Topics in Algebra – Math 702

Problem Set #4 Due Wed., April 15, 2020.

1. Let K and A be as in problem 2 of Problem Set 3, and preserve the notation from that problem.

a) Show that if A is a K-division algebra then A contains a maximal subfield E of degree n over K such that E is a cyclic Galois extension of K, i.e. a Galois extension of K whose Galois group is cyclic. (For this reason, A is referred to as a *cyclic algebra*.) Find the centralizer of E in A.

b) Show that if $b = 1$ then A is isomorphic to a matrix algebra over K. [Hint: Consider the matrices M, N.] What does this say if $n = 2$?

c) Given an example to show that A is not always isomorphic to a matrix algebra.

2. If σ is a permutation of $\{1, 2, 3, 4\}$, consider the map $f_\sigma : \mathbb{H} \to \mathbb{H}$ that takes $a_1 + a_2 i + a_3 j + a_4 k$ to $a_{\sigma(1)} + a_{\sigma(2)} i + a_{\sigma(3)} j + a_{\sigma(4)} k$, where each $a_i \in \mathbb{R}$.

a) For which permutations σ is f_σ an automorphism of $\mathbb{H}$?

b) Concerning each such σ, what assertion does the Skolem-Noether Theorem make?

c) Verify this assertion explicitly by finding an element as asserted in that theorem, for one such choice of σ (other than the identity).

3. Let F be a field of characteristic unequal to 2, let $a, b, c \in F^\times$, and let $q = \langle a, b, c\rangle$. Let $A = C(q)$ be the Clifford algebra of q. Describe A explicitly (in terms of generators and relations), and find its graded parts A_0, A_1. Also verify that A has the properties of being odd: that A is not a central simple algebra, but that A_0 is a central simple algebra.

4. Let Γ be a finite group acting on an abelian group A.

a) With respect to this action, explicitly describe the coboundary maps $d : C^1(\Gamma, A) \to C^2(\Gamma, A)$ and $d : C^0(\Gamma, A) \to C^0(\Gamma, A)$. Also describe the conditions to be a 1-cocycle and a 1-coboundary. Then find $H^1(\Gamma, A)$ explicitly in the special case that the action is trivial.

b) Suppose that $\Gamma = \mathrm{Gal}(L/K)$ for some finite Galois extension L/K, and take $A = L^\times$, under the natural action of Γ. For each 1-cocycle $\sigma \in Z^1(\Gamma, A)$, consider the map $\Phi_\sigma : L \to L$ taking each $a \in L$ to $\Phi_\sigma(a) := \sum_{\gamma\in\Gamma} \sigma(\gamma)\gamma(c)$. Show that the map Φ_σ is not identically zero, and so there exists some $c \in L$ such that $\Phi_\sigma(c) \neq 0$. [Hint: Use the result from Galois theory that says that the elements of $\mathrm{Gal}(L/K)$, when viewed as elements of the L-vector space of maps $L \to L$, form a linearly independent set.]

c) With notation as in part (b), let $b = \Phi_\sigma(c)$. Show that for every $g \in \Gamma$,

$$g(b) = \sum_{\gamma\in\Gamma} g(\sigma(\gamma))\ g\gamma(c) = \sum_{\gamma\in\Gamma} \sigma(g)^{-1}\ \sigma(g\gamma)\ g\gamma(c) = \sigma(g)^{-1}\ b,$$

and deduce that σ is a coboundary.

d) Explain why this shows that $H^1(\Gamma, A) = 0$ if Γ and A are as in part (b).

5. Use the result shown in problem 4(d) to give a proof of the classical form of Hilbert's Theorem 90: that if L/K is a finite Galois extension whose Galois group Γ is a cyclic group with generator g, and if $x \in L^\times$ has norm equal to 1, then there exists $y \in L^\times$ such that $x = g(y)/y$. (Recall that the *norm* of x is defined to be $\prod_{\gamma\in\Gamma} \gamma(x)$.) [Hint: Define $\sigma : \Gamma \to L^\times$ by $\sigma(g^i) = xg(x)g^2(x)\cdots g^{i-1}(x)$ for $0 \le i \le n-1$, where n is the order of Γ. Show that $\sigma \in Z^1(\Gamma, A)$ in the notation of problem 4, and then use the result in 4(d) to obtain the desired element y by considering $\sigma(g)$.] For this reason, the result in 4(d) is also often called "Hilbert's Theorem 90."