Topics in Algebra – Math 702

Problem Set #1 Due Wed., Feb. 26, 2020, in class.

1. Let r be a positive integer and let F be a field (as usual, of characteristic $\neq 2$). Show that the following two conditions are equivalent:
(i) Every regular quadratic form in r variables over F is universal.
(ii) Every regular quadratic form in $r+1$ variables over F is isotropic.

2. Let q be a regular quadratic form on $V = F^n$, corresponding to the invertible $n \times n$ symmetric matrix M. For $A \in \mathrm{M}_n(F)$, define $\tau(A) = M^{-1}A^{\mathrm{t}}M$, where A^{t} is the transpose of A.

a) Show that τ is an *involution* on $\mathrm{M}_n(F)$, in the sense that $\tau(A+B) = \tau(A) + \tau(B)$; τ^2 is the identity; and $\tau(AB) = \tau(B)\tau(A)$.

b) Show that

$$\mathrm{O}(q) = \{A \in \mathrm{GL}_n(F) \mid \tau(A) = A^{-1}\},$$

and deduce that

$$\mathrm{SO}(q) = \{A \in \mathrm{SL}_n(F) \mid \tau(A) = A^{-1}\}.$$

Explain why these equalities are familiar in the case that $q = \sum x_i^2$.

c) Let

$$\mathrm{Skew}(q) = \{A \in \mathrm{M}_n(F) \mid \tau(A) = -A\}.$$

What is $\mathrm{Skew}(q)$ if $q = \sum x_i^2$?

d) View $\mathrm{SO}(q)$ and $\mathrm{Skew}(q)$ as subsets of affine n^2-space $\mathbb{A}^{n^2}$ (where we identify $\mathrm{M}_n(F)$ with F^{n^2}). Show that each can be defined as the locus of common zeroes of a set of polynomials in n^2 variables; i.e., is a subvariety of $\mathbb{A}^{n^2}$. Is either one isomorphic to some affine m-space $\mathbb{A}^m$?

3. Retain the notation of problem 2.

a) Suppose that $A \in \mathrm{GL}_n(F)$ and that -1 is not an eigenvalue of A. Show that $A + I$ is invertible (where I is the identity).

b) If A is as in part (a), let $B = (A+I)^{-1}(A-I)$. Show that 1 is not an eigenvalue of B and that $I - B$ is invertible. Show also that $A = (I+B)(I-B)^{-1}$.

c) In the situation of part (b), show that if $A \in \mathrm{SO}(q)$ then $B \in \mathrm{Skew}(q)$, and conversely.

d) Let

$$\mathrm{SO}(q)^\circ = \{A \in \mathrm{SO}(q) \mid A + I \in \mathrm{GL}_n(F)\}$$

and let

$$\mathrm{Skew}(q)^\circ = \{A \in \mathrm{Skew}(q) \mid I - B \in \mathrm{GL}_n(F)\}.$$

Show that the association $A \mapsto B$ as above defines a bijection $C : \mathrm{O}(q)^\circ \to \mathrm{Skew}(q)^\circ$. Show moreover that the maps C and C^{-1} are defined by systems of polynomials in n^2 variables (i.e. they define an isomorphism of varieties).

e) Show that there is a polynomial in n^2 variables that does not vanish identically on $\mathrm{SO}(q)$ and such that $\mathrm{SO}(q)^\circ$ is the complement in $\mathrm{SO}(q)$ of the zero locus of this polynomial. Do the same for $\mathrm{Skew}(q)^\circ$ in $\mathrm{Skew}(q)$.

f) Deduce that some dense open subset of $\mathrm{SO}(q)$ is isomorphic to a dense open subset of some affine space $\mathbb{A}^m$ (in the Zariski topology). In algebraic geometry, one then says that $\mathrm{SO}(q)$ is a *rational variety*; its field of rational function is then isomorphic to the corresponding field for $\mathbb{A}^m$, i.e. $F(t_1, \ldots, t_m)$.

4. Let $F = \mathbb{C}((t))$.

a) Show that $F^\times/F^{\times 2}$ has exactly two elements, represented by $\{1, t\}$. [Hint: Show that if $f \in \mathbb{C}[[t]]$ has a non-zero constant term, then f is a square.]

b) Show that every binary quadratic form over F is universal. [Hint: Use part (a) to reduce to just a few possibilities.]

c) Deduce that $u(F) = 2$.

d) Describe the structure of $W(F)$, $Q(F)$, $I(F)$, and $I^2(F)$.