1. Let k be a field with algebraic closure $\overline{k}$. Let $Y \to \mathbb{P}^1_k$ be an irreducible cover of curves, corresponding to a field extension $k(x) \subset L$. Show that the following conditions are equivalent:

(i) The cover is *regular*, i.e. k is algebraically closed in L.

(ii) The cover is *geometrically irreducible*, i.e. the induced cover $Y_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ is irreducible.

2. (a) Let n be a positive integer, and let k be an algebraically closed field of characteristic 0 (e.g. $\mathbb{C}$ or $\bar{\mathbb{Q}}$). Find a Galois branched cover $Y \to \mathbb{P}^1_k$ with Galois group $\mathbb{Z}/n$ and with branch locus $\{0, \infty\}$. What is the group action? What are the inertia groups at the ramified points?

(b) In the case that $k = \bar{\mathbb{Q}}$, show that the cover $Y \to \mathbb{P}^1_k$ in part (a) is induced by a Galois branched cover of $\mathbb{P}^1_{\mathbb{Q}}$ with branch locus $\{0, \infty\}$ and Galois group $\mathbb{Z}/n$, if and only if $n = 2$. [Hint: If $\kappa =$ complex conjugation, then the generator of the Galois group of $Y^\kappa \to \mathbb{P}^1_k$ acts like the inverse of the generator of the Galois group of $Y \to \mathbb{P}^1_k$.]

(c) For any $n > 2$, find a Galois branched cover of $\mathbb{P}^1_{\mathbb{Q}}$ with group $\mathbb{Z}/n\mathbb{Z}$. What kind of branch locus are you forced to use?

3. Let $Y \to \mathbb{P}^1_{\mathbb{C}}$ be a G-Galois branched cover, with branch locus $\{1, 2, 3\}$. Let $x = 0$ be chosen as the base point of $\mathbb{P}^1_{\mathbb{C}}$, and choose a base point $\xi \in Y$ over $x = 0$. Choose disjoint counterclockwise loops $\sigma_1, \sigma_2, \sigma_3$ in $\mathbb{P}^1_{\mathbb{C}} - \{1, 2, 3\}$ at the base point $x = 0$, such that σ_j is a simple closed curve that winds once around $x = j$, and such that the region bounded by σ_j contains the semicircle in the upper half-plane connecting $x = 0$ to $x = j$. (Thus $\sigma_1\sigma_2\sigma_3$ is homotopic to the identity.) Let (c_1, c_2, c_3) be the branch cycle description of $Y \to \mathbb{P}^1_{\mathbb{C}}$ relative to these loops and the base point $\xi \in Y$. Let κ denote complex conjugation (on either $\mathbb{C}$ or $\bar{\mathbb{Q}}$), and let $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$ denote the induced G-Galois cover, with base point $\xi^\kappa \in Y^\kappa$ over $x = 0$.

(a) Determine the branch cycle description of $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$ as follows:

(i) Show that the description of $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$ with respect to the *clockwise* loops $\sigma_1^\kappa, \sigma_2^\kappa, \sigma_3^\kappa$ at ξ^κ is (c_1, c_2, c_3).

(ii) Show that σ_1 is homotopic to $(\sigma_1^\kappa)^{-1}$ in $\mathbb{P}^1_{\mathbb{C}} - \{1, 2, 3\}$. Similarly show that σ_3 is homotopic to $(\sigma_3^\kappa)^{-1}$. [Hint: You're working in $\mathbb{P}^1_{\mathbb{C}} - \{1, 2, 3\}$, not $\mathbb{A}^1_{\mathbb{C}} - \{1, 2, 3\}$.]

(iii) Deduce that the description of $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$ with respect to the loops $\sigma_1, \sigma_2, \sigma_3$ and the base point ξ^κ is $(c_1^{-1}, c_1c_3, c_3^{-1})$. [Hint: The product of the three entries is $1 \in G$.]

(b) Explain why a similar approach cannot be used to compute the branch cycle description of $Y^\omega \to \mathbb{P}^1_{\mathbb{C}}$, for elements $\omega \in G_{\mathbb{Q}}$ other than 1 and κ.

4. Let $H = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. Let $\pi : Y \to \mathbb{P}^1_{\mathbb{C}}$ be the H-Galois cover branched at $\{1, 2, 3\}$ with description $(i, j, -k)$ relative to a base point $\xi \in Y$ over $x = 0$ and relative to the loops $\sigma_1, \sigma_2, \sigma_3$ of problem 3 above. Let κ denote complex conjugation, and for each $\omega \in G_{\mathbb{Q}}$ let $Y^\omega \to \mathbb{P}^1_{\mathbb{C}}$ denote the cover induced by ω (where we may pass back and forth between covers of $\mathbb{P}^1_{\mathbb{C}}$ and covers of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$ branched at $\{1, 2, 3\}$).

(a) Find the description of $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$ relative to $\sigma_1, \sigma_2, \sigma_3$ and ξ^κ, and show that $Y \to \mathbb{P}^1_{\mathbb{C}}$ is isomorphic to $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$ as an H-Galois cover. [Hint: Problem 3.]

(b) Show that the isomorphism in (a) cannot take ξ to ξ^κ. [Hint: How does the description change, as the base point of the cover is allowed to vary in its fibre?]

(c) Show that the triple $(i, j, -k)$ is $\mathbb{Q}$-rational and rigid, and use this to find the field of moduli of $Y \to \mathbb{P}^1$. [Hint: Work explicitly.] How does this generalize part (a)?

(d) Suppose that $\pi : Y \to \mathbb{P}^1_{\mathbb{C}}$ is induced (as an H-Galois cover) by some $\pi_{\mathbb{R}} : Y_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$. Show the following:

(i) κ acts as an involution on the points of Y, and satisfies $\kappa \circ \pi = \pi \circ \kappa$ and $\kappa \circ g = g \circ \kappa$ for all $g \in H$. [Hint: By the above hypothesis, the equations of Y, π, and the g-actions all can be written with real coefficients.]

(ii) There is a unique element $h \in H$ such that $\kappa(\xi) = h(\xi)$. Moreover $h = \pm 1 \in H$. [Hint: Use (i) to show that $h^2(\xi) = \xi$.]

(iii) The description of $Y^\kappa \to \mathbb{P}^1_{\mathbb{C}}$, relative to $\sigma_1, \sigma_2, \sigma_3$ and to $h(\xi) = \xi^\kappa$, is $(i, j, -k)$. [Hint: Under the hypothesis of (d), we may identify Y with Y^κ, and so we're just changing base points from ξ to $h(\xi)$. Now use that (ii) to show that h is in the center of H.]

(iv) Combine (iii) with (a) above to derive a contradiction.

(e) Explain why the field of moduli of the H-Galois cover $\pi : Y \to \mathbb{P}^1$ is not a field of definition. (Hint: Parts (c) and (d). In particular, is $\mathbb{R}$ a field of definition?)

5. Let $G = A_5$, and let $c_0 = (1, 5, 2)$, $c_1 = (3, 5, 4)$, $c_2 = (1, 2, 3, 4, 5)$.

(a) Show that if $n \equiv \pm 1 \pmod 3$ then $c_i^n \sim c_i$ for $i = 0, 1$.

(b) Show that if $n \equiv \pm 1 \pmod 5$ then $c_2^n \sim c_2$ in G, but not if $n \equiv \pm 2 \pmod 5$.

(c) Deduce that (c_0, c_1, c_2) is rational over the field $K = \mathbb{Q}(\zeta_5 + \zeta_5^{-1})$, and that $K = \mathbb{Q}(\sqrt{5})$.

(d) Show that (c_0, c_1, c_2) is rigid. (Hint: Use the representation theory formula for $\#(\overline{\Sigma})$.)

(e) Conclude that the G-Galois cover of $\mathbb{P}^1 - \{0, 1, \infty\}$ with description (c_0, c_1, c_2) is defined over $\mathbb{Q}(\sqrt{5})$, but not over $\mathbb{Q}$. Conclude also that A_5 is a Galois group over $\mathbb{Q}(\sqrt{5})$.