Math 720 Problem Set #1 Due Wed., Oct. 9, 2013

1. Which of the following are Dedekind domains? Which are discrete valuation rings?
$\mathbb{Z}[\sqrt{21}]$; $\mathbb{Z}[x,1/x]$; $\mathbb{R}[x,y]/(y^2-x)$; $\mathbb{R}[x,y]/(y^2-x^3)$; $\mathbb{F}_3[[x,y]]$; $\mathbb{F}_3((x))[[y]]$;
$S^{-1}\mathbb{Z}[\sqrt{2}]$ where $S=\mathbb{Z}-5\mathbb{Z}$; $S^{-1}\mathbb{Z}[\sqrt{2}]$ where $S=\mathbb{Z}-7\mathbb{Z}$;
$S^{-1}\mathbb{C}[x,y]$ where S is the multiplicative set generated by $\mathbb{C}[x]-\{0\}$ and $\mathbb{C}[y]-\{0\}$;
$S^{-1}\mathbb{Z}[x,1/x]$ where S is the multiplicative set generated by $\{x^n-1 \mid n\geq 1\}$.

2. a) Where are the following number fields are ramified over $\mathbb{Q}$? That is, where are their rings of integers ramified over $\mathbb{Z}$? In each case, find the ramification indices, the degrees of the residue field extensions. Also discuss the behavior at infinity. Determine which of the extensions are Galois over $\mathbb{Q}$. For each one that is, give the decomposition groups and inertia groups at the ramified primes. For each one that is not, describe the Galois closure.

$$\mathbb{Q}(\zeta_{10}),\ \mathbb{Q}(\sqrt{21}),\ \mathbb{Q}(\sqrt[3]{3}),\ \mathbb{Q}(\sqrt[3]{10}),\ \mathbb{Q}(i,\sqrt{2}),\ \mathbb{Q}(\zeta_3)[y]/\big(y^3-\frac{2-\zeta_3}{2-\zeta_3^{-1}}\big).$$

b) Do the analogous problem over $k[x]$ and its fraction field $k(x)$, for the following rings and their fraction fields.
$k[x,y]/(y^2-x^3+x)$ where k is a field of characteristic $\neq 2$; $k[x,y]/(y^p-y-x)$ where $k=\mathbb{F}_p$; $k[x,y]/(y^p-x^{p-1}y-x)$ where $k=\mathbb{F}_p$; $k[x,y]/(y^p-x^{p-1}y-t)$ where $k=\mathbb{F}_p(t)$.

3. Show that there are infinitely many primes in $\mathbb{F}_p[x]$; and that the residue fields are precisely the finite fields of characteristic p, with each finite field of characteristic p appearing as a residue field finitely many times. What can you say about the number of times that each of these finite fields appears?

4. Let A be a Dedekind domain, with $\mathfrak{p}$ a non-zero prime ideal and $|\cdot|_{\mathfrak{p}}$ the associated absolute value on the fraction field K of A.

a) Show that for any $r>0$ and any $x,y\in K$, if y lies in the open disc $D:=D_r(x)$ then $D=D_r(y)$. Also prove the analogous assertion for closed discs. Which discs are equal to A? to $\mathfrak{p}^n$ for some $n\geq 1$?

b) Show that under the metric induced by $|\cdot|_{\mathfrak{p}}$, there are no non-constant paths in any open or closed disc of positive radius in K (i.e. continuous maps $[0,1]\to K$).

c) Show that under this metric, K is totally disconnected, and that the complement of any finite set is dense.

d) For each of the following choices of A and $\mathfrak{p}$, is A compact? Is K compact? Is A or K complete?
$A=\mathbb{Z}$, $\mathfrak{p}=(3)$; $A=\mathbb{Z}_{(3)}$, $\mathfrak{p}=3A$; $A=\mathbb{F}_3[x]$, $\mathfrak{p}=x$; $A=\mathbb{F}_3[x]_{(x)}$, $\mathfrak{p}=xA$; $A=\mathbb{F}_3[[x]]$, $\mathfrak{p}=xA$; $A=\mathbb{R}[[x]]$, $\mathfrak{p}=xA$.