, F_2, F_0) be a differential diamond with the factorization property. Assume that there exist Picard-Vessiot rings R_1/F_1 and R_2/F_2 with differential Galois groups the additive group \mathbb{G}_a and such that $R_1 \subseteq F_0$ and $R_2 \subseteq F_0$. Then there is a Picard-Vessiot ring S/F with differential Galois group SL_2 . Indeed, let H_1 and H_2 be the subgroups of SL_2 of upper and lower triangular matrices whose diagonal entries are equal to 1. These are known to generate SL_2 over any field. The groups H_1 and H_2 are isomorphic to \mathbb{G}_a and the Tannaka formalism implies that there are (2×2) -fundamental solution matrices for R_1/F_1 and R_2/F_2 such that the corresponding representations of the differential Galois groups of R_1/F_1 and R_2/F_2 yield H_1 and H_2 (see [BHH16, Prop. 3.2]). Since SL_2 is generated by H_1 and H_2 , Theorem 2.5 implies that there exists a Picard-Vessiot ring S/F with differential Galois group SL_2 .

Note that the assumptions on the existence of R_1/F_1 and R_2/F_2 are equivalent to the existence of “logarithmic elements” $y_1, y_2 \in F_0$ such that $y_1 \notin F_1$ and $\partial(y_1) \in F_1$ and similarly $y_2 \notin F_2$ and $\partial(y_2) \in F_2$.

Remark 2.7. If $F = k((t))(x)$ for some field k of characteristic zero and $\partial = d/dx$, a weaker version of Theorem 2.5 was proven in [BHH16, Thm 2.4(a)] using ad-hoc methods, on the way to solving the inverse differential Galois problem over F . However, that theorem only applies to factorization inverse systems $\{F_i\}_{i \in \mathcal{I}}$ over F where the corresponding graph Γ is star-shaped, where all fields of constants C_{F_i} equal C_F and where the Picard-Vessiot ring over F_i is trivial for the internal node i of the star. Theorem 2.5 does not rely on any of these assumptions and it can also be applied to more general factorization inverse systems, e.g. of the sort that arise in [HHK15].

2.3. Embedding problems. As in ordinary Galois theory, one can consider embedding problems in differential Galois theory. Using the above ideas, we prove a result about split differential embedding problems.

Let F be a differential field with field of constants K . In analogy with the case of ordinary Galois theory, a *differential embedding problem* over F consists of an epimorphism of linear algebraic groups $G \rightarrow H$ over K , say with kernel N , together with a Picard-Vessiot ring R/F with differential Galois group H . (Notice that we work with Picard-Vessiot rings here rather than Picard-Vessiot extensions, because those rings are needed to define the differential Galois groups as group schemes.) If the short exact sequence $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ is split (i.e., if G is a semi-direct product $N \rtimes H$), then we say that the embedding problem is *split* and abbreviate it by $(N \rtimes H, R)$.

A *proper solution* of a differential embedding problem as above is a Picard-Vessiot ring S/F with differential Galois group G and an embedding of differential rings $R \subseteq S$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \longrightarrow & H \\ \downarrow \cong & & \downarrow \cong \\ \underline{\mathrm{Aut}}^\partial(S/F) & \xrightarrow{\mathrm{res}} & \underline{\mathrm{Aut}}^\partial(R/F) \end{array}$$

Lemma 2.8. *Let $G \rightarrow H$ and R/F determine a differential embedding problem as above and let S/F be a Picard-Vessiot ring with differential Galois group G . Then there exists an embedding of differential rings $R \subseteq S$ constituting a proper solution to the differential embedding problem if and only if there exists an isomorphism of differential H_F -torsors $\text{Spec}(R) \cong \text{Spec}(S^{N_F})$.*

Proof. If $R \subseteq S$ constitutes a proper solution, then we have isomorphisms of differential H_F -torsors $\text{Spec}(R) \cong \text{Spec}(S)/N_F \cong \text{Spec}(S^{N_F})$ by Proposition 1.17 and Lemma A.6(b).

Conversely, an isomorphism of H_F -torsors $\text{Spec}(R) \cong \text{Spec}(S^{N_F})$ gives rise to a commutative diagram

$$(5) \quad \begin{array}{ccc} \underline{\text{Aut}}^\partial(S^{N_F}/F) & \xrightarrow{\cong} & \underline{\text{Aut}}^\partial(R/F) \\ & \searrow \cong & \nearrow \cong \\ & & H \end{array}$$

As the inner diagrams in

$$\begin{array}{ccc} & & \underline{\text{Aut}}^\partial(R/F) \\ & \nearrow & \uparrow \cong \\ \underline{\text{Aut}}^\partial(S/F) & \longrightarrow & \underline{\text{Aut}}^\partial(S^{N_F}/F) \\ \cong \uparrow & & \uparrow \cong \\ G & \longrightarrow & H \end{array}$$

commute, also the outer diagram commutes. It follows from (5) that we have solved the embedding problem. \square

Differential embedding problems not only provide information about which differential Galois groups arise over a given differential field, but also encode information about how the Picard-Vessiot extensions of that field fit together. As in ordinary Galois theory, the assertion that all split embedding problems over some field F have proper solutions implies that all groups occur as (differential) Galois groups, since one can take H to be trivial. In Theorem 2.12 below we show that proper solutions to split differential embedding problems can be obtained from solutions to patching problems.

Lemma 2.9. *Let (F, F_1, F_2, F_0) be a differential diamond such that $C_{F_0} = C_F$. Let R/F be a Picard-Vessiot ring such that $R \subseteq F_1$. Then the compositum $F_2R \subseteq F_0$ is a Picard-Vessiot ring over F_2 with the same differential Galois group as R/F . Moreover, F_2R is isomorphic to $F_2 \otimes_F R$ under the natural map $F_2 \otimes_F R \rightarrow F_2R$.*

Proof. Let G be the differential Galois group of R/F . By [BHH16, Lemma 1.7], the compositum F_2R is a Picard-Vessiot ring over F_2 and its differential Galois group H is a subgroup of G . By the differential Galois correspondence (see [Dyc08, Theorem 4.4] and the discussion at the beginning of Section 1.4), in order to prove $H = G$ it suffices to show that $\text{Frac}(R)^H = F$. This equality follows from the containments $\text{Frac}(R)^H \subseteq \text{Frac}(R) \subseteq F_1$ and $\text{Frac}(R)^H \subseteq \text{Frac}(F_2R)^H = F_2$, since $F_1 \cap F_2 = F$. So the first assertion of the lemma holds. The final assertion follows from the fact that the natural surjection $F_2 \otimes_F R \rightarrow F_2R$ corresponds to a homomorphism of G_{F_2} -torsors, which must therefore be an isomorphism. \square

Note that instead of citing the Galois correspondence from [Dyc08, Theorem 4.4] in the above proof, one could use the observation after Proposition 1.17. Namely, $X := \text{Spec}(R)$ is a G -torsor over F and $X_2 := \text{Spec}(F_2R)$ is an H -torsor over F_2 , where the actions of $H \subseteq G$ are compatible. By Proposition A.6(a) we may consider the quotient schemes X/H and $X_2/H = \text{Spec}(F_2)$. These are quasi-projective varieties, and are integral since X and X_2 are the spectra of integral domains. We have an inclusion of function fields $F(X/H) \subseteq F(X) = \text{Frac}(R) \subset F_1$, as well as an inclusion $F(X/H) \subseteq F_2(X_2/H) = F_2$. Thus $F \subseteq F(X/H) \subseteq F_1 \cap F_2 = F$; i.e., $F(X/H) = F$. By the second part of the observation after Proposition 1.17), $H = G$.

To avoid burdening the notation, we sometimes drop the base change subscripts on groups in the remainder of this section if there is no possibility of confusion, especially when the group appears in a subscript or superscript. For example, for field extensions L/F , we write expressions such as $\text{Ind}_H^G(Y_L)$ for $\text{Ind}_{H_L}^{G_L}(Y_L)$. We consider the following situation:

Hypothesis 2.10. Let (F, F_1, F_2, F_0) be a differential diamond with the factorization property and write $K = C_F$. Let G be a linear algebraic group over K of the form $N \rtimes H$. Let $S_1 = \text{Ind}_N^G(R_1)$ for some Picard-Vessiot ring R_1/F_1 with differential Galois group $N_{C_{F_1}}$ such that $R_1 \subseteq F_0$. Let R/F be a Picard-Vessiot ring with differential Galois group H such that $R \subseteq F_1 \subseteq F_0$; write $R_2 = F_2 \otimes_F R$; and let $S_2 = \text{Ind}_H^G(R_2) = F_2 \otimes_F \text{Ind}_H^G(R)$. For $i = 1, 2$ let $\Theta_i : F_0 \otimes_{F_i} S_i \rightarrow F_0[G]$ be the induced differential isomorphism defined in Lemma 1.18, using $R_i \subseteq F_0$.

Proposition 2.11. *In the situation of Hypothesis 2.10, let (S, Φ_1, Φ_2) be the solution of the patching problem $(S_1, S_2, F_0[G], \Theta_1, \Theta_2)$ of differential G -torsors over (F, F_1, F_2, F_0) (see Theorem 2.2). Then the ring of N_F -invariants S^{N_F} is isomorphic to R as a differential H_F -torsor.*

Proof. By taking N -invariants, the given patching problem gives rise to a patching problem $(S_1^N, S_2^N, F_0[H], \bar{\Theta}_1, \bar{\Theta}_2)$ of differential H -torsors over (F, F_1, F_2, F_0) , and the given solution gives rise to a solution $(S^N, \bar{\Phi}_1, \bar{\Phi}_2)$ to that patching problem. We also have a patching problem of differential H -torsors given by $(F_1 \otimes_F R, F_2 \otimes_F R, F_0 \otimes_F R, \Omega_1, \Omega_2)$, where Ω_i is the natural isomorphism $F_0 \otimes_{F_i} (F_i \otimes_F R) \rightarrow F_0 \otimes_F R$. A solution to this latter patching problem is (R, Ψ_1, Ψ_2) , where Ψ_i is the identity on $F_i \otimes_F R$, for $i = 1, 2$. But solutions to patching problems are unique up to isomorphism (given by inverse limit, and the restriction of the derivations; see Theorem 2.2). So it suffices to show that the above two patching problems of differential H -torsors are isomorphic. That is, for $j = 0, 1, 2$ we want to find differential isomorphisms Λ_j between the respective rings in the two H -patching problems that carry Ω_i to $\bar{\Theta}_i$ for $i = 1, 2$.

By Corollary A.14(a), $S_1^N = (\text{Ind}_N^G(R_1))^N = F_1[H]$, where we identify $F_1[H]$ with $F_1 \otimes_{F_1} F_1[H] \subseteq R_1 \otimes_{F_1} F_1[G]$. The differential isomorphism $\bar{\Theta}_1$ is given by $F_0 \otimes_{F_1} S_1^N = F_0 \otimes_{F_1} F_1[H] \rightarrow F_0[H]$. Meanwhile, since $R_2 = F_2 \otimes_F R$, we have $S_2 = F_2 \otimes_F \text{Ind}_H^G(R) = (R_2 \otimes_F F[G])^{H_{F_2}} = \text{Ind}_H^G(R_2)$, and so $S_2^N = (\text{Ind}_H^G(R_2))^N = \text{Ind}_H^H(R_2)$ by Corollary A.14(b). The differential isomorphism $\bar{\Theta}_2$ is given by $F_0 \otimes_{F_2} S_2^N = F_0 \otimes_{F_2} \text{Ind}_H^H(R_2) = F_0 \otimes_F \text{Ind}_H^H R \xrightarrow{\tilde{\Theta}_0} F_0[H]$, where $\tilde{\Theta}_0$ is the differential isomorphism given by Lemma 1.18 using that $R \subseteq F_0$. Also, since $R \subseteq F_1$, Lemma 1.18 yields a differential isomorphism $\tilde{\Theta}_1 : F_1 \otimes_F \text{Ind}_H^H(R) \rightarrow F_1[H] = S_1^N$; and this map is the restriction of $\tilde{\Theta}_0$. Let $\tilde{\Theta}_2$ be the identity map $F_2 \otimes_F$

$\text{Ind}_H^H(R) \rightarrow \text{Ind}_H^H R_2 = S_2^N$. We then have a commutative diagram

$$\begin{array}{ccccc}
F_0 \otimes_{F_1} S_1^N & \xrightarrow{\Theta_1} & F_0[H] & \xleftarrow{\Theta_2} & F_0 \otimes_{F_2} S_2^N \\
\uparrow \text{id}_{F_0} \otimes \tilde{\Theta}_1 & & \uparrow \tilde{\Theta}_0 & & \uparrow \text{id}_{F_0} \otimes \tilde{\Theta}_2 \\
F_0 \otimes_{F_1} (F_1 \otimes_F \text{Ind}_H^H(R)) & \xrightarrow{\sim} & F_0 \otimes_F \text{Ind}_H^H(R) & \xleftarrow{\sim} & F_0 \otimes_{F_2} (F_2 \otimes_F \text{Ind}_H^H(R)) \\
\uparrow \text{id}_{F_0} \otimes \text{id}_{F_1} \otimes \rho_H & & \uparrow \text{id}_{F_0} \otimes \rho_H & & \uparrow \text{id}_{F_0} \otimes \text{id}_{F_2} \otimes \rho_H \\
F_0 \otimes_{F_1} (F_1 \otimes_F R) & \xrightarrow{\Omega_1} & F_0 \otimes_F R & \xleftarrow{\Omega_2} & F_0 \otimes_{F_2} (F_2 \otimes_F R)
\end{array}$$

of differential isomorphisms, where $\rho_H : R \rightarrow \text{Ind}_H^H(R)$ is the co-action map associated to H (see Remark A.12). Setting $\Lambda_i = \tilde{\Theta}_i \circ (\text{id} \otimes \rho)$ for $i = 0, 1, 2$ then yields the assertion. \square

Theorem 2.12. *In the situation of Hypothesis 2.10, let (S, Φ_1, Φ_2) be a solution of the patching problem $(S_1, S_2, F_0[G], \Theta_1, \Theta_2)$ of differential G -torsors. Then S/F is a Picard-Vessiot ring with differential Galois group G , and hence is a proper solution to the split differential embedding problem given by $G = N \rtimes H$ and the Picard-Vessiot ring R/F .*

Proof. Once the first assertion is shown, the second assertion follows from Proposition 2.11 together with Lemma 2.8. By Corollary 1.14(a), to prove the first assertion it suffices to show that S has no new constants and is differentially simple.

We first prove that $C_S = K$. Recall that $S_1 = \text{Ind}_{N_{F_1}}^{G_{F_1}}(R_1) = (R_1 \otimes_{F_1} F_1[G])^{N_{F_1}} = (R_1 \otimes_{C_{F_1}} C_{F_1}[G])^{N_{F_1}}$. By the assumptions, R_1/F_1 is a Picard-Vessiot ring, hence $C_{R_1} = C_{F_1}$ and thus the ring of constants of $R_1 \otimes_{C_{F_1}} C_{F_1}[G]$ equals $C_{F_1} \otimes_{C_{F_1}} C_{F_1}[G]$. So $C_{S_1} = (C_{F_1} \otimes_{C_{F_1}} C_{F_1}[G])^{N_{C_{F_1}}}$, and the explicit description of Θ_1 in Lemma 1.18 implies $\Theta_1(C_{S_1}) = C_{F_1}[G]^{N_{C_{F_1}}}$. Hence $C_{\Theta_1(S_1)} = \Theta_1(C_{S_1}) = C_{F_1}[G]^{N_{C_{F_1}}} \subseteq F_0[G]^{N_{F_0}}$. Here the invariants are taken with respect to the action given in Lemma A.3(ii). But by Lemma A.3(b), since N is normal these are the same as the N -invariants under the action given in Lemma A.3(i) (which agrees with the torsor action on S). We thus have $C_S \subseteq S \cap C_{\Theta_1(S_1)} \subseteq S \cap F_0[G]^{N_{F_0}} = S^{N_F}$ and hence $C_S = C_{S^{N_F}}$. By Proposition 2.11, there is a differential isomorphism $S^{N_F} \cong R$, hence $C_{S^{N_F}} = C_R$. By the assumptions, R/F is a Picard-Vessiot ring, so $C_R = K$ and we conclude $C_S = K$.

Next, we show that S is a simple differential ring. Let I be a maximal differential ideal of S . By Lemma 2.4 (with $H_1 = N$), the ideal $I_0 := F_0 \otimes_F I \subseteq F_0 \otimes_F S$ is N -stable with respect to the action considered there; or equivalently with respect to the torsor action, by Lemma A.3(b). Since $F_0 \otimes_F S$ is faithfully flat over S , the ideal I is the contraction of its extension I_0 to $F_0 \otimes_F S$, and it is therefore N -stable. Moreover, $I \cap S^{N_F}$ is a proper differential ideal in S^{N_F} and so $I \cap S^N = (0)$, since $S^N \cong R$ is differentially simple. Therefore, Lemma A.7 implies $I = (0)$ and thus S is differentially simple. \square

Remark 2.13. In the situation that $C_{F_0} = C_F$, Theorem 2.12 follows from Theorem 2.5 (in the case of a differential diamond) via Lemma 2.9, since that lemma implies that R_2/F_2 is a Picard-Vessiot ring with differential Galois group H .

We conclude this section by summarizing the content of Proposition 2.11 and Theorem 2.12 in the following

Theorem 2.14. *Let (F, F_1, F_2, F_0) be a differential diamond with the factorization property and let $(N \rtimes H, R)$ be a split differential embedding problem over F with the property that $R \subseteq F_1$. Assume further that there exists a Picard-Vessiot ring R_1/F_1 with differential Galois group $N_{C_{F_1}}$ and with $R_1 \subseteq F_0$. Then there exists a proper solution to the differential embedding problem $(N \rtimes H, R)$ over F .*

Proof. Let $S_1 = \text{Ind}_{N_{F_1}}^{G_{F_1}}(R_1)$, let $R_2 = F_2 \otimes_F R$, let $S_2 = \text{Ind}_{H_{F_2}}^{G_{F_2}}(R_2) = F_2 \otimes_F \text{Ind}_{H_F}^{G_F}(R)$, and let $\Theta_i : F_0 \otimes_{F_i} S_i \rightarrow F_0[G]$ be the isomorphisms as explained in Hypothesis 2.10. Then $(S_1, S_2, F_0[G], \Theta_1, \Theta_2)$ defines a patching problem of differential G_F -torsors; and this has a solution (Z, Φ_1, Φ_2) that is unique up to isomorphism, by Theorem 2.2. Write $Z = \text{Spec}(S)$. Then Theorem 2.12 asserts that S is a Picard-Vessiot ring over F with differential Galois group G , and it is a proper solution to the given split differential embedding problem. \square

3. DIFFERENTIAL EMBEDDING PROBLEMS OVER COMPLEX FUNCTION FIELDS

Results about differential embedding problems, as considered in Section 2.3, were obtained in [Obe03] for the case $F = C(x)$ with $\partial = d/dx$ and C algebraically closed, by building on results of Kovacic ([Kov69], [Kov71]). In [Obe03], it was shown that there are proper solutions to certain types of differential embedding problems, including all those whose kernel is connected; but the general case has remained open.

As an application of Theorem 2.14, we show in this section that the assumption that H is connected can be dropped for $F = \mathbb{C}(x)$, and that more generally every differential embedding problem can be solved over the field of functions F of any compact Riemann surface (i.e., complex curve). Here we can take any non-trivial \mathbb{C} -linear derivation on F , or equivalently any derivation on F for which the constants are \mathbb{C} .

The next result, based on an idea that is often called the ‘‘Kovacic trick’’, is a more precise version of an assertion in [BHH16].

Proposition 3.1. *Let F be a differential field of characteristic zero and write $K = C_F$. Let L/F be a differential field extension that is finitely generated over F with $C_L = K$. Let L' be the algebraic closure of F in L and let L'' be the normal closure of L' in \bar{F} . Set $d = [L'' : F]$ and $m = \text{trdeg}(L/F) + 1$. Let G be a linear algebraic group defined over K and let R/F be a Picard-Vessiot ring with differential Galois group G^{2m+2d} . Then there is a subring R_0 of R such that R_0/F is a Picard-Vessiot ring with differential Galois group G , and such that $R_0 \otimes_F L$ is a Picard-Vessiot ring over L with differential Galois group G .*

Proof. This assertion was shown in the proof of Theorem 4.12 of [BHH16], though the statement of that result asserted a bit less. Namely, in the first step of that proof of that theorem, it was shown that with notation as above, R contains a subring R' which is a Picard-Vessiot ring having differential Galois group G^{2m} , with the additional properties that $F' \otimes_F L'$ is a field, where F' is the algebraic closure of F in $E := \text{Frac}(R')$; and that K is algebraically closed in $F' \otimes_F L'$. In the second step of that proof, it was shown that $E \otimes_F L$ is an integral domain and that K is algebraically closed in the fraction field \tilde{E} of that domain. In the third step, it was shown that R' contains a Picard-Vessiot ring R_0/F (called R_i there) with differential Galois group G , such that the fraction field \tilde{E}_0 of $\tilde{R}_0 := R_0 \otimes_F L \subseteq \tilde{E}$ is the compositum $E_0 L \subseteq \tilde{E}$, and such that $C_{\tilde{E}_0} = K$. Finally, in the conclusion of the proof, it was observed that \tilde{R}_0 is a Picard-Vessiot ring over L with differential Galois group G . \square

We also recall the following from [BHH16] (see the statement and proof of Lemma 4.2 there):

Lemma 3.2. *Let (F, ∂) be a differential field, let $a \in F^\times$, let $\partial' = a\partial$, and let K be the constants of (F, ∂) (or equivalently, of (F, ∂')). Let G be a linear algebraic group over K , let $A \in F^{n \times n}$, and let R be a Picard-Vessiot ring over (F, ∂) for the differential equation $\partial y = Ay$ with differential Galois group G . Then R is a Picard-Vessiot ring over (F, ∂') for the differential equation $\partial' y = aAy$ with differential Galois group G .*

In order to prove our main result, we consider fields of meromorphic functions on open subsets of Riemann surfaces, especially the Riemann sphere $\mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$, and we will apply our patching and embedding problem results using these fields. Consider the open disc $U \subset \mathbb{C}$ of radius $c > 0$ about the origin, and let F_1 be the field of meromorphic functions on U . Thus $\mathbb{C}(x) \subset F_1 \subset \mathbb{C}((x))$, and the standard derivation d/dx on $\mathbb{C}((x))$ restricts to the complex derivative d/dx on F_1 and to the standard derivation d/dx on $\mathbb{C}(x)$.

Lemma 3.3. *Let $c > 0$, and let ∂ be the derivation d/dx on the field F_1 of meromorphic functions on the open disc $U = \{x \in \mathbb{C} \mid |x| < c\}$. Let F be a differential subfield of F_1 with $C_F = \mathbb{C}$, let $A \in F^{n \times n}$, and let R/F be a Picard-Vessiot ring for the differential equation $\partial(y) = Ay$. If all the entries of A are holomorphic on U , then R embeds as a differential subring of F_1 .*

Proof. By [For77, Theorem 11.2], there exists a fundamental solution matrix Z for A with entries in F_1 (in fact holomorphic on U). Since $C_{F_1} = \mathbb{C}$, which equals the field of constants of F , the ring $F[Z, \det(Z)^{-1}] \subseteq F_1$ is a Picard-Vessiot ring over F for the differential equation $\partial(y) = Ay$ (notation as in Section 1.4). The assertion then follows from the uniqueness of Picard-Vessiot rings over algebraically closed fields of constants. \square

Lemma 3.4. *Let F be a one-variable function field over \mathbb{C} , or equivalently the field of meromorphic functions on a compact Riemann surface \mathcal{X} . Let O_1, O_2 be connected metric open subsets of \mathcal{X} such that $O_i \neq \mathcal{X}$, $O_1 \cup O_2 = \mathcal{X}$, and $O_0 := O_1 \cap O_2$ is connected. Let F_i be the field of meromorphic functions on O_i . Then*

- (a) *As subfields of F_0 , $F_1 \cap F_2 = F$, the field of meromorphic functions on \mathcal{X} .*
- (b) *For every $n \geq 1$, every element of $\mathrm{GL}_n(F_0)$ can be written as $A_2^{-1}A_1$ with $A_i \in \mathrm{GL}_n(F_i)$.*

Proof. Since being meromorphic is a local property, and since $O_1 \cup O_2 = \mathcal{X}$, the first assertion follows from the fact that every meromorphic function on \mathcal{X} is a rational function on \mathcal{X} . For the second assertion, let \mathcal{H} be the sheaf of holomorphic functions on \mathcal{X} in the complex metric topology. Then F_i is the fraction field of $R_i := \mathcal{H}(O_i)$, the ring of holomorphic functions on O_i . Let $A \in \mathrm{GL}_n(F_0)$.

Case 1: $A \in \mathrm{GL}_n(R_0)$.

Let M_i be a free R_i -module of rank n , for $i = 1, 2$, say with bases B_1, B_2 . Thus B_i is also an F_i -basis of the vector space $M_i \otimes_{R_i} F_i$. Consider the locally free \mathcal{H} -module \mathcal{M} on \mathcal{X} with $\mathcal{M}(O_i) = M_i$ for $i = 1, 2$, and with transition matrix $A \in \mathrm{GL}_n(R_0)$ on O_0 between B_1, B_2 (i.e., $B_1 = B_2A$). Since \mathcal{H} is coherent, so is \mathcal{M} , being locally free of finite rank. By [Ser56, Définition 2, Proposition 10, Théorème 3], there is an equivalence of categories $\mathcal{F} \mapsto \mathcal{F}^h$ from the coherent \mathcal{O} -modules on \mathcal{X} to coherent \mathcal{H} -modules on \mathcal{X} , satisfying $\mathcal{F}^h(U) = \mathcal{H}(U) \otimes_{\mathcal{O}(U)} \mathcal{F}(U)$ for every Zariski open subset $U \subseteq \mathcal{X}$. (Here \mathcal{O} is the sheaf of regular functions on

\mathcal{X} in the Zariski topology.) Thus $\mathcal{M} = \mathcal{F}^h$ for some coherent sheaf \mathcal{F} of \mathcal{O} -modules on \mathcal{X} . Moreover, since \mathcal{M} is locally free, so is \mathcal{F} (see [Ser56], bottom of page 31).

For $i = 1, 2$ choose a point $P_i \in \mathcal{X} \setminus O_i$, and let $U_i = \mathcal{X} \setminus \{P_i\}$. Also let $U_0 = U_1 \cap U_2 = \mathcal{X} \setminus \{P_1, P_2\}$. Since \mathcal{F} is locally free, there is a non-empty Zariski open subset $U \subset U_0$ such that $\mathcal{F}(U)$ is free of rank n over $\mathcal{O}(U)$, say with basis B . Each element of B has only finitely many poles on U_i , viewing $F \otimes_{\mathcal{O}(U)} \mathcal{F}(U) = F \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$. By Riemann-Roch, for $i = 1, 2$ there exists a non-zero element $f_i \in \mathcal{O}(U_i) \subset F$ such that the elements of $f_i B$ lie in $\mathcal{F}(U_i)$. Here $f_i B$ is an F -basis of $F \otimes_{\mathcal{O}(U_i)} \mathcal{F}(U_i)$. Since $O_i \subset U_i$, the set $f_i B$ is also an F_i -basis of $F_i \otimes_{\mathcal{H}(O_i)} \mathcal{M}(O_i)$. Let $C_i \in \mathrm{GL}_n(F_i)$ be the transition matrix between the bases B_i and $f_i B$ of $F_i \otimes_{\mathcal{H}(O_i)} \mathcal{M}(O_i)$; i.e., $B_i = (f_i B)C_i$. Thus $B_1 = f_1 B C_1 = f_1 (f_2^{-1} B_2 C_2^{-1}) C_1 \in \mathrm{GL}_n(F_0)$. Taking $A_1 = f_1 f_2^{-1} C_1 \in \mathrm{GL}_n(F_1)$ and $A_2 = C_2 \in \mathrm{GL}_n(F_2)$ yields $A_2^{-1} A_1 = B_2^{-1} B_1 = A$, completing the proof of Case 1.

Case 2: General case.

The entries of A are meromorphic functions on O_0 , as is $\det(A)$. So the set $\Sigma \subset O_0$ consisting of the poles of the entries of A and the zeroes of $\det(A)$ is a discrete subset of O_0 . Thus the limit points of Σ in \mathcal{X} lie in $\partial O_1 \cup \partial O_2$, where $\partial O_i = \bar{O}_i \setminus O_i$ is the boundary of O_i . Since O_i is open and $O_1 \cup O_2 = \mathcal{X}$, it follows that ∂O_1 and ∂O_2 are disjoint closed subsets of \mathcal{X} , with $\partial O_1 \subset O_2$ and $\partial O_2 \subset O_1$. Using the collar neighborhood theorem, we see that there are metric open neighborhoods $N_i \subset \mathcal{X}$ of ∂O_i (for $i = 1, 2$) such that the following properties hold:

- \bar{N}_1 is disjoint from \bar{N}_2 ;
- the open sets $\tilde{O}_i := O_i \setminus (O_i \cap \bar{N}_i)$ are connected for $i = 1, 2$;
- the three intersections $\tilde{O}_0 := \tilde{O}_1 \cap \tilde{O}_2$, $\tilde{O}_1 \cap O_2$, and $O_1 \cap \tilde{O}_2$ are each connected;
- $\tilde{O}_1 \cup \tilde{O}_2 = \mathcal{X}$; and
- $\tilde{\Sigma} := \Sigma \cap \tilde{O}_0$ is finite.

Let $O'_i := \tilde{O}_i \setminus \tilde{\Sigma}$ for $i = 0, 1$, and let $O'_2 := \tilde{O}_2$. Then O'_i is a connected open set for $i = 0, 1, 2$; $O'_1 \cap O'_2 = O'_0$; $O'_1 \cup O'_2 = \mathcal{X}$; and Σ is disjoint from O'_0 . Thus $A \in \mathrm{GL}_n(R'_0)$, where $R'_0 = \mathcal{H}(O'_0)$. Let $O^- = O'_1 \cap O_2$ and let $O^+ = O_1 \cap O'_2$; these are connected open subsets of \mathcal{X} . Write F'_i, F^\pm for the field of meromorphic functions on O'_i, O^\pm , respectively. By Case 1, there exist $A_i \in \mathrm{GL}_n(F'_i)$, for $i = 1, 2$, such that $A = A_2^{-1} A_1$ in $\mathrm{GL}_n(F'_0)$. It remains to show that $A_i \in \mathrm{GL}_n(F_i)$. Now $A \in \mathrm{GL}_n(F_0) \subset \mathrm{GL}_n(F^+)$; $A_1 \in \mathrm{GL}_n(F'_1)$; and $A_2 \in \mathrm{GL}_n(F'_2) \subset \mathrm{GL}_n(F^+)$. Since $A_1 = A_2 A \in \mathrm{GL}_n(F^+)$, and since $F'_1 \cap F^+ = F_1$, it follows that $A_1 \in \mathrm{GL}_n(F'_1 \cap F^+) = \mathrm{GL}_n(F_1)$. Similarly, $F'_2 \cap F^- = F_2$; so $A_2 = A_1 A^{-1}$ lies in $\mathrm{GL}_n(F'_2 \cap F^-) = \mathrm{GL}_n(F_2)$. \square

Proposition 3.5. *Let F be a one-variable function field over \mathbb{C} , and let ∂ be a derivation on F with constant field \mathbb{C} . Then every split differential embedding problem over (F, ∂) has a proper solution.*

Proof. A given split differential embedding problem consists of a semi-direct product $G = N \rtimes H$ of linear algebraic groups and a Picard-Vessiot ring R/F with differential Galois group H for some differential equation $\partial(y) = Ay$ over F .

Consider the smooth complex projective curve \mathcal{X} with function field F ; we may also view this as a compact Riemann surface. By taking a non-constant rational function on \mathcal{X} , we obtain a finite morphism $\phi : \mathcal{X} \rightarrow \mathbb{P}_{\mathbb{C}}^1$, corresponding to a branched cover of Riemann

surfaces, and also corresponding to an inclusion $\iota : \mathbb{C}(x) \hookrightarrow F$ of function fields. If we write ∂' for the derivation on F induced via ι from the derivation d/dx on $\mathbb{C}(x)$, then $\partial = g(x)\partial'$, where $g(x) := \partial(\iota(x)) \in F$ is non-zero (and where we use that the space of derivations of F over K is one dimensional). So by Lemma 3.2, in order to prove the result we may assume that ∂ is the derivation on F that extends the derivation d/dx on $\mathbb{C}(x)$.

Away from a finite subset of \mathcal{X} , ϕ is unramified and the entries of A are holomorphic. After a translation $x \mapsto x + c$, we may assume that the fiber over the point $(x = 0)$ on $\mathbb{P}_{\mathbb{C}}^1$ does not contain any point in that finite set. So there is an open disc D around $0 \in \mathbb{P}_{\mathbb{C}}^1$ such that $\phi^{-1}(D)$ also does not meet that finite set; and then $\phi^{-1}(D) \subset \mathcal{X}$, being unramified over D , consists of finitely many disjoint copies of D . Call one of those copies \hat{O} , and let P be the unique point in $\hat{O} \cap \phi^{-1}(0)$. The map ϕ then defines a differential isomorphism between the fields of meromorphic functions on \hat{O} and on D , which we identify and call \hat{F} . Applying Lemma 3.3 to the inclusion of differential fields $F \subset \hat{F}$, we obtain an inclusion $R \subseteq \hat{F}$.

Choose a non-constant regular function on the complex affine curve $\mathcal{X} \setminus \{P\}$; this defines a finite morphism $\pi : \mathcal{X} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ such that P is the unique point mapping to $\infty \in \mathbb{P}_{\mathbb{C}}^1$. So there is an open disc $D' \subset \mathbb{P}_{\mathbb{C}}^1$ centered about ∞ , whose closure contains no branch point of π other than ∞ , such that $O_1 := \pi^{-1}(D')$ is homeomorphic to an open disc, and such that the closure of O_1 is contained in \hat{O} . Since O_1 is contained in \hat{O} , the field of meromorphic functions F_1 on O_1 contains \hat{F} ; and thus $R \subseteq F_1$. Also, the map π defines an inclusion $j : \mathbb{C}(x) \hookrightarrow F$ (not the same as the above inclusion ι defined by ϕ). With respect to this inclusion, we may view F as a finite extension of $\mathbb{C}(x)$; let d be the degree of its Galois closure over $\mathbb{C}(x)$.

It is known that every linear algebraic group over \mathbb{C} is a differential Galois group over $\mathbb{C}(x)$ ([TT79]; see also [vdPS03, Theorem 5.12]). So there is a matrix A' over $\mathbb{C}(x)$, and a Picard-Vessiot ring \tilde{R}' over $(\mathbb{C}(x), d/dx)$ for the differential equation $\partial y = A'y$, with differential Galois group N^{2d} . Here A' is holomorphic on some open disc in $\mathbb{A}_{\mathbb{C}}^1$. A transformation of the form $x \mapsto ax + b$ (with $a \in \mathbb{C}^\times, b \in \mathbb{C}$) takes this open disc to an open disc D'' that is centered at the origin and contains $\mathbb{P}_{\mathbb{C}}^1 \setminus D'$. Thus $D' \cup D'' = \mathbb{P}_{\mathbb{C}}^1$. Note that $D' \cap D''$ is homeomorphic to an annulus, as is its inverse image O_0 under π . Also, $\mathbb{P}_{\mathbb{C}}^1 \setminus D''$ is homeomorphic to a closed disc, as is its inverse image under π . Hence $O_2 := \pi^{-1}(D'')$, which is the complement in \mathcal{X} of this inverse image, is connected.

The above transformation $x \mapsto ax + b$ carries d/dx to $a \cdot d/dx$; carries A' to a matrix A'' whose entries are holomorphic on D'' ; and carries \tilde{R}' to a Picard-Vessiot ring \tilde{R}'' over $(\mathbb{C}(x), a \cdot d/dx)$ with differential Galois group N^{2d} . By Lemma 3.2, \tilde{R}'' is also a Picard-Vessiot ring over $(\mathbb{C}(x), d/dx)$ for the differential equation $dy/dx = a^{-1}A''y$, with differential Galois group N^{2d} . Since the entries of $a^{-1}A''$ are also holomorphic on the open disc D'' , it follows from Lemma 3.3 that \tilde{R}'' is contained in the field of meromorphic functions on D'' . By Proposition 3.1, there is a subring \tilde{R}_0 of \tilde{R}'' such that \tilde{R}_0 is a Picard-Vessiot ring over $(\mathbb{C}(x), d/dx)$ with differential Galois group N , and such that $R' := \tilde{R}_0 \otimes_{\mathbb{C}(x)} F$ is a Picard-Vessiot ring over (F, ∂) with differential Galois group N . Since \tilde{R}'' is contained in the field F'' of meromorphic functions on D'' , so is its subring \tilde{R}_0 . Tensoring this latter containment of $\mathbb{C}(x)$ -algebras with F (which is flat over $\mathbb{C}(x)$), we obtain an inclusion $R' \subseteq F'' \otimes_{\mathbb{C}(x)} F = F_2$, where F_2 is the field of meromorphic functions on $O_2 = \pi^{-1}(D'')$.

Let F_0 denote the field of meromorphic functions on $O_0 := O_1 \cap O_2$. By Lemma 3.4, (F, F_1, F_2, F_0) is a differential diamond with the factorization property. Since $R' \subseteq F_2$ and $C_F = C_{F_0} = \mathbb{C}$, we may apply Lemma 2.9 and obtain that the compositum $R_1 = F_1 R' \subseteq F_0$ is a Picard-Vessiot ring over F_1 with differential Galois group N . Recall that $R \subseteq F_1$. We conclude that by Theorem 2.14, the differential embedding problem given by $G = N \rtimes H$ together with R has a proper solution. \square

Proposition 3.6. *Let F be a one-variable function field over an algebraically closed field K of characteristic zero, and let ∂ be a derivation on F with constant field K . Suppose that every split differential embedding problem over F has a proper solution. Then the same holds for **every** differential embedding problem over F .*

Proof. Consider a differential embedding problem given by an exact sequence of linear algebraic K -groups $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ and a Picard-Vessiot ring R for H over F . As in Proposition 1.12, $Z := \text{Spec}(R)$ is a simple differential H -torsor over F , corresponding to an element $\alpha \in H^1(F, H)$ under the functorial bijection between torsors and cohomology classes (e.g. see [Ser97, I.5.2 and III.1.1]). Since K is algebraically closed, it follows from [Ser97, II.3.3.Ex. 3 and II.3.1.Proposition 6(a)] that F is a field of dimension one (or equivalently, of cohomological dimension one, since $\text{char}(F) = 0$ [Ser97, II.3.1]). Hence F satisfies the hypotheses of [Ser97, III.2.4, Theorem 3]. Thus Corollary 2 of that theorem applies; i.e., the surjection $G \rightarrow H$ induces a surjection $H^1(F, G) \rightarrow H^1(F, H)$.

Let $\tilde{\alpha} \in H^1(F, G)$ be an element that maps to α , and let X be the corresponding G -torsor over F . We thus have a morphism of G -spaces $X \rightarrow Z$ that is constant on N -orbits (since G acts on Z through $H = G/N$). By the universal mapping property of the quotient torsor (see Proposition A.6 and the discussion just before), this morphism factors through X/N . The resulting map $\iota : X/N \rightarrow Z$ is an H -torsor morphism, which is automatically an isomorphism. The isomorphism ι defines a differential structure on X/N . By Proposition 1.8(b), there is a differential structure on X with respect to which the quotient morphism $\psi : X \rightarrow X/N$ is differential. Let $\pi : X \rightarrow Z$ be the composition $\iota \circ \psi$, which is then also differential.

By Proposition 1.15, there is a closed subgroup J of G such that $X = \text{Ind}_J^G Y$ for some simple differential J -torsor Y such that the fraction field of the coordinate ring $S = F[Y]$ has no new constants. Thus S is a Picard-Vessiot ring for J over F , by Proposition 1.12(b). By the defining property of induced torsors, we have a J -equivariant inclusion $Y \hookrightarrow X$. Applying π gives a $J/(N \cap J)$ -equivariant inclusion $Y/(N \cap J) \rightarrow X/N \xrightarrow{\sim} Z$. Here $Y/(N \cap J)$ is a torsor under $J/(N \cap J) = JN/N$, and Z is a torsor under $H = G/N$, and the above injection is equivariant with respect to the natural inclusion $JN/N \subseteq G/N$. (Here JN is the closed subgroup of G generated by J and N .) Also, $Z \cong X/N$ is of the form $(\text{Ind}_J^G Y)/N = \text{Ind}_{J/(N \cap J)}^{G/N}(Y/(N \cap J)) = \text{Ind}_{JN/N}^{G/N}(Y/(N \cap J))$, using Proposition A.13. But $Z = \text{Spec}(R)$ is a simple differential H -torsor with no new constants, since R is a Picard-Vessiot ring for H . So Lemma 1.16 implies that $JN/N = G/N$; i.e., $JN = G$ and the restriction of $G \rightarrow G/N = H$ to J is surjective. So the above JN/N -equivariant inclusion $Y/(N \cap J) \rightarrow Z$ is actually an inclusion of G/N -torsors, and it is therefore an isomorphism. Hence the composition $Y \rightarrow Y/(N \cap J) \xrightarrow{\sim} Z$ is surjective; i.e., Z is a quotient of Y as a differential torsor.

Let \tilde{G} be the semi-direct product $N \rtimes J$, where J acts on N via conjugation (as subgroups of G). Let M be the kernel of the surjection $\phi : \tilde{G} \rightarrow G$ given by $(n, j) \mapsto nj$. Let $\bar{\phi} : J \rightarrow H$ be the composition $J \hookrightarrow G \rightarrow G/N = H$. We then have the following commutative diagram of groups with exact rows and columns, and where the middle row is split exact:

$$\begin{array}{ccccccc}
& & & 1 & & 1 & \\
& & & \downarrow & & \downarrow & \\
& & & 1 & \longrightarrow & M & \longrightarrow & N \cap J & \longrightarrow & 1 \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & N & \longrightarrow & \tilde{G} & \longrightarrow & J & \longrightarrow & 1 \\
& & \downarrow = & & \downarrow \phi & & \downarrow \bar{\phi} & & \\
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 1 & & 1 & & 1 & &
\end{array}$$

Here the map $M \rightarrow N \cap J$ is an isomorphism, given by $(a^{-1}, a) \mapsto a$ for $a \in N \cap J$.

Consider the split differential embedding problem given by the middle row and the Picard-Vessiot ring S for J . By the hypothesis of the proposition, this has a proper solution, given by a Picard-Vessiot ring \tilde{R} for \tilde{G} that contains S as a differential subring. Its spectrum is a differential \tilde{G} -torsor \tilde{W} such that $\tilde{W}/N \xrightarrow{\sim} Y$ as differential J -torsors. Since $Y/(N \cap J) \rightarrow X/N \xrightarrow{\sim} Z$, and since the above diagram commutes, the differential G -torsor $W := \tilde{W}/M$ satisfies $W/N \xrightarrow{\sim} Z$. By the Galois correspondence ([Dyc08, Thm. 4.4] or the observation after Proposition 1.17), the coordinate ring \tilde{R}^M of W is a Picard-Vessiot ring for $G = \tilde{G}/M$ over F , and $(\tilde{R}^M)^N \xrightarrow{\sim} R$. So \tilde{R}^M is a proper solution to the given differential embedding problem. \square

Combining Proposition 3.5 with Proposition 3.6, we obtain:

Theorem 3.7. *Let F be any one-variable complex function field, together with a non-trivial \mathbb{C} -linear derivation ∂ . Then every differential embedding problem over (F, ∂) has a proper solution.*

Example 3.8. Let Γ be a lattice and let F be the field of elliptic functions with respect to Γ . Then every differential embedding problem over F has a proper solution.

Proof. Recall that F is generated over \mathbb{C} by the Weierstraß function \wp and its derivative \wp' which satisfy the equation $(\wp')^2 = 4\wp^3 - g_2(\Gamma)\wp - g_3(\Gamma)$. Therefore, F is a one-variable function field over \mathbb{C} and it has a non-trivial derivation, so the claim follows from Theorem 3.7. \square

APPENDIX A. TORSORS, QUOTIENTS, AND INDUCTION

This appendix contains basic results about torsors, group actions, and quotients, which are used in the body of the paper in a differential context. Many of these results are known, but for lack of a good reference we include them here. Throughout this appendix, K is an arbitrary field, and G is an affine group scheme of finite type over K .

A.1. G -spaces. An *affine G -space over K* is an affine scheme X of finite type over K together with a morphism $\alpha: X \times_K G \rightarrow X$ such that $\alpha_R: X(R) \times G(R) \rightarrow X(R)$ defines a right group action of $G(R)$ on $X(R)$ for every K -algebra R . We will usually write $x.g$ for $\alpha_R(x, g)$ when $x \in X(R)$ and $g \in G(R)$. A *morphism of G -spaces* is a G -equivariant morphism of schemes. The G -space structure X corresponds to a *co-action* homomorphism $\rho: K[X] \rightarrow K[X] \otimes_K K[G]$ (where $K[X]$ is the affine coordinate ring of X , and similarly for $K[G]$).

In addition to this co-action ρ , there is also a *functorial left group action* of G on the coordinate ring $K[X]$. That is, for every K -algebra R , there is a left action

$$(6) \quad G(R) \times (K[X] \otimes_K R) \rightarrow K[X] \otimes_K R, (h, f) \mapsto h(f)$$

of the group $G(R)$ on $K[X] \otimes_K R$, and these are compatible. To define the action in (6), first note that we can identify $K[X] \otimes_K R$ with the morphisms from X_R to \mathbb{A}_R^1 . Namely, an element $f \in K[X] \otimes_K R$ is interpreted as a morphism $f: X_R \rightarrow \mathbb{A}_R^1$ by sending an element $x \in X_R(S) = \text{Hom}_R(K[X] \otimes_K R, S)$ to $x(f)$ for every R -algebra S . So for $g \in G(R)$ and f as above, we can define $g(f)$ by

$$g(f)(x) = f(x.g)$$

for every R -algebra S and every $x \in X_R(S)$. Alternatively, $g(f)$ can be described as the image of $f \in K[X] \otimes_K R$ under the composition

$$(7) \quad K[X] \otimes_K R \xrightarrow{\rho \otimes \text{id}_R} K[X] \otimes_K K[G] \otimes_K R \rightarrow K[X] \otimes_K R,$$

where the last map is given by $a \otimes b \otimes r \mapsto a \otimes g(b)r$ for $a \in K[X]$, $b \in K[G]$, $r \in R$ and $g \in G(R) = \text{Hom}_K(K[G], R)$, and where $g(b)$ denotes the image of b under $g: K[G] \rightarrow R$.

Given an affine G -space X and a closed subgroup H of G , we may consider X as an H -space. We then obtain a functorial left action of H on $K[X]$; this is the restriction of the above functorial action of G . An element $f \in K[X]$ is *H -invariant* if $h(f \otimes 1) = f \otimes 1$ under the left action of $H(R)$ on $K[X] \otimes_K R$, for every K -algebra R and $h \in H(R)$. These H -invariant elements form a subring denoted by $K[X]^H$, the *(functorial) invariants* in $K[X]$.

Observe that $f \in K[X]$ is invariant under H if and only if $\rho_H(f) = f \otimes 1$, where $\rho_H: K[X] \rightarrow K[X] \otimes_K K[H]$ is the co-action corresponding to the action of H on X . The co-action maps for G and H are related by the identity $\rho_H = \pi \circ \rho$, where $\pi: K[G] \rightarrow K[H]$ is the homomorphism corresponding to the inclusion of H into G . This identity shows that the map in the display (7) above agrees with the corresponding map with G replaced by H and ρ by ρ_H , if the element $g \in G(R)$ that is used in (7) lies in $H(R)$.

Lemma A.1. *For an ideal $J \trianglelefteq K[X]$ the following conditions are equivalent:*

- (a) $H(R)(J \otimes_K R) \subseteq J \otimes_K R$ for all K -algebras R .
- (b) $Z = \text{Spec}(K[X]/J)$ is H -stable; i.e., $Z(R).H(R) \subseteq Z(R)$ for all K -algebras R .
- (c) $\rho_H(J) \subseteq J \otimes_K K[H]$.

Proof. To see (a) \Rightarrow (b), let R be a K -algebra and let $z \in Z(R)$, $h \in H(R)$ and $f \in J \otimes_K R$. Since

$$Z(R) = \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in J \otimes_K R\}$$

and $h(f) \in J \otimes_K R$, we have $f(z.h) = h(f)(z) = 0$. Therefore $z.h \in Z(R)$.

To prove (b) \Rightarrow (a), note that

$$J \otimes_K R = \{f \in K[X] \otimes_K R \mid f(z) = 0 \text{ for all } R\text{-algebras } S \text{ and all } z \in Z(S)\}$$

If $f \in J \otimes_K R$, $h \in H(R)$, $z \in Z(S)$, then $h(f)(z) = f(z.h) = 0$ because $z.h \in Z(S)$. Therefore $h(f) \in J \otimes_K R$.

In order to prove (a) \Rightarrow (c), consider display (7) with G replaced by H , taking $R = K[H]$, and letting g be the element $h \in H(R)$ that corresponds to the identity map in $\text{Hom}_K(K[H], R)$. Taking $j \in J$ and using the display in that situation, we see that $h(j \otimes 1) = \rho_H(j)$. But part (a) says in particular that $h(j \otimes 1) \in J \otimes_K R$. So (c) follows.

Finally, (c) \Rightarrow (a) follows from the description of the H -action in (7) with G replaced by H . \square

An ideal $J \trianglelefteq K[X]$ satisfying the equivalent conditions of Lemma A.1 is called H -stable. We will frequently use that every ideal generated by elements in $K[X]^H$ is H -stable.

If H_1, \dots, H_m are closed subgroups of G , the smallest closed subgroup of G that contains all H_i 's is called the closed subgroup *generated* by the H_i 's.

Lemma A.2. *Consider $X = G$ as a G -space via the action given by the formula $x.g = g^{-1}x$. Let H be the closed subgroup of G generated by the closed subgroups H_1, \dots, H_m of G . Then the following holds:*

- (a) $\bigcap_{i=1}^m K[G]^{H_i} = K[G]^H$ and $K[G]^G = K$.
- (b) Every ideal in $K[G]$ that is H_i -stable for all $i = 1, \dots, m$ is H -stable.
- (c) The only proper G -stable ideal in $K[G]$ is the zero ideal.

Proof. For part (a), observe that $K[G]^H \subseteq \bigcap_{i=1}^m K[G]^{H_i}$. For any K -algebra R write

$$H'(R) := \left\{ h \in G(R) \mid h(f \otimes 1) = f \otimes 1 \text{ for all } f \in \bigcap_{i=1}^m K[G]^{H_i} \right\}.$$

We claim that H' is a closed subgroup of H . To see this it suffices to show that the condition $h(f \otimes 1) = f \otimes 1$ is a closed condition for any non-zero $f \in K[G]$: Let (f_i) be a K -basis of $K[X] = K[G]$ such that $f_0 = f$ and write $\rho(f) = \sum f_i \otimes g_i \in K[X] \otimes_K K[G]$. Then $h(f \otimes 1) = \sum f_i \otimes h(g_i)$ for $h \in G(R) = \text{Hom}_K(K[G], R)$. Therefore, $h(f \otimes 1) = f \otimes 1$ if and only if $h(g_0) = 1$ and $h(g_i) = 0$ for $i \neq 0$. Thus H' is the closed subscheme of G defined by $g_0 - 1$ and g_i for $i \neq 0$, proving the claim.

Because H' is a closed subgroup of G containing H_1, \dots, H_m we must have $H \subseteq H'$; i.e., $\bigcap_{i=1}^m K[G]^{H_i} \subseteq K[G]^H$. This proves the first part of (a). For the second part of (a), if $f \in K[G]^G$, then $f(g) = f(1.g^{-1}) = g^{-1}(f)(1) = f(1) \in K$ for all $g \in G(R)$. So $f \in K$.

To prove (b), let $J \trianglelefteq K[G]$ be an ideal, and for any K -algebra R write

$$Z(R) := \{h \in G(R) \mid h(J \otimes_K R) \subseteq J \otimes_K R\}.$$

We claim that Z is closed in G , and hence so is $Z \cap Z^{-1} \subseteq G$. But

$$(Z \cap Z^{-1})(R) = \{h \in G(R) \mid h(J \otimes_K R) = J \otimes_K R\}$$

for any K -algebra R ; so $Z \cap Z^{-1}$ is a subgroup of G . Thus if J is H_i -stable for all i then J is also H -stable, by definition of H ; and (b) would then follow.

To prove the claim, let $(f_i)_{i \in \mathcal{I}}$ be a K -basis of $K[G]$ such that $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ is the disjoint union of \mathcal{I}_1 and \mathcal{I}_2 and $(f_i)_{i \in \mathcal{I}_1}$ is a basis of J . Let $f \in J$ and write $\rho(f) = \sum f_i \otimes g_i \in$

$K[G] \otimes_K K[G]$. Then $h(f \otimes 1) = \sum f_i \otimes h(g_i)$ for $h \in G(R) = \text{Hom}_K(K[G], K)$. Therefore $h(f \otimes 1) \in J \otimes_K R$ if and only if $h(g_i) = 0$ for $i \in \mathcal{I}_2$. This proves the claim, and part (b).

Part (c) is clear from Lemma A.1(b), since G is the only G -stable closed subscheme of G . \square

Lemma A.3. *Let N be a normal closed subgroup of G . We consider two right actions of N on G :*

- (i) $G \times N \rightarrow G$ given by the formula $(g, n) \mapsto g \cdot n$
- (ii) $G \times N \rightarrow G$ given by the formula $(g, n) \mapsto n^{-1} \cdot g$.

Let $\rho_1: K[G] \rightarrow K[G] \otimes K[N]$ and $\rho_2: K[G] \rightarrow K[G] \otimes K[N]$ denote the homomorphisms corresponding to (i) and (ii). Then the following hold.

- (a) The two actions give rise to the same ring of invariants $K[G]^N \subseteq K[G]$:

$$\{f \in K[G] \mid \rho_1(f) = f \otimes 1\} = \{f \in K[G] \mid \rho_2(f) = f \otimes 1\}.$$

- (b) Let $J \trianglelefteq K[G]$ be an ideal. Then J is N -stable with respect to the action (i) if and only if J is N -stable with respect to the action (ii), that is, $\rho_1(J) \subseteq J \otimes K[N]$ if and only if $\rho_2(J) \subseteq J \otimes K[N]$.

Proof. To prove (a) we have to show that for $f \in K[G]$ we have $f(gn) = f(g)$ for all $n \in N(R)$ and $g \in G(S)$ for all K -algebras R and R -algebras S if and only if $f(n^{-1}g) = f(g)$ for all $n \in N(R)$ and $g \in G(S)$ for all K -algebras R and R -algebras S .

First assume that $f(gn) = f(g)$. Since $g^{-1}n^{-1}g \in N(S)$ we find $f(n^{-1}g) = f(gg^{-1}n^{-1}g) = f(g)$. Conversely, if $f(n^{-1}g) = f(g)$, then $f(gn) = f(gng^{-1}g) = f(g)$ because $gng^{-1} \in N(S)$.

To prove (b), according to Lemma A.1, we have to show that for a closed subscheme Z of G we have $Z(R)N(R) \subseteq Z(R)$ for all K -algebras R if and only if $N(R)Z(R) \subseteq Z(R)$ for all K -algebras R .

First assume that $Z(R)N(R) \subseteq Z(R)$. For $z \in Z(R)$ and $n \in N(R)$ the element $z^{-1}nz$ belongs to $N(R)$ and therefore $nz = zz^{-1}nz \in Z(R)$.

Conversely, assume that $N(R)Z(R) \subseteq Z(R)$, then $zn = znz^{-1}z \in Z(R)$ because $znz^{-1} \in N(R)$. \square

A.2. Quotients. In order to discuss the quotient of a G -space X by the action of G , we recall some basic facts about sheaves in the faithfully flat topology. Let K be a field and let \mathcal{F} be a functor from the category of K -algebras to the category of sets. Then \mathcal{F} is a *sheaf* if $\mathcal{F}(A \times B) = \mathcal{F}(A) \times \mathcal{F}(B)$ for every pair of K -algebras A, B , and if for every faithfully flat homomorphism $R \rightarrow S$ of K -algebras the sequence $\mathcal{F}(R) \rightarrow \mathcal{F}(S) \rightrightarrows \mathcal{F}(S \otimes_R S)$ is exact (i.e., an equalizer of sets). *Morphisms of sheaves* are defined as morphisms of functors. Every quasi-projective scheme over K (in particular, every affine K -scheme) is a sheaf.

If \mathcal{F} is a functor from the category of K -algebras to the category of sets, then there exists a sheaf $\tilde{\mathcal{F}}$ and a morphism $\iota: \mathcal{F} \rightarrow \tilde{\mathcal{F}}$ that is universal among morphisms from \mathcal{F} to sheaves; i.e., for every morphism $\phi: \mathcal{F} \rightarrow \mathcal{F}'$ from \mathcal{F} to a sheaf \mathcal{F}' , there exists a unique morphism $\tilde{\phi}: \tilde{\mathcal{F}} \rightarrow \mathcal{F}'$ such that $\phi = \tilde{\phi} \circ \iota$. See for example [DG70, III, §1, Theorem 1.8] for a proof. The sheaf $\tilde{\mathcal{F}}$ is called the *sheafification of \mathcal{F}* . If $\phi: \mathcal{F} \rightarrow \mathcal{F}'$ is a morphism of functors, then the universal property of $\tilde{\mathcal{F}}$ applied to the morphism $\mathcal{F} \rightarrow \mathcal{F}' \rightarrow \tilde{\mathcal{F}}'$ yields a morphism $\tilde{\phi}: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}'$ that is called the *sheafification of ϕ* .

Let G be an affine group scheme of finite type over K and let X be an affine G -space over K . We view X as a sheaf. A morphism $\pi: X \rightarrow Y$ of sheaves is *constant on G -orbits* if the

following diagram commutes:

$$\begin{array}{ccc} X \times G & \xrightarrow{\alpha} & X \\ & \searrow \pi \circ \text{pr}_1 & \swarrow \pi \\ & & Y \end{array}$$

We write $X//G$ for the functor of G -orbits on X ; i.e., $(X//G)(R) = X(R)/G(R)$ for every K -algebra R . Let $\text{pr}: X \rightarrow X//G$ be the canonical projection. Set $Y = \widetilde{X//G}$ and let $\pi = \widetilde{\text{pr}}: X \rightarrow Y$ be the sheafification of pr . It is easy to see that Y together with $\pi: X \rightarrow Y$ is the *universal quotient in the category of sheaves*, that is, π is constant on G -orbits and for every other morphism $\phi: X \rightarrow Y'$ of sheaves that is constant on G -orbits, there exists a unique morphism $\psi: Y \rightarrow Y'$ such that $\psi \circ \pi = \phi$. In the situation where $\widetilde{X//G}$ is represented by a quasi-projective F -scheme, we will write X/G for that scheme. A case of particular interest is where X/G is an affine scheme; in that situation we have the following result:

Proposition A.4. *Let G be an affine group scheme of finite type over K and let X be an affine G -space over K . If the quotient*

$$Y = \widetilde{X//G}$$

is an affine scheme, then $K[Y] \cong K[X]^G$, and the projection $\pi: X \rightarrow Y$ is given by the inclusion $K[Y] \cong K[X]^G \subseteq K[X]$.

Proof. Let $\rho: K[X] \rightarrow K[X] \otimes_K K[G]$ be the homomorphism corresponding to the action $\alpha: X \times G \rightarrow X$; i.e., $\alpha^* = \rho$.

Set $Z = \text{Spec}(K[X]^G)$ and let $\pi_Z: X \rightarrow Z$ be the morphism defined by the inclusion $\pi_Z^*: K[X]^G \hookrightarrow K[X]$. By definition of $K[X]^G$, the composition $\rho \circ \pi_Z^*$ maps f to $f \otimes 1$ for every $f \in K[X]^G$. Hence $\pi_Z \circ \alpha: X \times G \rightarrow Z$ equals π_Z applied to the first factor, and π_Z is thus constant on G -orbits. It is easy to check that Z is the universal quotient of X by G in the category of affine schemes.

By assumption, Y is an affine scheme and it is the universal quotient in the category of sheaves, therefore it is also the universal quotient in the category of affine schemes. We conclude $Y \cong Z$ and thus $K[Y] \cong K[Z] = K[X]^G$. \square

Remark A.5. For every affine G -space X , there exists a universal quotient of $X \bmod G$ in the category of affine schemes, namely $\text{Spec}(K[X]^G)$. However, this does not imply that this is a quotient with “good properties”, since the category of affine schemes is in general too small for “good quotients”. For example the multiplicative group $G = \mathbb{G}_m$ acts on the affine line $X = \mathbb{A}^1$, but $K[X]^G = K$, so the affine quotient $\text{Spec}(K[X]^G)$ is trivial and does not contain any information on the G -orbits.

A non-empty affine G -space X is called a *G -torsor* if for every K -algebra R and all $x, y \in X(R)$ there exists a unique $g \in G(R)$ with $y = x.g$. Equivalently, the condition is that the morphism $(\alpha, \text{pr}_1): X \times G \rightarrow X \times X$ is an isomorphism, where pr_1 is the projection onto the first component. A torsor is called *trivial* if it is isomorphic to the G -torsor G (where the G -action on G is defined by right multiplication). A torsor X is trivial if and only if $X(K) \neq \emptyset$. Over an algebraically closed field, every torsor is trivial; and so for general K , there is a finite extension L/K such that the base change X_L is trivial as a G_L -torsor.

The next proposition assures that certain quotients of torsors are well-behaved. We first recall the basics of flat descent theory for schemes, in the special case of field extensions. Let L/K be a field extension. Then *descent data* for an L -scheme Z consists of an isomorphism $\phi : Z \times_K L \rightarrow L \times_K Z$ such that $\phi_2 = \phi_1 \circ \phi_3$, where $\phi_1 = \text{id}_L \times \phi : L \times_K Z \times_K L \rightarrow L \times_K L \times_K Z$ and similarly for $\phi_2 : Z \times_K L \times_K L \rightarrow L \times_K L \times_K Z$ and $\phi_3 : Z \times_K L \times_K L \rightarrow L \times_K Z \times_K L$ (where id_L is inserted into the second or third factor respectively). A K -scheme Y induces an L -scheme $Z := Y \times_K L$ together with descent data; and conversely an L -scheme Z together with descent data is induced by a K -scheme Y that is unique up to isomorphism. See [Gro71, VIII] and [Mil80, I.2] for more details. Similarly, a sheaf over K induces a sheaf over L together with descent data in the analogous sense. Concerning uniqueness, the sheaf axiom implies that a sheaf \mathcal{F} over K is determined by its base change to L together with the induced descent data, since for any K -algebra A , the set $\mathcal{F}(A)$ is the equalizer of $\mathcal{F}(A \otimes_K L) \rightrightarrows \mathcal{F}(A \otimes_K L \otimes_K L)$ and similarly for morphisms. See also [DG70, III, Section 3.4, Proposition 6.2].

Proposition A.6. *Let X be a G -torsor for an affine group scheme G of finite type over K .*

- (a) *If H is a closed subgroup of G , then X/H exists; i.e., $\widetilde{X//H}$ is represented by a quasi-projective scheme.*
- (b) *Let $N \trianglelefteq G$ be a closed normal subgroup. Then X/N is isomorphic to the affine scheme $\text{Spec}(K[X]^N)$, and it is a (G/N) -torsor. In addition, the homomorphism $\rho_X : K[X] \rightarrow K[X] \otimes K[G]$ corresponding to the G -action on X restricts to a homomorphism $\rho_Y : K[X]^N \rightarrow K[X]^N \otimes K[G]^N$ and this homomorphism corresponds to the (G/N) -action on Y . Furthermore, $K[X]$ is faithfully flat over $K[X]^N$.*

Proof. For part (a), the sheaf $\widetilde{G//H}$ is represented by a K -scheme (see [DG70, III, Section 3.5, Théorème 5.4]), and this scheme is quasi-projective by [Ray70, Corollaire VI.2.6]. That is, the quotient G/H exists. The G -torsor X becomes trivial over some finite field extension L/K , and this induces an isomorphism of sheaves between $(\widetilde{X//H})_L$ and $(G/H)_L$. The extension of constants from K to L induces descent data for the sheaf $(\widetilde{X//H})_L$. Since $(\widetilde{X//H})_L$ is isomorphic to a quasi-projective L -scheme (viz. $(G/H)_L$), it follows from faithfully flat descent for schemes ([Gro71, VIII, Corollaire 7.7]) that there is a quasi-projective K -scheme Y that induces this L -scheme together with the above descent data. But Y and $\widetilde{X//H}$ are then both sheaves that induce Y_L together with the above descent data. So by the uniqueness assertion before the statement of the proposition, it follows that Y and $\widetilde{X//H}$ are isomorphic as sheaves. That is, $Y = X/H$. This proves part (a).

Next we turn to part (b). Since N is normal, the quotient G/N is an affine K -group scheme of finite type (see [DG70, III, Section 3.5, Théorème 5.6]); hence so is $(G/N)_L$. Thus in the argument above (with $H = N$), the descended scheme $Y = X/N$ is also affine, by [Gro71, VIII, Théorème 2.1] (or [Mil80, I.2, Theorem 2.23]).

For every K -algebra R , the image of an element $x \in X(R)$ in $X(R)/N(R)$ will be denoted by $[x]$, and the image of $g \in G(R)$ in $G(R)/N(R)$ will be denoted by \bar{g} . Thus

$$X(R)/N(R) \times G(R)/N(R) \rightarrow X(R)/N(R), ([x], \bar{g}) \mapsto [x] \cdot \bar{g} := [x \cdot g]$$

is a well-defined group action for every K -algebra R and it defines a morphism of functors $(X//N) \times (G//N) \rightarrow (X//N)$. The sheafification $X/N \times (G/N) \rightarrow X/N$ of this morphism

defines a group action of G/N on X/N . For every K -algebra R , and for all $[x], [y] \in (X//N)(R)$, there exists a unique $\bar{g} \in (G//N)(R)$ with $[x] = [y].\bar{g}$. Therefore,

$$(X//N)(R) \times (G//N)(R) \rightarrow (X//N)(R) \times (X//N)(R), ([x], \bar{g}) \mapsto ([x], [x].\bar{g})$$

is bijective for every K -algebra R . Hence the sheafification $X/N \times (G/N) \rightarrow X/N \times X/N$ is an isomorphism, and we conclude that X/N is a (G/N) -torsor.

Proposition A.4 implies that $K[X/N] = K[X]^N$ (i.e., $X/N \cong \text{Spec}(K[X]^N)$) and that the quotient map $\pi: X \rightarrow X/N$ is induced from the inclusion $K[X]^N \subseteq K[X]$. Similarly, the quotient map $\pi_G: G \rightarrow G/N$ is induced from the inclusion $K[G]^N \subseteq K[G]$. By construction of the (G/N) -action on X/N , the following diagram commutes:

$$\begin{array}{ccc} X \times G & \longrightarrow & X \\ \downarrow \pi \times \pi_G & & \downarrow \pi \\ (X/N) \times (G/N) & \longrightarrow & X/N \end{array}$$

and we conclude that the homomorphism $\rho_X: K[X] \rightarrow K[X] \otimes K[G]$ corresponding to the G -action on X restricts to a homomorphism $\rho_{X/N}: K[X]^N \rightarrow K[X]^N \otimes K[G]^N$ corresponding to the (G/N) -action on X/N .

Finally, we show that $K[X]$ is faithfully flat over $K[X]^N$. Fix a finite field extension L/K such that $X(L)$ is non-empty. Let $x \in X(L)$ and define $y = \pi(x) \in (X/N)(L)$. Then $g \mapsto x.g$ defines an isomorphism $G_L \rightarrow X_L$, and $\bar{g} \mapsto y.\bar{g} = \pi(x.g)$ defines an isomorphism $(G/N)_L \rightarrow (X/N)_L$, in each case between trivial torsors. The isomorphism $(L \otimes_K K[X]) \rightarrow (L \otimes_K K[G])$ corresponding to $G_L \rightarrow X_L$ restricts to the isomorphism $(L \otimes_K K[X]^N) \rightarrow (L \otimes_K K[G]^N)$ corresponding to $(G/N)_L \rightarrow (X/N)_L$. By [DG70, III.3, Proposition 2.5], $K[G]$ is faithfully flat over $K[G]^N$. So $L \otimes_K K[G]$ is faithfully flat over $L \otimes_K K[G]^N$, and therefore $L \otimes_K K[X]$ is faithfully flat over $L \otimes_K K[X]^N$. Using that L/K is faithfully flat, we conclude that $K[X]/K[X]^N$ is faithfully flat. \square

The next result is used in Section 2.

Lemma A.7. *Let G be a smooth affine group scheme of finite type over a field K and let N be a normal closed subgroup. Let X be a G -torsor, and let $I \trianglelefteq K[X]$ be an N -stable, non-zero ideal. Then $I \cap K[X]^N \neq 0$.*

Proof. First note that by flatness of \bar{K} over K , we may assume that K is algebraically closed. Hence X is a trivial G -torsor and we may assume $X = G$. Let $I \trianglelefteq K[G]$ be an N -stable, non-zero ideal. We need to show that $I \cap K[G]^N \neq 0$. Let $Z \subsetneq G$ be the closed subset defined by I . By Proposition A.4, the quotient morphism $\pi: G \rightarrow G/N$ corresponds to the inclusion $K[G]^N \subseteq K[G]$ on the level of coordinate rings. Furthermore, π is open (see Theorem 5.5.5 in [Spr09] and its proof), so $\pi(G \setminus Z)$ is an open and non-empty subset of G/N . Also note that Z is a union of N -orbits, since I is N -stable. However, the fibers of π are the N -orbits in G ([Spr09, Cor. 5.5.4]) and so $\pi(Z) \cap \pi(G \setminus Z) = \emptyset$. Therefore, $\pi(Z) \subseteq (G/N) \setminus \pi(G \setminus Z)$ is not dense in G/N and hence $\sqrt{I} \cap K[G]^N \neq 0$. Choose $0 \neq f \in \sqrt{I} \cap K[G]^N$. Then some power f^n lies in $I \cap K[G]^N$. But G is reduced, so $f^n \neq 0$ and thus $I \cap K[G]^N \neq 0$. \square

A.3. Induced torsors. Let H be a closed subgroup of G and let Y be an H -torsor. There is a natural way to construct a G -torsor

$$X = \text{Ind}_H^G(Y)$$

from Y ([DG70, III, §4, 3.2]) which we recall in the proof of Proposition A.8 below. This follows directly from the identification of isomorphism classes of torsors with equivalence classes of 1-cocycles, and the fact that the inclusion map $H \hookrightarrow G$ induces a map $H^1(K, H) \rightarrow H^1(K, G)$. Alternatively, the induced torsor $\text{Ind}_H^G(Y)$ can be characterized by a universal property:

Proposition A.8. *Let H be a closed subgroup of G and let Y be an H -torsor. Then there exists a G -torsor $\text{Ind}_H^G(Y)$ together with an H -equivariant morphism $Y \rightarrow \text{Ind}_H^G(Y)$ such that for every H -equivariant morphism $Y \rightarrow Z$ from Y to some affine G -space Z , there exists a unique G -equivariant morphism $\text{Ind}_H^G(Y) \rightarrow Z$ making the diagram*

$$(8) \quad \begin{array}{ccc} Y & \longrightarrow & \text{Ind}_H^G(Y) \\ & \searrow & \swarrow \text{dotted} \\ & & Z \end{array}$$

commutative. This G -torsor is unique up to a unique isomorphism, and it is given by the quotient $(Y \times G)/H$, where $Y \times G$ is an H -space under the action given by $(y, g).h = (y.h, h^{-1}g)$ for every K -algebra R , and all $y \in Y(R)$, $g \in G(R)$, and $h \in H(R)$.

Proof. Set $\mathcal{F} = (Y \times G)//H$, and write $[y, g] \in \mathcal{F}(R)$ for the $H(R)$ -orbit of an element $(y, g) \in Y(R) \times G(R)$, where R is a K -algebra. We define a group action of G on the functor \mathcal{F} by

$$\mathcal{F}(R) \times G(R) \rightarrow \mathcal{F}(R), ([y, g], \tilde{g}) \mapsto [y, g\tilde{g}] \text{ for } y \in Y(R) \text{ and } g, \tilde{g} \in G(R).$$

Letting X be the sheafification $((Y \times G)//H)^\sim$, we obtain a group action of G on X given by the sheafification $X \times G \rightarrow X$ of the above morphism. For every K -algebra R and every two elements $[y, g], [y', g'] \in \mathcal{F}(R)$, there exists a unique $\tilde{g} \in G(R)$ with $[y, g\tilde{g}] = [y', g']$, since Y is an H -torsor. Therefore, the map $\mathcal{F}(R) \times G(R) \rightarrow \mathcal{F}(R) \times \mathcal{F}(R)$ given by $([y, g], \tilde{g}) \mapsto ([y, g], [y, g\tilde{g}])$ is bijective for every K -algebra R . Thus $\mathcal{F} \times G \rightarrow \mathcal{F} \times \mathcal{F}$ is an isomorphism of functors, and so its sheafification $X \times G \rightarrow X \times X$ is an isomorphism (here we used the canonical isomorphisms $\tilde{\mathcal{F}} \times \tilde{\mathcal{F}} \cong \widetilde{\mathcal{F} \times \mathcal{F}}$ and $\tilde{\mathcal{F}} \times G \cong \tilde{\mathcal{F}} \times \tilde{G} \cong \widetilde{\mathcal{F} \times G}$).

Since Y is non-empty, X is non-empty and we may fix a field extension L/K such that X, Y have an L -point. Pick such a point $x \in X(L)$. Then the formula $g \mapsto x.g$ defines an isomorphism $G_L \cong X_L$. In particular, the base change X_L of X from K to L is an affine scheme. As L is faithfully flat over K and X is a sheaf, it follows that X is an affine scheme, via faithfully flat descent, as in the proof of Proposition A.6(a). It also follows that X is of finite type over K . So we may write $X = (Y \times G)/H$. Proposition A.4 implies $K[X] = (K[Y] \otimes_K K[G])^H$. (Recall that $(K[Y] \otimes_K K[G])^H = \{f \in K[Y] \otimes_K K[G] \mid \rho(f) = f \otimes 1\}$, where $\rho : K[Y] \otimes_K K[G] \rightarrow K[Y] \otimes_K K[G] \otimes_K K[H]$ is the co-action corresponding to the action of H on $Y \times G$.) The isomorphism $X \times G \rightarrow X \times X$ shows that X is a G -torsor.

We define a morphism $Y \rightarrow \mathcal{F}$ by sending $y \in Y(R)$ to the equivalence class $[y, 1]$ of $(y, 1) \in (Y \times G)(R)$ for any K -algebra R . Composing with $\mathcal{F} \rightarrow \text{Ind}_H^G(Y)$ yields a morphism $\psi : Y \rightarrow \text{Ind}_H^G(Y)$. Since $[y.h, 1] = [y, h]$ for $h \in H(R)$, we see that ψ is H -equivariant. Now let $\psi' : Y \rightarrow Z$ be any H -equivariant morphism from Y to an affine G -space Z . The G -equivariant morphism $Y \times G \rightarrow Z$, $(y, g) \mapsto \psi'(y).g$ is constant on H -orbits and therefore induces a unique G -equivariant morphism $\text{Ind}_H^G(Y) \rightarrow Z$ making (8) commutative, by the

universal property of the quotient. This proves the universal property of induced torsors that was stated in the proposition.

Finally, the universal property satisfied by $\text{Ind}_H^G(Y)$ shows that this is unique up to a unique isomorphism. \square

Remark A.9. (a) Let H be a closed subgroup of G . If $Y \rightarrow X$ is an H -equivariant morphism from an H -torsor Y to a G -torsor X then $X = \text{Ind}_H^G(Y)$. This is clear because G -equivariant morphisms between G -torsors are isomorphisms.

(b) Consider the special case that G is a semi-direct product $N \rtimes H$ of closed subgroups with N normal; i.e., $G(A) = N(A) \rtimes H(A)$ for every K -algebra A . Here we may take $Z = Y$ and $Y \rightarrow Z$ the identity map in Proposition A.8, where G acts on Y through $H = G/N$. The proposition then yields a canonical G -equivariant morphism $\text{Ind}_H^G(Y) \rightarrow Y$, for which the given morphism $Y \rightarrow \text{Ind}_H^G(Y)$ is a section.

(c) Using the universal property, it is immediate that the G -torsor induced from the trivial H -torsor is trivial.

(d) As above, let H be a closed subgroup of G defined over K , and let Y be an H -torsor over K . If L is a field extension of K , the universal property of induced torsors implies that $(\text{Ind}_H^G(Y))_L$ is canonically isomorphic to $\text{Ind}_{H_L}^{G_L}(Y_L)$.

Lemma A.10. *The morphism $\psi: Y \rightarrow X = \text{Ind}_H^G(Y)$ given by Proposition A.8 is a closed embedding; i.e., the dual map $\psi^*: K[X] \rightarrow K[Y]$ is surjective.*

Proof. By flatness, we may assume that K is algebraically closed. Let $y \in Y(K)$ and define an isomorphism $H \rightarrow Y$ by $h \mapsto y.h$. Similarly, let $x = \psi(y)$ and define an isomorphism $G \rightarrow X$ by $g \mapsto x.g$. Then

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \cong \uparrow & & \uparrow \cong \\ H & \longrightarrow & G \end{array}$$

commutes. Since $H \rightarrow G$ is a closed embedding, $Y \rightarrow X$ is also a closed embedding. \square

Since $Y \rightarrow \text{Ind}_H^G(Y)$ is a closed embedding, we will in the sequel identify Y with a closed subscheme of $\text{Ind}_H^G(Y)$.

Let G be a finite (abstract) group. Recall (see e.g., [KMRT98, Section 18.B]) that a finite separable (commutative) K -algebra L equipped with a G -action from the left is called a *G -Galois algebra* if $\dim_K L = |G|$ and the invariant subfield L^G equals K . As noted there, these algebras are precisely the coordinate rings of G -torsors over K , where we view G as a constant group scheme over K (i.e., with trivial action of the absolute Galois group of K). The following example shows that the notion of induced Galois algebra can be seen as a special case of inducing torsors.

Example A.11. Let G be a finite group with a subgroup H . For an H -Galois algebra L , there exists an *induced G -Galois algebra* $\text{Ind}_H^G(L)$ together with an H -equivariant morphism $\pi: \text{Ind}_H^G(L) \rightarrow L$; and moreover the pair $(\text{Ind}_H^G(L), \pi)$ is unique up to isomorphism (see [KMRT98, Prop. 18.17] for details). Consider G and H as constant group schemes over K and let $Y = \text{Spec}(L)$ denote the H -torsor corresponding to L . We claim that

$$K[\text{Ind}_H^G(Y)] \cong \text{Ind}_H^G(L) \quad \text{as } G\text{-Galois algebras.}$$

To see this, note that if $H \rightarrow G$ is a morphism of algebraic groups, Y an H -torsor, X a G -torsor and $\psi : Y \rightarrow X$ an H -equivariant morphism, then $\psi^* \otimes_K R : F[Y] \otimes_K R \rightarrow F[X] \otimes_K R$ is $H(R)$ -equivariant for any K -algebra R . So, as $Y \rightarrow \text{Ind}_H^G(Y)$ is H -equivariant (choosing $R = K$), so is the dual map $K[\text{Ind}_H^G(Y)] \rightarrow L$. Therefore $K[\text{Ind}_H^G(Y)] \cong \text{Ind}_H^G(L)$.

Motivated by this example, if more generally $Y = \text{Spec}(R)$ in the situation of Proposition A.8, we will write $\text{Ind}_H^G(R)$ for the coordinate ring $K[\text{Ind}_H^G(Y)]$. Thus $\text{Ind}_H^G(Y) = \text{Spec}(\text{Ind}_H^G(R))$. By Proposition A.8, $\text{Ind}_H^G(R)$ is the invariant ring $(R \otimes_K K[G])^H$, where the action of H on $R \otimes_K K[G]$ corresponds to the geometric action of H on $Y \times G$ given in that proposition.

Remark A.12. If $Y = \text{Spec}(R)$ is a G -torsor over K , then by setting $H = G$ in Proposition A.8 we see that Y is canonically isomorphic to $\text{Ind}_G^G(Y)$. The isomorphism can be made explicit on the level of coordinate rings. Namely, the co-action $\rho : R \rightarrow R \otimes_K K[G]$ has image $\text{Ind}_G^G(R) = (R \otimes_K K[G])^G \subseteq R \otimes_K K[G]$, and ρ defines an isomorphism of R with $\text{Ind}_G^G(R)$.

Proposition A.13. *Let G be a smooth affine group scheme of finite type over a field K , let E, N be closed subgroups of G with N normal, and let $X = \text{Spec}(R)$ be an E -torsor over K . Consider the functorial G -action on $\text{Ind}_E^G(R)$ corresponding to the G -torsor structure of its spectrum. With respect to the restriction of this action to N , $(\text{Ind}_E^G(R))^N = \text{Ind}_{E/(E \cap N)}^{G/N}(R^{E \cap N})$ as subalgebras of $\text{Ind}_E^G(R)$.*

Proof. By Proposition A.8, $\text{Ind}_E^G(R) = (R \otimes_K K[G])^E$, where the E -invariants are taken with respect to the group action given geometrically by $(x, g).h = (x.h, h^{-1}g)$. By Proposition A.6, $X/(E \cap N) = \text{Spec}(R^{E \cap N})$, and this is a torsor for $E/(E \cap N)$. Moreover, $\text{Ind}_{E/(E \cap N)}^{G/N}(R^{E \cap N}) = (R^{E \cap N} \otimes_K K[G/N])^{E/(E \cap N)}$, with respect to the corresponding group action of $E/(E \cap N)$. Observe that the rings $(\text{Ind}_E^G(R))^N$ and $\text{Ind}_{E/(E \cap N)}^{G/N}(R^{E \cap N})$ in the statement of the proposition are indeed both contained in $\text{Ind}_E^G(R)$, whose spectrum is a G -torsor. The restriction to N of the corresponding functorial G -action on $\text{Ind}_E^G(R)$ extends to $R \otimes_K K[G]$, and is given geometrically by right multiplication on the second factor. Note that this action of N commutes with the above action of E . Hence $(\text{Ind}_E^G(R))^N$ is the ring of E -invariants of the K -algebra $R \otimes_K K[G]^N$. Equivalently, it is the ring of E -invariants of $(R \otimes_K K[G]^N)^{E \cap N}$, where $E \cap N$ acts via the restriction of the above action of E . But $(R \otimes_K K[G]^N)^{E \cap N} = R^{E \cap N} \otimes_K K[G]^N$, since the right-invariant subring $K[G]^N$ is also left-invariant under N by Lemma A.3(a), by normality of N . The actions of E on the spectra of $R^{E \cap N}$ and on $K[G]^N$ are functorially defined over each K -algebra, and their restrictions to $E \cap N$ are trivial. Hence the action of E factors through $E/(E \cap N)$, and the above E -invariant ring is the same as the ring of $E/(E \cap N)$ -invariants of $R^{E \cap N} \otimes_K K[G/N]$, viz. $\text{Ind}_{E/(E \cap N)}^{G/N}(R^{E \cap N})$. \square

Corollary A.14. *Consider a smooth affine group scheme G of finite type over a field K , with G a semi-direct product $N \rtimes H$ of a closed normal subgroup N and a closed subgroup H .*

- (a) *If $\text{Spec}(R)$ is an N -torsor over K , then $(\text{Ind}_N^G(R))^N$ equals $K[H]$; i.e., it is the subalgebra $\text{Ind}_1^{G/N}(K) = K \otimes_K K[H]$ of $\text{Ind}_N^G(R) \subseteq R \otimes_K K[G]$.*

(b) If $\text{Spec}(R)$ is an H -torsor over K , then $(\text{Ind}_H^G(R))^N$ equals R ; i.e., it is the subalgebra $\text{Ind}_H^H(R) \subseteq R \otimes_K K[H]$ of $\text{Ind}_H^G(R) \subseteq R \otimes_K K[G]$ (where R maps isomorphically to $\text{Ind}_H^H(R)$ by the co-action map ρ_H corresponding to the action of H ; cf. Remark A.12).

Proof. These are special cases of Proposition A.13, with E equal to N or H , respectively. \square

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