One day in 1989, I was surprised to receive a letter in the mail from Prof. Shreeram Abhyankar. Although I had known of his work since I was a graduate student, we had not met; and in those days before email was commonplace, an unsolicited letter from a well-known senior mathematician was quite unexpected. The contents were a surprise as well — he wrote that after a long hiatus in his work on algebraic fundamental groups, he wished to resume research in that direction, and he was asking if I could provide a summary of recent developments on that topic.

That letter led to many exchanges, including a number of visits of mine to West Lafayette, and of his to Philadelphia, where we discussed (and sometimes argued about) mathematics for long hours. Our discussions during my visits to Purdue took place not in his office in the mathematics department, but in the office/seminar room that was set up in his house, where grad students and others would gather for long mathematical sessions in front of his large blackboard. The discussions were lively to say the least, and at times even a bit unnerving, as he sought to see the underlying essence beneath the surface presentation of the mathematics.

Besides the mathematical discussions during my visits, he would also tell me tales of his "guru" (and my mathematical grandfather) Oscar Zariski, as well as about his other interests such as the characters in Indian lore. There were often crowds of mathematicians who would come to the house in the evening, and all would be graciously fed Indian food by his wife and loyal companion Yvonne.

Before I met Ram, I had been aware of two seemingly distinct threads in his research. One concerned the resolution of singularities on algebraic varieties, especially in finite and mixed characteristic. The other concerned the study of branched covers of varieties, and in particular the formulation of "Abhyankar's Conjecture" on the fundamental group of affine curves in characteristic p. But as I learned from him during our many talks, these two threads were in fact intimately intertwined, and both grew out of his Ph.D. thesis.

For the thesis, Zariski had proposed that he study resolution of singularities of algebraic surfaces in characteristic p. Zariski had proven resolution over algebraically closed fields of characteristic zero, and he had suggested that a proof in arbitrary characteristic could be obtained by translating into algebra an even earlier analytic argument of H.W.E. Jung that concerned complex surfaces. As Ram liked to tell the story, there were two parts to his thesis: the successful part and the "failure" part. What he meant was that he discovered that Jung's method breaks down in characteristic p [Abh55]; and as a result, he had to use another approach to prove resolution of singularities in that context [Abh56]. But the failure of Jung's method to generalize did not prove to be a dead-end. Instead, it led to Ram's work on branched coverings of varieties, and to his introduction of the idea of an algebraic fundamental group that can be viewed as analogous to the fundamental group in topology [Abh57].

To obtain resolution of singularities, what was needed was to prove "local uniformization," which asserted that the given variety could be written as a union of finitely many Zariski open sets, each of which is the image of a birational morphism from a smooth variety. In Jung's strategy for surfaces, the idea was to express the given variety as a branched cover of a linear space, and then to blow up so that the branch locus has only ordinary double

points. This relied on the fact that the variety is smooth at any point lying over a smooth point of the branch locus, with cyclic inertia group; and that over an ordinary double point of the branch locus, the inertia group is a product of two cyclic groups. This indeed holds over an arbitrary algebraically closed field of characteristic zero.

But as Abhyankar found in the "failure" part of his thesis, these properties of branched covers do not hold in characteristic p. To use his phrase, in characteristic p there can be "local splitting of a simple branch variety by itself" [Abh57, Sect. 1], meaning that the inertia group can increase over a smooth point of the branch locus, as distinct components of the ramification locus meet at a point on the cover lying over that smooth point. In addition, as he found, inertia groups over an ordinary double point of the branch locus can be quite complicated, even being non-solvable. Rather than simply abandoning this approach to resolution in favor of his successful desingularization strategy, he initiated a serious study of covers in characteristic p. This included not only surfaces and varieties of higher dimension [Abh59], but also of curves. What he observed was that by taking a slice of a cover of the plane that has non-solvable inertia, he could obtain an unramified cover of the affine line with non-solvable Galois group. Although it was well-known by Artin-Schreier theory that the affine line in characteristic p has non-trivial unramified covers with Galois group a p-group, the proliferation of covers with much more complicated Galois groups was a real surprise.

This work also led to his defining a "fundamental group" for varieties that would make sense even in characteristic p (and later, in mixed characteristic). For a variety over the complex numbers, the finite quotients of the (topological) fundamental group are the Galois groups of finite Galois (or "normal") covering spaces; and these quotients form an inverse system. Abhyankar defined the algebraic fundamental group to be the collection of Galois groups of finite unramified covers of the given variety. Of course this is a set, not a single group. But he added that "eventually one may have to consider the galois group ... of the compositum" of the function fields of the finite unramified covers [Abh57, 4.2]. This is equivalent to taking the inverse limit of the finite groups in the corresponding inverse system (which is what Grothendieck later did in his work on étale fundamental groups in SGA1).

This work also led him to make his conjecture stating which finite groups can be Galois groups over a curve in characteristic p of a given genus and with a given number of punctures. What he proposed is a type of "maximal conjecture," asserting that anything that cannot be ruled out must occur (a sort of Murphy's Law). Namely, he said that a group will occur as a Galois group of some unramified cover of a given affine curve if and only if its maximal prime-to-p quotient can occur over a characteristic zero curve of the same genus with the same number of punctures [Abh57, 4.2]. (The groups that occur in characteristic zero are well known by topology.) This condition is seen to be necessary, once one knows that the prime-to-p Galois groups are the same in characteristics 0 and p. But it was a non-obvious leap to conjecture, based on the examples he had found, that it is also sufficient.

Years later, after his conjecture was proven ([Ray94], [Har94]), his "maximal" philosophy led him to formulate possible analogous conjectures in related situations. These included higher dimensional varieties over an algebraically closed field of characteristic p; affine curves over finite fields; and local fundamental groups in the higher dimensional situation. In some

cases, further investigation led to additional obstructions to the Galois groups that can occur in those situations, leading to difficulties in formulating the correct "maximal conjecture"; and these problems remain wide open. Two other problems that also remain wide open are determining which groups can be inertia groups over infinity for a given Galois group over the affine line (the maximal conjecture in this setting was Ram's "inertia conjecture"); and determining the structure of the fundamental group of the affine line as a profinite group (on which he and Serre had some amusing exchanges). Also, motivated by the inexplicit nature of the proof of his conjecture, he worked on obtaining explicit realizations of interesting groups as Galois groups over the affine line in characteristic p, which often provided realizations as well in the case of the affine line over the field of p elements. These realizations appeared in a number of papers, particularly in his "nice equations" series in the 1990's (beginning with [Abh94]).

Ram liked to say that what he did was just "high school mathematics." But this would have to have been a *very* good high school indeed, to include his work on resolution of singularities on arithmetic surfaces [Abh65] in the curriculum! His point, though, was that one could get far by thinking concretely, e.g. in terms of polynomials, and that rushing to use abstractions like categories and functors (which he called "university mathematics") could well distract one from the deeper issues.

Still, there is a bit of a paradox here. As I had mentioned above, his work on the algebraic fundamental group led to the development and presentation of this topic in Grothendieck's SGA1, whose capstone result [Gro71, XIII, Corollaire 2.12] was similar in spirit to Abhyankar's results in his 1950's papers. The proof of Abhyankar's Conjecture on Galois groups over curves relied heavily on results in "university mathematics", though those results were analogs of the GAGA theorem of his friend J.-P. Serre. Ram's work on arithmetic surfaces would surely be viewed by most mathematicians today as being part of the abstract algebraic geometry of schemes. His efforts to find covers of the affine line given by polynomials whose coefficients are just 0 or 1 (as in his nice equations series) could be viewed as studying fundamental groups of curves over the highly speculative object $\mathbb{F}_1$. And his study of the structure of the fundamental group of affine curves in characteristic p, including how it depends on the curve and its base field, is related to Grothendieck's anabelian conjecture.

Regardless of one's own views on "high school mathematics" and "university mathematics", Shreeram Abhyankar's work has had a major impact on mathematics and the mathematical community, and continues to influence the direction of research in algebra and algebraic geometry. He certainly had a profound influence on my own mathematical career, for which I am grateful.

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