Generalizations of Abhyankar's Conjecture

An appendix to *Desingularization and modular Galois theory* by S.S. Abhyankar

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This appendix considers several possible generalizations of Abhyankar's Conjecture on fundamental groups in finite characteristic. The original version of the conjecture considered the case of affine curves over an algebraically closed field of characteristic p. In generalizing this, one may consider higher dimensional analogs in both local and global situations, and also the case of affine curves over a finite field. The preceding paper [A3] states several conjectures in those situations. This appendix regards these newer conjectures of Abhyankar as special cases of a more general conjecture, and discusses what is currently known about when that conjecture holds. In particular, it discusses why the local and global Normal Crossings Conjectures of the preceding paper must in general be modified.

The original version of Abhyankar's Conjecture [A1] describes $\pi_A(X)$, for X an affine curve over an algebraically closed field k of characteristic p; here $\pi_A(X)$ denotes the class of finite groups that are Galois groups of unramified covers of X. If X is the affine line over k, it says that $\pi_A(X)$ is the class of quasi-p groups (i.e. finite groups with no non-trivial prime-to-p quotients). More generally, if X is an affine curve obtained by deleting $t+1$ points from a smooth projective k-curve of genus g, it says that $\pi_A(X)$ consists of the finite groups G such that $G/p(G)$ can be generated by a set of at most $2g+t$ elements; here $p(G)$ is the subgroup of G generated by all the p-subgroups. Raynaud proved this conjecture in the affine k-line case [R], and afterwards I proved it for arbitrary affine k-curves [H].

Abhyankar's Conjecture is based on two principles: First, that the prime-to-p Galois groups over a characteristic p affine variety X should be the same as for an analogous complex variety. And second, that all quasi-p groups should occur as Galois groups over X. The precise meaning of the first of these two principles depends, of course, on the meaning of "an analogous complex variety". In the case of curves of genus g minus $t+1$ points, one simply interprets this to mean a complex curve of the same form. Similarly, one can interpret this in the higher dimensional global case of projective n-space minus $t+1$ hyperplanes crossing normally, and the higher dimensional local case of $\operatorname{Spec} k[[x_1,\ldots,x_n]][(x_1\cdots x_t)^{-1}]$ with $1 \le t \le n$. In these situations, the first principle does in fact hold; this was shown by Grothendieck [G] and Popp [P] in the case of curves, and by Abhyankar [A2] in the higher dimensional case. Namely, a prime-to-p group is a Galois group over a curve of genus g minus $t+1$ points if and only if it has a set of at most $2g+t$ generators; and it is a Galois group in the higher dimensional local or global case if and only if it is abelian and has a set of at most t generators.

In light of the agreement of π_A with the complex case for prime-to-p groups, one can rephrase Abhyankar's Conjecture in the following form, where G runs over finite groups:

(AC) $G \in \pi_A(X) \Leftrightarrow G/p(G) \in \pi_A(X)$.

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That is, a finite group G is a Galois group of an unramified Galois cover $Y \to X$ if and only if $G/p(G)$ is Galois group over X; and the nontrivial part is the reverse implication.

In the local and global higher dimensional situations described above, it is natural to ask whether this assertion (AC) still holds. There, the question is whether a finite group G is a Galois group if and only if $G/p(G)$ is abelian on t generators. This is precisely what Abhyankar has stated as his Normal Crossings Local and Global Conjectures in the preceding paper. Similarly, one can ask whether the assertion (AC) holds for affine curves over finite fields k of characteristic p, and in particular for open subsets U of the affine k-line. In the case of the affine line itself, the prime-to-p Galois groups are just the prime-to-p cyclic groups, with Frobenius as generator; and the corresponding covers are totally arithmetic, viz. the affine line over a larger finite field. Thus in this situation, assertion (AC) says that π_A of the affine line should consist of the finite groups G such that $G/p(G)$ is cyclic; and this is precisely the Affine Arithmetical Conjecture of the preceding paper. One can also consider the case of branched covers of the affine line over a finite field, with no restriction on which points are branched. This corresponds to $X = \operatorname{Spec} k(x)$; and *if* one knew that every prime-to-p finite group is a Galois group over $k(x)$, then assertion (AC) would say that *every* finite group is a Galois group over $k(x)$ — which is the Total Arithmetical Conjecture of the preceding paper. (At the moment, though, the realization of all prime-to-p groups over $k(x)$ is known only if $p = 2$.)

While the above statement (AC) thus combines various conjectures of the preceding paper, there is still the question of whether (or when) this statement actually holds. In fact, as discussed below, it often does not, and must be modified.

The motivation for modifying statement (AC) arises from the proof of the original form of Abhyankar's Conjecture, i.e. the case of affine curves over an algebraically closed field k of characteristic p. The proof in [H] used that if G is a finite group and $G/p(G)$ has $2g + t$ generators, then $p(G)$ has a prime-to-p supplement H which also has $2g + t$ generators. This suggests the following variant to (AC):

(AC)$'$ $G \in \pi_A(X) \Leftrightarrow p(G)$ has a prime-to-p supplement in $\pi_A(X)$.

Because of the above comment on the existence of a supplement, statements (AC) and (AC)$'$ are equivalent in the case of affine curves over algebraically closed fields of characteristic p. Unfortunately, (AC) and (AC)$'$ are generally *not* equivalent in the higher dimensional case, because $p(G)$ need not have an *abelian* supplement even if $G/p(G)$ is abelian. Similarly, (AC) and (AC)$'$ are generally not equivalent in the case of open subsets of the affine line over a finite field. In particular, there are examples due to R. Guralnick, appearing in the appendix to [HP], which show that (AC) and (AC)$'$ are inequivalent in the higher dimensional case if $n = t = 2$, and also for the once-punctured affine line over a finite field. Hence (AC) and (AC)$'$, which are both consistent with the two underlying principles discussed earlier, are competing ways of generalizing Abhyankar's Conjecture to other situations.

In fact, the proof of Abhyankar's Conjecture for affine curves over an algebraically closed field [H] used even more — that the supplement to $p(G)$ can be chosen so as to normalize a Sylow p-subgroup of G. (This is essentially the Schur-Zassenhaus Theorem in group theory; cf. [Go].) That in turn suggests the following alternative variant to (AC):

(AC)$''$ $G \in \pi_A(X) \Leftrightarrow p(G)$ has a prime-to-p supplement $H \in \pi_A(X)$ that normalizes a Sylow p-subgroup of G.

Again, (AC)$''$ is equivalent to (AC) for affine curves over algebraically closed fields of characteristic p. And again, (AC)$''$ is generally inequivalent to (AC) in the higher dimensional local and global cases and for open subsets of the affine line over a finite field.

At the 1997 Gentner Symposium on Field Arithmetic in Tel Aviv, I had posed (AC), (AC)$'$, and (AC)$''$ as three distinct possible generalizations of Abhyankar's Conjecture to the higher dimensional situation and to the situation of affine curves over finite fields. More recently, during the fall 1999 semester at MSRI on Galois groups and fundamental groups, Marius van der Put and I were able to show [HP] that the condition on supplements in (AC)$'$ is in fact necessary in order for G to be a Galois group over X, in these situations (assuming $t \le n$). Since (AC) and (AC)$'$ are generally inequivalent, this shows that (AC) does *not* in general hold. Thus, in the higher dimensional (local or global) case, if G is a finite group such that $G/p(G)$ is abelian on t generators but $p(G)$ does not have an abelian supplement on t generators, then G cannot be a Galois group over X. And for the once-punctured affine line X over a finite field k of characteristic p, if $G/p(G)$ is metacyclic (a necessary condition for a prime-to-p group to be a Galois group over X) but $p(G)$ does not have a prime-to-p metacyclic supplement, then G cannot be a Galois group over X.

While one might hope that (AC)$'$ is in fact the correct generalization of the original Abhyankar Conjecture, even this fails: n the higher dimensional case, there is yet another condition, lying between that of (AC)$'$ and that of (AC)$''$, which is also necessary in order for G to be a Galois group, and which generally lies *strictly* between those other two conditions. (This condition says that $p(G)$ is generated by "good" p-subgroups, viz. those that are normalized by a prime-to-p supplement to $p(G)$ lying in $\pi_A(X)$. Cf. [HP].) Whether this more involved condition is both necessary and sufficient for $G \in \pi_A(X)$ remains open, however. I suspect that some condition between this one and the one appearing in (AC)$''$ will turn out to be the correct necessary and sufficient condition.

On the other hand, in certain cases, assertions (AC), (AC)$'$, and (AC)$''$ are equivalent, and the above issue goes away. Specifically, they are equivalent for

(a) affine curves over an algebraically closed field (as already discussed);

(b) affine n-space over an algebraically closed field with at most one hyperplane deleted;

(c) $\operatorname{Spec} k[[x_1, \ldots, x_n]][x_1^{-1}]$, with k algebraically closed;

(d) the affine line over a finite field of characteristic p.

In situations (a) and (b), the equivalent assertions (AC), (AC)$'$, and (AC)$''$ are known to hold, while in (c) and (d) these equivalent assertions are open (and appear to be true). Of course (a) is the original Abhyankar Conjecture; (b) and (c) are special cases of the global and local Normal Crossings Conjectures of the preceding paper; and (d) is the Affine Arithmetical Conjecture of the preceding paper. Moreover, if it is shown that every prime-to-p group is a Galois group over $k(x)$ (as is known for $p = 2$), then the three assertions would also be equivalent for the case of $X = \operatorname{Spec} k(x)$. In that case they would then be the same as the Total Arithmetical Conjecture of the preceding paper, which asserts an affirmative solution to the Inverse Galois Problem over $k(x)$. Again, this is expected to be true, though it remains open.

The above observations suggest that (AC) holds for a given affine variety X of characteristic p if and only if (AC) is equivalent to (AC)$'$ and (AC)$''$ for X. (This equivalence depends solely on the set of prime-to-p Galois groups over X.) Here we need to exclude certain cases that are too simple, like $\operatorname{Spec} k[[x_1, \ldots, x_n]]$ and $\operatorname{Spec} k((x))$, where not all quasi-$p$ groups can occur as Galois groups. This is consistent with Abhyankar's philosophy that his conjecture should be generalizable to varieties in characteristic p in which there is "enough room" for sufficient ramification to take place.

Moreover, results of R. Guralnick (in the appendix to [HP]) essentially show that under some reasonable hypotheses, the three assertions (AC), (AC)$'$, and (AC)$''$ are equivalent if and only if the class of prime-to-p groups in $\pi_A(X)$ is closed under Frattini extensions. This last condition is related to (but is more general than) the condition of $\pi_1(X)$ being a projective profinite group, or equivalently of X having cohomological dimension 1. This relationship suggests a possible connection with work of M. Fried (see e.g. [F]) on Frattini covers of curves.

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