1. Let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-Galois branched cover, with branch locus $P_1, \ldots, P_r$, where P_j is at $x = j$. Let P be a base point on the positive imaginary axis. Choose a homotopy basis $\sigma_1, \ldots, \sigma_r$ of counterclockwise loops at P, where σ_j winds once around P_j, and where the loops (and their interiors) are disjoint except at P. Let $(g_1, \ldots, g_r)$ be the description of $Y \to \mathbb{P}^1_{\mathbb{C}}$ with respect to the loops σ_j. Let $\mathcal{H}$ be the associated Hurwitz space, and let $\xi \in \mathcal{H}$ be the point corresponding to $Y \to \mathbb{P}^1_{\mathbb{C}}$. Pick a j with $1 \le j < r$, and let C_j be the circle of radius $\frac{1}{2}$ centered at $x = j + \frac{1}{2}$.

a) Consider a path θ_j in $(\mathbb{P}^1)^r$ beginning at $(P_1, \ldots, P_r)$, in which the j^{th} and $(j+1)^{\text{th}}$ branch points of Y each move counterclockwise along half of C_j, and the other branch points remain fixed. (Thus the final point of θ_j is $(P_1, \ldots, P_{j-1}, P_{j+1}, P_j, P_{j+2}, \ldots, P_r)$.) Lift θ_j to a path Θ_j in $\mathcal{H}$ with initial point ξ, and let $Y_j \to \mathbb{P}^1_{\mathbb{C}}$ be the cover corresponding to the final point of Θ_j. Show that the description of $Y_j \to \mathbb{P}^1_{\mathbb{C}}$, relative to $\sigma_1, \ldots, \sigma_r$ (in that order), is $(g_1, \ldots, g_{j-1}, g_{j+1}, g_{j+1}^{-1} g_j g_{j+1}, g_{j+2}, \ldots, g_r)$. [Hint: Verify all the entries of the description other than $g_{j+1}^{-1} g_j g_{j+1}$, and then show that that one is forced.]

b) Now consider the loop θ'_j in $(\mathbb{P}^1)^r$ at $(P_1, \ldots, P_r)$, obtained by "doing θ_j twice." Lift θ'_j to a path Θ'_j in $\mathcal{H}$ with initial point ξ, and let $Y'_j \to \mathbb{P}^1_{\mathbb{C}}$ be the cover corresponding to the final point of Θ'. Show that the description of $Y'_j \to \mathbb{P}^1_{\mathbb{C}}$, relative to $\sigma_1, \ldots, \sigma_r$, is $(g'_1, \ldots, g'_r)$, where $g'_l = g_l$ for $l \neq j, j+1$, and $g'_l = (g_j g_{j+1})^{-1} g_l (g_j g_{j+1})$ for $l = j, j+1$. [Hint: Iterate part (a).]

c) Show that if the cover $Y \to \mathbb{P}^1_{\mathbb{C}}$ is deformed by allowing the $(j+1)^{\text{th}}$ point to wind once around P_j counterclockwise, then the resulting cover has description $(g'_1, \ldots, g'_r)$ as in (b). Show that the same happens in instead we allow the jth point to wind once around P_{j+1} clockwise. What happens if the j^{th} point winds once around P_{j+1} counterclockwise? [Hint: Use part (b).]

2. In problem 1, let $G = S_3$, $r = 4$, and $(g_1, \ldots, g_4) = ((12), (12), (13), (13))$. Thus $Y \to \mathbb{P}^1_{\mathbb{C}}$ is a slit cover. Let P_0 be the point $(x = 0)$. Consider paths θ in $(\mathbb{P}^1)^4$ with initial point (P_1, P_2, P_3, P_4) and final point (P_1, P_0, P_3, P_0), such that $\theta(t) \in (\mathbb{P}^1)^4 - \Delta$ for $0 \le t < 1$, and such that the first and third entries of $\theta(t)$ remain constant as t varies.

a) Show that each such path θ lifts to a unique path in the compactified Hurwitz space $\overline{\mathcal{H}}$ beginning at the point $\xi \in \mathcal{H}$ that corresponds to $Y \to \mathbb{P}^1$. Explain why the final point of the lifted path determines an unramified cover $Y_\theta \to \mathbb{P}^1 - \{P_0, P_1, P_3\}$.

b) Find a choice of θ such that Y_θ is connected, and find another choice of θ such that Y_θ is *not* connected. [Hint: Choose a homotopy basis for $\mathbb{P}^1 - \{P_0, P_1, P_3\}$, and use problem 1 to determine the description of a Y_θ. From the description, how can you tell if a cover is connected?]

3. Let $Y \to \mathbb{P}^1$ be as in problem 2, and let $\mathcal{P}$ be the space that parametrizes the covers obtained from $Y \to \mathbb{P}^1$ by allowing the third branch point to wander in $\mathbb{P}^1 - \{P_1, P_2, P_4\}$, while holding the other three branch points fixed. Let $\xi \in \mathcal{P}$ be the point corresponding to the cover $Y \to \mathbb{P}^1$.

a) Describe an unramified covering map $\pi : \mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$, and find the image of ξ.

b) Show that π is not an isomorphism. [Hint: Use problem 1 to show that $\deg(\pi) > 1$.]

c) Let $\overline{\pi} : \overline{\mathcal{P}} \to \mathbb{P}^1$ be the branched cover obtained by compactifying $\pi : \mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$. Show that there are points $\eta, \eta' \in \overline{\mathcal{P}}$ lying over $P_4 \in \mathbb{P}^1$ such that $\overline{\pi}$ is unramified at the point η but is ramified at η'. [Hint: Show that a loop τ at P_3 around P_4 takes Y to itself, but that τ takes some other cover in the family (also with branch locus P_1, P_2, P_3, P_4) to a cover other than itself.]

d) Deduce that $\pi : \mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$ is not Galois. [Hint: Use part (c).]

e) Conclude that the Hurwitz space associated to an S_3-Galois mock cover with generators $\{(12), (13)\}$ is not Galois over $(\mathbb{P}^1)^4 - \Delta$. [Hint: Relate the Hurwitz space $\mathcal{H} \to (\mathbb{P}^1)^4 - \Delta$ to the cover $\mathcal{P} \to \mathbb{P}^1 - \{P_1, P_2, P_4\}$.]

4. Let k be a field of characteristic $\neq 2$. Let $X = \mathbb{P}^1_{k[[t]]}$ (the projective x-line over $k[[t]]$), let $X_1 = \operatorname{Spec} k[x][[t]]$, let $X_2 = \operatorname{Spec} k[\bar{x}][[t]]$ (where $x\bar{x} = 1$), and let $X_0 = \operatorname{Spec} k[x, x^{-1}][[t]]$.

a) Prove that there is a 2-cyclic Galois branched cover of X whose pullback to X_1 is given by $y^2 = x(x - t)$; whose pullback to X_2 is given by $z^2 = \bar{x}(\bar{x} - t)$; and whose pullback to X_0 is trivial (i.e. a disjoint union of two copies of X_0).

b) Find such a cover explicitly.

5. For any ring A with an absolute value $|\cdot|$, let $A\{x\} \subset A[[x]]$ denote the ring of power series that converge on $|x| \leq 1$, and let $A\langle x\rangle \subset A[[x]]$ denote the ring of power series that converge on $|x| < 1$. Let k be a field, let $R = k[[t]]$, and let $K = k((t))$.

a) Show that $R\{x\} = k[x][[t]]$ and $K\{x\} = k[x][[t]][t^{-1}]$.

b) Show that $R\langle x\rangle = k[[x, t]]$ and $K\langle x\rangle = k[[x, t]][t^{-1}]$.