1. (a) Let b be an odd positive integer, and let $a = \pm 2b$. Show that there is a positive integer m such that the Jacobi symbol $(\frac{a}{m}) = -1$. [Hint: There is an m such that $m \equiv 5 \pmod 8$ and $m \equiv 1 \pmod b$.]

(b) Show that the same conclusion holds if instead $a = \pm qb$, where b is as in (a) and q is an odd prime not dividing b. [Hint: There is an n prime to q such that $(\frac{n}{q}) = -1$, and there is an m such that $m \equiv 1 \pmod{4b}$, $m \equiv n \pmod q$.]

(c) Conclude that if $a \in \mathbb{Z}$ is a square modulo p for every prime p, then a is a square in $\mathbb{Z}$.

2. Verify the assertion of Artin Reciprocity for the extension $\mathbb{Q} \subset K = \mathbb{Q}[\sqrt{n}]$, where $n \in \mathbb{Z}$ is not a square. That is, show that there is a conductor c supported only at the ramified and infinite primes such that the Artin map $\sigma : I^c \to \mathrm{Gal}(K/\mathbb{Q})$ is a surjective homomorphism with kernel equal to $P_c\mathcal{N}(c)$. [Hint: Use Quadratic Reciprocity, problem 1 above, and Dirichlet's theorem that every arithmetic progression $a, a+d, a+2d, \ldots$ with $(a,d) = 1$ contains (infinitely many) prime numbers.]

3. (a) Let $X = \mathbb{P}^1$, let $S = \{0, \infty\}$, and let $f : X - S \to \mathbb{G}_m$ be the identity map. Show explicitly from the definition that $(0) + (\infty)$ is a modulus for f.

(b) Let $X = \mathbb{P}^1$, let $S = \{\infty\}$, and let $f : X - S \to \mathbb{G}_a$ be the identity map. Show explicitly from the definition that $2 \cdot (\infty)$ is a modulus for f. [Hint: Make a change of variables to analyze the behavior of a rational function g at infinity.]

4. Let $X = \mathbb{P}^1$, let $\mathfrak{m}$ be an effective divisor on X, and let $X_\mathfrak{m}$ be the associated singular curve.

(a) Show that if $\mathfrak{m} = (0) + (\infty)$ then $X_\mathfrak{m}$ is isomorphic to the projective plane curve whose affine equation is $y^2 = x^3 + x^2$.

(b) Show that if $\mathfrak{m} = 2 \cdot (\infty)$ then $X_\mathfrak{m}$ is isomorphic to the projective plane curve whose affine equation is $y^2 = x^3$.

[Hint: Show the isomorphism away from the singular point of $X_\mathfrak{m}$ and then at the local ring at the singular point.]

5. Let p, q be prime numbers (possibly equal), let G be a finite p-group, and let $\mathbb{Q} \subset K$ be a G-Galois extension ramified only at q.

(a) Show that $\mathbb{Q} \subset K$ is totally ramified over q. [Hint: Let I be an inertia group. Use that every proper subgroup of a p-group is contained in a proper normal subgroup, and that $\mathbb{Q}$ has no non-trivial unramified extensions.]

(b) Show that K has no non-trivial abelian unramified extensions L of p-power degree over K. [Hint: Show that the maximal such extension is Galois over $\mathbb{Q}$, and then apply part (a) to this field.]

(c) Conclude that the class number of K (i.e. the order of the class group of K) is prime to p. [Hint: Use another characterization of the class group, and apply (b).]

6. Let p be an odd prime number, let G be a finite p-group, and let $\mathbb{Q} \subset K$ be a G-Galois extension ramified only at p.

(a) Show that if G is abelian then G is cyclic of p-power order. [Hint: What particular abelian extension of $\mathbb{Q}$ must K lie in? What is the Galois group of this extension of $\mathbb{Q}$?]

(b) Show that G is cyclic of p-power order even if we *don't* assume that G is abelian. [Hint: Apply part (a) to $G^{\mathrm{ab}} = G/[G,G]$ and then apply the Burnside Basis Theorem.]

(c) Conclude that the maximal pro-p-group quotient of $\pi_1(\operatorname{Spec} \mathbb{Z}[1/p])$ is $\mathbb{Z}_p$.