EMBEDDING PROBLEMS AND OPEN SUBGROUPS

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ABSTRACT. We study the properties of the fundamental group of an affine curve over an algebraically closed field of characteristic p, from the point of view of embedding problems. In characteristic zero, the fundamental group is free, but in characteristic p it is not even ω-free. In this paper we show that it is "almost ω-free," in the sense that each finite embedding problem has a proper solution when restricted to some open subgroup. We also prove that embedding problems can always be properly solved over the given curve if suitably many additional branch points are allowed, in locations that can be specified arbitrarily; this strengthens a result of the first author.

1. INTRODUCTION.

This paper studies the structure of the étale fundamental group of an affine curve over an algebraically closed field of characteristic p, and shows that in a certain sense it is "not too far from being free."

Consider an affine curve C over an algebraically closed field k. Thus $C = X - \Sigma$, where X is a smooth projective curve of genus $g \geq 0$ and $|\Sigma| = r > 0$. If k has characteristic zero, then $\pi_1(C)$ is a free profinite group on $2g + r - 1$ generators [SGA1, XIII, Cor. 2.12]. In contrast, if k has characteristic $p > 0$, then $\pi_1(C)$ is infinitely generated as a profinite group, since by Artin-Schreier theory there are infinitely many linearly disjoint $\mathbb{Z}/p\mathbb{Z}$-Galois étale covers of C. Moreover it cannot be free profinite (of infinite rank), since a prime-to-p group is a quotient of $\pi_1(C)$ if and only if it has at most $2g + r - 1$ generators [SGA1, XIII, Cor. 2.12]). For this same reason, $\pi_1(C)$ is not even ω-free; i.e. it is not the case that every finite embedding problem for $\pi_1(C)$ has a proper solution [FJ, p. 652]. (Infinitely generated free profinite groups are ω-free [FJ, Lemma 25.1.2]; the converse holds for countably generated groups [Iw, p. 167], but not in general [Ja, Example 3.1].)

The main result of this paper (Theorem 6) is to show that $\Pi = \pi_1(C)$ has the weaker property of being *almost ω-free*: for every finite embedding problem $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ there is an open subgroup $H \subset \Pi$ such that $\alpha|_H : H \to G$ is surjective and the induced embedding problem $\mathcal{E}_H = (\alpha|_H, f)$ has a proper solution.

This result can be interpreted geometrically as follows. We are given a G-Galois cover $\phi : Y \to X$ ramified at most over Σ. A proper solution to the original embedding problem is a Γ-Galois cover $Z \to X$ that dominates $\phi : Y \to X$ and is unramified away from Σ. The almost ω-free property asserts that there exists a cover $\psi : X' \to X$ ramified at most over Σ such that the normalized pullback $\phi' : Y' \to X'$ of ϕ via ψ is a G-Galois cover and moreover there exists a Γ-Galois cover $Z' \to X'$ that dominates $Y' \to X'$ and is branched only at

points of X' lying over $\Sigma \subset X$. Here the cover ψ corresponds to the restriction $\alpha|_H$, where $H \subset \pi_1(X - \Sigma)$ is the open subgroup $\pi_1(X' - \psi^{-1}(\Sigma))$.

This paper is motivated by the fact that the absolute Galois group $G_{k(X)}$ of the function field of a smooth projective k-curve X is free, for k an algebraically closed field of arbitrary characteristic ([Ha95], [Po]). In characteristic zero this is natural since each $\pi_1(C)$ is free; and in that case the freeness of $G_{k(X)}$ was proven earlier by viewing this group as the inverse limit of the free groups $\pi_1(C)$ for dense affine open subsets $C \subset X$ [Do]. But in non-zero characteristic, the freeness of $G_{k(X)}$ is a bit mysterious, since the groups $\pi_1(C)$ are not free. Still, this suggests that the groups $\pi_1(C)$ should satisfy some kind of freeness property; and this role is played here by the property of being almost ω-free. (The word *almost* is being used in the sense of "up to a finite discrepancy"; here the open subgroup H is necessarily of finite index.)

In the process of proving our main theorem, we also obtain a result (Theorem 5) that strengthens the main theorem of [Ha99]. That theorem, which provided a more precise form of a key result in [Ha95] and [Po], had asserted that an embedding problem for an affine curve over an algebraically closed field can be solved if a specific number of additional points are permitted. Our result here shows that the positions of the additional points can be chosen arbitrarily.

We also note the papers [Ku08] and [Ku09], which discuss another way in which $\pi_1(C)$ is "close to being free". Namely, those papers show that if k is an algebraically closed field of characteristic p then the commutator subgroup of $\pi_1(C)$ is free. For other papers on related themes, see also [Bo], [GS], [Os], [Pr], [Ta].

The structure of the current manuscript is as follows: In Section 2 we prove Proposition 2, which asserts that G-Galois branched covers have pullbacks that remain G-Galois but become unramified at sufficiently many points lying over the original branch locus. Section 3 proves results leading up to Theorem 5 on the solution to embedding problems with prescribed additional branch points. These are combined in Section 4 to obtain our main result, Theorem 6, that the fundamental group is almost ω-free. That section also discusses almost ω-freeness in the characteristic zero case, where the fundamental group is finitely generated.

Terminology and Notation: If G is a finite group and p is a prime number, let $p(G)$ denote the subgroup of G generated by its p-subgroups. This is a characteristic subgroup of G, and $G/p(G)$ is the maximal prime-to-p quotient of G. A finite group G is called *quasi-p* if $G = p(G)$. A *finite embedding problem* $\mathcal{E}$ for a group Π is a pair of surjections $(\alpha : \Pi \to G, f : \Gamma \to G)$, where Γ and G are finite groups. A *weak solution* to $\mathcal{E}$ is a homomorphism $\gamma : \Pi \to \Gamma$ such that $f \circ \gamma = \alpha$. We call γ a *proper solution* to $\mathcal{E}$ if in addition it is surjective. If H is a subgroup of Π and $\alpha|_H$ is a surjection onto G, then $\mathcal{E}' = (\alpha|_H : H \to G, f : \Gamma \to G)$ is the *induced* embedding problem for H.

In this paper we consider curves over an algebraically closed field k. A *cover* of k-curves is a morphism $\phi : V \to U$ of smooth connected k-curves that is finite and generically separable. If $\phi : V \to U$ is a cover, its *Galois group* $\mathrm{Gal}(V/U)$ is the group of k-automorphisms σ of V satisfying $\phi \circ \sigma = \phi$. If G is a finite group, then a *G-Galois cover* is a cover $\phi : V \to U$ *together* with an inclusion $\rho : G \hookrightarrow \mathrm{Gal}(V/U)$ such that G acts simply transitively on a generic geometric fibre of $\phi : V \to U$. If we fix a base point of U, then the pointed G-Galois étale covers of U correspond bijectively to the surjections $\alpha : \pi_1(U) \to G$, where $\pi_1(U)$ is the algebraic fundamental group of U. The proper solutions to an embedding problem

$\mathcal{E} = (\alpha : \pi_1(U) \to G, f : \Gamma \to G)$ for $\pi_1(U)$ then are in bijection to the pointed Γ-Galois covers $W \to U$ that dominate the pointed G-Galois cover $\phi : V \to U$ corresponding to α.

2. Removing branch points by pulling back.

Fix an algebraically closed field k of characteristic $p > 0$. The goal of this section is to prove Proposition 2 below, which says that for any branched cover of k-curves, there is a finite morphism to the base such that the inverse image of the original branch locus contains (many) points that are unramified in the normalized pullback of the given cover. This will be used in conjunction with the key result of the next section in order to prove our main theorem. The point is that since affine curves are not ω-free, not all embedding problems can be properly solved; but by proceeding to a pullback via Proposition 2, we may enlarge the branch locus and thereby obtain a solution to the induced embedding problem.

Lemma 1. *Let X be a smooth connected projective k-curve and let $\Delta \subset X$ be a non-empty subset of X. Given an integer $t > 1$ and a G-Galois cover $\phi : Y \to X$ étale away from Δ, there is a Galois cover $\pi : X_0 \to X$ of degree at least t that is linearly disjoint from $Y \to X$, is branched only over Δ, and such that $|\pi^{-1}(\Delta)| \geq t$.*

Proof. Fix a point $\sigma \in \Delta$, and choose an integer $n \geq \max(5, p)$ such that A_n is not a quotient of G and the largest prime-to-p factor m of $|A_n|$ is at least t. The group A_n is simple, so every A_n-Galois cover of X is linearly disjoint from Y over X, having no common subcovers.

Since A_n is simple of order divisible by p, it is a quasi-p group. So by Abhyankar's Conjecture [Ha94, Theorem 6.2] (or by [Ra, Corollaire 2.2.2], in the case of the affine line), there is an A_n-Galois cover $\pi : X_0 \to X$ that is branched only at $\sigma \in X$. Moreover, this cover may be chosen so that the inertia groups over σ are the Sylow p-subgroups of A_n, by [Po, Theorem B] (see also [Ha03a, Corollary 4.2]). The number of ramification points over σ is the index of such a Sylow p-subgroup Q. Since the index m of Q in A_n is greater than or equal to t, the set $\pi^{-1}(\sigma)$ contains at least t points; and hence so does $\pi^{-1}(\Delta)$, which contains it. $\square$

Proposition 2. *Let X be a smooth connected projective k-curve, let $\Sigma \subset X$ be a non-empty subset of X, and fix a point τ in Σ. Let G be a finite group and let $\phi : Y \to X$ be a G-Galois cover that is étale away from Σ. Then for every positive integer s there is a cover $\psi_s : X_s \to X$ that is étale away from Σ and is linearly disjoint from $\phi : Y \to X$ such that the normalized pullback $\phi_s : Y_s \to X_s$ of ϕ via ψ_s is a G-Galois cover with at least s unbranched points in $\psi_s^{-1}(\tau) \subset X_s$.*

Proof. Applying Lemma 1 to the cover $\phi : Y \to X$ with $t = 2$ and $\Delta = \Sigma$, there is a Galois cover $\pi : X_0 \to X$ of degree $d \geq 2$ that is linearly disjoint from $\phi : Y \to X$, is étale away from Σ, and such that $|\pi^{-1}(\Sigma)| \geq 2$. By linear disjointness, the normalized pullback $Y_0 \to X_0$ of $Y \to X$ is connected, and so it is a G-Galois cover. Replacing $Y \to X$ by $Y_0 \to X_0$ and Σ by $\pi^{-1}(\Sigma)$, we may assume that $|\Sigma| \geq 2$.

Next we reduce to the case that $s = 1$. Suppose that there exists a cover $\psi_1 : X_1 \to X$ étale away from Σ and linearly disjoint from $\phi : Y \to X$ such that the normalized pullback $\phi_1 : Y_1 \to X_1$ of ϕ via ψ_1 contains at least one point τ_1 in $\psi_1^{-1}(\tau)$ that is unbranched in ϕ_1.

Necessarily Y_1 is connected by linear disjointness. Let $\Delta = \psi_1^{-1}(\Sigma) - \{\tau_1\}$. This is non-empty by the assumption that $|\Sigma| \geq 2$. Applying Lemma 1 again, but this time to the cover ϕ_1 with $t = s$, there is a cover $\pi_s : X_s \to X_1$ such that its degree is at least s; it is linearly disjoint from $\phi_1 : Y_1 \to X_1$; it is étale away from Δ; and $|\pi_s^{-1}(\Delta)| \geq s$. Consider the normalized pullback $\phi_s : Y_s \to X_s$ of ϕ_1 via π_s; this is a connected G-Galois cover because X_s and Y_1 are linearly disjoint over X_1. Moreover, as $\tau_1 \notin \Delta$, the morphism π_s is unramified over τ_1. But π_s has degree at least s, and so $\pi_s^{-1}(\tau_1)$ contains at least s points. Each point in X_s lying over τ_1 is unbranched in ϕ_s, since τ_1 is unbranched in ϕ_1. Let $\psi_s := \psi_1 \circ \pi_s : X_s \to X$. Thus $\phi_s : Y_s \to X_s$ is the normalized pullback of $\phi : Y \to X$ via ψ_s, and X_s is linearly independent of Y over X by connectivity of Y_s. Finally, since τ_1 lies over τ in X_1, we have that X_s contains at least s points in $\psi_s^{-1}(\tau) \supset \pi_s^{-1}(\tau_1)$ that are unbranched in ϕ_s. Thus the cover $\psi_s : X_s \to X$ has the desired properties.

It now remains to prove the result for $s = 1$. We consider three cases.

Case 1. *The inertia groups of $Y \to X$ over τ are non-trivial p-groups, and there is point $\sigma \in \Sigma - \{\tau\}$ such that the inertia groups of $\phi : Y \to X$ over σ are the same as those over τ.*

Pick a point $\eta \in Y$ over τ and let $P \subset G$ be its inertia group. According to Proposition 2.7 of [Ha80], there is a P-Galois cover $W \to X$ that is unramified away from τ, totally ramified at τ, and such that the P-Galois extensions $\hat{\mathcal{O}}_{W,\omega}$ and $\hat{\mathcal{O}}_{Y,\eta}$ of $\hat{\mathcal{O}}_{X,\tau}$ are isomorphic, where ω is the unique point of W over τ. (In fact, according to the result cited, the number of such covers $W \to X$ is equal to the order of $H^1(X, P)$, though we do not need this here.)

Since $Y \to X$ has the same sets of inertia groups over τ and σ, whereas $W \to X$ is totally ramified over τ and unramified over σ, it follows that these two Galois covers are linearly disjoint (having no common subcovers). So the normalized pullback $\phi_W : Y_W \to W$ of $Y \to X$ via $W \to X$ is an irreducible G-Galois cover. This cover is unramified at ω, since it is trivial over the fraction field of $\hat{\mathcal{O}}_{W,\omega}$ (by the above isomorphism). Let $X_1 = W$, $Y_1 = Y_W$, and ϕ_1 be the cover ϕ_W. Now, $\omega \in W = X_1$ lies over $\tau \in X$. Thus $\psi_1 : X_1 \to X$ is étale away from Σ and is linearly disjoint from $\phi : Y \to X$, and the normalized pullback $\phi_1 : Y_1 \to X_1$ contains at least one point in $\psi_1^{-1}(\tau)$ that is unbranched in ϕ_1.

Case 2: *There are distinct points $\sigma_1, \sigma_2 \in \Sigma - \{\tau\}$ such that the inertia groups of $\phi : Y \to X$ over each σ_i are the same as those over τ.*

Pick a point over τ and let I be its inertia group. Then I is a semi-direct product of a p-group P and a cyclic prime-to-p group $C = \langle \gamma \rangle$. The other inertia groups over τ (and hence over σ_1 and σ_2) are conjugate to I. Let g be the genus of X. By [SGA1, XIII, Cor. 2.12], the prime-to-p fundamental group π_1' of $X - \{\tau, \sigma_1\}$ has generators a_i, b_i, c_j with $1 \leq i \leq g$ and $j = 0, 1$, subject to the single relation $(\prod_i [a_i, b_i]) c_0 c_1 = 1$.

Take the surjection $\pi_1' \to C$ given by sending each a_i and b_i to 1, sending c_0 to γ, and sending c_1 to γ^{-1}. The corresponding C-Galois cover $\psi_T : T \to X$ is branched just at τ and σ_1, where it is totally ramified. Each non-trivial subcover $T' \to X$ of $T \to X$ will also be totally ramified at τ and σ_1, and unramified at σ_2. On the other hand, each non-trivial subcover of $Y \to X$ will be branched at σ_2 if it is branched at τ and σ_1, because those three points have the same sets of inertia groups. So T is linearly disjoint from Y over X. Hence the normalized pullback $\phi_T : Y_T \to T$ of $Y \to X$ via $T \to X$ is an irreducible G-Galois cover. Moreover at the unique points $\tau', \sigma_1' \in T$ over $\tau, \sigma_1 \in X$, the inertia groups of $Y_T \to T$ have no tame part, by Abhyankar's Lemma. Also, the Galois group C of $T \to X$ has order

prime to p, whereas P is a p-group. Hence the inertia groups of $Y_T \to T$ over τ', σ'_1 are all isomorphic to P.

If P is trivial then the cover $\phi_T : Y_T \to T$ is unbranched over the unique point $\tau' \in T$ lying over $\tau \in X$. Taking $\psi_s : X_s \to X$ to be $\psi_T : T \to X$, the normalized pullback $\phi_T : Y_T \to T$ then plays the role of $\phi_s : Y_s \to X_s$, and the result is shown in this case.

On the other hand if P is non-trivial, we apply Case 1 to $\phi_T : Y_T \to T$ with Σ replaced by $\psi_T^{-1}(\Sigma)$, τ replaced by τ' and σ replaced by σ'_1. Doing so we obtain a cover $\psi : X_1 \to T$ that is étale away from $\phi_T^{-1}(\Sigma)$, linearly disjoint from $\phi_T : Y_T \to T$, and such that the normalized pullback $\phi_1 : Y_1 \to X_1$ of ϕ_T via ψ contains at least one point τ_1 in $\psi^{-1}(\tau')$ that is unbranched in ϕ_1. The cover $\phi_1 : Y_1 \to X_1$ is the normalized pullback of $\phi : Y \to X$ via $\psi_1 = \psi_T \circ \psi$ because ϕ_T is the normalized pullback of ϕ via ψ_T and ϕ_1 is the normalized pullback of ϕ_T via ψ. Similarly, X_1 is linearly disjoint from Y over X. Since $\psi_1(\tau_1) = \psi_T \circ \psi(\tau_1) = \psi_T(\tau') = \tau$, the cover ψ_1 has the asserted properties.

Case 3: *The general case when $s = 1$.*

Applying Lemma 1 to the cover $\phi : Y \to X$ with $t = 3$ and $\Delta = \{\tau\}$, there exists a cover $\pi : Z \to X$ of degree at least 3 that is linearly disjoint from $\phi : Y \to X$, is étale away from Δ, and such that $|\pi^{-1}(\Delta)| \geq 3$. Let Y_Z be the normalization of $Y \times_X Z$; this is irreducible because of linear disjointness. Hence the induced map $\phi_Z : Y_Z \to Z$ is G-Galois. Moreover, for any two points $\tau_1, \tau_2 \in \pi^{-1}(\tau)$, the sets of inertia groups of ϕ_Z over τ_1 and over τ_2 are the same since π is Galois. The result follows by applying Case 2 to the cover $Y_Z \to Z$ with Σ replaced by $\pi^{-1}(\Sigma)$, and τ, σ_1, and σ_2 replaced by any three points in $\pi^{-1}(\tau)$. $\qquad\square$

3. Solving embedding problems with restricted branch locus.

The goal of this section is to prove Theorem 5, which is a strengthening of the main theorem of [Ha99] (Theorem 5.4 of that manuscript) concerning embedding problems for branched covers over an algebraically closed field. That result gave a bound on the number of additional branch points that are needed in order to solve a given embedding problem. In the version we prove below, we also show that the positions of the additional branch points can be specified in advance (unlike [Ha99, Theorem 5.4], which allowed only one of these additional points to be specified).

In order to prove this strengthening, we will build upon the assertion of Theorem 5.4 of [Ha99], proceeding in two steps. First, we allow the additional branch points to move, obtaining a family of solutions to the given embedding problem each with the same number of additional branch points. Second, we show that the additional branch points can be specialized to the desired locations, while preserving the irreducibility and separability of the cover. The assertions needed to carry out these two steps appear in the next two propositions.

As before, k is an algebraically closed field of characteristic $p > 0$.

Proposition 3. *Let G be a finite group, let X be a smooth connected projective k-curve, and let $f : Y \to X$ be a G-Galois cover with branch locus B. Write B as a disjoint union $B_0 \cup B_1$ where $B_0 = \{\xi_1, \ldots, \xi_r\}$ is a set of $r > 0$ distinct k-points.*

  (a) *Then there is an integral affine k-variety T, a dominating morphism $\pi = (\pi_1, \ldots, \pi_r) : T \to X^r$, and a G-Galois cover $\tilde{f} : \tilde{Y} \to X \times T$, such that the fibre over each $\tau \in T$ is a G-Galois cover of X with branch locus $B_\tau \cup B_1$, where $B_\tau := \{\pi_1(\tau), \ldots, \pi_r(\tau)\}$*

is disjoint from B_1; and such that for some $\tau_0 \in \pi^{-1}(\xi_1, \ldots, \xi_r)$, the fibre over τ_0 is isomorphic to the given G-Galois branched cover $Y \to X$.

(b) *Moreover, if H is a quotient of G and $Y \to X$ dominates an H-Galois cover $g : Z \to X$ that is unramified over B_0, then the family $\tilde{f} : \tilde{Y} \to X \times T$ may be chosen to dominate the trivial family $\tilde{g} := g \times \mathrm{id} : Z \times T \to X \times T$.*

Proof. At each point ξ_i of X, choose a uniformizing parameter $u_i \in \mathcal{O}_{X,\xi_i}$. Embed the tensor square of $\mathcal{O}_{X,\xi_i}$ over k into the local ring of $X \times X$ at (ξ_i, ξ_i). For short, we write $x_i, t_i \in \mathcal{O}_{X \times X, (\xi_i, \xi_i)}$ for the elements identified with $u_i \otimes 1, 1 \otimes u_i$ respectively, and we also identify these elements with their images in the complete local ring $\hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}$. Consider the inclusion of complete local rings $\iota_i : \hat{\mathcal{O}}_{X,\xi_i} \hookrightarrow \hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}$ given by $u_i \mapsto x_i - t_i$. With respect to ι_i, for each point $\eta \in Y$ lying over ξ_i we may form the tensor product $A_\eta := \hat{\mathcal{O}}_{Y,\eta} \otimes_{\hat{\mathcal{O}}_{X,\xi_i}} \hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}$; here we regard $\hat{\mathcal{O}}_{Y,\eta}$ as an $\hat{\mathcal{O}}_{X,\xi_i}$-algebra via f. Then $A_i := \prod_{f(\eta)=\xi_i} A_\eta$ is a G-Galois algebra over $\hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}$ which is ramified precisely over the prime ideal $(x_i - t_i)$ of $\hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}$. Moreover the fibre of its spectrum mod t_i is isomorphic to the pullback of $Y \to X$ over $\hat{\mathcal{O}}_{X,\xi_i}$ as a G-Galois cover. (Here A_i is the induced algebra $\mathrm{Ind}_{G_\eta}^G A_\eta$, where G_η is the inertia group at η.)

Let $R^* = k[[t_1, \ldots, t_r]]$, which we identify with the complete local ring of X^r at the point $\underline{\xi} := (\xi_1, \ldots, \xi_r) \in X^r$, and whose fraction field will be denoted by F^*. For each i write $R_i^* = \hat{\mathcal{O}}_{X \times X^r, (\xi_i, \underline{\xi})}$ and let $A_i^* = A_i \otimes_{\hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}} R_i^*$; here we regard R_i^* as a module over $\hat{\mathcal{O}}_{X \times X, (\xi_i, \xi_i)}$ via the map $X \times X^r \to X \times X$ that projects X^r onto its i-th factor. Since $r > 0$, the curve $U = X - B_0$ is affine and may be written as $U = \mathrm{Spec}\, R_U$. Since the morphism f is finite, it is affine; and hence $f^{-1}(U) \subset Y$ is affine. Let A_U be its ring of functions, and let $A_U^* = A_U \otimes_{R_U} R_U^*$, where $R_U^* = R_U[[t_1, \ldots, t_r]]$. The mod $(t_1, \ldots, t_r)$-reductions of A_U^* and A_i^* can be identified with A_U and $A_i/t_i A_i$; and so we obtain isomorphisms between the G-Galois algebras that these reductions induce over $\hat{\mathcal{K}}_{X,\xi_i}$, the fraction field of $\hat{\mathcal{O}}_{X,\xi_i}$. Writing $\hat{\mathcal{K}}_{X,\xi_i}^* = \hat{\mathcal{K}}_{X,\xi_i}[[t_1, \ldots, t_r]]$, we thus obtain an identification $A_U^* \otimes_{R_U^*} \hat{\mathcal{K}}_{X,\xi_i}^* = A_i^* \otimes_{R_i^*} \hat{\mathcal{K}}_{X,\xi_i}^*$ of G-Galois étale algebras, for each i.
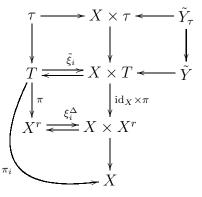
We now apply the patching result Theorem 3.2.8 of [Ha03b] in the context of the category of G-Galois covers (see also General Principle 2.2.4 of that paper). As a consequence, we obtain a G-Galois branched cover $Y^* \to X^* := X \times_k R^*$ whose restrictions to $\mathrm{Spec}\, R_U^*$ and to $\mathrm{Spec}\, R_i^*$ respectively agree with $\mathrm{Spec}\, A_U^*$ and $\mathrm{Spec}\, A_i^*$, and whose closed fibre is isomorphic to $Y \to X$. The cover is branched at the points of $B_1^* := B_1 \times_k R^*$ and at r distinct R^*-points $\xi_1^*, \ldots, \xi_r^*$. Here ξ_i^* is the pullback to $X^* = X \times_k \mathcal{O}_{X^r, \underline{\xi}}$ of the X^r-point ξ_i^Δ of $X \times X^r$, where ξ_i^Δ is the X^r-point at which the first coordinate (in X) is equal to the ith entry of the second coordinate. More formally, if $\mathrm{pr}_i : X^r \to X$ is the ith projection, then ξ_i^Δ is the inverse image of the diagonal $\Delta \subset X \times X$ under $\mathrm{id}_X \times \mathrm{pr}_i : X \times X^r \to X \times X$.

Observe that Y^* is smooth over R^*. Namely, Y^* is projective over R^* since X^* is; hence if the singular locus of Y^* is a (necessarily closed) non-empty subset, its image in $\mathrm{Spec}(R^*)$ contains the closed point. But then this point is in the image of the singular locus of Y over k, contradicting the smoothness of Y.

Note that in the situation of part (b), $Y^* \to X^*$ dominates the H-Galois cover $g^* = g \times \mathrm{id}_{\mathrm{Spec}\,R^*} : Z^* := Z \times_k R^* \to X^*$, since it does over R_U^*, R_i^*, and $\hat{\mathcal{K}}_{X,\xi_i}^*$, compatibly, and since the patching assertion of [Ha03b, Theorem 3.2.8] is an equivalence of categories.

Let $X' = \mathrm{Spec}\,S$ be an affine open neighborhood of $\{\xi_1, \ldots, \xi_r\}$ in X. We may regard $S^{\otimes r}$ as a subring of the local ring of X^r at ξ, and hence as a subring of R^*. The G-Galois cover $Y^* \to X^*$ is defined by equations involving only finitely many elements of R^*. The $S^{\otimes r}$-subalgebra $\tilde{R} \subset R^*$ generated by these elements is of finite type, and the G-Galois cover $Y^* \to X^*$ is induced by the G-Galois cover $\tilde{Y} \to X \times T$, where $T = \mathrm{Spec}(\tilde{R})$. Here $\tilde{R}$ is a domain and T is integral. The inclusion $S^{\otimes r} \hookrightarrow \tilde{R}$ corresponds to a morphism $T \to X'^r$; composing this with the embedding $X'^r \hookrightarrow X^r$ yields a morphism $\pi : T \to X^r$.

Possibly after enlarging the choice of $\tilde{R}$ by adjoining finitely many elements of R^* (which are inverses of elements of $\tilde{R}$), we may assume that the above properties of $Y^* \to X^*$ descend to $\tilde{Y} \to X \times T$. Specifically, since Y^* is smooth over R^*, we may choose $\tilde{Y}$ so as to be smooth over T. Similarly, we may choose $\tilde{Y} \to X \times T$ so that its fibre over $X \times \{\tau_0\}$ is isomorphic to $Y \to X$, where $\tau_0 \in T$ is the image of the closed point of $\mathrm{Spec}\,R^*$ (i.e. τ_0 corresponds to the contraction of the maximal ideal of R^* under the inclusion $\tilde{R} \hookrightarrow R^*$). Moreover we may choose $\tilde{Y} \to X \times T$ so that its branch locus consists of $\tilde{B}_1 := B_1 \times T$ together with r distinct T-points $\tilde{\xi}_i$, where $\tilde{\xi}_i : T \to X \times T$ is the pullback to $X \times T$, via $\pi : T \to X^r$, of the X^r-point ξ_i^Δ of $X \times X^r$ defined above. (Thus the image of $\tilde{\xi}_i$, which is a closed subset of $X \times T$ isomorphic to T, is a component of the branch locus of $\tilde{Y} \to X \times T$.) In addition we may assume that the loci $\tilde{\xi}_i$ are pairwise disjoint and are disjoint from $\tilde{B}_1$. Finally, in part (b), we may assume that $\tilde{Y} \to X \times T$ dominates the H-Galois cover $\tilde{g} = g \times \mathrm{id}_T : Z \times T \to X \times T$.

The morphism π is dominating since $S^{\otimes r} \hookrightarrow \tilde{R}$ is an inclusion. Write $\pi_i = \mathrm{pr}_i \circ \pi : T \to X$; so $\pi = (\pi_1, \ldots, \pi_r) : T \to X^r$. Let $p : X \times X^r \to X$ denote projection onto the first factor X. Then $p \circ \xi_i^\Delta = \mathrm{pr}_i : X^r \to X$. Also $\xi_i^\Delta \circ \pi = (\mathrm{id}_X \times \pi) \circ \tilde{\xi}_i : T \to X \times X^r$, since $\tilde{\xi}_i$ is the pullback of ξ_i^Δ. Thus $\pi_i = \mathrm{pr}_i \circ \pi = p \circ (\mathrm{id}_X \times \pi) \circ \tilde{\xi}_i : T \to X$. Also, $\Delta \circ \mathrm{pr}_i = (\mathrm{id}_X \times \mathrm{pr}_i) \circ \xi_i^\Delta : X^r \to X \times X$, where $\Delta : X \to X \times X$ is the diagonal map. So $\tilde{\xi}_i = (\pi_i, \mathrm{id}_T)$, and hence the restriction of $\tilde{\xi}_i$ to the fiber over τ is $\pi_i|_\tau : \tau \to X$ if we identify $X \times \tau$ with X.

So for each $\tau \in T$, the corresponding fibre of $\tilde{Y} \to X \times T$ is branched at B_1 and at the r points $\pi_i(\tau)$ of X which are respectively the restrictions of the T-points $\tilde{\xi}_i$ of $X \times T$. Since $\tilde{\xi}_i$ pulls back to ξ_i^* under $\mathrm{Spec}\,R^* \to T$, and since ξ_i^* restricts to ξ_i on the closed fibre of $X^* = X \times_k R^*$, we have that $\pi_i(\tau_0) = \xi_i$. Finally, the fibres of $\tilde{Y} \to X \times T \to T$ are connected by Zariski's Connectedness Theorem (see [Ht, III, Exercise 11.4] or [EGA III.1,
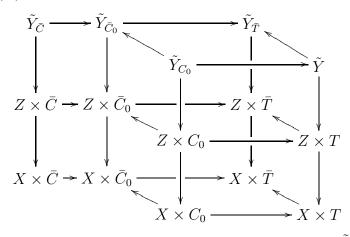
Théorème 4.3.1]) since the morphism is projective and the fibre over τ_0 is the connected curve Y. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 4. *In the situation of Proposition 3(b), assume that $N := \ker(G \to H)$ has order prime to p. Let $B' = \{\xi_1', \ldots, \xi_r'\}$ be a set of r distinct k-points of X that is disjoint from B_1. Then there is a G-Galois cover $f' : Y' \to X$ that has branch locus $B' \cup B_1$ and dominates $g : Z \to X$.*

Proof. Let $\tilde{f} : \tilde{Y} \to X \times T$ be as in the conclusion of Proposition 3(b). If $\underline{\xi}' := (\xi_1', \ldots, \xi_r')$ is in the image of the dominating morphism $\pi : T \to X^r$, then the conclusion follows from the properties of $\tilde{f}$ and its fibres. More generally, we proceed as follows:

Since T is an affine k-variety, we may regard T as a closed subset of some $\mathbb{A}_k^n$. The graph $\Gamma \subset T \times X^r \subset \mathbb{A}_k^n \times X^r$ of π is isomorphic to T by the first projection map; so replacing T by Γ we may assume that $T \subset \mathbb{A}_k^n \times X^r$ and that π is the second projection. Let $\bar{T}$ be the closure of T in $\mathbb{P}_k^n \times X^r$ and let $\bar{\pi} : \bar{T} \to X^r$ be the second projection map. Since the projective morphism $\bar{\pi}$ extends $\pi : T \to X^r$, it is dominating and hence surjective. Let $\tau_1 \in \bar{T}$ be a point that lies over $\underline{\xi}' \in X^r$. Since T is integral, there is an integral curve $\bar{C}_0$ in $\bar{T}$ that passes through τ_0 and τ_1. Let $C_0 = \bar{C}_0 \cap T$. Thus $\bar{C}_0$, being integral and closed in $\bar{T}$, is the closure of C_0 in $\bar{T}$.

The normalization of $X \times \bar{T}$ in the function field of $\tilde{Y}$ is a G-Galois branched cover $\tilde{Y}_{\bar{T}} \to Z \times \bar{T} \to X \times \bar{T}$ whose restriction to $X \times T$ is $\tilde{Y} \to Z \times T \to X \times T$. The pullback of this cover via $X \times \bar{C}_0 \to X \times \bar{T}$ is a G-Galois branched cover $\tilde{Y}_{\bar{C}_0} \to Z \times \bar{C}_0 \to X \times \bar{C}_0$ whose restriction to $X \times C_0$ is branched only over $B_1 \times C_0$ and at the C_0-points $\xi_{i,C_0} := (\pi_i, \mathrm{id}_{C_0}) : C_0 \to X \times C_0$ for $i = 1, \ldots, r$ (where, as before, $\pi_i : X^r \to X$ is the ith projection). Let $\bar{C}$ be the normalization of the curve $\bar{C}_0$, and let $\tilde{Y}_{\bar{C}} \to Z \times \bar{C} \to X \times \bar{C}$ be the normalized pullback of $\tilde{Y}_{\bar{C}_0} \to Z \times \bar{C}_0 \to X \times \bar{C}_0$ via $X \times \bar{C} \to X \times \bar{C}_0$. So $\tilde{Y}_{\bar{C}} \to X \times \bar{C}$ is unramified away from $B_1 \times \bar{C}$, the $\bar{C}$-points $\xi_{i,\bar{C}} := (\bar{\pi}_i, \mathrm{id}_{\bar{C}})$, and $X \times S$, where $S \subset \bar{C}$ is the finite set of points whose image in $\bar{C}_0$ does not lie in C_0. (Here $\bar{\pi}_i : \bar{C} \to X$ denotes the composition of the normalization map $\bar{C} \to \bar{C}_0$ with the projection map $\pi_i : \bar{C}_0 \to X$.) Hence the N-Galois cover $\tilde{Y}_{\bar{C}} \to Z \times \bar{C}$ is unramified away from $g^{-1}(B_1) \times \bar{C}$, the pullbacks $(g \times \mathrm{id}_{\bar{C}})^*(\xi_{i,\bar{C}})$ to $Z \times \bar{C}$ for $i = 1, \ldots, r$, and $Z \times S$.

Let m be the least common multiple of the ramification indices of $\tilde{Y}_{\bar{C}} \to Z \times \bar{C}$ over the generic points of $Z \times S$. Thus m is prime to p, since the order of N is prime to p. Choose a rational function f on $\bar{C}$ that has a simple pole at each point of S, and possibly other

poles. Adjoining an m-th root of f and normalizing yields a cyclic branched cover $\bar{C}' \to \bar{C}$ of degree m that is totally ramified at the points of S (and possibly elsewhere). Then by Abhyankar's Lemma, the normalized pullback $\tilde{Y}_{\bar{C}'} \to Z \times \bar{C}'$ of $\tilde{Y}_{\bar{C}} \to Z \times \bar{C}$ is unramified at the generic points of $Z \times S'$, where $S' \subset \bar{C}'$ is the inverse image of $S \subset \bar{C}$. So by Purity of Branch Locus ([Za, Proposition 2] or [Na, 41.1]), $\tilde{Y}_{\bar{C}'} \to Z \times \bar{C}'$ is branched only over $g^{-1}(B_1) \times \bar{C}'$ and at the pullbacks $(g \times \mathrm{id}_{\bar{C}'})^*(\xi_{i,\bar{C}'})$ to $Z \times \bar{C}'$ for $i = 1, \dots, r$, where $\xi_{i,\bar{C}'}$ is the pullback of $\xi_{i,\bar{C}}$ under $\bar{C}' \to \bar{C}$.

Let $\tau' \in \bar{C}'$ be a point mapping to $\tau_1 \in \bar{C}_0$ via $\bar{C}' \to \bar{C} \to \bar{C}_0$. The components of $B_1 \times \bar{C}'$ and the $\bar{C}'$-points $\xi_{i,\bar{C}'}$ are each smooth over $\bar{C}'$, as are their inverse images under $Z \times \bar{C}' \to X \times \bar{C}'$. In the fibre of $X \times \bar{C}'$ over τ', these loci specialize to distinct points of X (viz. to the points of B_1 and B' respectively); hence their inverse images in $Z \times \bar{C}'$ are also disjoint over τ'. So the union of these inverse images is smooth over τ', and there is an affine neighborhood $C' \subset \bar{C}'$ of τ' such that the (tame) branch locus of the restriction $\tilde{Y}_{C'} \to Z \times C'$ of $\tilde{Y}_{\bar{C}'} \to Z \times \bar{C}'$ is smooth over $\bar{C}'$. By Abhyankar's Lemma (in the form [GM, Theorem 2.3.2] or [SGA1, Proposition 5.5]), the cover $\tilde{Y}_{C'} \to Z \times C'$ is a Kummer cover étale locally, with smooth branch locus, hence is smooth over C'. So the fibre over τ' is smooth over k.

The intersection of C' with the inverse image of C_0 under $\bar{C}' \to \bar{C}_0$ is a dense open subset of C'. For each point τ in this dense open subset, the fibre of the projective morphism $\tilde{Y}_{C'} \to C'$ over τ is connected, by the conclusion of Proposition 3. By Zariski's Connectedness Theorem [Ht, III, Exercise 11.4], it follows that each fibre of $\tilde{Y}_{C'} \to C'$ is connected, in particular the fibre over τ'. So the fibre of the G-Galois cover $\tilde{Y}_{C'} \to X \times C'$ over the point $\tau' \in C'$ is a G-Galois cover of X with the desired properties. $\qquad \square$

We recall the definition of relative rank of a subgroup of a finite group (see Section 2 of [Ha99]). Let H be any finite group and let E be a subgroup of H. A subset $S \subset E$ will be called a *relative generating set* for E in H if for every subset $T \subset H$ such that $E \cup T$ generates H, the subset $S \cup T$ also generates H. We define *the relative rank of E in H* to be the smallest non-negative integer $s := \mathrm{rk}_H(E)$ such that there is a relative generating set for E in H consisting of s elements. Thus every generating set for E is a relative generating set, and so $0 \le \mathrm{rk}_H(E) \le \mathrm{rk}(E)$. Also, $\mathrm{rk}_H(E) = \mathrm{rk}(E)$ if E is trivial or $E = H$, while $\mathrm{rk}_H(E) = 0$ if and only if E is contained in the Frattini subgroup $\Phi(H)$ of H [Ha99, p. 122]. Also recall that $p(G)$ denotes the subgroup generated by the p-subgroups of a group G.

Theorem 5. *Let N be a normal subgroup of a finite group G, and let $H = G/N$. Let $\bar{N} = N/p(N)$ and $\bar{G} = G/p(N)$, and let $r = \mathrm{rk}_{\bar{G}}(\bar{N})$. Let $V \to U$ be an H-Galois étale cover of affine k-curves and let $\xi_1, \dots, \xi_r \in U$ be distinct points. Then there is a G-Galois cover $W \to U$ branched only at $\xi_1, \dots, \xi_r \in U$ such that $W/N \to U$ is isomorphic to $V \to U$ as an H-Galois cover.*

Proof. We may identify $\bar{G}/\bar{N}$ with H. Let X be the smooth completion of U; i.e., the unique smooth proper k-curve that contains U as an affine dense open subset. Let Z be the normalization of X in the function field of V. Thus the inclusion $U \hookrightarrow X$ lifts to an inclusion $V \hookrightarrow Z$. Also let $B_1 = X - U$, let $B_0 = \{\xi_1, \dots, \xi_r\}$, and let $B = B_0 \cup B_1$. We claim that there is a $\bar{G}$-Galois cover $\bar{Y} \to X$ that dominates $Z \to X$ and is branched only at B; or equivalently, a $\bar{G}$-Galois cover $\bar{W} \to U$ that dominates $V \to U$ and is branched only at B_0.

First suppose $r = 0$. For some subgroup $\bar{G}_0 \subset \bar{G}$, the cover H-Galois cover $V \to U$ is dominated by a $\bar{G}_0$-Galois étale cover $\bar{W} \to U$, because the fundamental group of the affine k-curve U has cohomological dimension 1 [Se90, Proposition 1] and hence is projective (by [Se73, I, §3.4 Prop. 16 and §5.9 Prop. 45]). Thus $\bar{G}/\bar{N} = H = \bar{G}_0/(\bar{G}_0 \cap \bar{N}) = \bar{G}_0\bar{N}/\bar{N}$ and so $\bar{G}$ is generated by $\bar{G}_0$ and $\bar{N}$. But $\bar{N}$ is contained in the Frattini subgroup of $\bar{G}$ because $r = 0$. Hence $\bar{G}_0 = \bar{G}$, and $\bar{W} \to U$ is as desired. (Cf. also [FJ, Proposition 22.5.9(a)].)

On the other hand if $r \geq 1$, then by [Ha99, Theorem 5.4] there is a smooth connected $\bar{G}$-Galois cover $\bar{W}_0 \to U$ branched at a set B' of r points such that $\bar{W}_0/\bar{N} \to U$ is isomorphic to $V \to U$ as an H-Galois cover. Let $\bar{Y}_0$ be the normalization of X in $\bar{W}_0$. So $Z = \bar{Y}_0/\bar{N}$; $Z \to X$ is unramified outside of $B_1 := X - U$; and $\bar{Y}_0 \to X$ is unramified outside of $B_1 \cup B'$. Since $\bar{N}$ has order prime to p, Proposition 4 above asserts that there is a smooth connected $\bar{G}$-Galois branched cover $\bar{Y} \to X$ that is branched only at B and that dominates $Z \to X$. This completes the proof of the claim.

Since $p(N)$ is a quasi-p group and B is non-empty, Theorem 4.4 of [Ha03a] asserts that the $\bar{G}$-Galois cover $\bar{Y} \to X$ is dominated by some smooth connected G-Galois branched cover $Y \to X$ that is also branched only over B. Since $\bar{Y} \to X$ dominates the H-Galois cover $Z \to X$, so does $Y \to X$; and hence the restriction $W \to U$ of $Y \to X$ over U has the desired properties. $\qquad\square$

4. Main Theorem

Using the results of the previous sections, we prove our main theorem, that the fundamental group of any affine curve in characteristic $p > 0$ is almost ω-free. To do so we combine the "newly unbranched" points in the covers obtained in Section 2 with the main result of Section 3 to solve the induced embedding problem.

Theorem 6. *Let k be an algebraically closed field of characteristic p, let X be a smooth connected projective curve, let $\Sigma \subset X$ be a non-empty set of closed points in X, and let $C = X - \Sigma$. Then $\pi_1(C)$ is almost ω-free.*

Proof. Explicitly, we wish to show the following. Let $\mathcal{E} = (\alpha : \pi_1(C) \to G, f : \Gamma \to G)$ be an embedding problem for $\pi_1(C)$. Then there exists an open subgroup $H \subset \pi_1(C)$ such that $\alpha_H := \alpha|_H$ is surjective and such that the induced embedding problem $\mathcal{E}_H = (\alpha_H, f)$ has a proper solution.

Let $N = \ker(f)$, and write $\bar{N} = N/p(N)$ and $\bar{\Gamma} = \Gamma/p(N)$. Let $\tau \in \Sigma$ and let $s = \mathrm{rk}_{\bar{\Gamma}}(\bar{N})$ (see the definition of relative rank just before Theorem 5). Let $\phi : Y \to X$ be the pointed G-Galois cover corresponding to α. By Proposition 2 there exists a cover $\psi_s : X_s \to X$ that is étale away from Σ; contains at least s points $\{\tau_1, \tau_2, ..., \tau_s\} \subset X_s$ over $\tau \in X$; and whose normalized pullback $\phi_s : Y_s \to X_s$ of ϕ over ψ_s is a G-Galois cover étale away from $\Delta := \psi_s^{-1}(\Sigma) - \{\tau_1, \tau_2, ..., \tau_s\}$. Let $\Sigma_s = \psi_s^{-1}(\Sigma)$. Applying Theorem 5 to the étale loci of $\phi_s : Y_s \to X_s$, there exists a smooth connected Γ-Galois cover $Z_s \to X_s$ that is étale away from $\Delta \cup \{\tau_1, ..., \tau_s\} = \Sigma_s$, such that Z_s/N is isomorphic to Y_s as G-Galois covers of X_s. Now, $\psi_s : X_s \to X$ is ramified at most over Σ, and the pullback $\phi_s : Y_s \to X_s$ of ϕ via ψ_s is a G-Galois cover. Moreover, the Γ-Galois cover $Z_s \to X_s$ dominates $Y_s \to X_s$. After some choice of base point, $\psi_s : X_s \to X$ corresponds to a subgroup $H \subset \pi_1(X - \Sigma)$ and a surjection $\alpha_H : H \to G$. We then find that $\alpha|_H = \alpha_H$ and that the induced embedding problem $\mathcal{E}_H = (\alpha_H, f)$ has a proper solution. $\qquad\square$

The situation in characteristic zero is somewhat different, because fundamental groups of curves in characteristic zero are finitely generated. Specifically, there is the following result:

Proposition 7. *Let X be a smooth connected projective curve of genus g defined over an algebraically closed field k of characteristic 0 and let Σ be a non-empty finite collection of points on X. The group $\pi_1(X - \Sigma)$ is almost ω-free if and only if $X - \Sigma$ is not isomorphic to the once or twice punctured projective line.*

Proof. Suppose first that $X - \Sigma$ is not isomorphic to the once or twice punctured projective line. Thus $g \geq 0$, $r := |\Sigma| \geq 1$, and either $g \geq 1$ or $r \geq 3$; hence $2g + r \geq 3$. Let $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ be a finite embedding problem for $\pi_1(X - \Sigma)$. Choose a prime number d that is greater than the order of Γ. The surjection $\alpha : \pi_1(X - \Sigma) \to G$ determines a G-Galois cover $Y \to X$ that is étale away from Σ. Pick $\tau \in \Sigma$ and set $\Sigma' = \Sigma - \{\tau\}$. Thus $X - \Sigma'$ is isomorphic either to the projective line with at least two points deleted, or else to a dense open subset of a curve of genus at least one. Hence $\pi_1(X - \Sigma')$ has a quotient that is cyclic of order d, corresponding to a d-cyclic cover $\phi_d : Z_d \to X$ that is étale away from Σ'. The unramified fibre $\phi_d^{-1}(\tau)$ contains at least d points, and $\Sigma_d := \phi_d^{-1}(\Sigma)$ contains at least $r - 1 + d > 0$ points. Since Z_d has genus at least g, the fundamental group $\Pi_0 = \pi_1(Z_d - \Sigma_d) \subset \pi_1(X - \Sigma)$ is free of rank at least $2g + (r - 1 + d) - 1 > d$. Also, $Z_d \to X$ is linearly disjoint from $Y \to X$ because they have no common subcovers, the degree of $Z_d \to X$ being a prime number greater than the degree of $Y \to X$. Thus $\alpha|_{\Pi_0} : \Pi_0 \to G$ is surjective. Since the rank of Π_0 is larger than the minimal number of generators of Γ, by [FJ, Proposition 17.7.3] there is a proper solution to the induced embedding problem $\mathcal{E}_0 = (\alpha|_{\Pi_0} : \Pi_0 \to G, f : \Gamma \to G)$. Thus $\pi_1(X - \Sigma)$ is almost ω-free.

Conversely, suppose that $X - \Sigma$ is the once or twice punctured projective line over k. In the former case $X - \Sigma$ has no non-trivial étale covers, while in the latter case every finite étale cover is itself isomorphic to the twice punctured projective line. Thus any open subgroup Π_0 of $\Pi = \pi_1(X - \Sigma)$ is either trivial or has rank one. So given a finite embedding problem $\mathcal{E} = (\alpha : \Pi \to G, f : \Gamma \to G)$ for Π such that Γ is not cyclic, there is no proper solution to the induced embedding problem $\mathcal{E}_0$ for any open subset Π_0. Thus $\pi_1(X - \Sigma)$ is not almost ω-free. $\qquad\square$

This result helps clarify the relationships among the conditions on a profinite group being free, ω-free, and almost ω-free. For infinitely generated profinite groups, every free group is ω-free (i.e. every finite embedding problem has a proper solution) but not conversely, as noted in the introduction. Also, every infinitely generated ω-free profinite group is (trivially) almost ω-free; and again the converse fails, since the fundamental group of an affine curve in characteristic p is almost ω-free by Theorem 6 but is not ω-free, as noted in the introduction. In contrast, finitely generated free profinite groups Π are not ω-free, since an embedding problem $\mathcal{E} = (\Pi \to G, \Gamma \to G)$ cannot have a proper solution if the rank of Γ is greater than that of Π. Concerning the almost ω-free condition, the above proposition has the following consequence:

Corollary 8. *A free profinite group of finite rank m is almost ω-free if and only if $m > 1$.*

Proof. A free profinite group of finite rank $m \geq 0$ is isomorphic to the algebraic fundamental group of the complex projective line minus $m + 1$ points. Thus the result follows from the proposition above. $\qquad\square$

Note that this corollary could alternatively be proved using that an open subgroup of index i in a free group of finite rank m is itself free of finite rank $1+i(m-1)$ [RiZa, Theorem 3.6.2(b)]. Proposition 7 could then be deduced from that using that the fundamental group of an r-punctured curve of genus g (for $r > 0$) is free of rank $2g + r - 1$. On the other hand, while the proof given in [RiZa, Theorem 3.6.2(b)] is purely group theoretic, that result can alternatively be deduced easily from the Riemann-Hurwitz formula, using fundamental groups of affine curves in characteristic zero, just as Corollary 8 was deduced above from Proposition 7.

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