Math 720 Problem Set #2 Due Mon., Oct. 21, 2013

1. a) For which primes p is there a $\sqrt{-1}$ in $\mathbb{Z}_p$? For the smallest such p, describe this element explicitly.

b) More generally, given a positive integer n, for which primes p is there a primitive n-th root of unity in $\mathbb{Z}_p$?

2. For each of the following local rings, is the completion an integral domain? a discrete valuation ring? a ring that is isomorphic to $k[[x]]$ for some field k? For the last of these, when the answer is yes, try to find an explicit isomorphism.

a) $\mathbb{Z}_{(p)}$

b) $\mathbb{R}[x]_{(x^2+1)}$

c) $(\mathbb{Q}[x,y]/(y^2-x^3))_{(x,y)}$

d) $(\mathbb{Q}[x,y]/(y^2-x^3-x^2))_{(x,y)}$

e) $(\mathbb{Q}[x,y]/(x^4+y^4-1))_{(x-1,y)}$

f) $\mathbb{Z}[x]_{(x)}$

g) $\mathbb{Z}[x]_{(2)}$

h) $\mathbb{C}[x,y]_{(x,y)}$

3. a) Suppose that p, p' are distinct prime numbers. Can the fields $\mathbb{Q}_p$ and $\mathbb{Q}_{p'}$ be isomorphic?

b) Let K be a finite extension of $\mathbb{Q}$, and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime lying over the rational prime p. Under what circumstances is the natural map $\mathbb{Q}_p \to K_\mathfrak{p}$ an isomorphism? (Here $K_\mathfrak{p}$ is the $\mathfrak{p}$-adic completion of K.)

4. Find an explicit ring isomorphism $W_3(\mathbb{F}_2) \to \mathbb{Z}/8\mathbb{Z}$.

5. Let R be a Noetherian domain, and let $\mathfrak{p}$ be a non-zero prime ideal of R. Let $R_\mathfrak{p}$ be the local ring of R at $\mathfrak{p}$. Let $\hat{R}_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of R, viz. $\varprojlim R/\mathfrak{p}^n$.

a) Show that if $\mathfrak{p}$ is maximal, then there is a natural inclusion $R_\mathfrak{p} \hookrightarrow \hat{R}_\mathfrak{p}$.

b) Show that this conclusion does not necessarily hold for more general prime ideals $\mathfrak{p}$. [Hint: Try $R = k[x,y]$ and $\mathfrak{p} = (x) \subset R$. What is $\hat{R}_\mathfrak{p}$? Is y a unit in $\hat{R}_\mathfrak{p}$?]

c) What is the relationship between $R_\mathfrak{p}$ and its completion with respect to the ideal $\mathfrak{p}R_\mathfrak{p}$?

6. a) Use Hensel's Lemma to show that there is a unique formal power series $F(t) = a_1 t + a_2 t^2 + \cdots$ ($a_i \in \mathbb{R}$) such that $3F(t)^2 + F(t)e^t + \sin t = 0$. Find the first few coefficients a_i.

b) Show that the power series $F(t)$ has a positive radius of convergence. (Hint: Use an appropriate version of the Implicit Function Theorem.)